

AC RESISTANCE OF RECTANGULAR COIL

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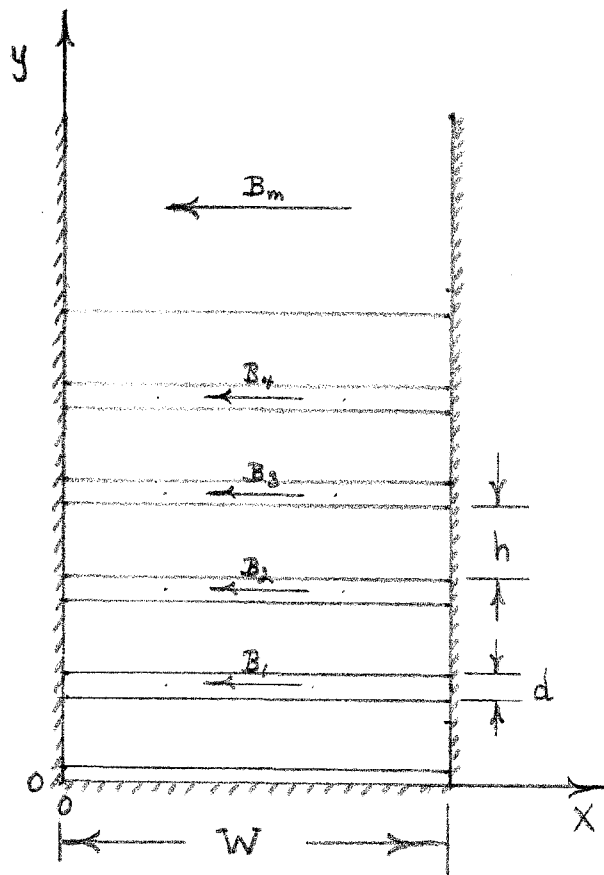
PURPOSE: To determine the AC resistance of rectangular conducting strip wound to form an excitation coil in a deep slot.

REFERENCES:

1. W. R. Smythe, Static and Dynamic Electricity, Chapter XI, McGraw-Hill Book Co., Inc., New York (1939).
2. S. C. Snowdon, Properties of the \mathcal{L} Operator, MURA Technical Note, TN-506 (1964).
3. T. L. Collins, AC Resistance of Coils, NAL Draft, July 1967.

FORMULATION OF PROBLEM

The following figure shows the quantities of interest in the problem.



Within a conductor of thickness h one has

$$\nabla \times \vec{B} = 4 \pi \mu \vec{J}, \quad \nabla \times \vec{E} = - \frac{\partial B}{\partial t}, \quad \vec{J} = \sigma \vec{E} \quad (\text{emu}) \quad (1)$$

For uniform σ , unity permeability and harmonic time variation ($e^{j\omega t}$) one has

$$\nabla \times \nabla \times \vec{B} = - 4 \pi j \sigma \omega \vec{B}. \quad (2)$$

Let

$$\vec{B} = \vec{\mathcal{L}} U, \quad \vec{\mathcal{L}} = k \times \nabla. \quad (3)$$

Then from Ref. 2

$$\vec{\mathcal{L}}(\nabla^2 U - 4 \pi j \sigma \omega U) = 0 \quad (4)$$

or

$$\nabla^2 U = 4 \pi j \sigma \omega U. \quad (5)$$

For the operator chosen, since B does not depend on x , U is a function of y apart from the $e^{i\omega t}$ variation. Letting

$$\lambda^2 = 4 \pi j \sigma \omega \quad (6)$$

one has

$$U = A \text{Ch } \lambda y + B \text{Sh } \lambda y \quad (7)$$

or

$$B_x = - \lambda (A \text{Sh } \lambda y + B \text{Ch } \lambda y).$$

The field is uniform in the gaps between conducting strips and above the coil. If there are m strips and a total current flowing of NI , then the increment in field from gap to gap between strips is

$$B_m - B_{m-1} = B_{m-1} - B_{m-2} = \cdots B_2 - B_1 = B_1 - 0 = \frac{4 \pi NI}{m W} \quad (8)$$

Hence

$$B_n = \frac{4 \pi NI}{m W} n, \quad 0 < n \leq m. \quad (9)$$

Thus within the n^{th} conductor

$$U_n(y) = \frac{4 \pi N I}{m W \lambda} \left\{ \frac{n - (n-1) \operatorname{Ch} \lambda h}{\operatorname{Sh} \lambda h} \operatorname{Ch} \lambda \left[y - \frac{2 n-1}{2} (h + d) + \frac{h}{2} \right] \right. \\ \left. + (n - 1) \operatorname{Sh} \lambda \left[y - \frac{2 n-1}{2} (h + d) + \frac{h}{2} \right] \right\} \quad (10)$$

for $\frac{2 n-1}{2} (h + d) - \frac{h}{2} < y < \frac{2 n-1}{2} (h + d) + \frac{h}{2} \quad n = 1, 2, 3, \dots m .$

AVERAGE POWER (AC RESISTANCE)

For direct current excitation the power consumed per unit length of each conducting strip is

$$P_o = \frac{N^2 I^2}{\sigma m^2 W h} . \quad (11)$$

In the alternating current case, the average power in the n^{th} strip is

$$P_n = \frac{1}{2 \sigma} \int \vec{J} \cdot \vec{J}^* dV = \frac{1}{32 \pi^2 \sigma} \int \nabla^2 U \nabla^2 U^* dS$$

or, from Eq. (5)

$$P_n = \frac{\lambda^2 \pi^2 \sigma}{32} \int_{\frac{2 n-1}{2} (h+d) - \frac{h}{2}}^{\frac{2 n+1}{2} (h+d) + \frac{h}{2}} U_n(y) U_n^*(y) dy . \quad (12)$$

Thus

$$\frac{P_n}{P_o} = 2 \pi \sigma \omega h \int_0^h [A_n \operatorname{Ch} \lambda u + D_n \operatorname{Sh} \lambda u] [A_n^* \operatorname{Ch} \lambda^* u + D_n^* \operatorname{Sh} \lambda^* u] du , \quad (13)$$

where

$$u = -\frac{2 n+1}{2} (h + d) + \frac{h}{2} + y \quad (14)$$

$$A_n = \frac{n - (n-1) \operatorname{Ch} \lambda h}{\operatorname{Sh} \lambda h} \quad (15)$$

$$D_n = n - 1 . \quad (16)$$

Equations (15) and (16) give

$$A_n A_n^* = \frac{n^2 - 2n(n-1) \text{Ch} \left[\frac{\lambda + \lambda^*}{2} h \right] \text{Ch} \left[\frac{\lambda - \lambda^*}{2} h \right] + \frac{1}{2} (n-1)^2 \{ \text{Ch} [(\lambda + \lambda^*) h] + \text{Ch} [(\lambda - \lambda^*) h] \}}{\frac{1}{2} \{ \text{Ch} [(\lambda + \lambda^*) h] - \text{Ch} [(\lambda - \lambda^*) h] \}} \quad (17)$$

$$A_n D_n^* + A_n^* D_n = \frac{2n(n-1) \text{Sh} \left[\frac{\lambda + \lambda^*}{2} h \right] \text{Ch} \left[\frac{\lambda - \lambda^*}{2} h \right] - (n-1)^2 \text{Sh} [(\lambda + \lambda^*) h]}{\frac{1}{2} \{ \text{Ch} [(\lambda + \lambda^*) h] - \text{Ch} [(\lambda - \lambda^*) h] \}} \quad (18)$$

$$(A_n D_n^* - A_n^* D_n) = \frac{-2n(n-1) \text{Sh} \left[\frac{\lambda - \lambda^*}{2} h \right] \text{Ch} \left[\frac{\lambda + \lambda^*}{2} h \right] + (n-1)^2 \text{Sh} [(\lambda - \lambda^*) h]}{\frac{1}{2} \{ \text{Ch} [(\lambda + \lambda^*) h] - \text{Ch} [(\lambda - \lambda^*) h] \}} \quad (19)$$

$$D_n D_n^* = (n-1)^2. \quad (20)$$

Using

$$\lambda = (1 + j) p, \quad p = \sqrt{2 \pi \sigma \omega}, \quad (21)$$

Eqs. (17-20) become

$$A_n A_n^* = \frac{2n^2 - 4n(n-1) \text{Ch } ph \text{ Cos } ph + (n-1)^2 [\text{Ch } 2 ph + \text{Cos } 2 ph]}{\text{Ch } 2 ph - \text{Cos } 2 ph}, \quad (22)$$

$$A_n D_n^* + A_n^* D_n = \frac{4n(n-1) \text{Sh } ph \text{ Cos } ph - 2(n-1)^2 \text{Sh } 2 ph}{\text{Ch } 2 ph - \text{Cos } 2 ph}, \quad (23)$$

$$j(A_n D_n^* - A_n^* D_n) = \frac{4n(n-1) \text{Ch } ph \text{ Sin } ph - 2(n-1)^2 \text{Sin } 2 ph}{\text{Ch } 2 ph - \text{Cos } 2 ph} \quad (24)$$

$$D_n D_n^* = (n-1)^2. \quad (25)$$

The resistance ratio (AC/DC) may be found from the power ratio.

Thus Eq. (13) gives after integrating

$$\begin{aligned} \frac{R_n}{R_o} = \frac{ph}{2} \{ & A_n A_n^* (\text{Sh } 2 ph + \text{Sin } 2 ph) + (A_n D_n^* + A_n^* D_n) (\text{Ch } 2 ph - 1) \\ & + j (A_n D_n^* - A_n^* D_n) (\text{Cos } 2 ph - 1) + D_n D_n^* (\text{Sh } 2 ph - \text{Sin } 2 ph) \} \end{aligned} \quad (26)$$

The average resistance ratio may be found by summing

$$\langle r \rangle = \frac{1}{m} \sum_{n=1}^m \frac{R_n}{R_o} \quad (27)$$

In this case, the average coefficients may be found. Thus

$$\frac{1}{m} \sum A_n A_n^* = \frac{\frac{1}{3}(m+1)(2m+1) - \frac{4}{3}(m^2-1) \text{Ch } ph \text{ Cos } ph + \frac{1}{6}(m-1)(2m-1)(\text{Ch } 2 \text{ ph} + \text{Cos } 2ph)}{\text{Ch } 2 \text{ ph} - \text{Cos } 2 \text{ ph}} \quad (28)$$

$$\frac{1}{m} \sum (A_n D_n^* + A_n^* D_n) = \frac{\frac{4}{3}(m^2-1) \text{Sh } ph \text{ Cos } ph - \frac{1}{3}(m-1)(2m-1) \text{Sh } 2 \text{ ph}}{\text{Ch } 2 \text{ ph} - \text{Cos } 2 \text{ ph}} \quad (29)$$

$$\frac{j}{m} \sum (A_n D_n^* - A_n^* D_n) = \frac{\frac{4}{3}(m^2-1) \text{Sin } ph \text{ Ch } ph - \frac{1}{3}(m-1)(2m-1) \text{Sin } 2 \text{ ph}}{\text{Ch } 2 \text{ ph} - \text{Cos } 2 \text{ ph}} \quad (30)$$

$$\frac{1}{m} \sum D_n D_n^* = \frac{1}{6} (m-1)(2m-1) \quad (31)$$

CALCULATIONS

The ratio of AC resistance to DC resistance for each layer as given by Eq. (26) has been computed at 15 Hz for several thicknesses ranging from 0.125 inch to 1.000 inch. In addition, for each layer the average ratio as given by Eq. (27) for all layers up to and including the layer in question is recorded. These results are shown in Table I.

TABLE I. AC RESISTANCE RATIOS

Frequency			15 Hz		
Resistivity			1.86 $\mu\Omega$ - cm		
Thickness (in) 0.125			0.250		
Layer	Ratio	Average Ratio	Layer	Ratio	Average Ratio
1	1.00	1.00	1	1.00	1.00
4	1.00	1.00	2	1.01	1.01
8	1.02	1.01	4	1.07	1.03
12	1.05	1.02	6	1.17	1.07
16	1.08	1.03	8	1.31	1.12
20	1.13	1.05	10	1.50	1.18
24	1.19	1.07	12	1.73	1.26
Thickness (in) 0.375			0.500		
1	1.01	1.01	1	1.02	1.02
2	1.06	1.04	2	1.20	1.11
3	1.17	1.08	3	1.55	1.26
4	1.34	1.15	4	2.07	1.46
5	1.56	1.23	5	2.76	1.72
6	1.84	1.33	6	3.63	2.04
7	2.17	1.45			
8	2.56	1.59			

TABLE I. AC RESISTANCE RATIOS (continued)

Frequency			15 Hz		
Resistivity			1.86 $\mu\Omega \cdot \text{cm}$		
Thickness (in) 0.625			0.750		
Layer	Ratio	Average Ratio	Layer	Ratio	Average Ratio
1	1.06	1.06	1	1.11	1.11
2	1.47	1.27	2	1.96	1.54
3	2.31	1.61	3	3.65	2.24
4	3.57	2.10	4	6.18	3.22
5	5.24	2.73			
Thickness (in) 1.000					
1	1.32	1.32			
2	3.73	2.53			
3	8.53	4.53			