Scalar-flat Kähler metrics with SU(2) symmetry

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Abstract.

We study Kähler metrics of zero scalar curvature in four real dimensions admitting an isometric action of SU(2).

0. Introduction.

A procedure for studying SU(2)-invariant anti-self-dual conformal structures on four-manifolds has recently been developed by Hitchin [H2]. The idea is to use the twistor correspondence of Penrose to associate to such a four-manifold a complex three-fold (the twistor space) with a holomorphic SU(2) action, and to show that the twistor lines of this space are determined by an isomonodromic family of connections on \( \mathbb{P}^1 \). In the generic case considered by Hitchin this family in turn is determined by a solution to the sixth Painlevé equation. This is one of a family of second order ODEs collectively known as the Painlevé equations, described for example in [AC]. The upshot is that the conformal structure on the four-manifold is specified by a solution of Painlevé VI.

On another front, Pedersen and Poon [PP] and Tod [T] have produced ansätze which give SU(2)-invariant scalar-flat Kähler metrics in four (real) dimensions. The ODEs arising from these ansätze are equivalent to a special case of the third Painlevé equation. It is known [G] that Kähler metrics of zero scalar curvature (in four real dimensions) are automatically anti-self-dual.

The purpose of this paper is to apply the twistor methods of Hitchin to attack the general problem of finding scalar-flat Kähler four-manifolds with SU(2) symmetry, and to interpret the results of [PP] and [T] in this framework. In the special case of diagonal Bianchi IX metrics we are also able to tackle this problem by direct methods.

We shall primarily work locally, although in the diagonal case we determine which of our metrics can be completed (even these complete examples are not compact). We assume throughout that the generic orbit of the SU(2) action is three-dimensional.

Our techniques are not well suited to the case when the metrics are Ricci-flat. For Kähler metrics in four real dimensions this condition is equivalent to the metric being locally hyperkähler. However there is no loss in excluding this case because SU(2)-invariant hyperkähler four-metrics have been completely classified [AH], [GP].

The layout of the paper is as follows. Section 1 describes the Penrose twistor construction and its application by Hitchin to SU(2)-invariant conformal structures. In section 2 we explain how the extra data of a Kähler structure on the four-manifold affects the twistor space, and in section 3 we show how this leads to an isomonodromy problem involving Painlevé III. In section 4 we consider the special case of diagonal Bianchi IX metrics, and in 5 we compare these results with those obtained by direct calculations. Section 6 contains an analysis of which of the diagonal metrics are complete.

1. Twistor spaces.

Let \( M \) be an oriented Riemannian four-manifold with metric \( g \) and let \( \Lambda_2^g \) denote the bundle of selfdual 2-forms. We define \( Z \) to be the sphere bundle of \( \Lambda_2^g \) and let \( \pi \) denote the projection from \( Z \) to \( M \). We can identify the fibre of \( \pi \) over a point \( m \) of \( M \) with the set of complex structures on \( T_mM \) compatible with the metric \( g \) and
with the orientation.

Using the Levi-Civita connection we can split the tangent space to $Z$ at a point $z$ into horizontal and vertical spaces $H$ and $V$, where $H$ can be identified with the tangent space to $M$ at $\pi(z)$ and $V$ is tangential to the two-sphere fibre of $\pi$ over $\pi(z)$. Now $z$ represents a complex structure on $T\pi(M)$ so induces a complex structure $J_z$ on $H$. Also we have the standard complex structure on $S^2$ which defines a complex structure $J_0$ on $V$. We can now define an almost complex structure $J$ on $Z$ by setting $J = J_0 \oplus J_z$.

The fundamental theorem of Penrose [Pe], [AHS] asserts that this almost complex structure is integrable if and only if the metric on $M$ is antisinelfdual. In this situation, $Z$ is called the twistor space of $M$.

There is a complex four-parameter family of projective lines in $Z$, each with normal bundle isomorphic to $O(1) \oplus O(1)$; these lines are called the twistor lines. Moreover $Z$ carries a free antiholomorphic involution $\tau$, called the real structure, and there is a real four-parameter family of twistor lines, called the real twistor lines which are preserved by $\tau$. In fact the restriction of $\tau$ to each real twistor line is the antipodal map. The real twistor lines are precisely the fibres of the projection $\pi$ from $Z$ to $M$.

The above construction depends only on the conformal class of the metric $g$ on $M$. Moreover, the complex manifold $Z$, its twistor lines, and the real structure together determine this conformal class.

Hitchin [H2] now assumes the existence of an isometric $SU(2)$ action on $M$. This lifts to an $SU(2)$ action on $Z$ with associated vector fields $X_1, X_2, X_3$. Using the complex structure of $Z$ we obtain holomorphic vector fields $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ satisfying $[\tilde{X}_1, \tilde{X}_2] = -\tilde{X}_3$ etc. These vector fields generate a local $SL(2, \mathbb{C})$ action on the twistor space. We have a holomorphic section $s = X_1 \wedge X_2 \wedge X_3$ of $K_Z^2$, the anti-canonical bundle of $Z$. Assuming that $s$ is not identically zero, it will vanish precisely on some anticanonical divisor $E$ on $Z$. Now we have the exact sequence

$$0 \to T\mathbb{P}^1 \to TZ |_E \to N \to 0$$

where $N$ is the normal bundle of a twistor line $\mathbb{P}^1$ and $T\mathbb{P}^1 \cong O(2)$. Taking determinants, and using the description of the normal bundle given above, we see that $K_Z^2 |_E \cong O(4) \otimes O(4)$ so each twistor line is either included in $E$ or meets $E$ at four points, counted with multiplicity.

On $Z - E$ the vector fields $\tilde{X}_i$ are linearly independent so we can define a one-form $\phi$ on this open set by

$$\phi = \tilde{\omega}_1 + \tilde{\omega}_2 + \tilde{\omega}_3$$

where $\tilde{\omega}_i$ are one-forms dual to $\tilde{X}_i$ and $\tilde{\omega}_i$ are the elements of the Lie algebra $sl(2, \mathbb{C})$ corresponding to $\tilde{X}_i$. If we use the local $SL(2, \mathbb{C})$ action to identify a neighbourhood of a point in $Z$ with an open set in $SL(2, \mathbb{C})$ then on this neighbourhood $\phi$ is just the Maurer-Cartan form, so is flat.

We thus obtain a flat meromorphic connection (which we also denote by $\phi$) on $Z$, holomorphic on $Z - E$ and with poles on $E$. Restricting to twistor lines gives us a family of flat connections $A(t)$, each meromorphic on $\mathbb{P}^1$ with poles at four points (with multiplicity). In the case considered by Hitchin, corresponding to a generic anti-self-dual conformal structure, we do in fact have four simple poles for generic $t$. Moreover, because each such connection is a restriction of our flat connection $\phi$ on $Z$, we see that the holonomy of the connections around the poles stays constant as we vary the connection within the family.

We can view the family of connections as defining a family of ordinary differential equations with four regular singularities on $\mathbb{P}^1$. The above statement about the holonomy remaining constant becomes, in this interpretation, the statement that this family of differential equations is isomonodromic. This condition was analysed by Painlevé and his school [AC] and shown to be equivalent to the condition that a certain function defined in terms of the coefficients of the connection form $A(t)$ should satisfy the sixth Painlevé equation. Here $t$ is the dependent variable of this differential equation.

Conversely, given a solution of a Painlevé VI equation we can recover the isomonodromic family of connections. This family in turn determines the restriction of the
twistor lines to the open set $Z - E$ of the twistor space $Z$, and this is enough to
determine the conformal structure of $M$.

To summarise, the conformal structure on $M$ is determined by the solution of
Painlevé VI.

2. The anticanonical divisor.

Let us now consider the case when $M$ admits a scalar-flat metric $g$ and a complex
structure $I$ with respect to which $g$ is Kähler. As noted earlier, $g$ must now be anti-
selfdual so we can associate to $M$ a twistor space. The twistor spaces of scalar-flat
Kähler surfaces have been studied by Pontecorvo [Po].

Now the complex structure $I$ on $M$ gives a global section of the twistor fibration,
and the image of this section is a divisor $D$ on $Z$ intersecting each real twistor line
once. Similarly the reverse complex structure $-I$ gives another divisor $\tilde{D}$ intersecting
each real twistor line at one point, and the two divisors are interchanged by the real
structure $T$ of $Z$.

In the case when $g$ is Kähler with respect to $I$ Pontecorvo shows
that the divisor class of $D + \tilde{D}$ is given by

$$[D + \tilde{D}] = K_Z^{-1/2}$$

We now relate the divisor $E$ to Pontecorvo's divisor. As mentioned in the intro-
duction, we are assuming throughout this paper that the generic $SU(2)$ orbif on the
eight-manifold is three-dimensional. We are interested principally in the local form of
the metric, so in what follows we assume, by restricting to an open set if necessary,
that the orbits of $SU(2)$ on $M$ are all three-dimensional.

**Lemma 2.1**

Let $M$ be a four-manifold with an $SU(2)$-invariant scalar-flat Kähler metric $g$,
not locally hyperkähler, where $SU(2)$ acts with three-dimensional orbits. Then the
section $s$ of $K_Z^{-1}$ is not identically zero.

**Proof**

The vector fields $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ on $Z$ span an integrable distribution. If $s$ is identi-
cally zero then these vector fields are everywhere linearly dependent so the rank of
the distribution is of complex dimension two or less. As we have assumed that the
$SU(2)$ orbits on $M$ all have real dimension equal to three we see that the rank of the
distribution is in fact precisely equal to two. For each point $z$ in the twistor space we
therefore have a complex surface $\Sigma$ passing through that point, such that $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$
are tangential to $\Sigma$. These surfaces are just the orbits of the local $SU(2,C)$ action
generated by the vector fields $\tilde{X}_i$.

The assumption that the metric is not hyperkähler implies that the space of co-
variant constant two-forms will have dimension less than three. However this space is
a real representation of the isometry group $SU(2)$ and hence will be acted on trivially
by this group. Hence the Kähler form $\Omega$ and complex structure $I$ on $M$ are preserved
by the $SU(2)$ action. It follows that $SU(2)$ acts on $D$, and so the vector fields gen-
erated by this action are tangential to $D$. As $D$ is a complex submanifold of $Z$ the
holomorphic vector fields $\tilde{X}_i$ are also tangential to $D$, so $D$ is a union of members of
the family of surfaces discussed above.

The members of this family define (locally) a family of sections of the twistor fibra-
tion, which induces a local hyperhermitian structure on $M$. That is, we have complex
structures $I_1, I_2, I_3$ (here $I_1$ is the given complex structure $I$ on $M$) multiplying like
the quaternions, such that $g$ is hermitian with respect to each of $I_1, I_2, I_3$.

Now there is a unique torsion-free connection, the Obata connection $\nabla$, such that

$$\nabla I_i = 0 \quad (i = 1, 2, 3).$$

As explained in [PS], [PPS] there is a 1-form $\omega$ such that

$$\nabla g = \omega \otimes g$$

and moreover

$$d\Omega_i = \omega \wedge \Omega_i \quad (i = 1, 2, 3).$$
where \( \Pi \) is the two-form defined by \( g \) and \( L \).

However by our hypothesis \( I = I_1 \) is Kähler so we have \( dI_1 = 0 \). As \( I_{11} \) is nondegenerate this means that \( \omega \) is zero, so \( dI_1 = dI_3 = 0 \) and all the complex structures are Kähler.

The metric on \( M \) is therefore locally hyperkähler and we have the required contradiction. □

Theorem 2.2

Let \( M \) be a four-manifold with an \( SU(2) \)-invariant scalar-flat Kähler metric, which is not locally hyperkähler; suppose also that \( SU(2) \) acts with three-dimensional orbits. Then the divisor \( E \) where \( s \) vanishes is equal to \( 2D + 2\bar{D} \).

Proof

Note first that because \( s \) is \( SU(2) \)-invariant, and because the \( SU(2) \) orbits are three-dimensional, if \( s \) vanishes on a real twistor line then it vanishes everywhere. It follows from Lemma 2.1 that this contradicts the hypotheses of the theorem, so we deduce that \( E \) contains no real twistor line and hence \( E \) meets each real twistor line at four points counted with multiplicity.

As shown in Lemma 2.1, the \( SU(2) \) action preserves the divisor \( D \), and so the holomorphic vector fields \( \tilde{X}_1, \tilde{X}_2, \tilde{X}_3 \) are tangential to \( D \), and hence are linearly dependent. It follows that our section \( s \) of \( K_Z^{-1/2} \) vanishes on \( D \) (and, using a similar argument, on \( D \)).

Consider now the two-dimensional space of \( SU(2) \)-invariant sections of \( K_Z^{-1/2} \) spanned by \( \phi_1, \phi_2 \). As above, two such sections define the same divisor if and only if they agree up to a constant, so we obtain a pencil of distinct \( SU(2) \)-invariant divisors in the linear system \( |K_Z^{-1/2}| \). But \( s \) must vanish on every member of this pencil, so \( s \) vanishes on some twistor line at more than four points with multiplicity, thus contradicting the hypotheses of the theorem. It follows that our assumption that \( F \) and \( D + D \) are unequal is false, and the theorem is proved. □

3. The isomonodromic family.

In the light of Theorem 2.2, we know that our flat meromorphic connection \( \phi \) has double poles on \( D, \bar{D} \) and is holomorphic elsewhere on \( Z \). We shall next show that restriction to the real twistor lines defines an isomonodromic family of connections, leading to the appearance of the third Painlevé equation.

Let \( C \) be a curve in \( M \) transverse to the \( SU(2) \) orbits, and consider the region \( U \) of the twistor space projecting onto \( C \). Let \( t \) be a coordinate along \( C \) and \( x \) a coordinate on the real twistor lines such that \( D, \bar{D} \) meet each twistor line at 0, \( \infty \) respectively.

The restriction of our connection form to \( U \) is given in these coordinates by

\[-A \, dx + B \, dt\]

where \( A, B \) are \( \mathbb{H} \)-valued functions of \( x, t \).

From our comments above about the poles of \( \phi \) we can write \( A, B \) as

\[ A = A_0 + \frac{A_{+1}}{x} + \frac{A_{-1}}{x^2} \]

and

\[ B = B_0 + \frac{B_{+1}}{x} + \frac{B_{-1}}{x^2} \]

of \( SU(2) \) and so is acted on trivially by \( SU(2) \). In particular \( p_1 \) is \( SU(2) \)-invariant. The same argument shows that \( p_2 \) is also \( SU(2) \)-invariant.

Consider now the two-dimensional space of \( SU(2) \)-invariant sections of \( K_Z^{-1/2} \) spanned by \( p_1, p_2 \). As above, two such sections define the same divisor if and only if they agree up to a constant, so we obtain a pencil of distinct \( SU(2) \)-invariant divisors in the linear system \( |K_Z^{-1/2}| \). But \( s \) must vanish on every member of this pencil, so \( s \) vanishes on some twistor line at more than four points with multiplicity, thus contradicting the hypotheses of the theorem. It follows that our assumption that \( F \) and \( D + D \) are unequal is false, and the theorem is proved. □
$B = B_2 x^2 + B_1 x + B_0 + B_{-1} x + B_{-2} x^2$

where $A_i, B_i$ are functions of $t$ taking values in $sl(2, \mathbb{C})$.

We shall assume that there is a range of $t$ where the eigenvalues of $A_0$ and $A_{-2}$ are nowhere zero, and restrict ourselves to this range in the following calculations.

By rescaling $x$ we can take $A_0$ and $A_{-2}$ to have the same eigenvalues and hence be conjugate. If the eigenvalues of $A_0$ and $A_{-2}$ are constant in $t$ it is easy to show, using the flatness of $-A dt - B dt$, that we can gauge $B$ to be zero and $A$ to be constant in $t$. Excluding this trivial case, and restricting the range of $t$ if necessary, we can choose $t$ so that the eigenvalues of $A_0$ and $A_{-2}$ are $\frac{1}{4} t, -\frac{1}{4} t$. Moreover, by a choice of gauge we can take $A_0$ to be diagonal.

We write $A_0, A_{-1}, A_{-2}$ as

$$A_0 = \begin{pmatrix} \frac{1}{4} t & 0 \\ 0 & -\frac{1}{4} t \end{pmatrix}, A_{-1} = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, A_{-2} = t \begin{pmatrix} u & v \\ w & -u \end{pmatrix}$$

where

$$u^2 + v w = \frac{1}{4}.$$  \hfill (2)

We can perform a gauge transformation by a diagonal matrix in $SL(2, \mathbb{C})$ (depending on $t$) to ensure that

$$B_0 = \begin{pmatrix} t^{-1} p & \delta \\ \epsilon & t^{-1} p \end{pmatrix}$$

for some functions $\delta, \epsilon$ of $t$.

We say that $A, B$ are in canonical form if (1)-(3) are satisfied.

The flatness condition for the connection is

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial x} + [A, B] = 0.$$  \hfill (4)

This is the isomonodromy condition for the ordinary differential equation

$$\frac{d z}{d t} = \left( A_0 + \frac{A_{-1}}{x} + \frac{A_{-2}}{x^2} \right) z$$

and it is well known that this condition is equivalent to the Painlevé III equation. We now give details of an argument to show this equivalence.

Equation (4) is equivalent to the relations

$$[A_{-2}, B_{-2}] = 0 \hfill (5)$$

$$2 B_{-2} + [A_{-3}, B_{-1}] + [A_{-1}, B_{-3}] = 0 \hfill (6)$$

$$\frac{d A_{-2}}{d t} + B_{-1} + [A_0, B_{-2}] + [A_{-1}, B_1] + [A_{-3}, B_0] = 0 \hfill (7)$$

$$\frac{d A_{-1}}{d t} + [A_0, B_{-1}] + [A_{-1}, B_0] + [A_{-2}, B_1] = 0 \hfill (8)$$

$$\frac{d A_{-3}}{d t} - B_1 + [A_0, B_3] + [A_{-1}, B_1] + [A_{-3}, B_0] = 0 \hfill (9)$$

$$2 B_2 + [A_0, B_1] + [A_{-1}, B_2] = 0 \hfill (10)$$

$$[A_0, B_3] = 0. \hfill (11)$$

Let us now analyse these equations.

The first and last equations imply that $B_{-3} = \psi A_{-3}$ and $B_3 = \phi A_0$ for scalars $\phi$ and $\psi$. Multiplying (10) by $A_0$ and (6) by $A_{-2}$ and taking the trace implies that $\phi$ and $\psi$ (and hence $B_1$ and $B_{-3}$) are both zero.

Equations (6) and (10) now show that $B_{-1} = \mu A_{-2}$ and $B_1 = \lambda A_0$, where $\lambda, \mu$ are scalar-valued functions of $t$.

We obtain the equations

$$\frac{d A_{-2}}{d t} + \mu A_{-2} + \mu [A_{-1}, A_{-2}] + [A_{-3}, B_0] = 0 \hfill (12)$$

$$\frac{d A_{-1}}{d t} + \mu A_0, A_{-2} + [A_{-1}, B_0] + \lambda [A_{-2}, A_0] = 0 \hfill (13)$$

$$\frac{d A_0}{d t} - \lambda A_0 + [A_0, B_3] + \lambda [A_{-1}, A_0] = 0. \hfill (14)$$

The last equation implies that $\lambda = t^{-1}$ (so $B_1 = t^{-3} A_0$) and that...
Observe from (12) that
\[ \frac{d}{dt} \text{Tr} A_{2}^{2} = -2\mu \text{Tr} A_{2}^{2}. \]
But we know that \( \text{Tr} A_{2}^{2} = \frac{1}{t^2} \), so \( \mu = -t^{-1} \) and \( B_{-1} = -t^{-1} A_{-2} \).

The flatness equation (4) now becomes
\[
\frac{dQ}{dt} = 2[Q, R], \quad \frac{dR}{dt} = 2[R, Q],
\]
where \( P = -t^{-1} A_{0}, Q = -A_{-1} \) and \( R = t^{-1} A_{-2} \).

This system of equations (together with the condition \( \frac{d}{dt} y = 0 \) which is automatic in our case) is a reduction of the self-dual Yang-Mills equations and has been studied by Mason and Woodhouse [MW]. They show that it is equivalent (if neither \( r \) nor \( w \) is identically zero) to the third Painlevé equation
\[ \frac{(dy)^2}{dt} = \frac{1}{y} \left( \frac{dy}{dt} \right)^3 - \frac{1}{y^2} \frac{dy}{dt} - \frac{1}{y} \frac{dy^2}{dt} + 8 \frac{dy}{dt} + \frac{8}{y} \frac{dy^2}{dt} \]
where \( y \) is defined by \( y = -\frac{t^{-2}}{2} \).

The constant parameters \( \kappa_i \) (\( i = 1, 2, 3, 4 \)) are given by
\[
\kappa_1 = 8 \text{Tr}(A_{-1}A_{-2}/t), \quad (15) \\
\kappa_2 = 4 - 8 \text{Tr}(A_{0}A_{-1}/t), \quad (16) \\
\kappa_3 = 4, \quad (17) \\
\kappa_4 = -4. \quad (18)
\]

Any Painlevé III equation with \( \kappa_3 \) and \( \kappa_4 \) nonzero may be brought to this form by scaling \( y \) and \( t \).

If \( r \) or \( w \) is identically zero we can use the substitution \( y = -\frac{t^{-2}}{4} \) to obtain Painlevé III, unless \( q \) or \( v \) is identically zero, when the flatness equation (4) becomes trivial. In the latter case both \( r, w, q \) and \( v \) are all in fact identically zero.

The next theorem summarises our findings.

**Theorem 3.1**

Let \( M \) be a scalar-flat Kähler, not locally hyperkähler, manifold of real dimension four, admitting an isometric action of \( SU(2) \) with three-dimensional orbits. Suppose that the matrices \( A_0, A_2 \) of (1) have nonzero, nonconstant eigenvalues, and that \( r, w, q \) and \( v \) are not all identically zero. Then we have an isomonodromic deformation problem leading to the Painlevé III equation. \( \square \)

Finally, it is straightforward to show that matrices \( A_i, B_j \) (\( i = 0, 1, 2; j = -2, -1, 0, 1, 2 \)) satisfying equations (5-11) and such that \( A, B \) are in canonical form are determined by \( y = -\frac{t^{-2}}{4} \) up to a gauge transformation.

\[ A_i \rightarrow \Theta A_i \Theta^{-1}, \quad B_j \rightarrow \Theta B_j \Theta^{-1} \quad (19) \]

where \( \Theta \) is a diagonal \( SU(2, \mathbb{C}) \)-valued matrix, constant in \( x \) and \( t \).

Therefore our solution of Painlevé III determines the connexion
\[ \Phi = -A dx - B dt \]
up to constant gauge transformations.

Now, the real twistor lines in \( U \) are embedded into \( Z \) by a family of maps \( f_\tau \). Equivalently we have a map from \( C \times \mathbb{P}^1 \) into \( Z \) given by \((t, x) \mapsto f_\tau(x) \). Restricting, we get a map
\[ \mathcal{F}: C \times \mathbb{P}^1 \rightarrow Z - E \]
the restriction of \( -Adx - Bdt \) to the domain of \( \mathcal{F} \) is the pullback by \( \mathcal{F} \) of the restriction of \( \Phi \) to \( Z - E \).
Locally we can identify \( \Phi \) with the restriction of the Maurer-Cartan form to an open set in \( SL(2, \mathbb{C}) \). Under this identification \( F \) is a fundamental solution for the equations

\[
\begin{align*}
\frac{\partial F}{\partial z} &= -F A \\
\frac{\partial F}{\partial t} &= -F B.
\end{align*}
\]

The general real twistor line in \( Z \) is given by

\[ x \mapsto F(x, t)G \]

for some \( t \) and some \( G \in SU(2) \).

We see that \( \Phi = -A \, dz - B \, dt \) determines \( F \) up to premultiplication by a constant matrix in \( SL(2, \mathbb{C}) \). Also, conjugating \( \Phi \) by a constant matrix \( \Theta \) as in (19) just corresponds to postmultiplying \( F \) by \( \Theta^{-1} \). These transformations do not affect the conformal structure; (see section 4 for a more detailed description of how to recover the conformal structure from \( A, B \)).

We have the following result.

Theorem 3.2

The conformal structure is determined by the solution \( y \) of Painlevé III, where \( y = -\frac{d}{dt} \) and \( \tau, w \) are as in (1). \( \square \)

4. The diagonal case.

In this section we shall consider the special case when the metric can be put in the form

\[ g = h^2dT^2 + \sigma_1^2 \sigma_2 + \delta^2 \sigma_3^2 \]

(20)

where \( \sigma_1, \sigma_2, \sigma_3 \) are invariant one-forms, satisfying the relations \( d\sigma_1 = \sigma_2 \wedge \sigma_3 \) and cyclically, \( T \) is a coordinate orthogonal to the orbits of \( SU(2) \), and \( h, \delta, \gamma, \alpha \) are functions of \( T \) only. We refer to this as diagonal Bianchi IX form. We can choose the coordinate \( T \) so that \( h = 1 \).

Let us now apply the techniques of the preceding sections to this case. Because the metric is in the diagonal form (20) we have, for each point \( m \) in \( M \), a copy of the four-group \( V \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \) which preserves the metric \( g \) and fixes \( m \). The non-identity elements of the group change the signs of two of the one-forms \( \sigma_1, \sigma_2, \sigma_3 \). This action lifts to a holomorphic effective action of \( V \) on the twistor space preserving the real twistor line over \( m \). So we have an injection of \( V \) into the group of Möbius transformations of this line.

As before, we choose a coordinate \( x \) on the twistor line so that \( D, \bar{D} \) intersect the line at \( 0, \infty \) respectively.

There are two possibilities; either \( V \) fixes the Kähler form or else two of the order two elements change the sign of the Kähler form and the third order two element fixes the form. But the first possibility means that the \( V \) action on the twistor line must fix \( 0, \infty \), and the only Möbius transformations with square equal to the identity which fix these two points are \( x \mapsto x \) and \( x \mapsto -x \). This contradicts the effectiveness of the action.

So we must have two order two elements interchanging \( 0 \) and \( \infty \), and two elements fixing these two points. This means that on the twistor line the action of the four-group must be by the Möbius transformations

\[
\begin{align*}
x &\mapsto x \\
x &\mapsto -x \\
x &\mapsto \Gamma \\
x &\mapsto -\Gamma/x
\end{align*}
\]

for some \( \Gamma \in \mathcal{C} \).
The \( \mathfrak{sl}(2, \mathbb{C}) \)-valued one-form \( \psi \) transforms by the adjoint representation under the \( \mathcal{V} \) action. So, restricting to a twistor line, we see that the connection form \(-\mathcal{A}(x,t)\, dx\) will be conjugated by an element of \( SU(2) \) when we apply the Möbius transformations above.

This implies that there exist \( \Theta_1, \Theta_2 \) such that
\[
\Theta_1(\mathcal{A}_0 - \frac{\mathcal{A}_{1}}{x} - \frac{\mathcal{A}_{2}}{x^2}) = \mathcal{A}_0 - \frac{\mathcal{A}_{1}}{x} + \frac{\mathcal{A}_{2}}{x^2}
\]
and
\[
\Theta_2(\mathcal{A}_0 - \frac{\mathcal{A}_{1}}{x} - \frac{\mathcal{A}_{2}}{x^2}) = \frac{\mathcal{A}_0}{x} + \frac{\mathcal{A}_1}{x^2} + \frac{\mathcal{A}_2}{x^3}.
\]

We deduce that \( \text{Tr} \mathcal{A}_0, \mathcal{A}_1, \text{and} \mathcal{A}_2 \) are zero. Referring to our expressions (15)-(18) for the parameters of the Painlevé equation in terms of \( \mathcal{A}_0, \mathcal{A}_1, \) and \( \mathcal{A}_2 \), we have the following result.

**Theorem 4.1**

The Painlevé III equation arising from a scalar-flat \( \mathcal{K} \)ähler, not locally \( \mathcal{K} \)ähler metric in diagonal Bianchi IX form has parameters \( \kappa_1 = 0, \kappa_2 = 4, \kappa_3 = 4, \kappa_4 = -4 \).

**Remark**

Tod [T] has found an ansatz to produce scalar-flat \( \mathcal{K} \)ähler metrics with \( S^3 \) action. A special case of this ansatz gives diagonal Bianchi IX solutions. The differential equation produced in this case is a rescaled version of the above Painlevé III equation.

If the metric is of diagonal Bianchi IX form we can find the conformal structure explicitly in terms of Painlevé transcendent.

Recall from section 3 that the real twistor lines are locally given by
\[
x \mapsto \mathcal{F}(x,t)\mathcal{G}
\]
where \( \mathcal{G} \in SU(2) \) and \( \mathcal{F} \) is a fundamental solution to the equations

\[
\frac{\partial \mathcal{F}}{\partial x} = -\mathcal{F} \mathcal{A} \\
\frac{\partial \mathcal{F}}{\partial t} = -\mathcal{F} \mathcal{B}.
\]

We can view \( t \) and \( \mathcal{G} \) as giving coordinates on the four-manifold \( M \). An element \((t, \mathcal{G})\) of the complexified tangent space at \((t, \mathcal{G})\) may be identified with an infinitesimal deformation
\[
\frac{\partial \mathcal{G}}{\partial t} \mathcal{G} + \mathcal{F} \mathcal{G}
\]
of the twistor line. Here \( \dot{\mathcal{G}} \) is an element of \( \mathfrak{sl}(2, \mathbb{C}) \).

We can rewrite this using (21), (22) as
\[
-\mathcal{F} \mathcal{B} \dot{\mathcal{G}} + \mathcal{F} \dot{\mathcal{G}}.
\]

The conformal structure is defined by declaring the tangent vector to be null if and only if the projection of the deformation (23) onto the normal bundle of the real twistor line vanishes for some \( x \). This is equivalent to (23) being tangential to the real twistor line for some \( x \). Now the tangent vectors to the real twistor line are multiples of \( (\partial \mathcal{F}/\partial x)\mathcal{G} = -\mathcal{F} \mathcal{A} \mathcal{G} \), so we see that \((t, \mathcal{G})\) is null if and only if
\[
-\mathcal{B} \mathcal{G} + \mathcal{G} = -\mathcal{A} \mathcal{G}
\]
for some \( x, \mathcal{A} \).

Using our explicit expressions for \( A, B \) from section 3 we find after some calculation that the conformal class of the (real) metric is represented by
\[
2\, dt^2 + \frac{\sigma_1^2}{2\Psi - 1} + \frac{\sigma_1^2}{2\Psi + 1} + \frac{8(\Psi^2 - 1)}{16} \chi^2
\]
where \( \Psi = -u \) and \( u \) is as in (1) of section 3. We can relate \( \Psi \) to the Painlevé transcendent by remarking that the equations (12-14) imply that
\[
y = \frac{1 - \Psi^2}{4\Psi^2 + 1} \frac{\partial}{\partial t} \left( \Psi^2 - \frac{1}{4} \right) \Psi - \frac{1}{4}
\]
where \( y \) is our solution to Painlevé III.
5. A direct approach.

In the case of a diagonal Bianchi IX metric we can also classify the scalar-flat Kähler examples by direct methods. These involve an argument used in [DS], which we first review.

Suppose that we have a Kähler, non-hyperkähler, diagonal Bianchi IX metric expressed in the form

\[ g = (abc)^2dT^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2 \]  

(24)

where \( a, b \) and \( c \) are functions of \( T \), and \( \sigma_i \) are the invariant one-forms satisfying

\[ d\sigma_i = \sigma_j \wedge \sigma_k \text{ etc.} \]

We denote the vector fields dual to \( dT, \sigma_1, \sigma_2, \sigma_3 \) by \( X_0, X_1, X_2, X_3 \).

These satisfy the relations

\[ [\frac{\partial}{\partial T}, X_i] = 0 \]

and

\[ [X_i, X_j] = -X_j \text{ and cyclically.} \]

We have an orthonormal coframe for \( g \) given by \( e_0 = abc \, dT, e_1 = a\sigma_1, e_2 = b\sigma_2, e_3 = c\sigma_3 \), and we choose the orientation so that

\[ \Omega_i^+ = e_0 \wedge e_1 + e_2 \wedge e_3 \]

\[ \Omega_i^- = e_0 \wedge e_2 + e_3 \wedge e_1 \]

\[ \Omega_i^0 = e_0 \wedge e_3 + e_1 \wedge e_2 \]

are self-dual two-forms.

As remarked in the proof of Lemma 2.1, the assumption that the metric is not hyperkähler means that the Kähler form \( \Omega \) is \( SU(2) \)-invariant.

Now the Kähler form on a complex surface is always self-dual, so we deduce from the above remarks that \( \Omega \) is given by

\[ \Omega = S_1\Omega_0^+ + S_2\Omega_1^- + S_3\Omega_2^0 \]  

(25)

where \( S_1, S_2, S_3 \) are functions of \( T \) only.

Now the Kähler condition \( d\Omega = 0 \) is equivalent to the equations

\[ (S_{1bc})' = S_0 a^3bc \]  

(26)

\[ (S_{2ac})' = S_0 ab^3c \]  

(27)

\[ (S_{3ab})' = S_0 abc^3 \]  

(28)

where we use a prime to denote differentiation with respect to \( T \).

If we now introduce the standard variables \( w_1 = bc, w_2 = ac, w_3 = ab \) and define functions \( \alpha, \beta, \gamma \) by

\[ w_1' = w_1w_3 + \alpha w_1 \]  

(29)

\[ w_2' = w_2w_1 + \beta w_2 \]  

(30)

\[ w_3' = w_3w_2 + \gamma w_3 \]  

(31)

then the equations (26-28) become

\[ S_1' = -\alpha S_1 \]  

(32)

\[ S_2' = -\beta S_2 \]  

(33)

\[ S_3' = -\gamma S_3 \]  

(34)

We see that for each metric \( g \) there is a 3-dimensional space of closed, self-dual, \( SU(2) \)-invariant two-forms, which are candidates for Kähler forms.

Given such a form \( \Omega \) we can use the metric to define an endomorphism \( I \) of the tangent bundle by

\[ g(I\mathbf{X}, \mathbf{X}) = \Omega(\mathbf{X}, \mathbf{X}). \]
With \( \Omega \) given by (25) the endomorphism is defined by

\[
I \frac{\partial}{\partial T} = S_1 w_1 X_1 + S_2 w_2 X_2 + S_3 w_3 X_3 \tag{35}
\]

\[
IX_1 = -S_1 \frac{\partial}{\partial u_1} + S_2 \frac{\partial}{\partial u_2} + S_3 \frac{\partial}{\partial u_3} \tag{36}
\]

\[
IX_2 = -S_1 \frac{\partial}{\partial u_1} + S_2 \frac{\partial}{\partial u_2} + S_3 \frac{\partial}{\partial u_3} X_3 \tag{37}
\]

\[
IX_3 = -S_1 \frac{\partial}{\partial u_1} + S_2 \frac{\partial}{\partial u_2} + S_3 \frac{\partial}{\partial u_3} X_3 \tag{38}
\]

Moreover \( I \) is an almost complex structure if and only if

\[
\alpha S_1 = -iS_2 S_3 (\gamma - \beta) \tag{39}
\]

\[
\alpha (1 + S_1^2) = (S_2^2 - S_3^2)(\gamma - \beta) \tag{40}
\]

\[
(S_2 - iS_3) S_1 (\alpha - \beta + \gamma) = 0 \tag{41}
\]

\[
(S_1 + iS_3) S_1 (\gamma - \alpha) = 0 \tag{42}
\]

hold.

It is now straightforward to show, using equations (32-34), that two of \( \alpha, \beta, \gamma \) must be equal and the third must be zero. This is also true if \( S_1 \) is identically 1 or -1 (in fact in this case we have \( \alpha = 0 \) and \( \beta = \gamma \)).

Moreover if \( \alpha = 0 \) and \( \beta = \gamma \) we must have either \( S_2 = S_3 = 0 \) or else \( \alpha = \beta = \gamma = 0 \) (in which case our metric is hyperkahler). Similar statements hold, with appropriate permutations, in the cases \( \beta = 0, \alpha = \gamma \) and \( \gamma = 0, \alpha = \beta \).

We summarise our conclusions as follows.

**Theorem 5.1 [DS]**

If the metric \( g \) of (24), with our choice of orientation, is Kähler and non-hyperkahler then one of the following three statements is true

(i) \( \alpha = 0, \beta = \gamma \)

(ii) \( \beta = 0, \alpha = \gamma \)

(iii) \( \gamma = 0, \alpha = \beta \).

Conversely if one of these statements holds then the metric is Kähler. The Kähler forms are

(i) \( \Omega = w_1 w_3 dT \wedge \sigma_1 + w_1 \sigma_2 \wedge \sigma_3 \)

(ii) \( \Omega = w_3 w_1 dT \wedge \sigma_2 + w_1 \sigma_3 \wedge \sigma_3 \)

(iii) \( \Omega = w_1 w_2 dT \wedge \sigma_1 + w_2 \sigma_2 \wedge \sigma_1 \)

respectively. \( \square \)
Hyperkähler structures with triholomorphic $SU(2)$ action correspond precisely to the case $\alpha = \beta = \gamma = 0$.

In the last section we considered the action of $V_4$ on the Kähler forms. It is clear from the expressions given in Theorem 5.1 that there will be two elements of $V_4$ fixing $\Omega$ and two elements changing the sign of $\Omega$, in accordance with our previous discussion.

Let us now require that the scalar curvature $R_{\text{scalar}}$ is zero.

Using the expressions of Pedersen-Poon [PP] for the connection forms of the metric we can calculate the Riemann curvature tensor and hence the scalar curvature. We find that

$$R_{\text{scalar}} = -\frac{1}{4\epsilon_0\epsilon_1\epsilon_2\epsilon_3}(2\alpha' + 2\beta' + 2\gamma' + \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\alpha\gamma).$$  (43)

Comparing with the expressions of (i),(ii),(iii) in the statement of Theorem 5.1, we find that the scalar-flat condition forces $\alpha,\beta,\gamma$ to be constant.

Theorem 5.2
The general scalar-flat Kähler, non-hyperkähler metric of diagonal Bianchi IX form compatible with our choice of orientation is given (up to permutations of $w_1, w_2, w_3$) by

$$g = w_1w_2w_3\,dt^2 + \frac{w_1w_2}{w_1}d\sigma_1^2 + \frac{w_1w_3}{w_2}d\sigma_2^2 + \frac{w_2w_3}{w_3}d\sigma_3^2$$

where

$$
\begin{align*}
w_1' &= w_1w_3 + \alpha w_1, \\
w_2' &= w_2w_3 + \alpha w_2, \\
w_3' &= w_3w_1 \\
\end{align*}
$$

and $\alpha$ is a nonzero constant.

Note
Choosing the opposite orientation just corresponds to reversing the sign of $T$.

Remark
The metrics of Theorem 5.2 are precisely those arising from the Pedersen-Poon ansatz [PP]. So we have shown that this ansatz produces all scalar-flat Kähler non-hyperkähler diagonal Bianchi IX metrics compatible with our choice of orientation.

As remarked in [PP], the metrics with $w_1 = w_2$ are the $U(2)$-invariant examples of LeBrun [L].

Let us compare the metrics of Theorem 5.2 with those which we obtained using the method of isomonodromic deformations in section 4.

As explained in [PP], if one makes the substitutions

$$w_1 = e^{\alpha T}F_1, \quad w_2 = e^{\gamma T}F_3, \quad w_3 = F_3$$

and

$$F_1 = \frac{\Delta}{2}(y + \frac{1}{y}), \quad F_2 = \frac{\Delta}{2}(y - \frac{1}{y})$$

then $\Delta$ is constant and $\tilde{y}$ satisfies

$$\frac{d^2\tilde{y}}{d\sigma_1^2} - \frac{1}{\tilde{y}}\left(\frac{d\tilde{y}}{d\sigma_1}\right)^2 = \frac{1}{\tilde{y}^2} + \frac{1}{\Delta^2}(\Delta^2 - 1).$$

This is the Painlevé III equation with parameters

$$\kappa_1 = \kappa_2 = 0, \quad \kappa_3 = -\kappa_4 = \frac{1}{4}\Delta^2.$$

Of course, this substitution is only valid where $w_1 \neq w_2$, but in fact it follows from the equations that if $w_1$ and $w_2$ are not identically equal then they are never equal,
so the above procedure is valid except in the special case $w_1 = w_2$ (when we get the metric of [E]).

Now, if we change variables by

$$z = -\frac{1}{2} \Delta Y^2, \quad Y = 1 + \frac{4}{\ell^2 + \phi^2 - 2}$$

we arrive at the equation

$$\frac{dY}{dz} = \left( \frac{1}{2} \frac{1}{Y - 1} \right) \left( \frac{dY}{dz} \right)^2 - \frac{1}{2} \frac{dY}{dz} + \frac{Y}{z}$$

which is Painlevé V with parameters

$$\kappa_1 = \kappa_2 = \kappa_4 = 0, \kappa_3 = 1.$$

(We are lead to this change of variables by a comparison of the calculations of [FP] and [T]).

In those variables the metric is

$$\frac{dY}{dz} \left( \frac{d^2}{dz^2} - \frac{z}{Y(Y-1)^2} - \frac{z^2 - 1}{Y(Y-1)^2} + 2Y \left( \frac{dY}{dz} \right)^{-1} \right).$$

Letting

$$z = 2t^2, \quad Y = \frac{\Phi}{\Phi + 1}$$

we find that the metric is

$$\frac{dY}{dt} \left( \frac{d^2}{dt^2} + \frac{\sigma_1^2}{2\Phi - 1} + \frac{\sigma_2^2}{2\Phi + 1} + \frac{8(\Phi - 1)}{2\Phi^2} \right)$$

and

$$y = \frac{1}{4} Y^{-1} \frac{dY}{dt} = \frac{1}{4} \left( \frac{\Phi - 1}{\Phi + 1} \right)^{-1} \frac{d}{dt} \left( \frac{\Phi - 1}{\Phi + 1} \right)$$

satisfies Painlevé III with parameters.

$$\kappa_1 = 0, \kappa_2 = 4, \kappa_3 = 4, \kappa_4 = -4.$$

This agrees with the expression for the conformal structure we derived by twistor methods in section 4.

6. Completeness analysis.

In the case of diagonal Bianchi IX metrics we can classify the complete scalar-flat Kähler examples. This analysis has already been performed for hyperkähler metrics [GP], [AH] so, as in the rest of this paper, we exclude this case.

Theorem 5.2 gives us a description of the metrics we are interested in. It is convenient to cast the equations of Theorem 5.2 in terms of $a, b, c$. They become

$$a' = \frac{1}{2} (b^2 + c^2 - a^2)$$

$$b' = \frac{1}{2} (c^2 + a^2 - b^2)$$

$$c' = \frac{1}{2} (a^2 + b^2 - c^2 + 2a).$$

As we are assuming the metric is not hyperkähler we take the constant $\alpha$ to be nonzero.

Consider a solution to (45-47), analytic on a maximal interval $(\xi, \eta)$. It is clear from the equations that if any one of $a, b, c$ is zero at some point in this open interval, then it is identically zero. As this will not give a metric with three-dimensional orbits we can exclude this case, and hence assume that $a, b, c$ are nowhere zero in $(\xi, \eta)$. It follows that the metric will be defined for $T \in (\xi, \eta)$, so to decide whether it is complete we need to study the behaviour of $a, b, c$ as $T$ approaches $\xi$ from above and $\eta$ from below.

As the equations and metric are invariant under sign changes of $a, b, c$ we shall take $a, b, c \geq 0$ from now on.

We record some useful facts about the equations (45)-(47) in the next lemma.

**Lemma 6.1**

$$(ab)' = abc^2$$

$$(bc)' = bc(a^2 + \alpha)$$
\[
\begin{align*}
(ac)' &= a(b' + \alpha) \\
(ab)' &= (a^2 - b^2)(c^2 - a^2 - b^2) \\
(c)' &= \frac{c}{b}(b^2 - a^2) \\
(abc)' &= \frac{1}{2} abc(a^2 + b^2 + c^2 + 2a).
\end{align*}
\]

The critical points of the equations (45-47) are the points \((a, b, c)\) satisfying

(i) \(a = b, c = 0\) or

(ii) \(a = b = 0, c = \sqrt{\alpha\omega} \) (if \(\alpha > 0\)).

We see that either \(a\) is identically equal to \(b\) or else \(a\) is never equal to \(b\) on \((\xi, \eta)\).

In the former case the metrics are those of LeBrun [L]. If \(a\) is never equal to \(b\) then, by the symmetry of the equations, we can without loss of generality take \(a > b\).

From the above remarks, we can take \(a \geq b\). It follows from (45-47) and Lemma 6.1 that \(b\) and \(ab\) are increasing; moreover \(\xi\) is greater than or equal to 1 and either identically equal to 1 or else strictly decreasing on \((\xi, \eta)\). In particular, note that if \(a\) tends to 1 as \(T\) tends to \(\xi\), then \(a\) is identically equal to \(b\).

We first consider the situation when \(\alpha\) is positive.

It follows from Lemma 6.1 that \(a, b, c, ac\) are increasing on \((\xi, \eta)\).

Case 1. Suppose that \(\xi\) is finite. From Lemma 6.1 we see that \(abc\) is increasing on \((\xi, \eta)\), so tends to a finite limit as \(T \to \xi\). Hence the geodesic distance

\[\int_\xi^T abc\]

is finite, and to get a complete metric we would have to add a nut (point orbit of \(SU(2)\)) or bolt (two-dimensional orbit of \(SU(2)\)) at \(T = \xi\). In the former case we would have \(a, b, c = 0\) at \(\xi\); in the latter case one of \(a, b, c\) would be zero and the other two would attain nonzero finite limits at \(\xi\). Both cases would force at least one of \(a, b, c\) to be identically zero, giving a contradiction.

Case 2. So complete metrics can only arise if the maximal interval is \((-\infty, \eta)\). Let us now see when this can happen.

As \(a, b, c, ac\) are increasing they tend to finite nonnegative limits \(\lambda_1, \lambda_2, \lambda_3, \lambda_4\) as \(T\) tends to \(-\infty\).

(i) If the limit \(\lambda_4\) of \(b\) is nonzero then \(a, c\) also tend to finite limits, so \((a, b, c)\) tends to a critical point \((p, \mu, \nu)\) with \(\lambda_1\) positive. By Lemma 6.1 we must have \(\lambda_1 = \mu\) and \(\nu > 0\), so \(\xi\) tends to 1 as \(T\) tends to \(-\infty\). From our comments following Lemma 6.1 we deduce that \(a\) is identically equal to \(b\). The trajectory is an unstable curve of the critical point \((\mu, \nu, 0)\).

(ii) Let us assume therefore that \(\lambda_4 = 0\), that is, \(b\) tends to 0 as \(T\) tends to \(-\infty\). Using Lemma 6.1 we now find that \((ac)'\) tends to \(\lambda_2c\) as \(T\) tends to \(-\infty\). However we know that \(ac\) tends to a finite limit as \(T\) tends to \(-\infty\) so, as \(\alpha\) is nonzero, it follows that \(\lambda_4 = 0\); that is, \(ac\) tends to 0 as \(T\) tends to \(-\infty\).

If \(b^2 + c^2 - a^2\) is negative at some \(T_0\) then it follows from the equations that the derivative of this expression is positive at \(T_0\). Hence \(b^2 + c^2 - a^2\) is negative on \((-\infty, T_0)\) and \(a, c\) are increasing and \(a\) decreasing on this interval. We deduce that

\[a \to \nu, b \to 0, c \to \mu\]

as \(T \to -\infty\)

where \(\nu\) is positive and may be \(\infty\), while \(\mu\) is nonnegative and finite.

Since \(ac\) tends to zero as \(T\) tends to \(-\infty\) we have \(\mu = 0\). By considering the limit of \(a^2\) as \(T\) tends to \(-\infty\) we find that \(\nu = \infty\) and it readily follows from the equations that \(a\) becomes infinite at a finite value of \(T\) less than \(T_0\), giving a contradiction.

Similar arguments show that if \(a^2 = b^2 - c^2 + 2a\) is negative at some point \(T_0\) then \(c\) becomes infinite at a finite value of \(T\) less than \(T_0\), again giving a contradiction.

So we see that we need

\[a^2 \leq b^2 + c^2, c^2 \leq a^2 + b^2 + 2a\]

on \((-\infty, \eta)\).

This implies that \(a, b, c\) are increasing so tend to finite limits as \(T\) tends to \(-\infty\). That is, \((a, b, c)\) tends to a critical point. Moreover, recall that \(b\) tends to 0 so this critical point is either \((0, 0, 0)\) or \((0, 0, \sqrt{2}\alpha)\). If the critical point is \((0, 0, 0)\) then the
equations imply that $abc$ decays exponentially fast as $T$ tends to $-\infty$. Hence the geodesic distance

$$
\int_{-\infty}^{T} abc \\
$$

to $-\infty$ is finite and to obtain a complete metric we must have a nut at $T = -\infty$.

This means that as $T$ tends to $-\infty$ the terms $a$, $b$ and $c$ all tend to zero like $1/\sqrt{T}$ times the square of the geodesic distance, and it is easy to see that this is inconsistent with the equations.

If, on the other hand, the critical point is $(0,0,\sqrt{2\alpha})$ then for $T$ large and negative we have

$$a = \tau e^{\alpha T}, b = \tau e^{\alpha T}, c = \sqrt{2\alpha}$$

and it easily follows that the metric is incomplete.

To recapitulate, if $\alpha > 0$ there are no complete metrics except those arising from unstable curves of $(\mu,\mu,0)$ for positive $\mu$. Such metrics all have $a$ identically equal to $b$.

Let us now consider the situation when $\alpha$ is negative.

Case 1. Suppose first that $\xi$ is finite.

As above, $b$ and $ab$ are increasing so tend to finite limits $\lambda_1, \lambda_2$ as $T$ tends to $\xi$.

(i) First assume $\lambda_1 > 0$. We deduce that $a$ tends to a finite positive limit also, as $T$ tends to $\xi$.

If $\lambda^2_1 + \alpha \geq 0$ then $ac$ is increasing so $ac$ and hence $c$, tend to finite limits at $\xi$. This contradicts the fact that $\xi$ is a singularity.

If $\lambda^2_1 + \alpha < 0$ then $ac$ is decreasing for $T$ near $\xi$ so $ac$ tends to a limit, possibly $\infty$ as $T$ tends to $\xi$. As $\xi$ is a singularity this limit is in fact $\infty$ and $c$ tends to $\infty$ at $\xi$. Therefore $abc$ is increasing near $\xi$ so tends to a finite limit at $\xi$. It follows that the geodesic distance

$$\int_{\xi}^{T} abc$$

is finite, so the metric is incomplete.

(ii) Suppose on the other hand that $\lambda_1 = 0$, that is, $b$ tends to $0$ as $T$ tends to $\xi$.

So $(ac)^2 \approx ac$ near $\xi$ and $ac$ tends to a finite limit at $\xi$. Hence $abc$ tends to $0$ at $\xi$ and again the geodesic distance is finite. To obtain a complete metric we must have a nut or bolt at $T = \xi$, and as in the case of positive $\alpha$ this leads to a contradiction.

Case 2. Let us consider, therefore, trajectories defined on $(-\infty, \eta)$.

Again $b, ab$ tend to $\lambda_1, \lambda_2$ as $T$ tends to $-\infty$.

(i) If $\lambda_1 > 0$ then $a$ tends to a finite positive limit.

Arguing as before we see that if $\lambda^2_1 + \alpha \geq 0$ then $c$ also tends to a finite limit. So $(a,b,c)$ tends to a critical point which must be $(\mu,\mu,0)$ for some $\mu > 0$. As before this implies that $a$ is identically equal to $b$.

If, on the other hand, $\lambda^2_1 + \alpha < 0$ then we find that either $(a,b,c)$ tends to a critical point (in which case $a \equiv b$) or else $c$ tends to $\infty$. The latter case implies that $c$ becomes infinite at a finite value of $T$ less than $\eta$, giving a contradiction.

(ii) We can therefore assume that $\lambda_1 = 0$, so $b$ tends to $0$ as $T$ tends to $-\infty$. As $(ac)^2 \approx ac$ and $\alpha$ is negative we see that $ac$ tends to $0$ as $T$ tends to $-\infty$.

If $a^2 + b^2 - c^2 < 0$ at $T_0$ then its derivative is positive at $T_0$ and so $a^2 + b^2 - c^2 < 0$ on $(-\infty, T_0)$. It follows that $a, b$ are increasing and $c$ is decreasing on $(-\infty, T_0)$. Combined with the fact that $ac$ tends to $\infty$ we see that

$$a \to \mu, b \to 0, c \to \infty$$

as $T \to -\infty$

where $\mu$ is finite. It easily follows that our solution cannot be finite on all of $(-\infty, T_0)$, giving a contradiction.

Similarly, we find that if $a^2 - b^2 - c^2 + 2\alpha > 0$ at any point $T_0$, then the solution is not finite on all of $(-\infty, T_0)$.

So the remaining case to consider is if

$$c^2 \leq a^2 + b^2, a^2 \leq b^2 + c^2 - 2\alpha \quad (18)$$

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on \((-\infty, \eta)\). Since \(b\) tends to 0 and \(ac\) tends to \(\infty\) we deduce from these inequalities that \(a\) and \(c\) both tend to \(\infty\) as \(T\) tends to \(-\infty\).

It follows that \(abc\) is increasing for \(T\) large and negative so tends to a finite limit \(L\) as \(T\) tends to \(-\infty\). Moreover \(\frac{abc}{\eta}\) tends to \(\infty\). If \(L > 0\) then \(\frac{abc}{\eta}\) tends to \(\infty\) as \(T\) tends to \(-\infty\), which gives a contradiction. So we have

\[
abc \to 0, \quad \frac{abc}{ac} \to \infty \quad \text{as} \quad T \to -\infty.
\]

It easily follows by considering \(\log(abc)\) that \(abc\) decays exponentially fast as \(T\) tends to \(-\infty\). Therefore the geodesic distance to \(T = -\infty\) is finite, and, since \(a, c\) become infinite at \(T = -\infty\), the metric is incomplete.

We summarise our results in the final theorem.

**Theorem 6.2**

The only scalar-flat Kähler, non-hyperkähler, diagonal Bianchi IX metrics which are complete are the complete examples with \(a = b\) (and hence with \(U(2)\) symmetry).

In fact our discussion shows that all the complete metrics with \(a = b\) arise from the unstable curves of points \((\mu, \nu, 0)\) for \(\nu > 0\). It follows from (45-47) that in this situation we have

\[
a \simeq \mu, \quad b \simeq \mu, \quad c \simeq 2\mu(\mu^3 + \alpha^3)T
\]

as \(T\) tends to \(-\infty\), for some constant \(k\).

Taking

\[
v = \frac{k\mu^3}{\mu^3 + \alpha}
\]

as a new coordinate, we find that the metric is asymptotically

\[
dv^2 + \mu^2(\sigma_1^2 + \sigma_2^2) + (1 + \frac{\alpha}{\mu^3})^2e^2
\]

as \(v \to 0\).

This metric can be completed (by adding a bolt) precisely when \(1 + \frac{\alpha}{\mu^3}\) equals \(\frac{1}{n}\) for some positive integer \(n\).

If we put \(r = 2\sqrt{ab}\) then the metric corresponding to a solution of (45-47) with \(a \equiv b\) becomes

\[
\left(1 + \frac{8\mu^4}{r^2} - \frac{16(\mu^4 + 2\mu^3)}{r^4}\right)^{-1} dv^2 + \frac{1}{4} \left(\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{8\mu^4}{r^2} - \frac{16(\mu^4 + 2\mu^3)}{r^4}\right)\sigma_3^2\right)
\]

As explained above, we obtain the complete examples by setting

\[
1 + \frac{\alpha}{\mu^3} = \frac{1}{2^n}
\]

and the resulting metrics are

\[
\left(1 + \frac{4\mu^2(n - 2)}{r^2} - \frac{16\mu(n - 1)}{r^4}\right)^{-1} dv^2 + \frac{1}{4} \left(\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{4\mu^2(n - 2)}{r^2} - \frac{16\mu(n - 1)}{r^4}\right)\sigma_3^2\right)
\]

where \(\mu \in \mathbb{R}\).

These are the complete \(U(2)\)-invariant metrics found by LeBrun [L].

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