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Quantum Mechanics on the Quantum Sphere

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ABSTRACT

A discussion on the quantum mechanics of one- and many-particle systems on the quantum spheres is presented. The rôle of quantum group symmetry is fully exploited and developed. Generalized dynamical systems are introduced and discussed. We present the concept of generalized homogeneous spaces, and obtain the analogue of topological charges through inequivalent quantizations. The importance and origin of braiding in multi-particle systems is explained.

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1. Introduction

The concept of quantum groups has been introduced in physics from two different sets of motivations: one coming from integrable models ([9],[17]), and another from non-commutative geometry ([39],[27]). In non-commutative geometry it is desired to define properties that have been up to now adscribed to smooth manifolds to more general kinds of spaces ([6]). Special interest must then be assigned to generalizations of the concept of symmetry that apply to these new non-commutative "manifolds". In particular, since Lie groups are themselves manifolds, it is important to look at non-commutative generalizations of them, and such are the pseudogroups proposed by Woronowicz in [39].

From the point of view of physics, such a generalization might prove very important if it would provide a new form of regularization in which symmetries of the physical system under consideration were to be respected to a greater extent than is currently possible with the available methods. This idea of q-regularization (see [23]) is not yet fully feasible, and toy models are needed where physical systems are deformed in the sense of quantum groups. The integrable models approach has clearly vindicated quantum groups as symmetries of physical systems, the most famous example being statistical models such as, for instance, the XXZ chain [32]. Also important is the quantum group symmetry underlying Kač-Moody algebras uncovered by Faddeev and colaborators [2],[3]. A very interesting discussion of quasi-quantum groups as symmetries of quantum theories has been presented by Mack and Schomerus in [20]. But not so many models are available where the non-commutative geometrical aspects come to the fore.

In a previous paper ([10]), though, we have presented a set of one dimensional statistical models where the site variables take their values in the algebra of functions over a quantum sphere ([34]), thus providing a deformation of the one dimensional $O(3)$ sigma model.

We here proceed to continue this line of work by introducing a good set of definitions for the quantum mechanics of a particle moving on the quantum sphere. This system has been considered previously by Podleš in [35] from the point of view of differential calculus. Our approach is totally different and far more direct, making use of ideas derived from C* quantization, and is easily generalizable to other non-commutative "manifolds" related to quantum groups. Furthermore, the nontriviality of systems consisting of two or more particles moving on the quantum sphere, even for "free" hamiltonians, first proposed in [36], is seen in this context to be a natural consequence of the braiding of the tensor product of the algebra of functions on the quantum sphere, necessary if we are to keep within the braided category of $U_q(su_2)$-modules ([24]), as was already presented in [10].

The main guiding principle of this paper has been to consider the quantum group as a physical symmetry of the model to be defined. This naturally leads us to the definition of generalized dynamical system, very closely related to that natural in the C*-algebra context, and similar to the one discussed in [13] for actions of Kac algebras on von Neumann algebras. The study of the covariant representations of such generalized dynamical systems provides us with the "kinematics" of a particle moving on a quantum sphere, through an explicit construction of the Hilbert spaces related to such representations. A completely new development is the appearance of inequivalent quantizations, which, in the classical setting, correspond to topological charges. It is thus seen that, although topological in origin, phenomena like monopoles have an algebraic treatment that carries over to the deformed case. We then proceed to consider the dynamics given by introducing hamiltonians that commute with the quantum group symmetry.

A next step is to consider multiparticle systems. The construction of the relevant Hilbert spaces follows immediately from having mantained a "quantum group" as the symmetry of the system, and we obtain braided Hilbert spaces. It has to be pointed out that this is the only place where the quasitriangularity of quantum groups comes into play. We present a discussion of the physical significance of this result.

The organization of this paper is as follows. In order to make it as self-contained as possible, section two is a review of the relevant definitions and conventions of Hopf algebra theory, including a careful presentation of * structures. We also review the algebras of functions on the quantum spheres, and their represention theory, and the Hopf algebras $U_q(su_2)$ and $\mathrm{Fun}_q(su_2)$. In section three there is a short discussion of C*-dynamical systems and quantum mechanics on homogenous spaces, which naturally leads us to consider cross product algebras and to introduce the definition of generalized dynamical systems and their covariant representations. We propose a general construction for these representations. We prove that for compact matrix pseudogroups with faithful Haar measure and corresponding coset spaces, the covariant representations induced by irreducible (co-)representations of the invariance subgroup are themselves irreducible. Section four analyzes what the dynamics must be if they are to preserve the Hopf algebra symmetry we start with, and extends this result to multiparticle systems, in the case of quasitriangular Hopf algebras. In section five we apply the general theory to the case of the quantum sphere, starting with the kinematics, going on to the dynamics and then proceeding to multiparticle hamiltonians. The connection with Podleš' laplacians is studied. The paper is ended by a short conclusion where a number of remaining questions are posed. We concentrate throughout on the algebraic and physical concepts, and obviate some of the functional analytical and topological aspects.

2. Definitions and Conventions

2.1. Hopf algebras

A **coalgebra** C is a vector space (over some field k), with an additional linear mapping $\Delta : C \to C \otimes C$, called the coproduct, which is coassociative, and a linear map, the counit, $\epsilon : C \to k$. Coassociativity means that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. The properties of the counit are that $(\epsilon \otimes \mathrm{id}) \circ \Delta(c) = (\mathrm{id} \otimes \epsilon) \circ \Delta(c) = c$. We shall in what follows use Sweedler's notation for the coproduct: $\Delta(c)$ is, for all $c \in C$, a sum of terms of the form $c_{i(1)} \otimes c_{i(2)}$. For compactness of the formulae, both the symbol for sum over i and the subindex i will be omitted, thus representing $\Delta(c)$ as $c_{(1)} \otimes c_{(2)}$. Similarly for consecutive applications of the coproduct.

In this notation, coassociativity reads as follows:

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}\,, \tag{1}$$

and the counit leads us to

$$\epsilon(c_{(1)})c_{(2)} = \epsilon(c_{(2)})c_{(1)} = c\,. \tag{2}$$

Define also a coalgebra map as a map between coalgebras that respects the coalgebra structure, i.e., f is a coalgebra map if $(f \otimes f) \circ \Delta = \Delta \circ f$.

A **Hopf algebra,** H, is an associative algebra (over a field k), with identity (the product being denoted by juxtaposition), and a coassociative coalgebra with counit, in a manner compatible with the algebra structure, and such that it possess an antialgebra and anticoalgebra morphism, S, called the antipode. The compatibility of algebra and coalgebra structures correspond to saying that the product and the unit are coalgebra maps, or, alternatively, that the coproduct and the counit are algebra maps. These compatibility conditions, and the properties of the antipode, are as follows:

$$\begin{aligned} &\Delta(hg) = \Delta(h)\Delta(g)\,,\ \Delta(1) = 1 \otimes 1\,,\ \epsilon(hg) = \epsilon(h)\epsilon(g)\,,\\ &S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \epsilon(h)1\,, \quad \epsilon(S(g)) = \epsilon(g)\,. \end{aligned} \tag{3}$$

The elements g of H such that $\Delta(g) = g \otimes g$ are called grouplike. Those elements p of H such that $\Delta(p) = p \otimes 1 + 1 \otimes p$ are called primitive.

The usual examples of (cocommutative) Hopf algebras are the algebra of functions on a group, the group algebra, the universal enveloping Hopf algebra of a group, etc. For comprehensive accounts of the theory of Hopf algebras, see [38],[1].

The dual Hopf algebra H° of a Hopf algebra H is defined as the subspace of the dual of H such that for any element of H° its kernel contains an ideal of H of finite codimension. If H is finite dimensional, it is clear that the dual linear space can be endowed with the structure of a Hopf algebra in the obvious way: the multiplication in H induces the comultiplication in the dual, comultiplication in H leads to multiplication in the dual, etc. Similarly for H° in the infinite dimensional case. Denote the duality pairing by $\langle\,,\rangle$, and let $h \in H$, $u, v \in H^\circ$, etc.. Then we have that $\langle h, uv\rangle = \langle h_{(1)}, u\rangle\langle h_{(2)}, v\rangle$, and so on. For details, see [24] or the references above. If H is a topological Hopf algebra [30], the dual can be endowed with topological Hopf algebra structure.

A very important element of the dual of a Hopf algebra, if it exists, is the integral. An element ω of the dual H^* is a (right) integral if for all $f \in H^*$, $\omega f = \langle f, 1\rangle \omega$. Left integrals are analogously defined. If an integral exists, it is unique up to normalization [1].

An important class of Hopf algebras is that of quasitriangular Hopf algebras. They have the additional property that there exists an element of $H \otimes H$, called $\mathcal{R}$, such that $\mathcal{R}\Delta(h) = ((\tau \circ \Delta)(h))\mathcal{R}$, with $(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ and $(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$, where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, and, if we write $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, we have that $\mathcal{R}_{13} = \mathcal{R}^{(1)} \otimes 1 \otimes \mathcal{R}^{(2)}$, and similarly for $\mathcal{R}_{23}$.

2.2. *-Hopf algebras

A ***-Hopf algebra**, H, is a Hopf algebra over the complex numbers with an additional internal operation, denoted $*$, such that, for $h \in H$ and λ a complex number ($\bar{\lambda}$ being complex conjugation),

$$\begin{gathered} (\lambda h)^* = \bar{\lambda} h^*, \quad * \circ \cdot = \cdot \circ \tau \circ (* \otimes *) = \cdot \circ (* \otimes *) \circ \tau, \\ (* \otimes *) \circ \Delta = \Delta \circ *, \quad \epsilon(h^*) = \overline{\epsilon(h)}, \quad * \circ * = \mathrm{id}, \\ (S \circ *)^2 = \mathrm{id}, \end{gathered} \tag{4}$$

where τ is the permutation, $\tau(v \otimes w) = w \otimes v$. It is clear that the * and the antipode do not in general commute. The dual Hopf algebra of a *-Hopf algebra can be given a * structure by $\langle h^*, v\rangle = \overline{\langle h, (S(v))^*\rangle}$, and viceversa. For any element f of the dual of a *-Hopf algebra H, the adjoint is defined as $\tilde{f} \in H^*$, such that $\langle \tilde{f}, h\rangle = \overline{\langle f, h^*\rangle}$ for all $h \in H$, and the statement above is that, for all $u \in H^o$, $\tilde{u} = (Su)^*$. An important object to look at is the adjoint $\tilde{\omega}$ of the (right) integral ω, if the latter exists. It can be proved that, if ω exists, for all $f \in H^*$, $\tilde{\omega} f = \langle f, 1\rangle \tilde{\omega}$. The uniqueness of the integral then forces the adjoint of the integral to be proportional to the integral.

2.3. $U_q(su_2)$ and $\mathrm{Fun}_q(su_2)$

The best known example of dually paired algebras is that of $U_q(su_2)$ and $\mathrm{Fun}_q(su_2)$ (see [17][9][39][14] for their introduction and pairing). A good treatment of this pairing from the point of view of topology can be found in [30]. Here we simply write the algebraic definitions, in order to set our conventions, and refer to the literature for a fuller discussion on the character of this pairing.

$\mathrm{Fun}_q(su_2)$ is defined as the associative algebra over the complex numbers, with identity, generated by $\{d^{1/2}_{ij}\}_{i,j=\pm 1/2}$ under the relations (not writing the superindex)

$$\begin{gathered} d_{++}d_{+-} = q d_{+-}d_{++}, \quad d_{++}d_{-+} = q d_{-+}d_{++}, \quad d_{+-}d_{-+} = d_{-+}d_{+-}, \\ d_{+-}d_{--} = q d_{--}d_{+-}, \quad d_{-+}d_{--} = q d_{--}d_{-+}, \quad d_{--}d_{++} - q^{-1} d_{-+}d_{+-} = 1, \\ d_{++}d_{--} - q d_{+-}d_{-+} = 1. \end{gathered} \tag{5}$$

Here, and in what follows, q is a real positive constant (this algebra could be defined for q any complex number, and, in fact, q being a root of unity is a specially important case, but

in order to make it into a *-algebra we have to require reality of q). The comultiplication is $\Delta(d^{1/2}_{ij}) = \sum_k d^{1/2}_{ik} \otimes d^{1/2}_{kj}$, the counit $\epsilon(d^{1/2}_{ij}) = \delta_{ij}$, and the antipode

$$S\begin{pmatrix} d^{1/2}_{++} & d^{1/2}_{+-} \\ d^{1/2}_{-+} & d^{1/2}_{--} \end{pmatrix} = \begin{pmatrix} d^{1/2}_{--} & -q^{-1} d^{1/2}_{+-} \\ -q d^{1/2}_{-+} & d^{1/2}_{++} \end{pmatrix}. \tag{6}$$

This algebra can be given an * structure that makes it into a *-Hopf algebra:

$$\begin{gathered} (d^{1/2}_{++})^* = d^{1/2}_{--}, \quad (d^{1/2}_{+-})^* = -q d^{1/2}_{-+}, \\ (d^{1/2}_{-+})^* = -q^{-1} d^{1/2}_{+-}, \quad (d^{1/2}_{--})^* = d^{1/2}_{++}. \end{gathered} \tag{7}$$

There are a set of elements $\{d^l_{ij}\}_{i,j=-l,\ldots,l}$ such that $\Delta(d^l_{ij}) = \sum_k d^l_{ik} \otimes d^l_{kj}$, for $l = 1/2, 1, 3/2, \ldots$ These elements are the deformed analogue of the matrix element functions of the spin l representation. They are constructed out of products of generators of $\mathrm{Fun}_q(su_2)$ by means of deformed Clebsch-Gordan coefficients.

The algebra $\mathrm{Fun}_q(su_2)$ is dually paired to $U_q(su_2)$, which is defined to be the associative algebra over the complex numbers, with identity, generated by j_+, j_- and j_z, under the relations

$$[j_z, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = \frac{k^2 - k^{-2}}{q - q^{-1}}, \tag{8}$$

where $k = q^{j_z}$, $k^{-1} = q^{-j_z}$. These last elements do not belong to the algebra as we have defined it, but this can be solved by setting the definition in a topological framework. The other operations that make $U_q(su_2)$ into a *-Hopf algebra are given by

$$\begin{gathered} \Delta(j_\pm) = j_\pm \otimes k^{-1} + k \otimes j_\pm, \quad \Delta(j_z) = j_z \otimes 1 + 1 \otimes j_z, \\ S(j_\pm) = -q^{\mp 1} j_\pm, \quad S(j_z) = -j_z, \quad \epsilon(j_\pm) = 0, \quad \epsilon(j_z) = 0, \\ (j_\pm)^* = j_\mp, \quad (j_z)^* = j_z. \end{gathered} \tag{9}$$

It is important to notice that $U_q(su_2)$ is a quasitriangular Hopf algebra (when a suitable topological completion has been taken). The representation theory of this algebra follows that of $U(sl_2)$, the universal enveloping algebra of $sl(2, \mathbb{C})$, which is the complexification of su_2, and each of the irreducible representations is labelled by j, the spin, with $2j$ an integer. For instance, the spin one representation is, in ket notation, as follows:

$$\begin{gathered} j_+|11\rangle = 0 \quad j_+|10\rangle = [2]^{1/2}|11\rangle \quad j_+|1-1\rangle = [2]^{1/2}|10\rangle \\ j_-|11\rangle = [2]^{1/2}|10\rangle \quad j_-|10\rangle = [2]^{1/2}|1-1\rangle \quad j_-|1-1\rangle = 0 \\ j_z|1m\rangle = m|1m\rangle \end{gathered} \tag{10}$$

where the square brackets indicate q-deformed numbers: $[x] = (q^x - q^{-x})/(q - q^{-1})$

The duality pairing is given for the generators of these *-Hopf algebras by

$$\begin{gathered} \langle j_+, d^{1/2}_{ij}\rangle = \delta_{i+}\delta_{j-}, \quad \langle j_-, d^{1/2}_{ij}\rangle = \delta_{i-}\delta_{j+}, \\ \langle j_z, d^{1/2}_{ij}\rangle = \pm\delta_{ij}, \quad \langle 1, d^{1/2}_{ij}\rangle = \epsilon(d^{1/2}_{ij}) = \delta_{ij}, \\ \langle h, 1\rangle = \epsilon(h) \text{ for all } h \in U_q(su_2). \end{gathered} \tag{11}$$

2.4. *Hopf algebra module algebras, module *-algebras*

Let H be a Hopf algebra. Let A be an associative algebra with identity, not necessarily commutative. We say that A is a (left) H-module algebra if there exists an action α of H on A (i.e., an algebra homomorphism from H into $\mathrm{End}(A)$), such that

$$\alpha_h(a \cdot b) = \alpha_{h_{(1)}}(a) \cdot \alpha_{h_{(2)}}(b)\,, \quad \alpha_h(1) = \epsilon(h)1\,, \tag{12}$$

for all $h \in H$, $a, b \in A$. If A is a locally finite H-module, which means that $\dim(\alpha_H(a)) < \infty$ for all $a \in A$ (dimension over the field), then A will be an H^o-comodule [1]. In the topological setting of [30], it can be seen that a locally finite continuous module of H will be a continuous comodule of the dual topological Hopf algebra. In what follows we shall assume that we are always dealing with locally finite H-modules. Let H be now a *-Hopf algebra, and A an *-algebra with identity. We say that A is a (left) unitary H-module *-algebra if it is a (left) H-module algebra, and we further have

$$(\alpha_h(a))^* = \alpha_{(Sh)^*}(a^*)\,, \tag{13}$$

for all $h \in H$, $a \in A$. It is easy to check that this definition is consistent. Notice that it corresponds to the star structure on A being consistent with the comodule structure Δ_A that sends A to $A \otimes H^o$, where H^o is the dual Hopf algebra to H:

$$\Delta_A \circ *_A = (*_A \otimes *_{H^o}) \circ \Delta_A\,. \tag{14}$$

In fact, the dual Hopf algebra H^o to a *-Hopf algebra H, with the star structure defined above, and the action α of H on H^o defined by $\alpha_h(v) = \langle h, v_{(2)}\rangle v_{(1)}$, is a unitary H-module *-algebra.

An important question is the study of the tensor product of *-algebras, of relevance, for instance, in the study of multiparticle systems. In particular, for an unitary H-module *-algebra A , with action α, is it feasible to define a tensor product algebra $A \otimes A$ such that it is also an unitary H-module *-algebra? Let us just point out here that if H is quasitriangular, there exist a natural concept of braiding given by $\mathcal{R}$. Let us now define the (braided!) product for $A \otimes A$ by

$$(a \otimes b) \cdot (c \otimes d) = a(\alpha_{\mathcal{R}^{(2)}}(c)) \otimes (\alpha_{\mathcal{R}^{(1)}}(b))d\,, \tag{15}$$

for all $a, b, c, d \in A$, where $\mathcal{R}$, which belongs to $H \otimes H$, is written as $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$ (See [22]).

Then, if $\mathcal{R}$ satisfies

$$\mathcal{R} = (\tau \circ (* \otimes *))(\mathcal{R})\,, \tag{16}$$

the involution $*_2 = \tau \circ (* \otimes *)$ and the action α', given by

$$\alpha'_h(a \otimes b) = \alpha_{h_{(1)}}(a) \otimes \alpha_{h_{(2)}}(b)\,, \tag{17}$$

make $A \otimes A$ an unitary H-module *-algebra.

For example, the quasitriangular *-Hopf algebra $U_q(su_2)$ satisfies condition (16), and thus we have a way of constructing tensor products of unitary $U_q(su_2)$-module *-algebras.

2.5. *The quantum sphere $A_{q,\mu}$*

In the spirit of non-commutative geometry [6], a non-commutative analogue of the sphere will be given by the spectrum of a non-commutative C*-algebra, which is in some sense related to the commutative C*-algebra of continuous function on the (normal) sphere. This analogue of the sphere has been introduced by Podleś [34]. Our conventions will be somewhat different, but correspond to the same algebra.

To justify the introduction of these algebras, observe first that, since the sphere is a compact space, the ring of polynomials over the coordinate functions is dense in the space of continuous functions, by the Stone-Weierstrass theorem. So, let us introduce the algebra A, which is an associative algebra, with identity, generated by the commuting coordinate functions Φ_+, Φ_- and Φ_z, under the relation $-\Phi_+\Phi_- - \Phi_-\Phi_+ + \Phi_z\Phi_z = \lambda$. λ will then be the radius squared. The sphere is a homogenous space: the group of rotations acts transitively on it. This translates into an action of $SU(2)$ on the algebra A, given by the generators transforming under the spin one representation of $SU(2)$, and extended to the rest of A as $g(\psi.\xi) = g(\psi).g(\xi)$ for $g \in SU(2)$, $\psi, \xi \in A$. In fact, the sphere is the coset space $SO(3)/SO(2)$, and the algebra of functions on the sphere is seen to be isomorphic to the subspace of functions on $SU(2)$ invariant under the (right) action of $U(1)$: $\mathrm{Fun}(S^2) \cong \mathrm{Fun}(SU(2))^{U(1)}$.

Now, we have an analogue of the group of rotations, as shown in sections 2.3 and 2.4. So we consider the associative but not commutative algebra with identity, generated by Φ_+, Φ_- and Φ_z, under some (at most) quadratic relations, and such that the generators transform under the action of $U_q(su_2)$ with the spin one representation, and the action of $U_q(su_2)$ on the rest of the algebra is determined by it being a module algebra, i.e.,

$$h(\psi.\xi) = h_{(1)}(\psi).h_{(2)}(\xi) \quad \text{for } h \in U_q(su_2) \text{ and } \psi, \xi \text{ in the algebra.} \tag{18}$$

(We have omitted writing the action α and represent it as above) Then the following relations are obtained (uniquely if we also impose that there be a sum of quadratic elements that is set to be equal to the unit):

$$\begin{aligned} \Phi_+\Phi_z &= q^2\Phi_z\Phi_+ + \mu\Phi_+, \\ \Phi_z\Phi_- &= q^2\Phi_-\Phi_z + \mu\Phi_-, \\ \Phi_+\Phi_- &= \Phi_-\Phi_+ + (q - q^{-1})\Phi_z\Phi_z + q^{-1}\mu\Phi_z, \\ -q^{-1}\Phi_+\Phi_- - q\Phi_-\Phi_+ + \Phi_z\Phi_z &= \lambda 1. \end{aligned} \tag{19}$$

Here λ and μ are real constants. An involution compatible with these relations can be defined by

$$(\Phi_\pm)^* = -q^{\mp 1}\Phi_\mp, \quad \Phi_z^* = \Phi_z, \quad 1^* = 1, \tag{20}$$

extending to the whole algebra by $(\psi\xi)^* = \xi^*\psi^*$. With this *-structure, it is a unitary $U_q(su_2)$-module *-algebra. We shall call this algebra $A_{q,\mu}$, for given values of q and μ, and we will refer to it as the quantum sphere algebra, or the algebra of functions over the quantum sphere, for these values of q and μ. Notice that $A_{q,\mu}$ is a locally finite $U_q(su_2)$ module, and therefore a comodule of the dual Hopf algebra.

It is useful to define the element $\zeta = \Phi_z + \sigma 1$, where $\sigma = q^{-1}\mu/(q-q^{-1})$, since

$$\Phi_+\zeta = q^2\zeta\Phi_+, \quad \zeta\Phi_- = q^2\Phi_-\zeta. \tag{21}$$

Let us look at finite dimensional representations of this algebra $A_{q,\mu}$. Inspection of equations (19) suggests that we shall obtain one if we represent Φ_+ by an upper triangular matrix with no diagonal entries, Φ_- by a lower triangular matrix, also with no diagonal entries, and Φ_z by a diagonal matrix. There will then exist a lowest weight vector $|\Omega\rangle$ such that $\pi(\Phi_-)|\Omega\rangle = 0$, and orthogonal vectors to this one are obtained by repeatedly applying $\pi(\Phi_+)$ to $|\Omega\rangle$. Since Φ_+ is represented by an upper triangular matrix with no diagonal entries, there exists a number n such that $\pi(\Phi_+^n) = 0$. This n will label this representation. Let us call a_n the eigenvalue of ζ corresponding to $|\Omega\rangle$. Then $\pi(\zeta)\pi(\Phi_+^k)|\Omega\rangle = q^{-2k}a_n\pi(\Phi_+^k)|\Omega\rangle$.

Then, the relations (19),the fact that $|\Omega\rangle$ is annihilated by $\pi(\Phi_-)$, and the existence of a minimum n such that $\pi(\Phi_+^n) = 0$ lead us to the following result:

$$\begin{aligned} a_n &= \frac{q^{n-1}[2]\sigma}{q^n+q^{-n}}, \\ \mu^2 = \mu_n^2 &= \frac{q^2\lambda(q^n+q^{-n})^2}{[n+1][n-1]}. \end{aligned} \tag{22}$$

Defining $\{|m\rangle\}_{m=0,1,\ldots,n-1}$ to be the orthonormal basis obtained from $|\Omega\rangle$ by repeated action of $\pi(\Phi_+)$, the n-dimensional representation of $A_{q,\mu}$ (for $\mu = \mu_n$), π_n is given by

$$\begin{aligned} \pi_n(\Phi_+)|m\rangle &= \left(\frac{q^{-1}\lambda[2][n-m-1][m+1]q^{n-2m-2}}{[n+1][n-1]}\right)^{1/2}|m+1\rangle, \\ \pi_n(\Phi_-)|m\rangle &= -\left(\frac{q\lambda[2][n-m][m]q^{n-2m}}{[n+1][n-1]}\right)^{1/2}|m-1\rangle, \\ \pi_n(\zeta)|m\rangle &= q^{-2m}a_n|m\rangle. \end{aligned} \tag{23}$$

It can be checked that this is indeed a representation of $A_{q,\mu}$, when μ is given by equation (22). Notice that π_n is a star representation: $\pi_n(\psi^*) = \pi_n(\psi)^*$.

In order to find the other possible representations, consider an eigenvector $|b\rangle$ of $\pi(\zeta)$ for some representation π, with eigenvalue b. The norm of $\pi(\Phi_+)|b\rangle$ will be

$$\|\pi(\Phi_+)|b\rangle\|^2 = -\frac{q^{-1}}{[2]}\left(q^{-2}b^2 - q^{-1}[2]\sigma b + \sigma^2 - \lambda\right). \tag{24}$$

If $\lambda > \sigma^2$ and $0 < q < 1$, this forces the eigenvalues of $\pi(\zeta)$ to be of the form $q^{2n}a_\pm$, for $n = 0, 1, \ldots$, and $a_\pm = q\left(\frac{[2]}{2}\sigma \pm \frac{1}{2}\sqrt{(q-q^{-1})^2\sigma^2 + 4\lambda}\right)$. We see we obtain two infinite dimensional irreducible representations, $\pi_\pm$, such that the representatives of the generators of $A_{q,\mu}$ act on the basis $\{|n,\pm\rangle\}_{n=0,1,\ldots}$ as follows:

$$\begin{aligned} \pi_\pm(\zeta)|n,\pm\rangle &= q^{2n}a_\pm|n,\pm\rangle, \\ \pi_\pm(\Phi_+)|n,\pm\rangle &= q^{-1/2}[2]^{-1/2}|P(q^{2n-1}a_\pm)|^{1/2}|n-1,\pm\rangle, \\ \pi_\pm(\Phi_-)|n,\pm\rangle &= -q^{1/2}[2]^{1/2}|P(q^{2n+1}a_\pm)|^{1/2}|n+1,\pm\rangle, \end{aligned} \tag{25}$$

where we have defined

$$P(x) = x^2 - [2]\sigma x + \sigma^2 - \lambda.$$

For $\lambda > \sigma^2$ we also have one-dimensional representations given by $\pi_\gamma(\zeta) = 0$ and $\pi_\gamma(\Phi_+) = q^{-1/2}[2]^{-1/2}(\lambda-\sigma^2)^{1/2}\gamma$, where $|\gamma| = 1$.

If $\lambda = \sigma^2$ we only get one irreducible infinite dimensional representation (since either a_+ or a_- are equal to zero). If $\lambda < \sigma^2$, the only possibilities of irreducible representations are the finite dimensional ones expressed above (always assuming that $\lambda > 0$).

A further important fact concerning the quantum sphere is the theorem of Podleś ([34]) which expresses the fact that for $\lambda > \sigma^2$ the algebra $A_{q,\mu}$ can be decomposed as the direct sum of $U_q(su_2)$ modules of dimension $2n+1$, for $n = 0, 1, \ldots$. This theorem has been developed by Noumi and Mimachi, who obtain the spherical functions on the quantum spheres [31]. The study of the representations of the quantum spheres was first undertaken and completed by Podleś [34]. From his work and that of [29], it can be seen that the norm closure of the adequate (faithful) representation of the algebra of functions on a quantum sphere is a C*-algebra.

From the general construction presented in section 2.4, we see that $A_{q,\mu} \otimes A_{q,\mu}$ with braided product and involution $*_2 = \tau \circ (* \otimes *)$ is an unitary $U_q(su_2)$-module *-algebra, as was announced in [10].

The braiding is given in the usual way,

$$\Psi(a \otimes b) = \mathcal{R}^{(2)}(b) \otimes \mathcal{R}^{(1)}(a), \tag{26}$$

which, for the generators, corresponds to the R^{11} matrix of $U_q(su_2)$.

3. Quantum Mechanics on Homogeneous Spaces

3.1. C*-dynamical systems

The concept of C*-dynamical system is the basis for the general algebraic theory of symmetries in quantum systems. It has special relevance in the process of quantization on homogenous spaces [8], but also for time evolution in quantum statistical mechanics, and for the study of continuous symmetries in algebraic quantum field theory [16], [5].

A **C*-dynamical system** is defined as the triple (A, G, α), where A is a C*-algebra, G is a locally compact group, and α a strongly continuous representation of G on A. A **covariant representation** of a C*-dynamical system is a triple $(\mathcal{H}, \pi, \rho)$, where $\mathcal{H}$ is a Hilbert space, π a non-degenerate representation of the C*-algebra A in $\mathcal{H}$, and ρ is a strongly continuous unitary representation of G in $\mathcal{H}$, such that for all $g \in G$ and all $a \in A$,

$$\pi(\alpha_g(a)) = \rho(g)\pi(a)\rho(g^{-1}). \tag{27}$$

A functional ω on the algebra A is called G-invariant if

$$\omega(\alpha_g(a)) = \omega(a) \quad \text{for all } g \in G,\, a \in A. \tag{28}$$

One can use the Gel'fand-Naimark-Segal (GNS) construction to generate in a canonical fashion cyclic covariant representations for each G-invariant state, and the cyclic vector Ω_ω is invariant under $\rho(G)$.

The representation theoy of a C*-dynamical system (A, G, α) is equivalent to the representation theory of the crossed product $A \rtimes_\alpha C^*(G)$.

The relevance of C*-dynamical systems in the context of quantization of particles moving on homogeneous spaces lies in equation (27), which corresponds to the Heisenberg commutations relations, or, in the classical context, to the Poisson algebra structure on the phase space T^*Q. Since, if the C*-algebra A is the algebra of functions on a configuration space Q which is a homogeneous space for the group G, the momentum observables will be given by the differentials of the group action, and thus the canonical commutation relations between momenta and coordinate functions will correspond to equation (27). From the point of view of representation theory, equation (27) corresponds to the definition of system of imprimitivity introduced by Mackey [21].

3.2. *Generalized dynamical systems*

Several extensions of the concept of C*-dynamical system are present in the literature. One first important instance is that of [12],[13], concentrated on the action of Kač algebras on von Neumann algebras. Another point of view, inspired by questions of duality and selfduality in physics, has been that of Majid in [25],[26].

We now introduce our definition of generalized dynamical system: it consists of a triplet (A, U, α), where A is a unital C*-algebra, not necessarily commutative, U a *-Hopf algebra, and α a left action of U on A, such that A is a locally finite unitary U-module *-algebra. Associated with this definition, we have that of a covariant representation of a generalized dynamical system, which is a triplet $(\mathcal{H}, \pi, \rho)$, where $\mathcal{H}$ is a Hilbert space, π is a nondegenerate representation of the C*-algebra A in $\mathcal{H}$, and ρ is a Hermitian representation of U in $\mathcal{H}$, and this triple satisfies

$$\pi(\alpha_h(a)) = \rho(h_{(1)})\pi(a)\rho(Sh_{(2)})\,. \tag{29}$$

It must be noted that the elements of U will in general be represented by unbounded operators on the Hilbert space. This immediately produces problems of domains and selfadjointness. We shall demand that the domain of $\rho(U)$, $\mathcal{D}(\rho)$, be dense in $\mathcal{H}$. For ρ to be a representation it is also needed that Range$(\rho(h)) \subset \mathcal{D}(\rho)$ for all $h \in U$. A representation ρ of U will be Hermitian if

$$\rho(h^*) \subset \rho(h)^*\,. \tag{30}$$

That is, $\rho(h)^*$ is an extension of $\rho(h^*)$ (the domain of $\rho(h)^*$ contains that of $\rho(h^*)$, and they are equal on the latter). Equation (29) has to be understood as defined on the domain of $\rho(U)$, or, alternatively, change the equal sign to an inclusion of the right hand side in the left hand side.

We could relax the demand that A be a C*-algebra to it being an involutive algebra, and then ask for π to be Hermitian and the domain of π to be dense in $\mathcal{H}$. The assumption that it be a C*-algebra is necessary to make contact with non-commutative geometric

notions, though. In fact, in what follows we shall mostly ignore the norm of A, and proceed in a purely algebraic manner. Similarly with the problems derived from unboundedness. The assumption the A be a unital algebra is in principle a restriction to "compact" spaces, but it can be lifted without too much difficulty.

Let us look at the possible physical significance of such a definition. In fact, a little familiarity with the notion that the coproduct corresponds to the distribution of a quantum observable between two subsystems immediately gives us a clue as to its meaning. If we were to evaluate both sides of equation (29) between two vectors, the left hand side would entail a transformation of the physical system, whereas the right hand side corresponds to an equivalent transformation of the two vectors, leaving the system unchanged. That is to say, we are moving from an active to a passive transformation and viceversa. The reason for the appearance of the coproduct is now seen to correspond to the sharing out of the transformation between the two vectors. It follows from this analysis that we can correctly identify equation (29) as the main equation for the implementation of symmetries in a quantum system.

3.3. *Cross product algebras*

It is a well known fact that the irreducible covariant representations of a C* dynamical system are in one-one correspondence with the irreducible representations of the crossed product algebra $A \rtimes_\alpha C^*(G)$. But the definition of crossed product algebra is a standard construction in Hopf algebra theory [24]. Let A be a U-module algebra, with action α. The cross product algebra, $A \rtimes_\alpha U$ is defined to be $A \otimes U$ as the underlying vector space (the tensor product being over the complex numbers, of course), with product given by

$$(a \otimes h)\cdot(b \otimes g) = a\,\alpha_{h_{(1)}}(b) \otimes h_{(2)}g\,, \tag{31}$$

Let us now consider A to be a unitary U-module *-algebra. Then we can give $A \rtimes_\alpha U$ a * structure, as follows:

$$(a \otimes h)^* = \alpha_{h^*_{(1)}}(a^*) \otimes h^*_{(2)}\,. \tag{32}$$

With this * structure, $A \rtimes_\alpha U$ is a *-algebra. It is also a U-comodule *-algebra, with comodule structure $\tilde\Delta$ given by

$$\tilde\Delta(a \otimes h) = a \otimes \Delta(h) = a \otimes h_{(1)} \otimes h_{(2)}\,, \tag{33}$$

and the star structure obeys

$$\tilde\Delta \circ * = (* \otimes *_U) \circ \tilde\Delta\,. \tag{34}$$

Both A and U are *-subalgebras of this * algebra, with the obvious inclusion and restriction maps.

It is now clear that, granted sufficient conditions on the domains of definition of the representing operators, the covariant representations of a generalized dynamical system (A, U, α) will be equivalent to the representations of the crossed product algebra $A \rtimes_\alpha U$, as follows. To each covariant representation of the generalized dynamical system $(\mathcal{H}, \pi, \rho)$, assign the hermitian representation π_ρ of $A \rtimes_\alpha U$ on the same Hilbert space, with domain $\mathcal{D}(\pi_\rho) = \mathcal{D}(\pi) \cap \mathcal{D}(\rho)$ given by

$$\pi_\rho(a \otimes h) = \pi(a)\rho(h)\,, \tag{35}$$

which can be seen to be a *-algebra morphism, and, conversely, given a hermitian representation of the crossed product algebra, we obtain a covariant representation of (A, U, α) on the same Hilbert space by restricting to the subalgebras A and U and domains $\mathcal{D}(\rho) = \mathcal{D}(\pi_\rho)$ and $\mathcal{D}(\pi) = \cap_{a\in A}\mathcal{D}(\pi_\rho(a\otimes 1))$, possibly extending $\pi(A)$ to a closed self-adjoint extension (feasible if it is a C*-algebra). The covariance of such a representation is ensured by the algebra structure of the crossed product algebra, and by its star structure.

3.4. Construction of covariant representations

We start with the archetypical example. Suppose there exists a U-invariant positive linear functional, L, on the unitary U-module *-algebra A. U-invariance means that, for all $h \in U$ and all $a \in A$,

$$L(\alpha_h(a)) = \epsilon(h)L(a).$$

It can be used to define a hermitian form,

$$\langle a, b\rangle = L(a^* b), \tag{36}$$

and the standard GNS construction can be applied. The unitarity of the action of U on A, and the invariance of the linear functional ensure that

$$\langle a, \alpha_h(b)\rangle = L(a^*.\alpha_h(b)) = L((\alpha_{h^*}(a))^*.b) = \langle \alpha_{h^*}(a), b\rangle. \tag{37}$$

This, together with the positivity of L, makes the ideal $\mathcal{I}$ of elements I of A such that $L(i^*i) = 0$ invariant under the action of U. In this way, the Hilbert space $\mathcal{H}_0$ obtained by completing the pre-Hilbert space $A/\mathcal{I}$ (in fact, $A_f/\mathcal{I}$, where $A_f = \{a \in A | L(a^*a) < \infty\}$) will carry a hermitian representation ρ_0 of U given by $\rho_0(h)(a+\mathcal{I}) = \alpha_h(a) + \mathcal{I}$, and a hermitian representation π_0 of A given by $\pi_0(b)(a+\mathcal{I}) = ba + \mathcal{I}$. That this is a covariant representation can readily be seen:

$$\begin{aligned}\pi_0(\alpha_h(a))(b+\mathcal{I}) &= \alpha_h(a).b + \mathcal{I} = \alpha_{h_{(1)}}(a).\epsilon(h_{(2)})b + \mathcal{I} = \\ &= \alpha_{h_{(1)}}(a).\alpha_{h_{(2)(1)}S(h_{(2)(2)})}(b) + \mathcal{I} = \alpha_{h_{(1)}}(a.\alpha_{S(h_{(2)})}(b)) + \mathcal{I} = \\ &= \rho_0(h_{(1)})\pi_0(a)\rho_0(S(h_{(2)}))(b+\mathcal{I}).\end{aligned} \tag{38}$$

In the case of the quantum sphere this representation corresponds to Podleś' $L^2(S^2_{\mu c})$ [34].

This idea can be extended as follows. Suppose there exists a unitary U-module *-algebra B such that A is a * subalgebra of B, and such that there exists a U-invariant positive linear functional on it. Then the GNS construction will immediately give us a covariant representation of (A, U, α) (Note that the null space corresponding to the linear functional is invariant under the action of A and U). Now, $A \rtimes_\alpha U$ is a U-module algebra if we define the action γ as

$$\gamma_h(a\otimes g) = \alpha_{h_{(1)}}(a)\otimes h_{(2)}gS(h_{(3)}). \tag{39}$$

Even more, it is a unitary U-module * algebra, with the previously defined * structure. The remaining question is the existence of a U-invariant positive linear functional. But

for the interesting cases the Haar measure ω of the quantum group is ad-invariant, and therefore, if there exists a linear positive U-invariant linear functional L on A, the linear functional $\tilde{L}$ on $A \rtimes_\alpha U$ defined by

$$\tilde{L}(a\otimes h) = L(a)\omega(h) \tag{40}$$

is U-invariant. It is obvious that the GNS representation will be a reducible representation of $A \rtimes_\alpha U$ (where we use the correspondence between covariant representations and represention of $A \rtimes_\alpha U$). For example, the previously defined Hilbert space $\mathcal{H}_0$ is isomorphic to the invariant subspace obtained from $A\otimes 1$.

3.5. Left regular representation

Another, possibly more interesting representation, is obtained when we impose on the generalized dynamical system (A, U, α) the condition that there exists an *-algebra homomorphism from A to the dual Hopf algebra U°, which is also a (right) comodule morphism. If this map is also an injection we shall say that A is an algebra of functions on a generalized homogeneous space of U (or we shall call A a generalized homogeneous space of U). It can readily be seen that the algebras of functions on the quantum spheres are algebras of functions on generalized homogeneous spaces of $U_q(su_2)$, and the *-algebra and comodule homomorphism i from $A_{q,\mu}$ to the dual Hopf algebra U°, which in this case is (essentially) the algebra of functions over the quantum group, $\mathrm{Fun}_q(SU_2)$, is defined on the generators by ([34])

$$\begin{gathered} i(\Phi_k) = \sum c_j d^1_{jk}, \\ c_1 = q^{1/2}[2]^{-1/2}(\lambda-\sigma^2)^{1/2}, \quad c_0 = -\sigma, \quad c_{-1} = -q^{-1}c_1, \end{gathered} \tag{41}$$

where d^1_{jk} are the q-analogues of the matrix element functions corresponding to the spin one representation (cf. [39],[40] and section 2.3).

Going back to the general case $i: A \to U^\circ$, we have already seen that U° is a unitary U-module *-algebra, and if A is an algebra of functions on a generalized homogeneous space of U, there also exists an action β of A on U° given by $\beta_a(v) = i(a)v$, with the product in U° denoted by juxtaposition.

If we make the further assumption that there exists an linear functional ω on U° that is positive and invariant under the action of U (which immediately implies the existence of an U-invariant linear functional on A, $L = \omega\circ i$, and allows us to construct $\mathcal{H}_0$ as before), we can make use of the GNS construction once more, and the resulting Hilbert space, $\mathcal{H}_L$, will carry a covariant representation of $A \rtimes_\alpha U$, with the representations π_L and ρ_L being induced by the respective actions on U°. That this representation satisfies the "imprimitivity" equation (29) is due to the crucial fact that i is a comodule and algebra morphism, and the hermiticity is due to it being a * morphism. It is clear that $U^\circ \subset \mathcal{D}(\rho_L)$ (when we assume that, for all $v \in U^\circ$, $\omega(v), \infty$). A is represented by bounded operators on $\mathcal{H}$, the usual operator norm being given by

$$\|\pi_L(a)\| = \omega(i(a^*a)).$$

Let us call this representation the left regular representation of $A \rtimes_\alpha U$, due to the obvious analogy to the left regular representations of groups and transformation group C*-algebras (cf. [18][19]). If such a linear functional exists, and U is dense in $(U^\circ)^*$ (i.e., U° is proper), then it must be the integral on U°, and is therefore unique up to normalization.

It would be desirable to know whether this representation is faithful and to desintegrate it (cf. [7]), preferably in an extremal way (i.e., into irreducible representations). Since there exists an invariant linear subspace given by $\mathrm{Im}(i)$: $i(A)$ is a right subcomodule of U° because i is a comodule morphism. Therefore, $i(A)$ is invariant under the action of U. It is also invariant under the action of A, because i is a *-algebra morphism.

But if such a positive linear invariant functional exists, it is a right integral for the Hopf algebra U°, and the existence of a right integral implies that U° is completely reducible as a right comodule of U° [1]. Consider then any subcomodule N of U° in the complement to $i(A)$. It will be invariant under the action of U by its being a U°-comodule. As to the invariance under the action of $i(A)$, if N is orthogonal to $i(A)$ with respect to the interior product in the pre-Hilbert space U°, then it will remain orthogonal under the action of $i(A)$.

In general the elements of the Hopf algebra U will be represented by unbounded operators. This prevents us from being able to assert that the projection onto the closure of an invariant subspace is going to commute with the representatives of the algebra; see [33].

It is therefore not an easy task in general to carry out this decomposition. Let us first concentrate in a case we shall call the generalized coset homogeneous spaces. First, observe that the Hilbert space $\mathcal{H}_L$ also carries the right regular representation of U, ρ_R, which is given by the action α^R of U on U°: $\alpha^R_h(v) = \langle S(h), v_{(1)}\rangle v_{(2)}$. The antipode in a *-Hopf algebra is invertible, the inverse being given by $S^{-1} = * \circ S \circ *$, so the domain of ρ_R, $\mathcal{D}(\rho_R)$, is equal to the domain of ρ_L. Suppose then that there exists a subalgebra of elements of U, called V, such that, for all $k \in V$ and all $a \in A$, $\alpha^R_k(i(a)) = \epsilon(k)i(a)$. Then $\rho_R(V)$ commutes with $\pi_L(A)$ over its domain of definition, $\mathcal{D}(\rho_R(V))$, as well as commuting with $\rho_L(U)$ over their common domain, which is obvious since $\rho_L(U)$ and $\rho_R(U)$ commute. The operators in $\rho_R(V)$ are affiliated to the von Neumann algebra $\mathcal{M}$ which is the commutant of the weak bounded commutant of $\rho_R(V)$. Diagonalising a maximal abelian subalgebra of $\mathcal{M}$ then gives us a decomposition of $\mathcal{H}_L$ into invariant subspaces. The extremality of this decomposition would have to be ascertained by inspection.

This is completely analogous to the disintegration of the left regular representation for transformation group C*-algebras through a maximal abelian subalgebra of the von Neuman algebra generated by the right regular representation of the little group on the carrier Hilbert space, although in that case the extremality of the decomposition is guaranteed once some technical assumptions are made. The reason for the name of generalized coset homogeneous spaces is that if such a subalgebra V does indeed exist, $i(A)$ will be a subalgebra of the right V-invariant subalgebra of U°, $(U^\circ)^V$, as will become more apparent in the analysis of the left regular representation for the quantum spheres presented in the penultimate section of this paper.

The argument for the existence of inequivalent quantizations can also be presented in a purely algebraic manner if the maximal commutative subalgebra of U, V, such that

α^R_V commutes with β_A, is generated by a set of grouplike and selfadjoint elements $\{v_i\}$. The counit of a grouplike element is the unit in the complex numbers, and its antipode is its inverse. Let $\psi_m \in U^\circ$ be a common eigenvector to this commuting subalgebra, with eigenvalues λ^m_i: $\alpha^R_{v_i}(\psi_m) = \lambda^m_i \psi_m$. λ^m_i must be different from 0, since if ψ_m is an eigenvector of v_i with such eigenvalue, ψ_m must also be an eigenvector of $S(v_i) = v_i^{-1}$, with eigenvalue $(\lambda^m_i)^{-1}$. Then $\alpha^R_{v_i}(\psi^*_m) = (\alpha^R_{S(v_i)^*}(\psi_m))^* = (\lambda^m_i)^{-1}\psi^*_m$. It is seen that ψ_m and ψ_p are orthogonal in the pre-Hilbert interior product if the corresponding sets of eigenvalues are different. By continuity, the completion of the subspace U°_m of eigenvectors ψ_m with eigenvalues λ^m_i will be orthogonal to the analogous U°_p, and they are invariant under the action of U and A. They are carrier spaces for inequivalent quantizations of the motion of a free particle on the generalized coset homogeneous space.

3.6. Induced Representations

The standard construction of induced representations is carried over to the case of Hopf algebras very easily, and immediately provides us with covariant representations. Recall that the different presentations of the representations induced by a compact subgroup H of a locally compact group G are equivalent to that obtained by constructing a Hilbert bundle over the manifold of the homogenous space G/H, with typical fibre the Hilbert space $\mathcal{H}_\chi$ carrying a representation of H, and with points the equivalence classes $[x, \psi_\chi]$, $x \in G, \psi_\chi \in \mathcal{H}_\chi$, under the equivalence relations $(x, \psi_\chi) \sim (xh^{-1}, \pi_\chi(h)\psi_\chi)$ for all $h \in H$.

Accordingly, consider a Hopf algebra U, with Hopf subalgebra V, and let $A = (U^\circ)^V$, the invariant subspace under the right action of V. Let N be a V left module. Then $M = U^\circ \otimes_V N$ is a left $A \rtimes_\alpha U$ module, with the action $\bar\beta$ induced by the left action β of $A \rtimes_\alpha U$ on U°: $\bar\beta_{(a\otimes h)}([r,n]) = [\beta_{(a\otimes h)}(r), n]$ for $a \in A, h \in U, r \in U^\circ, n \in N$, and $\beta_{(a\otimes h)}(r) = i(a) r_{(1)}\langle h, r_{(2)}\rangle$. This is so because U° is a right V module algebra, the right and left actions of U on U° commute, and A is the invariant subalgebra under the right action of V. This definition of induced representations is a specialization of the standard construction for arbitrary algebras.

The important imprimitivity theorem for locally compact groups suggests that an analogous result should hold for induced representations of Hopf algebras. Furthermore, the algebraic results concerning strong Morita equivalence of some C*-algebras given by Rieffel ([37]) suggests a possible route. We conjecture that for a proper *-Hopf algebra U with integral and a proper *-Hopf subalgebra V, the algebras $(U^\circ)^V \rtimes U$ and V are Morita equivalent (i.e., there exists an equivalence functor between the respective categories of left modules). No proof nor counterexample of this statement is known to the author at the time of writing, but see [11]. Some pointers in that direction have already been given in the previous subsection.

3.7. Compact Matrix Pseudogroups

In the case of the compact matrix pseudogroups defined by Woronowicz [39], the previous exposition can be tightened up considerably. Let G be a compact matrix pseudogroup generated by (u) with dense subalgebra $\mathcal{G}$, and faithful Haar measure ω (We will not distinguish notationally the compact matrix pseudogroup (G, u) and the underlying C*-algebra G). We use Δ, S, ϵ to denote coproduct, coinverse and counit. We first have to translate the definition of covariant representation to this context. In order to do that, consider A a right G-comodule C* algebra, with coaction Ψ_A: $\Psi_A(a) = \sum_a a_0 \otimes a_1 \in A \otimes G$. Then a covariant representation of the triple (G, A, Ψ_A) will be a triple $(\mathcal{H}, \pi, \Psi_{\mathcal{H}})$, where $\mathcal{H}$ is a Hilbert space, π is a representation of the C*-algebra A on $\mathcal{H}$, and $\Psi_{\mathcal{H}}$ is a (right co-)representation of the compact matrix pseudogroup G, given by $\Psi_{\mathcal{H}}(\varphi) = \sum_\varphi \varphi_0 \otimes \varphi_1 \in \mathcal{H} \otimes G$, for $\varphi \in \mathcal{H}$, and the following equation is satisfied for all $a \in A$ and all $\varphi \in \mathcal{H}$ (omitting the sum signs for ease of notation):

$$\Psi_{\mathcal{H}}(\pi(a)\varphi) = \pi(a_0)\varphi_0 \otimes a_1\varphi_1 . \tag{42}$$

This equation corresponds to equation (29) when we move from actions to coactions, and has the same physical significance. It is also equivalent if we only consider rational modules in equation (29). It can also be understood as meaning that the coaction is such that $\pi(a)\varphi$ transforms with the tensored (co-)representation.

If there exists a C*-algebra morphism $i: A \to G$, such that it also is a comodule morphism (i.e., $(i \otimes 1) \circ \Psi_A = \Delta \circ i$)), the right regular (co-)representation is immediately built as in the general case, with the GNS construction using the faithful Haar measure, and providing us with the Hilbert space $\mathcal{H}_R$. The representation of A is then given by the map i, and the (co-)representation of G by Δ. It is immediately seen to satisfy equation (42).

Let now H be a sub-(compact matrix pseudogroup) of G, with coproduct, coinverse and counit denoted by $\Delta_H, S_H, \epsilon_H$, and faithful Haar measure given by the restriction $\omega_H = \omega|_H$. Let $\hat{H}$ be the set of equivalence classes of smooth irreducible (co-)representations of H. Let $\mathbf{m}, \mathbf{n} \ldots$ be used to indicate the elements of $\hat{H}$. By Theorem 4.7 of [39],

$$H = \overline{\oplus_{\mathbf{m}\in\hat{H}} H_{\mathbf{m}}} , \tag{43}$$

where $H_{\mathbf{m}}$ are the linear spans of the "matrix elements" $u^{\mathbf{m}}_{ij} \in \mathcal{G}$ corresponding to a unitary representation in the class $\mathbf{m} \in \hat{H}$. There will exist a C*-algebra epimorphism $j: G \to H$ such that $(j \otimes j) \circ \Delta = \Delta_H \circ j$. Let $A_{\mathbf{m}}$ be defined by

$$A_{\mathbf{m}} = \{a \in \mathcal{G} \mid ((j \otimes 1) \circ \Delta)(a) \subset H_{\mathbf{m}} \otimes G\} . \tag{44}$$

In particular, A_0 denotes the subspace of G invariant under the left coaction of H:

$$A_0 = \{a \in \mathcal{G} \mid ((j \otimes 1) \circ \Delta)(a) = 1_H \otimes a\} . \tag{45}$$

The restriction of the Haar measure to $A_{\mathbf{m}}$ is a positive linear functional on $A_{\mathbf{m}}$, and although $A_{\mathbf{m}}$ are not algebras, the GNS construction can be applied again to obtain a Hilbert space $\mathcal{H}_{\mathbf{m}}$, which will carry a covariant representation of (G, A, Ψ_A), if $A = A_0$.

The (co-)representation map $\Psi_{\mathcal{H}_{\mathbf{m}}}$ is given by Δ, because the coassociativity of the coproduct guarantees that if $a \in A_{\mathbf{m}}$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$, then $(a_{(1)}) \subset A_{\mathbf{m}}$. The representation $\pi_{\mathbf{m}}$ is given by $\pi_{\mathbf{m}}(a)\varphi_{\mathbf{m}} = i(a)\varphi_{\mathbf{m}}$, which is indeed a representation, since $i(a)A_{\mathbf{m}} \subset A_{\mathbf{m}}$ because of Δ and j being algebra morphisms, and i an algebra map. The covariance is immediate by i being a comodule morphism and Δ an algebra morphism.

Let us now prove that these Hilbert spaces $\mathcal{H}_{\mathbf{m}}$ (which are clearly Hilbert subspaces of $\mathcal{H}_R$) are orthogonal. For this purpose it is necessary to remember that the Haar measure has the property that

$$(1 \otimes \omega) \circ \Delta = (\omega \otimes 1) \circ \Delta = 1\omega . \tag{46}$$

Correspondingly, $(j \otimes \omega) \circ \Delta = 1_H\omega$. Let us apply this last object to $\alpha^*\beta$, where $\alpha \in A_{\mathbf{m}}$ and $\beta \in A_{\mathbf{n}}$.

$$1_H\omega(\alpha^*\beta) = (j \otimes \omega) \circ \Delta(\alpha^*\beta) = ((1 \otimes \omega) \circ (j \otimes 1) \circ \Delta)(\alpha^*\beta) = j(\alpha^*_{(1)}\beta_{(1)})\omega(\alpha^*_{(2)}\beta_{(2)}) , \tag{47}$$

and this last expression will be different from zero only if $1 \in \bar{H}_{\mathbf{m}} H_{\mathbf{n}}$, since $(j(\alpha^*_{(1)}\beta_{(1)})) \subset \bar{H}_{\mathbf{m}} H_{\mathbf{n}}$. But the orthogonality of the unitary representations with respect to the Haar measure ω_H implies that that will only happen if $\mathbf{m} = \mathbf{n}$. By continuity of the interior product defined by the Haar measure ω, the orthogonality extends to $\mathcal{H}_{\mathbf{m}}, \mathcal{H}_{\mathbf{n}}$.

Notice now that $\oplus_{\mathbf{m}\in\hat{H}} A_{\mathbf{m}} = \mathcal{G}$, and, therefore, the completion with respect to the inner product given by the Haar measure, $\mathcal{H}_R = \oplus_{\mathbf{m}\in\hat{H}} \mathcal{H}_{\mathbf{m}}$.

We can now prove that every $\mathcal{H}_{\mathbf{m}}$ carries an irreducible covariant representation of the triple (G, A, Ψ_A), for $A = A_0$. That it carries a covariant representation has been proved, so only the irreducibility is left. First, suppose that there existed an invariant subspace of $A_{\mathbf{m}}$, let us call it B, and its linear complement, also an invariant subspace, will be called C. Observe that the epimorphism j sends $A_{\mathbf{m}}$ into $H_{\mathbf{m}}$, since $\Delta(H_{\mathbf{m}}) \subset H_{\mathbf{m}} \otimes H_{\mathbf{m}}$. The image of B under j is then an invariant subspace of $H_{\mathbf{m}}$. But the representation $\mathbf{m}$ is irreducible, so it must be that $j(B) = H_{\mathbf{m}}$. We therefore have that $j(C) = 0$. But the faithfulness of the Haar measure and $(j \otimes \omega) \circ \Delta = 1_H\omega$ imply that $c = 0$ for all $c \in C$; therefore, $B = A_{\mathbf{m}}$. From the faithfulness of the Haar measure and the density of $A_{\mathbf{m}}$ in $\mathcal{H}_{\mathbf{m}}$, it can be seen that there will be no invariant subspace of $\mathcal{H}_{\mathbf{m}}$, as we had set out to prove.

3.8. *Physical significance*

The question of the physical significance of a similar construction to the one presented above, but only for cocommutative Hopf algebras and homogeneous spaces, has been addressed by Landsman in a recent series of papers [18],[19], although not making use of the Hopf algebra language. It had been found some time ago [8] that the inequivalent irreducible covariant representations correspond to inequivalent quantizations of the free motion of a particle on a homogeneous space. Landsman adds to that the interpretation that they correspond to different topological charges of monopole or Aharonov-Bohm type. This goes hand in hand with the question of lifting the action of a group G on a homogenous space G/H to the different principal fiber bundles with base space G/H and typical fiber H. Alternatively, with the ideas stemming from geometric quantization that inequivalent quantizations correspond to inequivalent principal fibre bundles, which are tied up with the nontrivial topology of the base space, and correspond to topological charges.

From this we can conclude that our construction is uncovering the "topology" of the non-commutative space underlying the C*-algebra A. Equivalently, it is pointing towards the existence of gauge theories over these non-commutative "manifolds". This is keeping in line with the notion that deforming a manifold in a sensible way will mantain most of its interesting properties. It is interesting at this point to refer to [4].

It was only to be expected that these inequivalent quantizations were to exist. The studies which have been carried out on cyclic homology of quantum groups ([15],[28]), which is the noncommutative analogue of De Rham cohomology [6], point out to their occurrence. In fact, in the case of quantum groups the main part of cyclic homology is carried by the maximal tori [15], which indicates maximality of the decomposition of the left regular representation of $A \rtimes_\alpha U$ for the case of quantum groups, when the abelian subalgebra we are diagonalising is the maximal torus T, and $A = (U^\circ)^T$.

We must point out, though, that the interpretation of these inequivalent quantizations as topological charges is very much connected with the correct introduction of a radial coordinate, or a corresponding "expansion" of configuration space, and the selection of the adequate Hamiltonian [18]. Neither of these tasks has been addressed here so far. The matter of the adequate Hamiltonian is taken up in the next section, but we shall not look into the addition of a radial coordinate, which would correspond to an understanding of the quantum spheres as embedded in an euclidean 3-dimensional quantum space, of the kind introduced by Manin [27].

4. Dynamics and multiparticle systems

4.1. *The choice of a Hamiltonian*

The principal concept guiding the inception of the quantum sphere itself, and of the covariant representations introduced here, has been that of mantaining the symmetry under the Hopf algebra $U_q(su_2)$. Therefore, when looking for Hamiltonians that correspond to the free motion of a particle on the non-commutative spaces of which A is the algebra of functions, we have to look for operators acting on the Hilbert space that carries the particular irreducible covariant representation we are considering, such that they commute with the action of U on that Hilbert space.

For a particular irreducible covariant representation $(\mathcal{H}, \pi, \rho)$ there might exist many more objects in the commutant of $\rho(U)$ in the algebra of operators on $\mathcal{H}$ than in the representation of the center of U, $Z(U)$, but $Z(U)$ is the natural place to look for objects whose representatives might be the adequate Hamiltonian.

A further way of selecting among the different elements of $Z(U)$ is to identify those that go over to the Laplace-Beltrami operator when the deformation is removed, in the case that A is a deformation of the algebra of functions on a homogeneous space and U is a "quantum group".

If U is a quasitriangular Hopf algebra of quantized universal enveloping algebra type and corresponds to the Cartan matrix of a simple compact Lie algebra, the natural element to consider is the quadratic Casimir, which does indeed go over to the Laplace-Beltrami operator on homogeneous spaces, when the deformation parameter is removed.

4.2. *Multiparticle dynamics*

So far we have only considered the motion of one particle on the non-commutative space Spec(A). In order to address the multiparticle dynamics, it will prove worthwhile to restrict ourselves to the case that U is a quasitriangular Hopf algebra. We have seen in section 2.5 that this allows for the definition of an action of U and an involution in $A \otimes A$, which, with the algebra structure given by the braiding, make it into a unitary U-module *-algebra. The analysis of the previous section then goes through, although substituting this algebra $A \otimes A$ for the algebra A. If we look at the zero chage sector, the invariant linear functional L has to be substituted for L_2, by which is meant that $L_2(a \otimes b) = L(a)L(b)$. It can be seen to be positive, faithful if L is faithful, and U-invariant. The interior product in $A \otimes A$ is braided:

$$(a \otimes b, c \otimes d) = L_2((a \otimes b)^*(c \otimes d)) = L_2(b^* \alpha_{\mathcal{R}(2)}(c) \otimes \alpha_{\mathcal{R}(1)}(a^*) d) . \qquad (48)$$

All this, which is of course adequate for the description of just two particles on non-commutative space, can be extended to multiparticle dynamics, as long as the braiding is taken into account.

It is necessary to introduce the braiding because the symmetry under the Hopf algebra must be preserved. This corresponds to being able to decompose total momenta (which are the generators of the Hopf algebra U acting on the configuration space algebra) into momenta for each one of the particles (i.e. the generators of U acting on each of the A

algebras). Consequently, the action of h, an element of U, on $a\otimes b$, an element of $A\otimes A$, has to be given by $\alpha'_h(a\otimes b) = \alpha_{h_{(1)}}(a)\otimes\alpha_{h_{(2)}}(b)$. Additionally, in order to have "propagation" of the particles on the quantum sphere that preserves the Hopf algebra symmetry, it seems that the configuration space algebra has to be a U-module algebra. This is because such a propagation can be understood as taking place by maps of the form $C \to C\otimes C \to C$, where C is the configuration space algebra, analogously to dilations in Markov processes, etc. Then the symmetry has to be preserved under these maps, and the condition that C be a U-module algebra is sufficient for it.

The unitarity, of course, is necessary for physical reasons too, since we want to understand expectation values as probabilities. Putting all these elements together, we look for an algebra C such that it is equal to $A\otimes A$ (for the two-particle case) as a linear space; that it is a U-module algebra, and such that the action of U on it is as given above. The braided product provides us with exactly those conditions.

This gives us the "kinematics" of multiparticle systems. We have already discussed the introduction of dynamics for a single free particle. If, analogously, we consider the dynamics for a system of free particles with no interaction term, we might expect triviality; but, as was already pointed out by Podleś [36], the braiding introduces a non trivial "exchange" between the particles. It must be pointed, however, that if the one-particle hamiltonian H commutes with the action of the symmetry Hopf algebra U, the braiding induces a mere twist on $H\otimes 1$. If the one particle hamiltonian does not commute with the symmetry Hopf algebra, then the braiding will act non-trivially, and even if there is no interaction term there will be interaction between particles due to the braiding. But only, we must insist, if the one particle hamiltonian is not U-invariant.

We shall now see how this works in the particular example of the quantum spheres.

5. Quantum Mechanics on Quantum Spheres

5.1. Kinematics on Quantum Spheres

We have already seen that the algebras $A_{q,\mu}$ are locally finite unitary $U_q(su_2)$-module *-algebras. It is then clear that $(A_{q,\mu}, U_q(su_2), \alpha)$ is a generalized dynamical system. Furthermore, we have also established that $A_{q,\mu}$ are algebras of functions on generalized homogenous spaces of $U_q(su_2)$. Since the work of Woronowicz [39], it is well known that there exists an integral on $\mathrm{Fun}_q(SU_2)$, the Haar measure of the quantum group. Let us first examine the induced positive invariant linear functional on $A_{q,\mu}$ for the case when μ belongs to the discrete series, μ_n. The representation π_n is faithful, and the invariant functional turns out to be given by a deformed trace:

$$L_n(\psi) = \frac{q^{n-1}}{[n]}\sum_{m=0}^{n-1} q^{-2m}(m|\pi_n(\psi)|m)\,. \tag{49}$$

This linear functional is faithful, and when the GNS construction is applied, we obtain an n^2 dimensional vector space on which a covariant representation of $(A_{q,\mu}, U_q(su_2), \alpha)$ for $\mu=\mu_n$ is defined.The finite dimensionality of the representation makes it perfectly well

behaved. The Hamiltonian can be written in (finite) matrix form. This is not so when we consider $\lambda \geq \sigma^2$. Then the Hilbert space obtained is $L^2(A_{q,\mu})$.

It is more interesting to look at the left regular representation. Consider the case $\lambda = \sigma^2$. This corresponds to $A_{q,\mu} = \mathrm{Fun}_q(SU_2)^{U(1)}$. Let us look at the left regular representation and its decomposition in irreducible subspaces. The injective *-algebra and comodule morphism $i: A_{q,\mu} \to \mathrm{Fun}_q(SU_2)$ reduces to $i(\Phi_k) = -\sigma d^1_{0k}$. Consequently, let us look at elements of $\mathrm{Fun}_q(SU_2)$ such that $\rho_R(k)\psi_m = q^m\psi_m$, for m an integer. Let A_m denote the linear space of such elements. It can readily seen that it is invariant under the action of $U_q(su_2)$, because $\rho_R(U(1))$ commutes with the left action of $U_q(su_2)$. And it is also invariant under the action of $A_{q,\mu}$. Moreover, it is clear that A_m is orthogonal to any other A_l if $m\neq l$ (since $\omega(\alpha^R_k(\psi_l^*\psi_m)) = \omega(\psi_l^*\psi_m) = q^{(m-l)}\omega(\psi_l^*\psi_m)$, and therefore $\omega(\psi_l^*\psi_m)=0$ if $m\neq l$). So the completion of each A_m in the Hilbert space norm gives rise to a Hilbert subspace $\mathcal{H}^{(m)}$ which carries a covariant representation $\pi^{(m)}, \rho^{(m)}$ of $A_{q,\mu} \rtimes_\alpha U_q(su_2)$. These (inequivalent) representations correspond to different quantizations of the motion of a free particle on the quantum sphere, and can be understood as "monopole charges". We have thus uncovered a topological property of the noncommutative object we have been calling the quantum sphere.

The case $\lambda > \sigma^2$ is somewhat more complicated, since the algebra of functions on the quantum sphere is no longer an invariant subalgebra of $\mathrm{Fun}_q(SU_2)$. Remember that there exists a *-algebra an comodule morphism i, with coefficients c_j. We can produce inequivalent quantizations for this case as well, in the following way: choose two complex 3-vectors orthogonal to c_j and to each other, called c'_j and c''_j, such that when σ^2 tends to λ we have that c'_j tends to $-\sigma\delta_{j,1}$ and c''_j tends to $-\sigma\delta_{j,-1}$. Let $\psi'_k = \sum c'_j d^1_{jk}$ and $\psi''_k = \sum c''_j d^1_{jk}$. Let A' be the space spanned by the ψ'_k and by the action of $U_q(su_2)$ and $A_{q,\mu}$ on this span. It can clearly be seen to be orthogonal to $i(A)$, and, by its very definition, invariant under the action of $A_{q,\mu} \rtimes_\alpha U_q(su_2)$. Similarly with A''. These two invariant subspaces correspond to $m=\pm 2$ in the previous case, and higher charge representations can also be built. We have then obtained inequivalent quantizations for this case that correspond to monopole charges as well.

5.2. Dynamics on Quantum Spheres

As has been discussed above, the natural element to consider is the quadratic Casimir element of $U_q(su_2)$, $C_q = j_-j_+ + (q-q^{-1})^{-2}(qk^2 - q - q^{-1} + q^{-1}k^{-2})$. This object, by its own definition, commutes with the elements of $U_q(su_2)$, and is such that will be valued $[j][j+1]$ for the j-th representation of $U_q(su_2)$. Consequently, we know its action on $A_{q,\mu}$. Since it commutes with all the operators in $U_q(su_2)$, including itself, and since the algebra $A_{q,\mu}$ has a PBW decomposition, with orthogonal basis given by the polynomials found in [31], we have a complete hamiltonian evolution for the charge 0 representation. But we also have it in general for any subspace of the left regular representation carrier space, since the algebra $\mathrm{Fun}_q(SU_2)$ is generated by the (deformed) matrix element functions of the different representations of $U_q(su_2)$. Thus the question of the spectrum of the Hamiltonian is completely settled. Notice in this respect that although two deformed laplacians had been proposed in [35], one is a function of the other.

6. Conclusion

In this paper we have shown that the idea of inequivalent quantizations can be extended to the motion of particles on non-commutative manifolds. This can be understood as the surfacing of "topological" properties of such non-commutative manifolds. It has been proved that non-trivial braiding induces non-trivial interactions between particles if the Hamiltonian is not invariant under the quantum group that induces the braiding.

We have also defined the concept of generalized dynamical system, which extends the usual one. Since the idea of quantum group invariance has been the guiding principle in this construction, the generalized dynamical systems we consider can be thought of as deformations of C^*-dynamical systems, with broken symmetry that survives in a special (deformed) way. It is hoped that this will allow us to develop further the idea of q-regularization.

The particular case we have been looking at as a first non-trivial example has been the motion of particles on quantum spheres, and we have proved the existence of monopole-like charges in this situation, and how the braiding affects multiparticle systems. In order to make this monopole-like charges more like "the real thing", the embedding of quantum spheres in 3-dimensional quantum space has to be carried out. This is left for future work.

The relationship between our construction and cyclic homology of quantum spaces is also matter of ongoing research. This connection will illuminate how to write down systematically "gauge" theories corresponding to quantum groups, thus carrying us closer to our goal of q-regularization.

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