Einstein's Principle of Equivalence

and

The Einstein-Minkowski Condition

C. Y. Lo

17 Newcastle Drive, Nashua, NH 03060

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Abstract

Einstein insisted on the fundamental importance of his equivalence principle to general relativity. However, many theorists do not share this view because: 1) his principle had not been defined rigorously with space-time coordinate systems, 2) many are confused Einstein's equivalence principle declared in 1921 with Einstein's earlier 1911 preliminary application of equivalence with Newtonian gravity, 3) the Einstein-Minkowski condition for local spaces at free fall has not been illustrated with a verified example, 4) the crucial role of Einstein's equivalence principle in theoretical predictions was not recognized, 5) the compatibility of a uniformly accelerated frame and relativity is only a faith, and 6) efforts to obtain the space-time metric of a uniform gravity had failed. It is found that such confusion, related misunderstanding, and naïve intuition were the causes of the failure in the acceptance of Einstein's equivalence principle. These issues are addressed and the metric for a uniform gravity is found to be time-dependent. Moreover, Einstein' s equivalence principle can be rigorously defined and illustrated since the theoretical framework of general relativity has actually provided the physical meaning of coordinates.

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1. Introduction

It is generally agreed, as pointed out by Einstein [1], Eddington [2], Pauli [3], Weinberg [4], Misner, Thorne & Wheeler [5], and Straumann [6] that Einstein's equivalence principle is the theoretical foundation of general relativity. However, as Einstein [7] saw it, few like Eddington [2] understand the crucial role of Einstein's equivalence principle. In fact, theorists are so confused on this principle that they have different versions [1-3] that Einstein [7] considered as invalid and/or objected as misinterpretations of his principle. However, you will soon see that Einstein is also responsible for this situation. On the other hand, Einstein insisted, throughout his life, on the fundamental importance of this principle to his general theory of relativity [7].

Einstein claimed that his equivalence principle (1921) requires the Einstein-Minkowski condition that a free fall results in a co-moving local Minkowski space that the time dilation and space contractions are obtained [1]. However, in spite of Einstein's objection [7], his equivalence principle was commonly but mistakenly regarded the same as Pauli's [3] version:

> "For every infinitely small world region (i.e. a world region which is so small that the space- and time-variation of gravity can be neglected in it) there always exists a coordinate system K_0 (X_1, X_2, X_3, X_4) in which gravitation has no influence either in the motion of particles or any physical process."

Thus, Pauli regards the equivalence principle as merely the mathematical existence of locally constant spaces, which may not be locally Minkowski. In addition, Pauli invalidly extended the removal of uniform gravity to the removal of gravity in general.

Apparently, Pauli, Straumann [6], and Will [8], overlooked (or disagreed with) Einstein's [9, p.144] remark, "For it is clear that, e.g., the gravitational field generated by a material point in its environment certainly cannot be 'transformed away' by any choice of the system of coordinates..." Pauli's version has the consequence that unphysical solutions have been accepted as valid [10, 11]. Moreover, the price for ignoring the Einstein-Minkowski condition is often not free. For instance, Misner et al [5] and Ohanian & Ruffini [12], different from others, have made the incorrect conclusion on the local time of the earth.[1)]

Furthermore, some theorists rejected Einstein's equivalence principle explicitly because they have not been able to get the facts right and/or its physical meaning in full. For instance, Synge [13] claimed,

> "...I have never been able to understand this principle...Does it mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does or does not vanish. This is an absolute property; it has nothing to do with any observer's world line...The Principle of Equivalence performed the essential office of midwife at the birth of general relativity...I suggest that the midwife be now buried with appropriate honours and the facts of absolute spacetime be faced."

From these statements, it is clear that Synge does not fully understand the physics underlying Einstein's equivalence principle.[4)] In fact, Einstein's equivalence principle states only that the effects of an accelerated frame are equivalent to a related uniform gravity [1, 9]. Moreover, a gravitational field need not relate to a non-vanishing curvature. Einstein [7] explained to Laue, "What characterizes the existence of a gravitational field, from the empirical standpoint, is the non-vanishing of the Γ^l_{ik} (field strength), not the non-vanishing of the R_{iklm}." and no gravity is a special case of gravity. This view is important because it justifies that the geodesic equation is also the equation of motion of a massive particle under the influence of only gravity. Moreover, from the Einstein-Minkowski condition, Einstein [1; p. 91] obtained the time dilation and space contractions.

For a perceived philosophical difference, Fock [14] rejected Einstein's principles. He claimed, with the support of his calculation, that it is not possible to have a metric for a uniform gravity that satisfies Einstein's equivalence principle. His calculation converted many followers, including Wheeler and his students Ohanian and Ruffini [12], to reject both Einstein's principles. However, Fock's rejection is actually invalid because he confused Einstein's equivalence principle of 1921 with the 1911 preliminary application of the notion of equivalence with Newtonian gravity (see also Section 2).

A major problem in Einstein's theory, as pointed out by Whitehead [15] and Fock [14], is that the physical meaning of space-time coordinates is ambiguous. Consequently, both Einstein's equivalence principle and the principle of general relativity, which require a clear meaning of coordinates, cannot be rigorously defined. For instance, a uniform acceleration can be defined and the meaning of space contractions is clear only if the physical meaning of space coordinates is unambiguous. Such a situation created a frustration that some theorists simply regarded his equivalence principle as merely a heuristic argument [16].

Recently, it has been shown that Einstein's equivalence principle necessarily implies an existence of the Euclidean-like structure for a physical space [17-19]. Thus, the physical meaning of space coordinates has been clarified. Nevertheless, the metric for a uniform gravity has not yet obtained. Moreover, the crucial role of Einstein's equivalence principle is still not recognized [20]. Perhaps, a reason is that the notion of space contraction is still not well understood. Although some work has been done by Landau & Lifshitz [21], their work has not been referred to in the textbooks of the West. Recently, it has been found that their formula of local distance is not always valid [20] because Pauli's version and implicit assumptions were used.

In this paper, the space time-time metric of a uniform gravity will be derived, and thus the objection of Fock [14] and Ohanian and Ruffini [12] against Einstein's equivalence principle, is proven incorrect. Concurrently, it is shown also, as an improvement of the work of Landau & Lifshitz [21], that the Einstein-Minkowski condition is crucial in deriving the space contractions. Moreover, Einstein's notion of local time and local clock rates are illustrated, and thus general relativity is further clarified [20]. Now, it is clear that Einstein's equivalence principle is not just a midwife and should not be buried.

2. Einstein's Principle of Equivalence, the Einstein-Minkowski Condition

First, let us state Einstein's principle, which is based on the equivalence of inert mass and gravitational mass due to Galileo, with Einstein's own words. In his book, "The Meaning of Relativity" [1], Einstein wrote:

> 'Let now K be an inertial system. Masses which are sufficiently far from each other and from other bodies are then, with respect to K, free from acceleration. We shall also refer these masses to a system of co-ordinates K', uniformly accelerated with respect to K. Relatively to K' all the masses have equal and parallel accelerations; with respect to K' they behave just as if a gravitational field were present and K' were unaccelerated. Overlooking for the present the question as to the "cause' of such a gravitational field, which will occupy us latter, there is nothing to prevent our conceiving this gravitational field as real, that is , the conception that K'; is "at rest" and a gravitational field is present we may consider as equivalent to the conception that only K is an "allowable" system of coordinates and no gravitational field is present. The assumption of the complete physical equivalence of the systems of coordinates, K and K', we call the "principle of equivalence;" this principle is evidently intimately connected with the law of the equality between the inert and the gravitational mass, and signifies an extension of the principle of relativity to coordinate systems which are non-uniform motion relatively to each other.'

In the above statements, no Newtonian theory of gravity was mentioned. Later, Einstein made clear that a gravitational field is generated from a space-time metric. Thus, his principle was proposed with the gravity as an integral part of the physical space. (In hinder sight, Einstein should have provided an example of the metric for a uniform gravity.)

Thus, Einstein's equivalence principle should not be confused with Einstein's 1911 preliminary application on the notion of equivalence between uniform acceleration with uniform Newtonian gravity [9]. However, Fock managed to contain almost all the characteristics of Newtonian gravity in his calculation. He [14] attempted to show such a metric is of the following form,

$$ds^2 = g_{tt}(x)\, c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad (1)$$

where $g_{tt}(x)$ is the only non-zero component of the space-time metric $g_{\mu\nu}$. As in the 1911 preliminary application, the physical space has a Euclidean subspace, and only the time-time component of the metric is non-zero. Moreover, he also assumed that the metric is static and this would guarantee his failure. The calculation of Tolman [24] on uniform gravity was not trusted because it seems to be against Einstein's earlier analysis, in addition to errors in his calculation [20].

What is new in Einstein's equivalence principle in 1916 is the claim of the Einstein-Minkowski condition as a consequence [9, p. 161]. While Einstein's 1911 preliminary application was based essentially on his intuition, the Einstein-Minkowski condition additionally has its foundation from theorems [13] in Riemannian geometry as follows:

Theorem 1. Given any point P in any Lorentz manifold (whose metric signature is the same as a Minkowski space) there always exist coordinate systems (x^μ) in which $\partial g_{\mu\nu}/\partial x^\lambda = 0$ at P.

Theorem 2. Given any time-like geodesic curve Γ there always exists a coordinate system (so-called Fermi coordinates) (x^μ) in which $\partial g_{\mu\nu}/\partial x^\lambda = 0$ along Γ.

In these theorems, the local space of a particle is locally constant, but not necessarily Minkowski. However, after some algebra, a local Minkowski metric exists at any given point and that along any time-like geodesic curve Γ, since a moving local constant metric exists. The only condition is that the space-time metric has a proper Minkowski signature. Accordingly, Pauli's version is essentially a simplified but corrupted version of these theorems since a locally constant metric may not be constant, though changing very little, in a small region.

What Einstein added to theses theorems is that physically such a locally constant metric must be Minkowski. Such a condition is needed for special relativity as a special case [1]. In a uniformly accelerated frame, the local space in a free fall is a Minkowski space according to special relativity.5) Einstein also used the Einstein-Minkowski condition for his calculations [1; p. 91, 9; p. 161]. However, the existence of local spaces being Minkowski is assumed only in his calculations.

3. The Einstein-Minkowski Condition and the Canonical Form of the Space-time Metric

To obtain the time dilation and space contractions, Einstein's 1916 approach described by Ohanian and Ruffini [12; p. 164] is as follows:

> "... and the spacetime interval becomes $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. This expression determines the spacestime distances. For example, a coordinate displacement dx^1 along the x-axis has a length $\sqrt{(-g_{11})}\, dx^1$, that is, the measured distance differs from dx^1 by a factor $\sqrt{(-g_{11})}$. Likewise, a coordinate time (t time) displacement dx^0 has a duration$\sqrt{g_{00}}\, dx^0$ when measured by the proper time of a clock at rest."

However, such an approach is not generally valid. For instance the metric of Einstein's rotating disk K'(x', y', z') [21] is

$$ds^2 = (c^2 - \Omega^2 r^2)\, dt^2 - 2\Omega r^2\, d\phi' dt - dr^2 - r^2\, d\phi'^2 - dz'^2. \qquad (2)$$

where Ω is an angular velocity relative to an inertial system K (x, y, z, t). Then one would have claimed, in disagreement with Einstein, that there is no space contraction in the presence of gravity. Thus, to obtain the correct space contractions, one must first transform the metric to a canonical form such that the space contractions are clear.

The Einstein-Minkowski condition states that a free fall results in a co-moving local Minkowski space [1]. Since a free fall can start at rest from anywhere, the space-time metric of a physical space with a frame (x^1, x^2, x^3) must have a canonical form,

$$ds^2 = g_{t't'}c^2dt'^2 - dl^2 \tag{3a}$$

where

$$dl^2 = -\gamma_{\alpha\beta}\, dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, 2, 3) \quad \text{and} \quad g_{t't'} < 1 \tag{3b}$$

because of the time dilation. However, dt' could be only a local time, but t' is not a global time. Then, the time dilation and space contractions can be obtained because of the Einstein-Minkowski condition.

For a given frame of reference, in general a space-time metric has the form,

$$ds^2 = g_{00}(dx^0)^2 + 2g_{0\alpha}dx^0dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta \tag{4}$$

Since the space coordinates are given and determined by the frame of reference [17, 18], the only way to transform (3) to a canonical form (2) is by absorbing the cross term in a new differential dt'

$$dt' = dx^0 + \frac{g_{0\alpha}dx^\alpha}{g_{00}}, \qquad \text{and} \qquad \gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{00}}. \tag{5a}$$

Thus

$$ds^2 = g_{00}(dt')^2 - (dl)^2, \quad \text{where} \quad dl^2 = -\left(g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{\alpha\beta}}\right)dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3) \tag{5b}$$

which is also the local distance formula of Landau & Lifshitz [21]. However, Pauli version does not guarantee the frame is realizable although does imply the conditions $g_{00} > 0$,

$$\begin{vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{vmatrix}(-1) > 0, \quad \begin{vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix}(-1) > 0. \tag{6}$$

A mistake of Landau & Lifshitz [21] is that they assumed the local frame is always realizable. An implicit problematic assumption is that the metric is valid in physics. For instance, the canonical form of the metric,

$$ds^2 = [dx + (c - v)dt][-dx + (c + v)dt] - dy^2 - dz^2, \tag{7a}$$

would be

$$ds^2 = (c^2 - v^2)\, dt'^2 - (1 - v^2/c^2)^{-1}dx^2 - dy^2 - dz^2, \tag{7b}$$

where v is a constant. Both metric (7a) and metric (7b) are not valid in physics because the constant metrics are not Minkowski.

In fact, any metric (valid in physics or not) can have a canonical form. However, the canonical form does allow one to see the physics clearer. For instance, the local space contraction requires that

$$dl^2 \geq (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{8}$$

which may not be clear from metric (2). For example, from the metric (2) of Einstein's rotating disk K' [21], one would get a superficial light speed larger than c. However, the canonical form of metric (2) is

$$ds^2 = (c^2 - \Omega^2 r'^2)dt'^2 - dr'^2 - (1 - \Omega^2 r'^2/c^2)^{-1} r'^2\, d\phi'^2 - dz^2, \tag{9a}$$

where

$$cdt' = cdt - (r\Omega/c)rd\phi'[1 - (r\Omega/c)^2]^{-1}, \tag{9b}$$

Then, it is clear that the local light speed cannot be larger than c, and that there is no Euclidean subspace. However, (9b) is not integrable [6)] because dt' is related to different inertial systems at different r or different time.

Thus, this example illustrates that the Einstein-Minkowski condition is crucial in the consideration of space contractions and the time dilation. It is interesting to note that in this calculation the notion of uniform rotation can be proven to be feasible.

4. Uniform Acceleration and Einstein's Equivalence Principle in Riemannian Space

The case of Einstein's uniformly rotating disk suggests that there would be similarities in the metric of a uniformly accelerated system. Note that the notion of a constant acceleration depends on the inertial frame since the acceleration with respect to another frame with relative velocity u (for simplicity, in the x-direction) would have an additional factor $(1 - u^2/c^2)^{1/2}/(1 + uv/c^2)^2$ [24, p. 27], and thus is no longer a constant.

Note that a uniform acceleration cannot exist forever, otherwise the resulting speed would exceed the velocity of light. It follows that a uniform acceleration must be started at some time, and then decreased some time afterward. Moreover, a uniform gravity must be confined in a finite region; otherwise the light speed as the maximum velocity would be violated. In other words, uniform gravity like an electromagnetic plane wave, also does not really exist in nature. Thus, the equivalence of acceleration and uniform gravity is best described, as Einstein did, in terms of an accelerated chest [25].

Now, consider a system K'(x', y', z') accelerated with an acceleration a relative to an inertial system K (x, y, z, t). Then, if the coordinates of the origin of K' in system K is $(X_{0'}(t), 0, 0, t)$, we have

$$\frac{d^2 X_{0'}}{dt^2} = a, \tag{10a}$$

and

$$X_{0'}(t) = at^2/2; \quad y = y', \quad z = z', \quad \text{and} \quad v(t) = at \quad \text{if} \quad v(0) = 0, \quad \text{and} \quad X_{0'}(0) = 0. \tag{10b}$$

In general, for an arbitrary fixed point at (x', y' z') of K', its coordinate in K is,

$$X(t) = X(0) + at^2/2; \qquad X(0) = x', \qquad y = y'; \qquad z = z'; \qquad (10c)$$

This is the fundamental equation for a uniform acceleration. Then, one would obtain,

$$dx = dx' + vdt, \quad dy = dy', \quad \text{and} \quad dz = dz'. \qquad (11)$$

and

$$ds^2 = (c^2 - a^2t^2)dt^2 - 2at\, dtdx' - dx'^2 - (dy'^2 + dz'^2), \qquad (12)$$

for the system K*(x', y', z', t). So far, the calculation agrees with Tolman's [24; p. 177], who got the metric (12). A careful reader will see that the derivation of metric (12) simply follows the steps of Landau & Lifshitz [21] in deriving metric (2) for Einstein's uniformly rotating disk. The main problems are to justify the derivation of metric (12) and to understand the related physics that Tolman and many others had failed to identify. In fact, Tolman's [24] presentation is so inadequate that it has been regarded as an example of misunderstanding Einstein's equivalence principle [20].

Note that although Tolman's book was first published in 1934, many theorists [12] believed the later invalid calculation of Fock [14] published in 1961. The main problems are that the new metric is incomplete and that the conceptual difficulties were not identified. There are several main reasons that Tolman's calculation is being ignored, and they are as follows:

1) Tolman has not shown that the fundamental relation (10c) is justified. In special relativity, a rod with a relative velocity does not have the same length as at rest. The local time in an accelerated frame has not been clarified. Thus, the work to obtain a metric for a uniformly accelerated frame is incomplete.
2) According to the 1916 argument of Einstein [9], it seems, for metric (12) there are no space contractions for gravity although the time dilation would be dt $[1 - (v/c)^2]^{1/2}$.
3) From metric (12), the superficial coordinate light speed could be larger than c. One may argue that t is not the local time of (x', y', z'). Then, a related question is what is the local time of such an accelerated frame.
4) Metric (12) has not been related to the conventional form of gravity started from Newton and supported by the work of Einstein. Thus, his metric had been regarded as an example of misunderstanding Einstein's equivalence principle.
5) The nature of a uniform gravity has not been clarified.

These issues must be solved, before one can be convinced that metric (12) is valid for a uniformly accelerated frame. For example, based the approach of an extended Lorentz transformation, J.-P. Hsu and L. Hsu [28] had attempted in 1997 to get a coordinate transformation between an inertial frame and a uniformly accelerated frame. It is clear that Tolman's work has no influence in their paper. It will be shown that the first three issues will be solved with calculations based on the Einstein-Minkowski condition, and the other two issues in the next section.

Now, consider a particle P resting at K*(x', y', z', t), the state vector of P is (0, 0, 0, dt), and its local space has a Minkowski metric. P has a velocity v (= at) in the x-direction in K. Thus, since y = y' and z = z', a local vector (dx, dy, dz, dt) in K is related to a local vector (dX, dy', dz', dT) in the local space of P where dX is in the x'-direction and t" is the time of the inertial system (x", y, z). Then, the two local spaces relate to each other by the Lorentz transformation in special relativity as follows:

$$dx = [1-(v/c)^2]^{-1/2}\,[dx'' + v\,dt''] \; ; \qquad (13a)$$

and

$$c\,dt = [1-(v/c)^2]^{-1/2}\,[c\,dt'' + (v/c)dx''];. \qquad (13b)$$

or

$$dX = [1-(v/c)^2]^{-1/2}\,[dx - v\,dt], \qquad (13c)$$

and

$$dT = [1-(v/c)^2]^{-1/2}\,[dt - (v/c^2)\,dx] \qquad (13d)$$

From (13c) and (11), one obtains

$$dX = [1-(v/c)^2]^{-1/2}\,dx', \qquad (14a)$$

and

$$dT = [1-(v/c)^2]^{1/2}\{dt - [1-(v/c)^2]^{-1}(v/c^2)\,dx'\} \qquad (14b)$$

Eq. (14a) shows that, the local distance in the x'-direction is measured from a local space at a free fall, and therefore its is not measured in K'(x', y' z'). To obtain (14a), although the measuring rod is at rest, they must be in a free fall state. When dx' is measured from K, it becomes $dX = [1-(v/c)^2]^{-1/2}\,dx'$ in (x", y, z) first. Then $dx = [1-(v/c)^2]^{1/2}\,dX = dx'$. This explains (11), and thus the notion of constant uniform acceleration is justified and valid in general relativity.

In (14b) we have replace dx with dx' because a local clock at K' has dx'= 0. Now, it is clear that the term for space contractions in the x'-direction can be generated by completing the square of the dtdx' and the dt^2 terms, i.e.,

$$ds^2 = (c^2 - v^2)\,\{dt - [1-(v/c)^2]^{-1}(v/c^2)\,dx'\}^2 - dy'^2 - [1-(v/c)^2]^{-1}dx'^2 - dz'^2 \qquad (15)$$

This suggests a local time dt', and locally the metic [7] is,

$$ds^2 = (c^2 - v^2)\,dt'^2 - dy'^2 - (1 - v^2/c^2)^{-1}dx'^2 - dz'^2. \qquad (16a)$$

where

$$c\,dt' = c\,dt - (v/c)dx'[1-(v/c)^2]^{-1}. \qquad (16b)$$

Metric (16) shows clearly that there is space contraction in the x'-direction. Then one would have

$$dT = [1 - (v/c)^2]^{1/2} dt' \qquad (17)$$

The local clock in K' is identified with dx' = 0 and its rate is $[1 - (v/c)^2]^{1/2}dt$ (and at t = 0, dt' = dt). Thus, the local clock rate is independent of the location in K' (x', y', z'). Now, it is clear that local light speeds are not larger than c [17].

However, dt' is the local time only and one cannot integrate (16b) into a coordinate transformation because, at different time, dT is the time of different inertial coordinate system. Thus, the approach of J.-P. Hsu and L. Hsu [28] to get a coordinate transformation between an inertial frame and a uniformly accelerated frame has been proven as not attainable.

Nevertheless, for a clock rest at K' (dx' = 0), according to Einstein [1, 9], the observed clock rate is dT. Thus, we have

$$dt = [1 - (v/c)^2]^{-1/2} dT \qquad (18)$$

if observed from K. Eq. (18) is the same result as derived from (13b) in special relativity with dX = 0 (i.e., dx" = 0). On the other hand, if observed from K', one would have $dT = [1 - (v/c)^2]^{-1/2} dt$. Please note that the discussion is on the rate of a clock fixed in a uniformly accelerated frame. However, there is no global time coordinate transformation [6)] although (11) has an integrated form x = x' + vt + C, where C is a constant.

Thus, in terms of the final measurable results, metric (16) would be considered just as a midwife that identifies the space contractions and the time dilation clearly. Since (16) is derived from Einstein-Minkowski condition, its crucial role is clear. Thus, Einstein's equivalence principle has been illustrated with a space-time metric of a uniform gravity.

5. Local coordinate Transformation and Comparison with the Conventional Form of Uniform Gravity

The coordinate transformation is incomplete because there is no transformation for the local time of frame (x', y', z'). Nevertheless, the local coordinates of the frames (x, y, z) and (x', y', z') are related as follows:

$$dx' = dx - v\,dt, \qquad x = x' + at^2/2 \qquad (19a)$$

and

$$cdt' = [1 - (v/c)^2]^{-1}[cdt - vc^{-1}dx]\,; \qquad (19b)$$

or

$$dx = [1 - (v/c)^2]^{-1}dx' + vc^{-1}cdt', \qquad (20a)$$

and

$$cdt = cdt' + [1 - (v/c)^2]^{-1}vc^{-1}dx'. \qquad (20b)$$

Thus, for a clock and a rod in the x'-direction attached to K', if measured from K, one has respectively

$$dt = dt', \quad \text{and} \quad dx = dx'. \tag{21a}$$

On the other hand, for a clock and a rod in the x-direction attached to K, if measured from K', one has respectively

$$dt' = [1 - (v/c)^2]^{-1} dt, \quad \text{and} \quad dx' = [1 - (v/c)^2] dx. \tag{21b}$$

However, according to Einstein [1], the observed clock rate for clock attached to K' is

$$dT = dt'[1 - (v/c)^2]^{1/2}. \tag{22a}$$

Thus the observed clock rates observed from K and K' respectively are actually,

$$dt = dT\,[1 - (v/c)^2]^{-1/2}, \quad \text{and} \quad dT = dt\,[1 - (v/c)^2]^{-1/2}. \tag{22b}$$

This difference from special relativity is that the velocity v (= at) increases. For the case of space contractions, they differ from special relativity, because the effects are either accumulated together or cancel each other.

Now, it is clear that Synge's [13] suggestion of the "midwife" retirement is due to inadequate understanding of Einstein's equivalence principle. The application of the arguments of Ohanian & Ruffini to metric (19) is valid because the Einstein-Minkowski condition has been applied to make sure dt' is a local time, but is not valid if such arguments are applied to metric (12) because dt is not local time of K'(x', y', z'). However, the implications are clear only after an adequate analysis.

The calculation of Tolmann [24] was not trusted because the usual gravity is related to a partial derivative of g_{tt} with respect to a space variable in analogy to Newtonian theory. Moreover, as we shall see that Tolmann actually made some mistakes probably because of his conceptual errors. The then popular belief was that Einstein's equivalence principle related two Euclidean subspaces. Apparently, his notion of Euclidean subspace was so strong that he made errors in his calculation. This shows that a bias opinion would affect the judgment of a scientist who is otherwise excellent.

To establish a relation between metric (12) and the conventional form of gravity, let us consider metric (12) (i.e., metric [16]) in the following form,

$$ds^2 = (c^2 + 2U)\,dt'^2 - (1 + 2U/c^2)^{-1} dx'^2 - (dy'^2 + dz'^2), \quad \text{and} \quad c^2/2 > -U(x', t') \geq 0. \tag{23}$$

where $-U(x', t') = (at)^2/2$. Metric shows the time dilation and space contractions clearly. Although t' is not a global variable, dt' is defined locally. Thus, it would also be possible to calculate the geodesic equation locally in terms of local coordinates.

For a particle P' attached to K' (i.e., $dx' \equiv 0$) we have $dx/dt = v$. On the other hand, for a particle P_0 attached to K (i.e., $dx \equiv 0$) we have $dx'/dt = -v$. The physical meaning of v would relate to the coordinate system that one refers to. To check our calculation, a simple way is to calculate the equation related to eq. (74.9) of Tolman [24]

The condition for his eq. (74.9) is for the accelerated particle attached to K is as follows:

$$\frac{d^2x}{ds^2} = \frac{d^2y}{ds^2} = \frac{d^2z}{ds^2} = \frac{d^2t}{ds^2} = 0 \qquad (24a)$$

These conditions imply just

$$dx = dy = dz \equiv 0, \qquad ds = cdt, \qquad dx' = -v\,dt, \quad and \qquad dt' = [1 - (v/c)^2]^{-1}dt. \qquad (24b)$$

Therefore

$$\frac{d^2x'}{ds^2} = -\frac{dv}{c^2dt} = \frac{-a}{c^2}, \qquad -\mathrm{dU} = \mathrm{avdt} = -\mathrm{a\ dx'}, \quad \text{and} \quad -\frac{\partial U}{\partial x'} = -a \qquad (25)$$

On the other hand,

$$\Gamma^{x'}{}_{t't'}\frac{cdt'}{ds}\frac{cdt'}{ds} + \Gamma^{x'}{}_{x'x'}\frac{dx'}{ds}\frac{dx'}{ds} + 2\,\Gamma^{x'}{}_{x't'}\frac{dx'}{ds}\frac{cdt'}{ds} = \frac{1}{c^2}\frac{\partial U}{\partial x'}, \qquad (26a)$$

and thus

$$\frac{d^2x'}{ds^2} = \frac{-1}{c^2}\frac{\partial U}{\partial x'}, \qquad (26b)$$

in agreement with (25). Since ds = cdt, (25) is the expected equation. However, eq. (74.9) of Tolman is

$$\frac{d^2x'}{ds^2} = \frac{-a}{c^2}\frac{1}{1-(at/c)^2}. \qquad (27)$$

The derivation of (25) is so simple that it almost impossible to be wrong.

Now, one can see why Tolman's calculation was ignored. The most probably reason is that he related a uniform gravity with a Euclidean subspace. The correct general equation, based on metric (12), is

$$\frac{d^2x'}{ds^2} = \frac{-a}{c^2}\frac{1}{1-V^2/c^2}. \qquad \text{where} \quad V^2 = [\frac{dx}{dt}]^2 + [\frac{dy}{dt}]^2 + [\frac{dz}{dt}]^2 \qquad (28a)$$

and V is the velocity of the particle in the inertial system (x, y, z, t). Therefore, eq. (27) is proven invalid unless $V^2 = (at)^2$. On the other hand, Eq. (28a) agrees with eq. (25) since it has $V^2 = 0$. Moreover, (28a) can be reduced to

$$\frac{d^2x'}{dt^2} = -a. \qquad (28b)$$

Now, consider a particle resting at K'. We have $dx'/dt' = dy'/dt' = dz'/dt' = 0$, and the geodesic equation is

$$\frac{d^2y'}{ds^2} = \frac{d^2z'}{ds^2} = 0, \qquad \text{and} \qquad \frac{d^2x'}{ds^2} = -\Gamma^{x'}{}_{t't'}\frac{cdt'}{ds}\frac{cdt'}{ds} = -\frac{1}{c^2}\frac{\partial U}{\partial x'} \qquad (29)$$

i.e., eq. (26) is still valid. On the other hand, a direct calculation shows that

$$\frac{d^2x'}{ds^2} = -\frac{1}{c^2}\frac{\partial U}{\partial x'} = -\frac{a}{c^2}\frac{1}{[1-(v/c)^2]}. \tag{30}$$

This is the same equation as (27). However, to determine ∂U/∂x' directly is more complicated. In eq. (25), the derivation is simple because dx = 0. The condition dx' = 0 gives an uncertainty to ∂U/∂x' because dt' is not a global variable.

In short, Einstein's equivalence principle is indeed compatible with Einstein's notion of Riemannian space. In this verification, there are two main conceptual problems: 1) the Euclidean-like structure is independent of a metric [17-19]; 2) there is a time-dependent in the potential $U \equiv -v(t)^2/2$, this is a surprise probably to those, who consider the uniform gravity in terms of Newtonian theory and thus it would have had an analogy with electromagnetism.

6. Discussions and Conclusions

Currently, a uniform gravity is still understood in terms of Newtonian gravity, and is used as a local approximation. This was what Einstein [9] did in his 1911 derivation of the gravitational redshifts. Nevertheless, Einstein's arguments for such a derivation are essentially valid because its result is only a first order approximation. First, because of the Euclidean-like structure [17-19], for this situation a frame of reference of a curved space-time can be treated as if a Euclidean space. Second, since light travels much faster than an accelerated frame, the potential for a uniform gravity can be considered as if static. However, raising such an approximation to the level of a physical principle and resulting the equivalence of gravity and acceleration, is far more serious. As Einstein pointed out, this is misleading in physics and mathematically incorrect [7, 9].

Currently, the meaning of Einstein's equivalence principle was mistaken [26] to be the same as the preliminary application of equivalence with Newtonian gravity. As P. Morrison [27] pointed out, this is incorrect. The gravitation potential is a scalar in Newtonian theory, but is a second rank tensor in general relativity. A uniform gravity must be time-dependent because of Einstein's equivalence principle, but can be static in Newtonian theory. Metric (16) confirms these and clearly there is no Euclidean subspace. Moreover, a foundation of Einstein's equivalence principle is the theorems that the local metric of a "free fall" particle is locally constant [13], but generally cannot be a constant metric. Einstein proposes the Einstein-Minkowski condition that for a physical situation such a locally constant metric must be a local Minkowski metric [1, 9].

The potential U in metric (19) of a uniform gravity, must be linear with respect to x'. For a particle at rest with the inertial system K, ∂U/∂x' is the expected constant. For a particle with a velocity, ∂U/∂x' is not expected to be the same constant, but it should have no explicit x'-dependence. Nevertheless, the various situations can be summarized to the same equation since the acceleration is created by the uniformly accelerated frame of reference. Moreover, because of Einstein's equivalence principle, the metric is time-dependent. On the other hand, the gravitational redshifts of lights emitted from the accelerated K' should be independent of its location. Since U has no explicit space dependence, such a requirement is satisfied. These are proven with the Einstein-Minkowski condition that for such a case the resulting local metric in a free fall is Minkowski.

In this paper, it is shown that the metric form (1) used by Fock [14] is invalid. This is expected since he incorrectly assumed the metric is static. However, it was not expected that the local time transformation (16b) for an accelerated frame has no integrated form. J. P. Hsu and L. Hsu [28] had made an attempt to obtain a coordinate transformation between an inertial frame and a uniformly accelerated fame. They correctly assumed that the time-time component of the metric is time dependent, but they did not see that the spatial component of the metric is also time dependent. This demonstrates that the physical meaning of a metric is far from obvious if it is not transformed into the canonical form

A major problem was that the space-time coordinates are ambiguous as pointed out by Whitehead [15] and Fock [14]. Einstein believed that the space coordinates are decided by the metric as in a space embedded in a higher dimensional flat space. Although Einstein was aware that mathematics might not be related to physical reality [28], Einstein or Dirac [29] did not see that the definition of space-time coordinates is necessarily independent of the space-time metric.

This ambiguity of space coordinates forced Einstein [8] to propose an interim assumption, the so-called "covariance principle" that has also been proven to be over extended the principle of general relativity [30]. Weinberg [4] illustrated, however, a curved space need not be embedded in a higher dimensional flat space. In fact, the theoretical framework of general relativity implies that a frame of reference must have the Euclidean-like structure [17-19].

However, a metric for uniform gravity was not provided since Einstein did not need such a metric in his subsequent calculations. Moreover, in a uniformly accelerated frame, the local space in a free fall is clearly Minkowski. Einstein and subsequent theorists seem to be unaware that the feasibility of the notion of uniform acceleration in general relativity must be proved. Moreover, since many theorists focus their attention to uniform gravity as a local approximation, it would be useful to see how conventional a metric for uniform gravity could be after all.

Einstein's equivalence principle implies that the time dilation and space contractions are measurable [1]. However, such measurements seem to be trivial since Einstein addressed only the diagonal metrics or metrics without a crossing space-time element. This creates a false impression that the Einstein-Minkowski condition is trivial. For instance, Synge [13], who is an excellent mathematician, failed to investigate the physics of Einstein's equivalence principle.

Among the textbooks, only Landau & Lifshitz [19] first took the trouble of addressing the important issue of space contractions for the general case. Unfortunately, because their arguments are based on Pauli's version, the derivation of their local distance formula is not faultless [20]. In fact, almost all the textbooks in the West ignored their work. Moreover, theorists, who do not understand Einstein's notion of local time [1, 8], rejected the formula of Landau & Lifshitz totally as invalid.

In this paper, the metric for a uniformly rotating disk is derived rigorously with the Einstein-Minkowski condition. Then the physical meanings of the metric and the local time are clear. For the given assumptions of Einstein, validity of the metric is firmly established since such a calculation is based on special relativity. Concurrently, the calculated space contractions of Landau & Lifshitz [21] are proven valid for the case of the rotating disk. Thus, the crucial role of the Einstein-Minkowski condition in the space contractions is illustrated. More important, oversights of Einstein on measurement are rectified.

Einstein's equivalence principle is important because it is crucial in the question of space contractions and the time dilation. This principle would play a central role in Einstein's theory of measurements [27], and would involve in many important issues of application. For instance, this principle and the observed light bending would clarify [31] validity of the receding velocity in the model of expanding universe [5, 12, 32].

In physics, the fundamental concepts are often difficult to grasp. Now, the faith on the feasibility of a uniformly accelerated frame is proven valid. It is clear that the metric related to uniform gravity must be time-dependent, but it was difficult to see this intuitively right at the beginning. Einstein's equivalence principle is important because it has physics beyond the mathematical theorems in Riemannian space [11, 19]. In this year of celebrating the birth of relativity, it would be meaningful to understand correctly what is Einstein's equivalence principle and its importance in general relativity.

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ENDNOTES

1) Misner, Thorne, & Wheeler [5; p. 386] claimed that Einstein's equivalence principle is as follows: -

> "In any and every local Lorentz frame, anywhere and anytime in the universe, all the (Nongravitational) laws of physics must take on their familiar special-relativistic form. Equivalently, there is no way, by experiments confined to infinitestimally small regions of spacetime, to distinguish one local Lorentz frame in one region of spacetime frame any other local Lorentz frame in the same or any other region."

However, this is only an alternative version of Pauli's because the Einstein-Minkowski condition is ignored. In their eq. (40.14) [5; p. 1107], they got a physically incorrect conclusion on the local time of the earth in the solar system because they did not understand Einstein's equivalence principle and related theorems in Riemannian space. Unfortunately, Ohanian & Ruffini [12; p. 198] also ignored the Einstein-Minkowski condition and had the same problems as shown in their eq. (50). However, Eddington [2], Liu [23], Straumann [6], Wald [32], and Weinberg [4] did not make the same mistake.

2) Straumann [6] incorrectly claimed that Einstein's principle of equivalence is, "In any arbitrary gravitational field no local experiment can distinguish a freely falling nonrotating system (local inertial system) from a uniformly moving system in the absence of a gravitational field." He recognized that the local space in a free fall is Minkowski, but fail to see that such a metric is not the same metric in the absence of a gravitational field.

3) Will [8, p. 20] incorrectly claimed "'Equivalence' came from the idea that life in a free falling laboratory was equivalent to life without gravity. It also came from the converse idea that a laboratory in distant empty space that was being accelerated by a rocket was equivalent to one at rest in a gravitational field." Thus, Will's understanding [33] is essentially as Pauli's.

4) Synge [13] is an excellent mathematician. However, the issues of the time dilation and the space contractions were not sufficiently discussed in his book. Understandably, he did not see the crucial role of Einstein's equivalence principle.

5) This simple fact, as shown in Section 3, is crucial to derive a space-time metric for a uniform gravity.

6) In general, if a metric has a non-zero irreducible space-time cross element, we can synchronize clocks along any open curve. However, it is not possible to synchronize all the local clocks along a closed contour [21, 23, 30].

7) Metrics (16) and (7b) appear to have the same form, but they differ in three aspects: 1) the velocity v (= at) in (16) is time-dependent; 2) the time dt' in (16) is a local time only; 3) the frame of reference for metric (7b) is not physically realizable.

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