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UCLA/92/TEP/30

Tasi Lectures on Critical String Theory*

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ABSTRACT

In these lectures, we present an introduction to bosonic and super string theory in the critical dimension. We review the operator formalism, the Polyakov approach and their connection with conformal field theory. These lectures were presented at the TASI summer school, Boulder, June 1992.

* Research supported in part by NSF grant PHY-89-15286.

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Acknowledgements

I would like to thank the organizers of TASI and in particular K.T. Mahanthappa, and the students at the school for providing the interesting and pleasant atmosphere in which these lectures were delivered.

1. MOTIVATIONS FOR STRING THEORY

The long and tortuous history of string theory takes us back to the late 1960's when no compelling theory of the strong interactions had emerged yet. Quantum field theory did not seem to provide with any useful tools for the description of strongly interacting hadrons. This situation appeared in sharp contrast with the theory of electromagnetic and weak interactions which can be described by weakly coupled quantum field theory. In attempts at constructing alternatives to field theory, S-matrix theory gained perhaps the strongest support. The problem with S-matrix theory though is that it embodies no dynamics, and merely provides the most general realization of a relativistic quantum theory consistent with unitarity and causality, but little more unless specific extra dynamical input is supplied.

A remarkable dynamical principle, "nuclear democracy", was proposed in this vain. Nuclear democracy proclaims that all hadrons are to be viewed as composites of all other hadrons, a principle motivated by the empirical fact that all hadrons interact very strongly with one another. This proposal when implemented on scattering amplitudes, naturally leads to the property of duality of strong interaction scattering amplitudes. Duality means that the states (resonances) observed in any channel of a scattering amplitude should be the same as in any other channel. In particular, the two particle into two particle scattering amplitude should be invariant under interchange of s and t , the two Lorentz invariant Mandelstam variables characterizing the kinematics of the process. The equations for duality when combined with the principles of analyticity and factorization are surprisingly restrictive. They were first solved in a famous formula of Veneziano (1968) and later in an alternative but different formula of Virasoro (1969) and Shapiro (1970), as we shall see later on in these lectures.

In a search for a mechanistic realization that would produce the Veneziano amplitude as a result of its dynamics, it was found by Nambu (1970) and Susskind (1970) that the corresponding starting point is strings. If the hadrons of the strong interactions are strings, *i.e.* one dimensional extended objects obeying the laws of relativity and quantum mechanics, then the Veneziano amplitude results directly from the quantum equations of motion of these strings.

The problem with any such theories though, it was soon found out, is that consistency with quantum mechanics and special relativity leads to additional requirements on these string theories. They tend to require more than 4 space-time dimensions and also naturally contain massless particles of spin 1 and spin 2 which are definitely not seen in the spectrum of hadrons.

Moreover, in the early 1970's it became clear that successful theories of electromagnetic, weak and strong interactions can be formulated in terms of local quantum field theory, more precisely Yang-Mills gauge theories. Even for the

strong interactions, the corresponding Yang-Mills theory, called Quantum Chromodynamics (QCD) is relatively weakly coupled at high energies due to the property of asymptotic freedom. String theory as a model for hadron physics was soon abandoned.

At about the same time, Scherk and Schwarz (1974-75) and independently Yoneya (1973), observed that precisely those properties of string theory that had previously created problems if viewed as a model of strong interaction, can be used to one's advantage for a theory of gravitation. The massless spin 2 that had been unobserved as hadrons can be reinterpreted as a graviton as in Einstein gravity. As a bonus, there are also massless spin 1 particles characteristic of unbroken Yang-Mills theory. Even the extra space-time dimensions can be viewed along the lines of the extra fifth dimension introduced long ago by Kaluza (1921) and Klein (1926). As such, these extra dimensions can live on a small internal space, whose typical scale is so minute as to escape present day experimental tests.

String theory does not couple consistently to pointlike particle field theory. If one single elementary particle or fundamental interaction is described by a string theory, then all particles and interactions must be fit within the string picture. Thus, if string theory is to be applied to the quantization of gravity, then naturally, all four fundamental interactions, electroweak, strong and gravity must be described by string theory; whence the common term "the theory of everything".

It is this set of observations about some of the fundamental properties of string theory together with the realization that more or less realistic models of grand unified Yang-Mills gauge theories emerge that has given rise to the tremendous excitement about string theory as a model of all fundamental particles and interactions over the past 10 years.

The field of string theory was created almost 25 years ago now, and Qspires as of the time these lecture notes are being written shows almost 4000 papers mentioning the word "string" in the title. In these lectures, we shall address only some aspects of string theory, and we shall indicate references only sporadically.

A number of books and reviews on the subject have appeared, and I shall mention just a few of them here. The oldest reviews on dual models and then on string theory are in Alessandrini, Amati, Le Bellac and Olive (1970), Schwarz (1973), Frampton (1974), Mandelstam (1974), Rebbi (1974), Veneziano (1974) and perhaps the most accessible one for a modern audience, Scherk (1975). Many of the older papers on the Ramond (1971) and Neveu-Schwarz (1971) theories are reprinted in Schwarz (1986), where also the paper by Scherk (1975) appears.

Reviews and books in the modern era of string theory include the following. There are two review papers by Schwarz (1982), Green (1983). There is a graduate level textbook by Green, Schwarz and Witten (1987) which is a good starting point for learning string theory, and which has an extended list of references as

well. There is a beautiful book dealing with string theory amongst other things by Polyakov (1987). There are a number of conference proceedings held at Argonne (ed. Bardeen and White, 1985), Santa Barbara (ed. Green and Gross, 1986), San Diego (ed. Yau, 1987), Maryland (ed. Gates and Mohapatra, 1987), Stony Brook (ed. Berkovits et al., 1991) and many others of course.

There is a review paper by D'Hoker and Phong (1988), which is on super-string perturbation theory in the critical dimension, and which is used as the basis for these lectures. This review also has an exhaustive bibliography up to 1988. There is also a nice review on conformal field theory by Ginsparg (1988), a book of reprints in conformal field theory and statistical mechanics by Itzykson, Saleur and Zuber (1988), and a book on the interface between statistical mechanics and (conformal) quantum field theory in two dimensions by Drouffe and Itzykson (1989).

A. Problems with quantizing Einstein gravity

Before entering completely into the subject of superstring theory, we point out some of the problems that arise when attempting to quantize Einstein gravity within the context of local quantum field theory, and we discuss some further ramifications and possible applications of string theory. To date, there is no successful quantization scheme of Einstein gravity based on quantum field theory. It seems impossible to meet the following basic physical requirements all at once:

1. pointlike elementary particles
2. local pointlike interactions, i.e. no actions at a distance
3. unitarity or quantum mechanical evolution that conserves probability
4. causality or commutativity of space-like separated observables
5. general coordinate invariance, as postulated by general relativity
6. renormalizability as a quantum field theory

Item 1, 2 and 5 are just the basic assumptions of quantum field theories like Einstein gravity, so conflicts arise in these theories between items 3, 4 and 6. To illustrate the problem of renormalizability, consider the Einstein-Hilbert action

$$S_{EH} = -M_p^2 \int d^4x \sqrt{g} R \qquad M_p^2 \equiv \frac{\hbar c}{16\pi G_N}$$

where G_N is Newton's constant and M_p is the so-called Planck mass ($M_p = 1.710^{-6}g = 10^{18}GeV$). Expanding this action around a flat Minkowski space-time

metric $\eta_{\mu\nu}$, we get schematically

$$S_{EH} = \int d^4x \left[h \square h + \frac{1}{M_p} h \square h^2 + \frac{1}{M_p^2} h \square h^3 + \dots \right] \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \frac{1}{M_p}$$

In this action, the field h has dimension 1, and the effective coupling is $1/M_p$ which has negative dimension. Thus, the Einstein-Hilbert action is not power counting renormalizable.

An interesting alternative to Einstein-Hilbert action is the famous R^2 gravity, with action

$$\begin{aligned} S_{R^2} &= \int d^4x \sqrt{g} [-M_p^2 R + R_{\mu\nu} R^{\mu\nu}] \\ &= \int d^4x \left[h \square^2 h + M_p^2 h \square h + M_p h \square^2 h^2 + \dots \right] \end{aligned}$$

This model is power counting renormalizable, but by analyzing the propagator at tree level

$$\frac{1}{k^2(k^2 - M_p^2)} = \frac{1}{M_p^2} \left[\frac{1}{k^2} - \frac{1}{k^2 - M_p^2} \right]$$

we see that a massive ghost propagates, which renders the quantum field theory acausal and non-unitary.

B. Further ramifications and possible applications of string theory

In a string theory of elementary particles, each particle is described at any fixed instant in time as a “string” *i.e.* a one dimensional extended object of zero thickness. This picture is in contrast with local quantum field theory, where the fundamental building blocks are viewed as point in space. In the next lecture, we shall show how from a purely geometrical point of view the string picture is more natural than the picture of pointlike elementary particles. Along these lines, we may perhaps expect a theory of strings to be a more suitable theory of particles. One may also ask why strings and not membranes or solitons? Whereas we know that we can build consistent theories of strings, it is very unclear to this date that theories of interacting membranes or solitons could be made to work.

There is an important mathematical and physical connection between the elementary particle theory of strings and the statistical mechanical theory of random surfaces; this was put forward especially by Polyakov (1981). Recall that a point particle sweeps out a worldline during its time evolution. Analogously, the time evolution of a string sweeps out a surface or a worldsheet. There will be intimate connections between the kinematics and dynamics of random surfaces and strings.

Besides the application of string theory to the quantization of gravity, the connection with surface statistical mechanics opens the door to many further applications as well. In strong interaction theory, hadronic bound states may be viewed as extended chromoelectric flux tubes ending on quarks and antiquarks. In the case of mesons for example, such flux tubes can be approximated by open string, and it is believed that in some way the dynamics of the hadron of QCD can be approximated by true string description. Some even believe that QCD is equivalent at all levels to a string model. We shall see that indeed in certain more complicated string theories, no massless spin 1 or spin 2 particles are present, and could be used as models for hadrons.

Certain statistical mechanics problem on fixed lattices can be reinterpreted in terms of statistical mechanics problems for random surfaces. Perhaps the most important one is the 3-dimensional Ising model, in which the + and - signs on the sites can be separated by 2-dimensional surfaces. It has long been conjectured that this model might be equivalent to a string dynamics model with fermionic or superstrings; see Polyakov (1987).

There are also statistical mechanics problems that from the beginning involve spins on a random lattice. When the time scales for dynamics of the random lattice and of the spins are comparable, the randomness is said to be annealed and naturally leads to a string description as we shall outline it in these lectures.

In the present set of lectures, we shall provide an introduction to the quantization of bosonic and superstring theory, both from the point of view of its applications as a theory of quantum gravity as well as of a theory of random surfaces. In fact, it turns out that the random surface formulation lends itself most naturally to a geometrical setting of the string theory approach to particle physics. This approach was the guiding theme in the review by D'Hoker and Phong (1988).

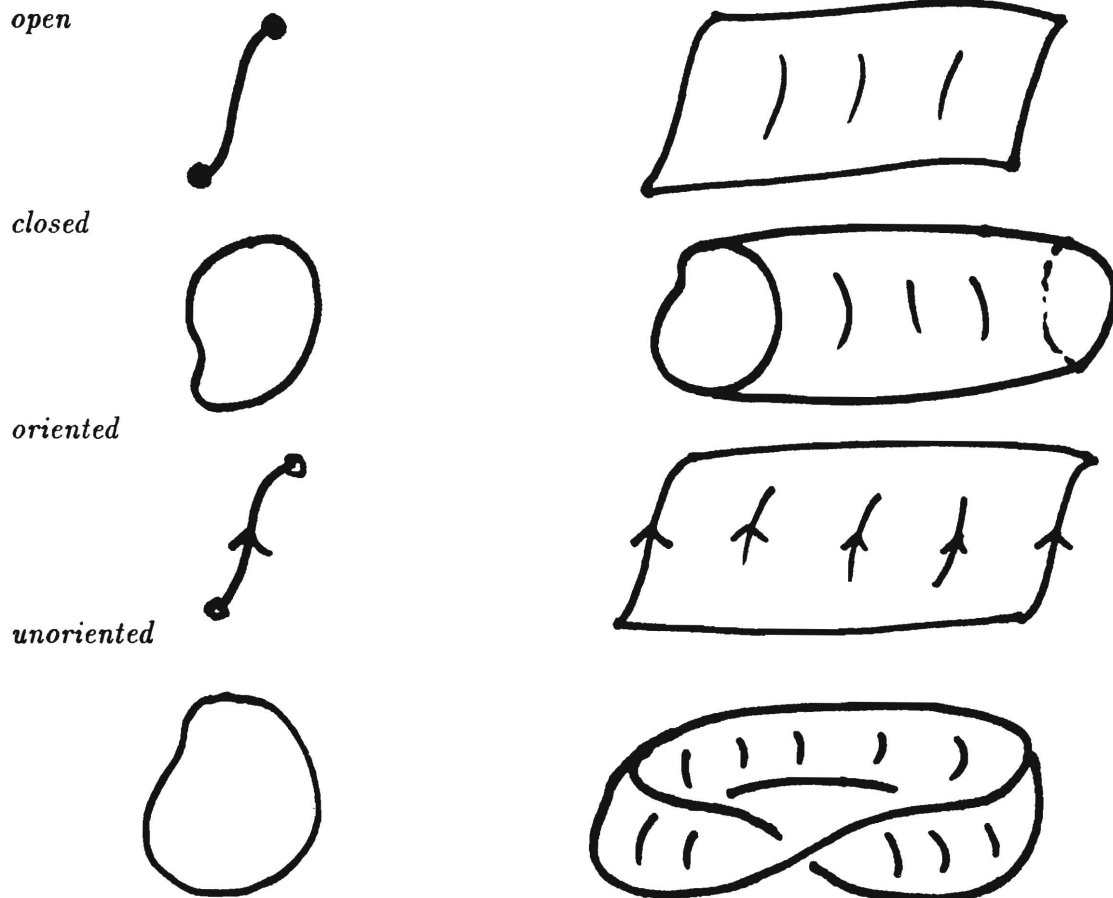
We limit ourselves most of the time to string theory in the critical dimension, where gravitons and spin 1 Yang-Mills gauge fields naturally have 0 mass. Compactifications to lower dimensions, and the full dynamics of non-critical strings including the Liouville model will be discussed by other lecturers at this school.

II. BASICS OF STRING THEORY

In this lecture, we explain the connection between string evolution and the geometry of surfaces. We review the topology and geometry of surfaces and introduce the classical actions describing string dynamics.

A. Strings Versus Two-dimensional Surfaces

A string at a fixed time is a one-dimensional extended object. It can be open or closed, oriented (with a preferred direction) or unoriented. Under time evolution, the one dimensional object in space sweeps out a two-dimensional worldsheet or surface in space-time. A few pictures will make clear what we have in mind. On the left is the string viewed at fixed time while on the right is a piece of worldsheet swept out by the string during its evolution.

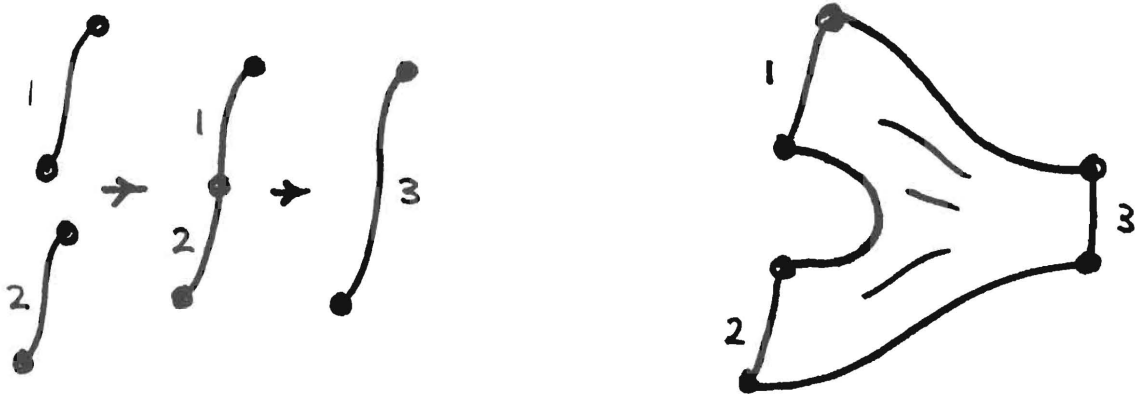


Examples of unoriented surfaces are the Möbius strip open strings or the Klein bottle for closed strings.

B. Interactions Between Strings

In quantum field theory of pointlike particles, an interaction between two particles only occurs when they overlap at the same point in space at the same time. This was required by causality, otherwise we would have fundamental actions at a distance. Through the exchange of particles one may of course effectively get an interaction at a distance like for example the Coulomb interaction in electromagnetism. In quantum field theory, one must put the interactions by hand as specified by the Lagrangian of the theory.

In string theory, even though the basic objects are extended, causality demands that their interactions nonetheless be local. This means that two strings can interact only at one point at one particular moment in time. This requirement is very restrictive. Open strings can interact by joining their end points and merging into a single string, as depicted below.

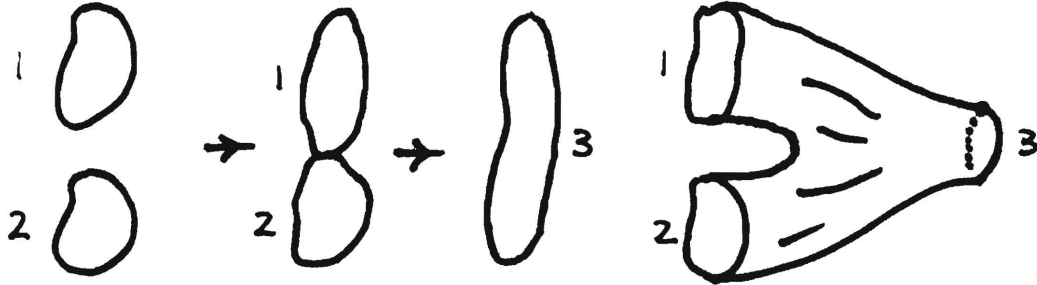


Of course, an open string would not know the difference between the end of another strings and its own opposite end, so for consistency, one should allow an open string to also close upon itself, as depicted below.



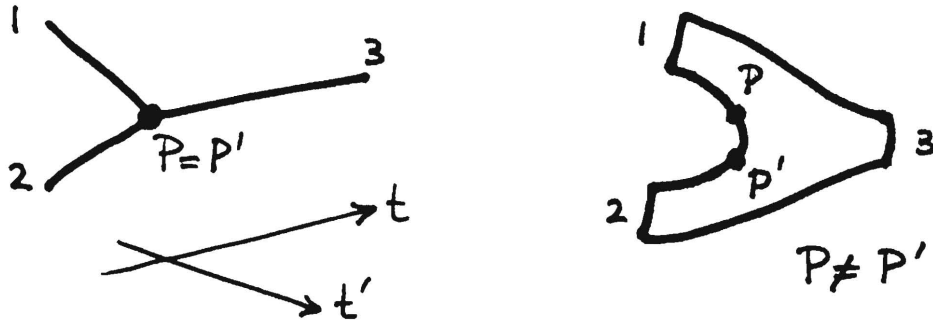
Thus, any string theory of interacting open strings must also contain closed strings for consistency, and there is a non-zero transition probability from open to closed strings and vice versa.

Closed strings may also directly interact with closed strings, as shown in the figure below.



A theory containing only closed strings can be consistent all by itself without introducing open strings, and the only interaction between closed strings is then the one shown above. You may wonder whether open strings can also interact in their middle parts or interact directly with closed strings. They may, but this interaction is already contained in the ones we have listed above; so to avoid double counting, it should not be introduced separately.

Whereas in a fixed time picture, there is a clear interaction site for two strings, this is not so when one considers the full time evolution of the string by examining the corresponding worldsheet. This should be contrasted with the situation in point-like particle field theory, as illustrated by the figure below.



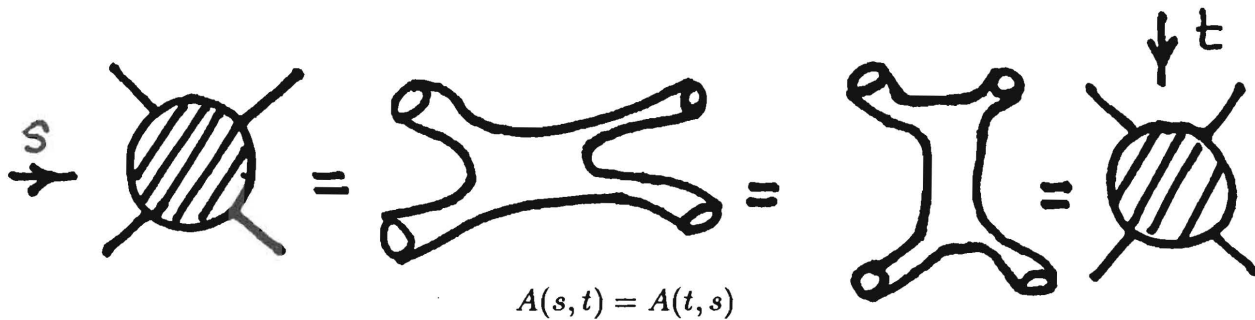
As we mentioned before, in quantum field theory of pointlike particles, one must put in the interaction by hand. The geometry around the interaction point looks different from the geometry of free propagation. In fact the worldlines are geometrically singular at the interaction point in that the worldlines cease to be a manifold there. If the interaction point is observed from different Lorentz frames, the geometrical interaction point is unchanged. The interaction is an intrinsic special point in the evolution of the particles.

In string theory, in any given Lorentz frame, the interaction is local. But the geometrical point depends upon the Lorentz frame chosen. There is nothing geometrical about the point at which the interaction occurs. Lorentz invariance prevents one from putting in an interaction vertex that depends on any geometrical point, because from a different Lorentz frame this geometrical point would not look

like it occurs at the right time and causality would be violated.

In fact, in string theory, it is natural to put in no explicit interaction at all. Interactions result from joining and splitting. Locally on the surface one cannot distinguish where the interaction takes place. One might imagine putting in an interaction vertex that does not depend (locally) on where you put it. We'll see that this indeed occurs for the superstring.

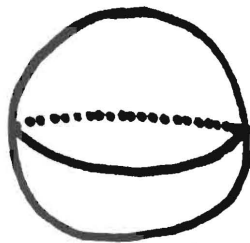
From this geometrical formulation, string amplitudes appears naturally dual, as can be visualized by the figure below.



C. Topology of Oriented Surfaces

It will turn out that the most promising string theories are those constructed on closed oriented surfaces, and henceforth in these lectures, we shall restrict our study to the theory of closed oriented strings only. However, it must be noted that open-closed strings are potentially equally interesting. They are however more complicated, and somewhat less well understood. Many of the techniques we shall develop in these lectures for dealing with closed strings carry over to the case of open strings with only minor modifications.

Topologically, oriented surfaces are classified by the number of handles h and the number of boundaries b .



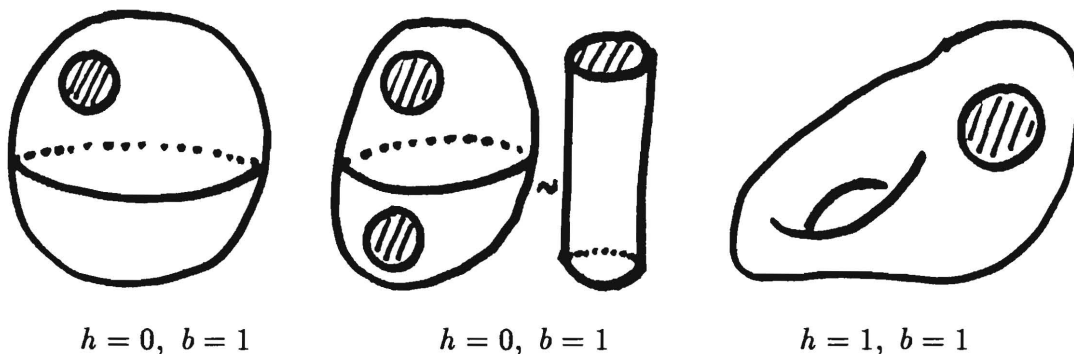
$$h = 0, b = 0$$



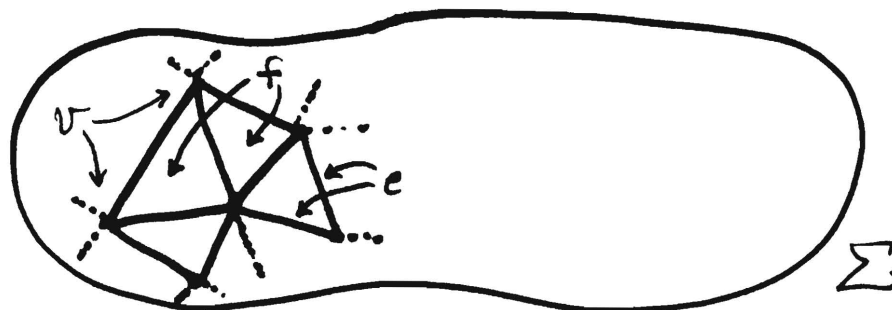
$$h = 1, b = 0$$



$$h = 2, b = 0$$



There is a topological invariant, called the Euler number, defined as follows. We consider a surface Σ , and we triangulate its surface by laying down on the surface v vertices, connected by e edges and exhibiting f faces, as pictured in the figure below.



Then, the Euler number is defined by

$$\chi(\Sigma) = v - e + f = 2 - 2h - b \quad (2C.1)$$

and its importance is derived from the fact that the actual number $\chi(\Sigma)$ is independent of the triangulation. It is easy to prove this by examining elementary moves that remove vertices or edges, as exemplified in the figure below.



Considering also unoriented surfaces, there would be a third integer in the classification, the number of so-called cross caps. A cross cap is a boundary on which opposite points on the boundary circle are identified pairwise. This renders the surface non-orientable.

D. Basic Differential Geometry of Surfaces

Here, we focus on the *intrinsic geometry* of the surface. Geometry on an oriented surface is specified by the following data

- a positive Riemannian metric $g_{\mu\nu}$
- the antisymmetric tensor $\sqrt{g}\epsilon_{mn}$ (since the surface is oriented)
- the Christoffel connection Γ_{mn}^p

There are a number of basic concepts that we shall here list

- Reparametrizations or diffeomorphisms, forming a group $\text{Diff}(\Sigma)$

$$\xi^m \rightarrow \xi'^m(\xi) \quad g_{mn}(\xi) \rightarrow g'_{mn}(\xi') = \frac{\partial \xi^p}{\partial \xi'^m} \frac{\partial \xi^q}{\partial \xi'^n} g_{pq}(\xi) \quad (2D.1)$$

Infinitesimally, this transformation acts as follows

$$\delta \xi^m = v^m \quad \delta g_{mn} = \nabla_m v_n + \nabla_n v_m \quad (2D.2)$$

- Covariant differentiation

$$\begin{aligned} \nabla_m v^n &= \partial_m v^n + \Gamma_{mp}^n v^p && \text{on vector fields} \\ \nabla_n v_n &= \partial_m v_n - \Gamma_{mn}^p v_p && \text{on one forms} \end{aligned} \quad (2D.3)$$

- Riemannian geometry requires compatible metric and connection and zero torsion

$$\left. \begin{aligned} \nabla_m g_{pq} &= 0 \\ \Gamma_{mn}^p &= \Gamma_{nm}^p \end{aligned} \right\} \quad \begin{aligned} \Gamma_{mn}^p &= \frac{1}{2} g^{pq} (\partial_m g_{nq} + \partial_n g_{mq} - \partial_q g_{mn}) \\ &\text{zero torsion} \end{aligned} \quad (2D.4)$$

- Riemann curvature tensor

$$\begin{aligned} [\nabla_m, \nabla_n] V_k &\equiv R_{kmn}^\ell V_\ell \quad \text{for any } V \\ R_{\ell mnk} &= R_{nk\ell m}; R_{\ell mnk} = -R_{m\ell nk} \\ R_{\ell mnk} + R_{\ell nkm} + R_{\ell mkn} &= 0 \end{aligned} \quad (2D.5)$$

- Ricci curvature tensor and Gaussian curvature scalar

$$\begin{aligned} R_{mn} &= R_{m\ell n}^\ell \\ R &= -\frac{1}{2} g^{mn} R_{mn} \end{aligned} \quad (2D.6)$$

In these conventions, the Gaussian curvature is 1 for a sphere with the standard round metric and radius 1. In two dimensions the only independent curvature

components are

$$\begin{aligned} R_{k\ell mn} &= -R(g_{km}g_{\ell n} - g_{\ell m}g_{kn}) \\ R_{mn} &= -Rg_{mn} \end{aligned} \quad (2D.7)$$

- Complex structure, Riemann surfaces

On *oriented* surfaces, there exists a *complex structure*

$$J_m{}^n = \sqrt{g}\epsilon_{pm}g^{np} \quad \text{with} \quad J_m{}^n J_n{}^p = -\delta_m{}^p \quad (2D.8)$$

i.e. a map from $T(\Sigma) \rightarrow T(\Sigma)$ whose square is -1 . This defines a (covariant) Cauchy-Riemann equation

$$J_m{}^n \partial_n f = i \partial_m f \quad (2D.9)$$

whose solutions are complex analytic functions $f(\xi) = f(z)$. Thus, an oriented surface equipped with a metric becomes automatically a complex analytic surface or *Riemann surface*.

- Gauss' Theorem states the following crucial fact.

By a reparametrization (i.e. a Diffeomorphism), one can always find a *local* coordinate system in any simply connected domain such that the metric is *conformally flat*.

$$g_{mn} = e^{2\phi} \delta_{mn} \quad (2D.10)$$

For the plane, the cylinder and the sphere (without boundaries), this result in fact holds *globally*.

- Local complex conformal coordinates

Using Gauss' theorem and the complex nature of Riemann surfaces, the metric is locally conformally flat in a complex coordinate system $(z, \bar{z})$, so that we have

$$ds^2 = 2g_{z\bar{z}} dz d\bar{z} \quad g_{z\bar{z}} = e^{2\phi}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad (2D.11)$$

- Conformal tensors

Under locally analytic or *conformal reparametrizations*

$$z \rightarrow z'(z) \quad \bar{z} \rightarrow \bar{z}'(\bar{z})$$

tensors decompose into 1-dimensional irreducible representations, e.g.

$$V^m \longrightarrow V^z \oplus V^{\bar{z}}$$

General irreducible tensors in 2 dimensions are symmetric, traceless (antisymmetric with 2 indices is already a pseudo scalar). To see this, we decompose a general

symmetric tensor as follows

$$V_{m_1 \dots m_n} \longrightarrow V_{\underbrace{z \dots z}_{\nu_1} \underbrace{\bar{z} \dots \bar{z}}_{\nu_2}} \sim V_{z \dots z}^{\nu_1 \dots \nu_2} \quad (2D.12)$$

where $\nu_1 + \nu_2 = n$. Tracelessness then requires either $\nu_1 = 0$ or $\nu_2 = 0$. Also, by an abuse of notation

$$\underbrace{V_{\bar{z} \dots \bar{z}}}_n \sim \underbrace{V_{z \dots z}}_{-n}$$

So we have the vector spaces of conformal tensors of weight n :

$$T_n = \left\{ \underbrace{V_{z \dots z}}_n \sim \underbrace{V_{\bar{z} \dots \bar{z}}}_{-n} \right\} \quad (2D.13)$$

Under conformal reparametrizations, they transform as

$$\underbrace{V'_{z \dots z}}_n(z') = \left(\frac{\partial z}{\partial z'} \right)^n \underbrace{V_{z \dots z}}_n(z) \quad (2D.14)$$

- There is a natural *inner product* on these spaces

$$\langle T_1 | T_2 \rangle = \int d^2 z \sqrt{g} (g^{z\bar{z}})^n T_{1z \dots z}^* T_{2z \dots z} \quad (2D.15)$$

- Covariant differentiation in local conformal coordinate systems

$$\begin{aligned} \nabla_m &\longrightarrow \nabla_z \oplus \nabla_{\bar{z}} \\ \nabla_z^{(n)} : T_n &\longrightarrow T_{n+1} & \nabla_z^{(n)} T &= (g_{z\bar{z}})^n \frac{\partial}{\partial z} (g^{z\bar{z}})^n T \\ \nabla_{\bar{z}}^{(n)} : T_n &\longrightarrow T_{n-1} & \nabla_{\bar{z}}^{(n)} T &= \partial_{\bar{z}} T \end{aligned} \quad (2D.16)$$

- Adjointness with respect to the inner product of (2D.15)

$$\left(\nabla_z^{(n)} \right)^\dagger = -\nabla_{\bar{z}}^{(n+1)}; \quad \left(\nabla_{\bar{z}}^{(n)} \right)^\dagger = -\nabla_z^{(n-1)} \quad (2D.17)$$

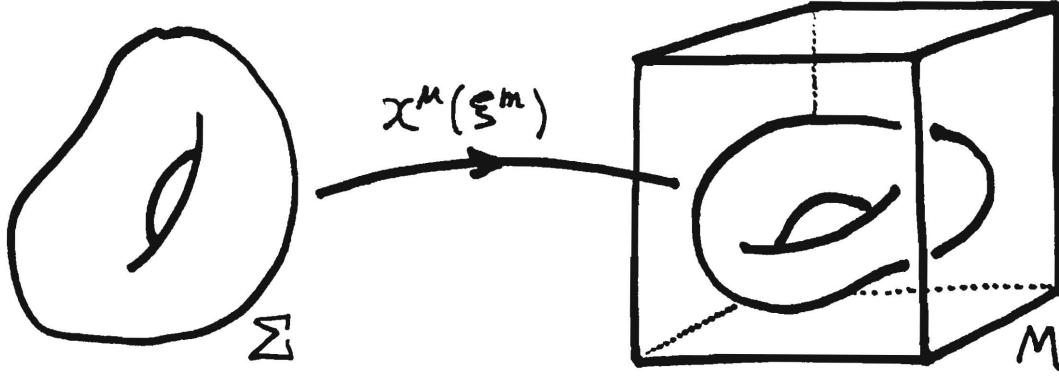
- Weyl transformations, forming a group $\text{Weyl}(\Sigma)$

$$g_{mn} = e^{2\phi} \hat{g}_{mn} \quad R_g = e^{-2\phi} (R_{\hat{g}} + \Delta_{\hat{g}} \phi) \quad (2D.18)$$

All of the above definitions and results on differential geometry of surfaces will be used ubiquitously throughout these lectures.

E. Basic Variables For The Bosonic String

A picture is often worth a thousand words; so to exhibit the relevant *intrinsic* and *space-time* or extrinsic geometry, examine the figure below.



On the left, we have the *intrinsic* (Riemann) surface, as described in §2D with a metric g_{mn} , an orientation ϵ_{mn} and a set of local coordinates ξ^m . On the right, we have the *embedding* or *target* space-time M , which we assume to be of dimension d . In the target space-time, we have a local coordinate system x^μ with $\mu = 1, \dots, d$, and we shall assume that space-time also has a metric $G_{\mu\nu}(x)$. The intrinsic surface Σ on the left is embedded in the target space-time by specifying how each point on Σ is mapped into a point in M . This may be achieved in local coordinates by specifying a set of functions $x^\mu(\xi^m)$.

On the image of Σ in M , we have a metric induced by G , since the line element can be expressed as

$$ds^2 = G_{\mu\nu}(x)dx^\mu dx^\nu = h_{mn}d\xi^m d\xi^n$$

so that the components of the induced metric read

$$h_{mn} = G_{\mu\nu}(x)\partial_m x^\mu \partial_n x^\nu \quad (2E.1)$$

The physical degrees of freedom of the string correspond to the fluctuation modes of the embedding function x .

We should note at this point that the physical string propagates in Minkowski space-time, so M should really be *pseudo Riemannian*. Often though, we consider the Euclidean problem, *i.e.* analytically continue to imaginary time.

F. Dynamics of Strings, Classical Actions

A sensible quantum mechanical theory of strings should define a sensible S-matrix, that is relativistic invariant, unitary and causal. We have already provided with a heuristic argument that the only interactions between strings should result from their joining and splitting, and no extra interaction vertex ought to be put in. A vertex whose local position is immaterial would be allowed, but this does not occur for the bosonic string; it does occur for the fermionic string in the component formulation as we shall see. Thus, all string dynamics must be governed by the dynamics of a local action on the worldsheet.

The following requirements on this action are fundamental

- locality on the worldsheet
- Diff (Σ) - invariant
- Diff (M) - invariant or general coordinate invariance in target space-time
- renormalizable as a worldsheet quantum field theory
- ultimately one really has to check the unitarity, causality etc. of the S-matrix

Are these requirements unavoidable for a consistent S-matrix theory ? The requirement that is perhaps the least obvious is that the worldsheet action should be local. Indeed some very special proposals have been made that might avoid this requirement; see *e.g.* Green (1991). However, I believe that in these cases it must somehow be possible to introduce extra degrees of freedom in terms of which the final action is again local. This is for example what will happen for the superstring case, where a local action is obtained in superfield formulation, though the dynamics appears non-local in the component formulation. We now describe some of the basic actions that have been proposed for bosonic strings.

- Nambu-Goto action; see Nambu (1970) and Goto (1971).

$$I_{NG}[x] = \frac{T}{4\pi} \int_{\Sigma} d^2\xi \sqrt{\det h_{mn}} \quad (2F.1)$$

Here h_{mn} is the induced metric on the surface from (2E.1). This action is seemingly independent of the intrinsic geometry of the surface and in particular independent of g_{mn} . In quantizing this action though, one finds that part of the intrinsic geometry does in fact appear in a subtle way.

- Polyakov random surface action; see Polyakov (1981).

$$I_o[x, g] = \frac{T}{8\pi} \int_{\Sigma} d^2\xi \sqrt{g} g^{mn} h_{mn} \quad (2F.2)$$

Classically, using the equations of motion for g_{mn} one finds that it must be proportional to h_{mn} : $g_{mn}^o = \rho h_{mn}$. On this solution, the Polyakov action coincides

with the Nambu-Goto action.

$$I_o[x, g^o] = I_{NG}[x] \quad (2F.3)$$

- Extrinsic curvature action; see Polyakov (1987).

To define this action, we appeal to the so-called *Second Fundamental Form* of the embedded surface. The derivatives of the tangent vectors $\partial_m x^\mu$ to the embedded surface can be decomposed onto the two vectors along the surface and the $d - 2$ (ortho-) normal vectors, denoted by n_i^μ . We have the following orthonormality conditions

$$\begin{cases} n_i^\mu G_{\mu\nu} \partial_m x^\nu = 0 \\ n_i^\mu G_{\mu\nu} n_j^\nu = \delta_{ij} \end{cases} \quad (2F.4)$$

The second fundamental form K_{mn}^i is defined by

$$\partial_m \partial_n x^\mu = \gamma_{mn}^p \partial_p x^\mu + K_{mn}^i n_i^\mu \quad (2F.5)$$

The extrinsic curvature action was introduced by Polyakov

$$I_E[x] = \frac{T}{4\pi} \int_\Sigma d^2\xi \sqrt{h} + \frac{1}{\alpha_o} \int_\Sigma d^2\xi \sqrt{h} K_{mn}^i K_{pq}^i h^{mp} h^{nq} \quad (2F.6)$$

The first term opposes tension, and the second one opposes friction of the string.

- The stress tensor.

For any classical action, or any quantum mechanical effective action, we may define the so-called stress tensor, which expresses the response of the quantum field theory to a change in the background metric. The stress tensor T_{mn} is related to the (classical or effective quantum) action as follows

$$T_{mn} = -\frac{4\pi}{\sqrt{g}} \frac{\delta W}{\delta g^{mn}} \quad (2F.7)$$

The stress tensor will play a fundamental role in capturing conformal symmetry in string theory.

G. Sum over all Surfaces Formulation

To define scattering amplitudes, we proceed by analogy with particle physics. We have a number of incoming strings, which are thought of as asymptotically freely propagating and sweeping out cylinders in space-time. At the interaction region, strings interact by joining and splitting as described above, until a number of strings appear in the final state that propagate freely out to infinity, as depicted schematically in the figure below.



The partial amplitude for such a process with given intrinsic geometry and embedding is weighted only by the exponential of action. The full amplitude for given initial and final string states is the sum over all partial amplitudes contributing. But this is precisely the sum over all surfaces that connect the initial and final string states; see Polyakov (1981).

$$\begin{aligned}
 \text{Amplitude} &\sim \sum_{\text{surfaces}} e^{-I} \\
 &= \sum_{\text{topologies}} \frac{1}{\mathcal{N}} \int Dg_{mn} \int Dx^\mu e^{-I[x,g]}
 \end{aligned}
 \tag{2G.1}$$

As we shall see much more explicitly later on, the sum over all surfaces can be decomposed into a sum over topologies and then for each topology a sum over worldsheet intrinsic metrics, represented here by the functional integral over g_{mn} . Finally, one sums over all embeddings of the string by functionally integrating over the position field x^μ .

In analogy with the Lehmann-Symanzik-Zimmermann (LSZ) procedure in quantum field theory, the external propagators in the above amplitudes may be canceled.

This procedure was shown to hold for the bosonic string by D'Hoker and Giddings (1987). We are left with a representation for the same amplitudes in which incoming and outgoing strings are replaced by certain local vertex operators on the worldsheet, one for each incoming or outgoing string, as illustrated in the figure below.



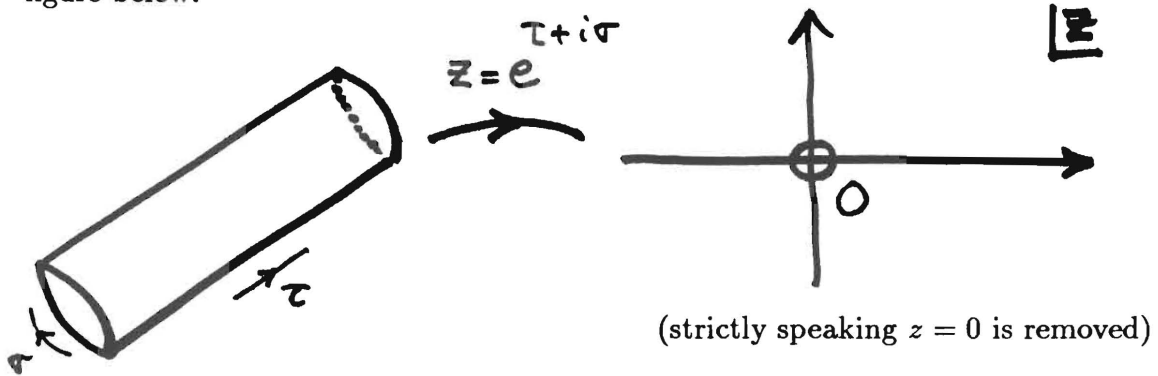
These vertex operators may be thought of as interpolating fields for the corresponding states, and the expression for the amplitude we find then is

$$\text{Amplitude} = \sum_{\text{topologies}} \frac{1}{\mathcal{N}} \int Dg_{mn} \int Dx^\mu V_1 \cdots V_n e^{-I[x,g]} \quad (2G.2)$$

The major topic to be treated in the remainder of these lectures is the understanding of this integral representation of this amplitude, and its generalization to the superstring.

III. CANONICAL QUANTIZATION OF FREE BOSONIC STRINGS

In this lecture, we canonically quantize the Polyakov string, which is classically equivalent to the Nambu-Goto string, as we have shown in the preceding chapter. Space-time will be taken to be flat and Minkowskian, with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$. We quantize the free string so that the worldsheet has the topology of a cylinder, with the axis of the cylinder along the time direction. By conformal mapping the cylinder is equivalent to the plane, as illustrated in the figure below.



A. Classical Symmetries and Constraints

In flat target space time, the Polyakov string action becomes

$$I_0[x, g] = \frac{T}{8\pi} \int_{\Sigma} d^2\xi \sqrt{g} \ g^{mn} \partial_m x^\mu \partial_n x^\mu \quad (3A.1)$$

Henceforth we shall set $T = 1$. The classical action has the following symmetries

- Diff (Σ) preserved
- Diff (M) becomes Poincaré invariance in $\mathbf{R}^d$
- Weyl (Σ) maps $g_{mn} \rightarrow e^{2\phi} g_{mn}$ for any $\phi(\xi)$.

The stress tensor on the worldsheet, defined by (2F.7), is given by

$$T_{mn} = -\frac{1}{2} \partial_m x^\mu \partial_n x^\mu + \frac{1}{4} g_{mn} g^{pq} \partial_p x^\mu \partial_q x^\mu \quad (3A.2)$$

The intrinsic metric g_{mn} is not a dynamical variable; hence the equation of motion for g_{mn} is a constraint, and the metric can be eliminated from the action by using the equations of motion. In a more familiar setting, Gauss' law in electrodynamics

is a constraint which coincides with the equation of motion for the timelike component of the electromagnetic potential, which is also not dynamical. In our case, we get the constraint

$$T_{mn} = 0 \quad (3A.3)$$

B. Equations of Motion on the Cylinder, Mode Expansion

The equation of motion for x^μ is Laplace's equation:

$$\Delta_g x^\mu = 0 \quad \Delta_g \equiv -\frac{1}{\sqrt{g}} \partial_m \sqrt{g} g^{mn} \partial_n \quad (3B.1)$$

Using Gauss' theorem, we choose the metric to be conformally flat on the cylinder so that the constraint and the equations of motion become

$$\begin{cases} (\partial_\tau^2 - \partial_\sigma^2) x^\mu = 0 \\ \frac{1}{4} [(\partial_\sigma \pm \partial_\tau) x^\mu]^2 = 0 \end{cases} \quad \text{or} \quad \begin{cases} \partial_z \partial_{\bar{z}} x^\mu = 0 \\ T_{zz} = -\frac{1}{2} (\partial_z x^\mu)^2 = 0 \end{cases} \quad (3B.2)$$

If the constraints are imposed on the initial data, they are preserved by time evolution under the equation of motion for x^μ . This is seen from the fact that

$$(\partial_\tau - \partial_\sigma) [(\partial_\sigma + \partial_\tau) x^\mu]^2 = (\partial_\sigma + \partial_\tau) [(\partial_\tau^2 - \partial_\sigma^2) x^\mu] = 0 \quad (3B.3)$$

If the constraint is imposed at a given time, then the σ derivative will vanish and hence the constraint is preserved in view of (3B.2). Thus, to solve (3B.2), we first solve the equation of motion for x^μ and then impose the constraint on the initial data. On the cylinder, where σ is an angle variable, the dynamical variables must be periodic in σ with period 2π . The solution is

$$\begin{aligned} x^\mu(\tau, \sigma) &= q^\mu + 2p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \{ x_n^\mu e^{-in(\tau-\sigma)} + \tilde{x}_n^\mu e^{-in(\tau+\sigma)} \} \\ P^\mu(\tau, \sigma) &= \partial_\tau x^\mu = 2p^\mu + \sum_{n \neq 0} \{ x_n^\mu e^{-in(\tau-\sigma)} + \tilde{x}_n^\mu e^{-in(\tau+\sigma)} \} \end{aligned} \quad (3B.4)$$

We still have to enforce the constraint on these solutions. The constraints in the

normal mode expansion take on the following expression

$$\begin{aligned}\frac{1}{8}(\partial_\sigma x^\mu - \partial_\tau x^\mu)^2(\tau, \sigma) &= \sum_n L_n e^{-in(\tau-\sigma)} \\ \frac{1}{8}(\partial_\sigma x^\mu + \partial_\tau x^\mu)^2(\tau, \sigma) &= \sum_n \tilde{L}_n e^{-in(\tau+\sigma)}\end{aligned}\tag{3B.5}$$

define the fundamental Virasoro operators L_n

$$L_n \equiv \sum_p \frac{1}{2} x_{n-p}^\mu x_p^\mu \quad \tilde{L}_n \equiv \sum_p \frac{1}{2} \tilde{x}_{n-p}^\mu \tilde{x}_p^\mu \tag{3B.6}$$

Enforcing the constraint requires setting $L_n = \tilde{L}_n = 0$ on the initial data.

The Poisson brackets structure is easily set up

$$\begin{aligned}\{x^\mu(\tau, \sigma), P^\nu(\tau, \sigma')\} &= -4\pi\delta(\sigma - \sigma')\eta^{\mu\nu} \\ \{x^\mu(\tau, \sigma), x^\nu(\tau, \sigma')\} &= 0\end{aligned}\tag{3B.7}$$

which implies the Poisson relations on the modes x_n^μ and $\tilde{x}_n^\mu$

$$\begin{cases} \{x_m^\mu, x_n^\nu\} = im\eta^{\mu\nu}\delta_{m+n,0} \\ \{q^\mu, p^\nu\} = \eta^{\mu\nu} \\ \{\tilde{x}_m^\mu, \tilde{x}_n^\nu\} = im\eta^{\mu\nu}\delta_{m+n,0} \end{cases}\tag{3B.8}$$

The Virasoro generators satisfy the following Poisson relations

$$\{L_m, x_n^\mu\} = -in x_{m+n}^\mu \quad \{L_m, \tilde{x}_n^\mu\} = 0 \tag{3B.9}$$

From these, we derive the transformation law of x^μ under the Virasoro generators

$$\{L_m, x^\mu(\tau, \sigma)\} = \frac{1}{2}e^{in(\tau-\sigma)}(\partial_\tau - \partial_\sigma)x^\mu(\tau, \sigma) \tag{3B.10}$$

$$\{\tilde{L}_m, x^\mu(\sigma, \tau)\} = \frac{1}{2}e^{in(\tau+\sigma)}(\partial_\tau + \partial_\sigma)x^\mu(\tau, \sigma)$$

The Virasoro generators thus act as conformal reparametrizations with vector fields $e^{in(\tau-\sigma)}, e^{in(\tau+\sigma)}$. Finally, the L_n and $\tilde{L}_n$'s satisfy the Witt-Virasoro algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n} \tag{3B.11}$$

The Hamiltonian (i.e. the generator of worldsheet time translations) and the shift

operator (i.e. the generator of worldsheet space translations) are particular generators of the Witt-Virasoro algebra

$$H = L_0 + \tilde{L}_0 = \frac{1}{8\pi} \int_0^{2\pi} d\sigma \{P^\mu P^\mu + \partial_\sigma x^\mu \partial_\sigma x^\mu\} \quad (3B.12)$$

$$\{H, x^\mu(\sigma, \tau)\} = \partial_\tau x^\mu \quad \{L_0 - \tilde{L}_0, x^\mu\} = \partial_\sigma x^\mu$$

It is easy to see that the canonical equations reduce to the equations of motion (3B.2).

C. Canonical Quantization

We canonically quantize by replacing Poisson brackets by commutators

$$[x^\mu(\tau, \sigma), P^\nu(\tau, \sigma')] = 4\pi i \eta^{\mu\nu} \delta(\sigma - \sigma') \quad (3C.1)$$

$$[x_m^\mu, x_n^\nu] = \eta^{\mu\nu} m \delta_{m+n,0} \quad [\tilde{x}_m^\mu, \tilde{x}_n^\nu] = \eta^{\mu\nu} m \delta_{m+n,0} \quad [q^\mu, p^\nu] = i\eta^{\mu\nu}$$

In addition, we impose the following adjointness conditions

$$(x_n^\mu)^\dagger = x_{-n}^\mu \quad (\tilde{x}_n^\mu)^\dagger = \tilde{x}_{-n}^\mu \quad (3C.2)$$

Operators x_n and $\tilde{x}_n$ with $n > 0$ are termed annihilation operators, those with $n < 0$ are creation operators. As the constraints were preserved under time evolution, we may construct a Fock space from all x_m^μ 's, and then enforce the constraint on the states afterwards. This restricts the states to lie in the *physical Hilbert space* only. The one free string ground state is labeled by the string collective momentum k^μ and defined by the standard Fock construction

$$\begin{cases} p^\mu |0, k^\mu\rangle = k^\mu |0, k^\mu\rangle \\ x_n^\mu |0, k^\mu\rangle = \tilde{x}_n^\mu |0, k^\mu\rangle = 0 \quad \text{for all } n > 0 \end{cases} \quad (3C.3)$$

The full Fock space is obtained by applying creation operators

$$|\epsilon(k)\rangle = \epsilon_{\mu_1 \dots \mu_p \bar{\mu}_1 \dots \bar{\mu}_q} x_{-n_1}^{\mu_1} \dots x_{-n_p}^{\mu_p} \tilde{x}_{-\bar{n}_1}^{\bar{\mu}_1} \dots \tilde{x}_{-\bar{n}_q}^{\bar{\mu}_q} |0, k^\mu\rangle \quad (3C.4)$$

Some of these states have negative norm! This can be seen from the following example

$$||x_{-n}^\mu |0, k^\mu\rangle||^2 = \langle 0, k^\mu | x_n^\mu x_{-n}^\mu |0, k^\mu\rangle = n\eta^{\mu\mu}$$

which for $n > 0$ and $\mu = \nu = 0$ is negative. Clearly such states are physically unacceptable within the context of a consistent quantum theory. However, the full

Fock space is not the space of physical states, it is much larger. We must still select only those Fock space states that are annihilated by the constraints, *i.e.* that are physical.

D. Virasoro Algebra

Upon quantization, the algebra of (3B.12) is modified. In particular an operator ordering must be prescribed for L_o and $\tilde{L}_o$ which we take to be normal ordering. Any other prescription would differ from the one with normal ordering by an additive constant.

$$L_o \equiv \frac{1}{2}p^2 + \sum_{m>0} x_{-m}^\mu x_m^\mu \quad \tilde{L}_o \equiv \frac{1}{2}p^2 + \sum_{m>0} \tilde{x}_{-m}^\mu \tilde{x}_m^\mu \quad (3D.1)$$

All other operators are well defined as they stand. Due to ordering of operators, the L_o one gets upon commutation of L_n and L_{-n} may not precisely coincide with the one defined above:

$$[L_m, L_n] = (m - n)L_{m+n} + A(m)\delta_{m+n,0}$$

Requiring that this structure relation should satisfy the Jacobi identity plus anti-symmetry $A(-m) = -A(m)$, we get

$$A(m) = \alpha m^3 + \beta m$$

To determine α and β we evaluate 2 commutators, and use a state with $p = 0$

$$\begin{aligned} \langle 0, 0 | [L_1, L_{-1}] | 0, 0 \rangle &= \alpha + \beta = 0 \\ \langle 0, 0 | [L_2, L_{-2}] | 0, 0 \rangle &= 8\alpha + 2\beta = \frac{1}{2}d \end{aligned}$$

We find that the quantum Virasoro generators satisfy the Virasoro algebra with central charge $c = d$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (3D.2)$$

We also obtain the adjointness conditions from the definition of the L_n

$$L_n^\dagger = L_{-n} \quad \tilde{L}_n^\dagger = \tilde{L}_{-n} \quad (3D.3)$$

As one attempts to impose the Virasoro operators as constraints on the states in the Fock space, one notices that as long as $c \neq 0$, one *cannot* impose the conditions

all the Virasoro constraints at once. Indeed if

$$L_n|\phi\rangle = 0 \quad \text{for all } n \in \mathbf{Z},$$

then, by the Virasoro algebra plus the adjointness conditions, we get

$$\begin{aligned} 0 &= \langle \phi | L_n^\dagger L_n | \phi \rangle = \langle \phi | L_{-n} L_n | \phi \rangle \\ &= \langle \phi | [L_{-n}, L_n] | \phi \rangle \\ &= -2n \langle \phi | L_o | \phi \rangle - \frac{c}{12} (n^3 - n) \langle \phi | \phi \rangle \end{aligned}$$

Hence we would get $\langle \phi | \phi \rangle = 0$, which means that these states must all have zero norm, and this is physically unacceptable as well.

E. Physical Spectrum, no ghost theorem

The correct *physical state conditions* are instead weaker

$$\begin{aligned} L_n |\psi\rangle &= 0 & \tilde{L}_n |\psi\rangle &= 0 & n > 0 \\ (L_o + \tilde{L}_o - 2\alpha) |\psi\rangle &= 0 & (L_o - \tilde{L}_o) |\psi\rangle &= 0 \end{aligned} \quad (3E.1)$$

The last condition imposes constant translation invariance in σ on the physical states. The next to last equation can be rewritten in terms of the mass operators M as

$$\begin{aligned} (p^2 + M^2) |\psi\rangle &= 0 \\ M^2 &= \sum_{m>0} (x_{-m}^\mu x_m^\mu + \tilde{x}_{-m}^\mu \tilde{x}_m^\mu) - 2\alpha \end{aligned} \quad (3E.2)$$

For example in the states $|\epsilon\rangle$, the $L_o - \tilde{L}_o = 0$ condition enforces that

$$\begin{cases} n_1 + n_2 + \cdots + n_p = \bar{n}_1 + \cdots + \bar{n}_q \\ M^2 |\epsilon\rangle = n_1 + n_2 + \cdots + n_p + \bar{n}_1 + \cdots + \bar{n}_q - 2\alpha. \end{cases}$$

A *spurious state* is a state orthogonal to *all* physical states. Spurious states are of the form $L_{-n}|s\rangle$ $n > 0$ and they completely decouple from the spectrum. They are *pure gauge or longitudinal states* in analogy with Yang-Mills theories.

The first few states are given as follows

(1.) $|\psi\rangle = |0, p^\mu\rangle \quad -p^2 = M^2 = -2\alpha$

This is the string ground state. We assume that its norm is positive.

(2.) $|\psi\rangle = \epsilon_{\mu\bar{\mu}} x_{-1}^\mu \tilde{x}_{-1}^{\bar{\mu}} |0, p\rangle$ with $-p^2 = M^2 = -2\alpha + 2$ and $p^\mu \epsilon_{\mu\bar{\mu}} = p^{\bar{\mu}} \epsilon_{\mu\bar{\mu}} = 0$

The norm of this state is given by $\langle\psi|\psi\rangle = \epsilon_{\mu\bar{\mu}} \epsilon^{\mu\bar{\mu}} \langle 0, p | 0, p \rangle$.

To have acceptable physical states, this norm must be positive. We have the following cases :

- $\alpha > 1 \quad p^2 > 0 \quad (p^\mu(0; p, \dots 0)) \quad \epsilon_{oi}$ produces *negative norm* states
- $\alpha = 1 \quad p^2 = 0 \quad (p^\mu(p; p, \dots 0)) \quad \epsilon_{\mu\nu}$ has $d - 2$ positive norm states plus one null state
- $\alpha < 1 \quad p^2 < 0 \quad (p^\mu(p; 0 \dots o)) \quad \epsilon_{ij}$ only has *positive norm* states

(3.) $|\psi\rangle = (\epsilon_{\mu\nu} x_{-1}^\mu x_{-1}^\nu + \eta_\mu x_{-2}^\mu) |0, p\rangle$ with $-p^2 = M^2 = -2\alpha + 4$

As a special case consider $|\psi\rangle = (L_{-2} + aL_{-1}^2) |0, p\rangle$

Imposing $L_{1,2}|\psi\rangle = 0$ we find $3 = 2a(3 - 2\alpha)$. The norm of this special state is

$$\langle\psi|\psi\rangle = \frac{d}{2} - \left(4 + \frac{9}{3 - 2\alpha}\right)(2 - \alpha)$$

Assuming $\alpha \leq 1$ we must have $d \leq 26$.

More generally, there is a *no ghost* theorem.

- there are negative norm states when $d > 26$ or $\alpha > 1$
- there are *no* negative norm states when $d \leq 26$ and $\alpha \leq 1$.
- there are null states (zero norm) when $d = 26$ and $\alpha = 1$ because an extra null state appears.

The *free* bosonic string is consistent in $d \leq 26$ and $\alpha \leq 1$, but it is special in the critical dimension $d = 26, \alpha = 1$.

In that case, the physical states at the two lowest mass levels are

$ 0, p\rangle$	$M^2 = -2$	<i>tachyon</i>
$\epsilon_{\mu\bar{\mu}} x_{-1}^\mu \tilde{x}_{-1}^{\bar{\mu}} 0, p\rangle$	$M^2 = 0$	
	$\epsilon_{\mu\bar{\mu}}$	symmetric, traceless <i>graviton</i>
	$\epsilon_{\mu\bar{\mu}}$	scalar <i>dilaton</i>
	$\epsilon_{\mu\bar{\mu}}$	<i>antisymmetric tensor field</i>

One really only has to show absence of negative norm states for $d = 26, \alpha = 1$,

because the rest follows from that

$$p_{26}^2 + M^2 = 0 \qquad p_d^2 + p_{26-d}^2 + M^2 = 0$$

F. Conformal Tensors and Vertex Operators for Physical States

Operators x_{-n}^μ create states in Fock space that are not physical in general. Here we construct operators that generalize x_{-n}^μ , but send physical states into physical states. Since physical states satisfy $L_n|\psi\rangle = 0 \quad n > 0$, such physical state vertex operators are those that commute with the Virasoro generators :

$$[L_n, V] = 0 \qquad [\tilde{L}_n, V] = 0 \qquad (3F.1)$$

for all n . As a result, spurious states are also sent into spurious.

Recall that L_n induces conformal reparametrizations; so we can view V as a conformal reparametrization invariant. To construct such invariants, we consider general conformal tensors first and then construct invariants by forming appropriate tensor products. To do so, we shall from now on slowly shift over into the standard notations of complex analysis. We define the complex variables

$$z = e^{-i(\tau-\sigma)} \qquad \bar{z} = e^{-i(\tau+\sigma)}$$

and you may think of τ as imaginary, so that z spans out the full complex plane. Tensors in general are defined by

$$V'^m(\xi') = \frac{\partial \xi^m}{\partial \xi'^n} V^n(\xi) \qquad (3F.2)$$

Under conformal reparametrizations

$$\begin{cases} z \rightarrow z'(z) \\ \bar{z} \rightarrow \bar{z}'(\bar{z}) \end{cases} \qquad (3F.3)$$

tensors split $V^m = V^z \oplus V^{\bar{z}}$ and all irreducible representations are of dimension

one, and labeled by $(n, \bar{n})$. These transform as *conformal tensors* or *primary fields*

$$V'(z') = \left(\frac{\partial z}{\partial z'} \right)^n \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right)^{\bar{n}} V(z) \quad (3F.4)$$

with *conformal weight* $(n, \bar{n})$! Infinitesimally:

$$\begin{aligned} z' &= z - v^z \\ \delta V(z, \bar{z}) &= v^z \partial_z V + n(\partial_z v^z) V \end{aligned} \quad (3F.5)$$

For the special reparametrizations $v^z = z^m$,

$$\begin{aligned} \delta V &= -i[L_m, V] = -i(z^m \partial_z V + m n z^{m-1} V) \\ &= -i \partial_z (z^m V) - i m (n-1) z^{m-1} V \end{aligned} \quad (3F.6)$$

Analogously

$$\begin{aligned} \delta V &= -i[\tilde{L}_m, V] = -i(\bar{z}^m \partial_{\bar{z}} V + m \bar{n} z^{m-1} V) \\ &= -i \partial_{\bar{z}} (\bar{z}^m V) - i m (\bar{n}-1) \bar{z}^{m-1} V \end{aligned} \quad (3F.7)$$

When $(n, \bar{n}) = (1, 1)$ the conformal reparametrizations of V are total derivatives; so we have the following *physical vertex operators*

$$\int dz^2 V_{(1,1)} \quad \oint dz V_{(1,0)} \quad \oint d\bar{z} V_{(0,1)}. \quad (3F.8)$$

Examples are

$$\begin{aligned} \partial_z x^\mu & \quad (1, 0) \\ \partial_{\bar{z}} x^\mu & \quad (0, 1) \end{aligned}$$

$$\begin{aligned} V(z, \bar{z}; k) &\equiv: e^{i k_\mu x^\mu} := e^{i k_\mu x_+^\mu} e^{i k_\mu (q^\mu + 2p^\mu \tau)} e^{i k_\mu x_-^\mu} \\ x_\pm^\mu &= i \sum_{\mp n > 0} \frac{1}{n} \{ x_n^\mu z^n + \tilde{x}_n^\mu \bar{z}^n \} \end{aligned} \quad (3F.9)$$

Because of the normal ordering, V has non-trivial conformal dimension:

$$\left(\frac{k^2}{2}, \frac{k^2}{2} \right) \quad (3F.10)$$

Now look at $d = 26, \alpha = 1$ free string theory.

$$\begin{aligned}
k^2 = -M^2 : & \text{ tachyon} & V^T &= \int dz^2 : e^{ik \cdot x} \\
k^2 = 0 : & \text{ graviton etc.} & V^G &= \epsilon_{\mu\bar{\mu}} \int dz^2 \partial_z x^\mu \partial_{\bar{z}} x^{\bar{\mu}} e^{ikz} \\
& \text{etc.} & &
\end{aligned} \tag{3F.11}$$

All the physical states of the string can be gotten this way! The scattering amplitude for four tachyons can now easily be computed

$$\begin{aligned}
\langle 0 | V_{k_1}^T V_{k_2}^T V_{k_3}^T V_{k_4}^T | 0 \rangle &= (2\pi)^{26} \delta(k) \int dz^2 |z|^{2p_1 p_2} |1-z|^{2p_3 p_4} \\
&= (2\pi)^{26} \delta(k) \frac{\Gamma(-1-s/2)\Gamma(-1-t/2)\Gamma(-1-u/2)}{\Gamma(2+s/2)\Gamma(2+t/2)\Gamma(2+u/2)} \tag{3F.12}
\end{aligned}$$

This is the famous formula for the tree level scattering of bosonic closed strings that appeared in the work of Virasoro - Shapiro.

What we have learned from this simple example of strings propagating freely in flat Minkowski space-time can be summarized as follows

1. In the full Fock space of all states generated by the fields on the surface, negative norm states always arise due to the Minkowski signature of space-time
2. To restrict to the physical Hilbert space, we need a local gauge invariance : general coordinate invariance on the worldsheet or conformal reparametrization symmetry in the conformal gauge
3. Strings can propagate freely without negative norm states only when the underlying field theory on the worldsheet is conformally invariant
4. We started out by formulating a string theory in a general target space-time, with space-time metric $G_{\mu\nu}(x)$. Then we specialized to considering only Flat Minkowski space-time. Amongst the excitations for the string, we have found for $d = 26$ and $\alpha = 1$, the graviton particle.
5. The important thing to realize is that one should *not* sum over the background metric field $G_{\mu\nu}(x)$. Rather, since we are doing first quantized string theory, you should think of $G_{\mu\nu}$ as a point of expansion of the theory: compare with the Higgs vacuum expectation value in a Higgs-Yang Mills theory. Summing also over the background $G_{\mu\nu}$ would be double counting: one would have 2 gravitons!

Some particularly useful references for a review on the operator quantization are in Scherk (1975) and Green, Schwarz and Witten (1987).

IV. CONFORMAL FIELD THEORY ON A FIXED SURFACE

We shall now present a more general and more abstract discussion of the salient features that have emerged from canonical quantization. The basis was the Virasoro or conformal algebra, which is the algebra of conformal reparametrizations. Conformal reparametrizations change the metric, but leave the angles invariant; so the metric is just scaled up.

$$\delta g_{mn} = \nabla_m v_n + \nabla_n v_m \sim g_{mn} \quad (4.1)$$

So conformal reparametrizations satisfy the following equation in 2 dimensions

$$\nabla_m v_n + \nabla_n v_m - g_{mn} \nabla^p v_p = 0 \quad (4.2)$$

or in a locally flat complex coordinate system

$$\nabla_z v_z = 0 \quad \text{with solution} \quad v^z(z)$$

For any classical theory, which is reparametrization invariant, we have

$$\nabla^m T_{mn} = 0; \quad T_{mn} = T_{nm} \quad (4.3)$$

If in addition we have scale invariance, then

$$T_m{}^m = 0 \quad (4.4)$$

which automatically implies invariance under conformal transformations and under Weyl transformations

$$g_{mn} \rightarrow e^{2\phi} g_{mn} \quad (4.5)$$

Classically then, such theories are conformal invariant. In quantum field theory however, scale anomalies occur, and more care is needed.

A. Scale Invariant Quantum Field Theories

In general, the procedure of renormalization in relativistic quantum field theory, breaks scale invariance at the classical level and the anomalous Ward identities are expressed by the renormalization group equations. To illustrate how this works, consider *e.g.* a field theory in four space-time dimensions with one real scalar field ϕ and renormalized coupling constant λ . We mainly think of a ϕ^4 theory here. The n -point Green functions for the field ϕ satisfy the following renormalization group equations.

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} - n \gamma_\phi(\lambda) \right] G_R^{(n)}(\xi_i; \lambda; \mu) = 0 \quad (4A.1)$$

$$\beta(\lambda) = \mu \frac{\partial \lambda}{\partial \mu} \Big|_{\lambda_o, \Lambda} \quad \gamma_\phi(\lambda) = \mu \frac{\partial \ell n Z_\phi}{\partial \mu} \Big|_{\lambda_o, \Lambda}.$$

The bare coupling is denoted by λ_o , the cutoff by Λ , and the renormalization scale by μ . The functions β and γ_ϕ are common to all Green functions and called the *beta function* and the *anomalous dimension of ϕ* respectively. The running coupling $\bar{\lambda}$ is defined by the implicit equation

$$\ell n \frac{\bar{\mu}}{\mu} = \int_{\lambda}^{\bar{\lambda}} \frac{dx}{\beta(x)} \quad \mu e^t = \bar{\mu} \quad (4A.2)$$

The differential equations (4A.1) can be solved, and one finds the following scaling equation

$$G_R^{(n)}(e^t \xi_i; \lambda; \mu) = \exp \left[nt + n \int_0^t dt' \gamma(\bar{\lambda}(t')) \right] G_R^{(n)}(\xi_i; \bar{\lambda}(t); \mu) \quad (4A.3)$$

Scale invariance of the *renormalized quantum theory* is achieved at a *fixed point* where the beta function vanishes.

$$\begin{aligned} \beta(\lambda_*) = 0 & \Rightarrow \bar{\lambda} = \lambda_* \quad \text{fixed point.} \\ \Delta_\phi = 1 + \gamma(\lambda_*) \end{aligned} \quad (4A.4)$$

$$G_R^{(n)}(e^t \xi_i; \lambda_*; \mu) = e^{nt \Delta_\phi} G_R^{(n)}(\xi_i; \lambda_*; \mu).$$

The effect is that ϕ has acquired a *scaling dimension* Δ_ϕ , which need not be an integer. Similarly, every composite operator will acquire some non-trivial dimension.

B. Trace Anomaly for Conformal Field Theories

We consider a conformal field theory on a general metric background g_{mn} . Assuming general coordinate invariance, the stress tensor must satisfy (4.4). In a local inertial frame these equations guarantee Lorentz and translation invariance. In general, the quantum theory produces a Weyl or conformal anomaly. Classically, we had (4.5) and this condition continues to hold at the quantum level provided we are at a fixed point of the renormalized theory and provided we are on a flat surface.

However, even for a fixed point theory, due to the metric background, T^m_m can receive a non-trivial contribution. Since T^m_m is of dimension 2, and the anomaly is local and of dimension 2, we must have, on general grounds that

$$T^m_m = -\frac{c}{3}R_g + \mu^2 \quad (4B.1)$$

with c and μ^2 constants. We will establish that this c is the same as the central charge in the Virasoro algebra for this conformal field theory. The above equation for the trace of the stress tensor can be viewed as the *definition of a conformal field theory*, in $d = 2$ on an arbitrary Riemann surface, either classically or quantum mechanically.

Using the definition of the stress tensor in (2F.7), we may derive the complete dependence of the conformal field theory effective action W on the Weyl scaling factor.

$$\delta W[g] = \int_{\Sigma} \sqrt{g} \left[-\frac{c}{3}R_g + \mu^2 \right] \delta \sigma \quad (4B.2)$$

To solve the equation, we parametrize $g_{mn} = e^{2\sigma} \hat{g}_{mn}$, and use a Weyl rescaling formula for the curvature given in (2D.18).

$$\delta W[\hat{g}, \sigma] = \frac{1}{4\pi} \int_{\Sigma} \sqrt{\hat{g}} \left[-\frac{c}{3}(R_{\hat{g}} + \Delta_{\hat{g}}\sigma) + \mu^2 e^{2\sigma} \right] \delta \sigma \quad (4B.3)$$

Now this is easily integrated since all the σ -dependence has been rendered explicit and we have

$$W[\hat{g}, \sigma] = W[\hat{g}, 0] + S_L[\hat{g}, \sigma] \quad (4B.4)$$

with the “Liouville action” defined by

$$S_L[\hat{g}, \sigma] = -\frac{c}{12\pi} \int_{\Sigma} \sqrt{\hat{g}} \left[\frac{1}{2} \sigma \Delta_{\hat{g}} \sigma + R_{\hat{g}} \sigma + \mu_o^2 (e^{2\sigma} - 1) \right] \quad (4B.5)$$

It was Polyakov’s proposal that one way to avoid the inconsistencies that arise in canonical quantization away from the critical dimension is to retain the Liou-

ville mode. Unfortunately, quantization of the Liouville action is very tricky. The first attempts were by Polyakov (1981), D'Hoker and Jackiw (1982), Curtright and Thorn (1982) and Gervais and Neveu (1984). The pioneering work by Belavin, Polyakov and Zamolodchikov (1984) and Friedan, Qiu and Shenker (1984) on integrable or so-called minimal conformal field theories originated in another attempt to quantizing Liouville type theories. More recently, this quantization was carried further in the functional formulation by David (1988) and by Distler and Kawai (1989), with special emphasis on the critical exponents in the model. A review of some of this work is in D'Hoker (1992).

Examples of conformal field theories are provided by quantization of the following actions

1.

$$S = \int_{\Sigma} \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\mu$$

This is the action for d free scalar fields each with central charge 1, so the total central charge equals the dimension of the target space-time. These fields may also be compactified on a flat torus without changing the central charge.

2.

$$S = \int_{\Sigma} \sqrt{g} \bar{\psi} \gamma^a e_a^m (i\partial_m - eA_m) \psi$$

This is the action for a Majorana spinor (fermion), possibly coupled to an external gauge field. The central charge is $\frac{1}{2}$.

3.

$$S = \int_{\Sigma} \sqrt{g} b \nabla_{\bar{z}} c$$

for $c \in T_{\lambda}$ and $b \in T_{1-\lambda}$. This is the so-called $b - c$ system, where b and c may be quantized as bosons or fermions respectively. The corresponding central charge is $\pm 2(6\lambda^2 + 6\lambda + 1)$. The structure of the ghost system was very nicely discussed by Friedan, Martinec and Shenker (1986).

4.

$$S = \int_{\Sigma} \sqrt{g} \left(\frac{1}{2} g^{mn} \partial_m \phi \partial_n \phi + k R \phi + \mu^2 e^{2\alpha \phi} \right)$$

This is the Liouville action, which is classically conformally invariant for $\kappa = \alpha = 1$ and quantum mechanically conformal invariant when $2\alpha^2 - 2\kappa\alpha + 1 = 0$. Even classically, Liouville theory has a non-zero central charge. References were already given above.

5.

$$S = \frac{T}{8\pi} \int_{\Sigma} \sqrt{g} g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x)$$

This is the so-called non-linear sigma model and it is *classically* conformally invariant for all $G_{\mu\nu}$. Quantum mechanically, one should view $G_{\mu\nu}(x)$ as a collection of an infinite number of coupling constants, so there is a beta function $\beta_{\mu\nu}(G)$. To leading order we have

$$\beta_{\mu\nu}(G) = \frac{1}{T} R_{\mu\nu} + \mathcal{O}\left(\frac{1}{T^2}\right)$$

Thus, the vanishing of the beta function occurs when the space-time metric describes an *Einstein space* with $R_{\mu\nu} = 0$. Many special cases of non-linear sigma models were considered, and their beta functions evaluated; see e.g. Polyakov (1987). The general approach, viewing the metric as a collection of coupling constants was pioneered by Friedan (1980), and has become extremely important in the development of string theory.

6.

$$S = \int_{\Sigma} \sqrt{g} \left[g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) + \epsilon^{mn} \partial_m x^\mu \partial_n x^\nu H_{\mu\nu}(x) + R^{(2)} \Phi(x) + T(x) \right]$$

This is a very important generalization of the non-linear sigma model that incorporates space-time fields not only for the graviton (G) but also for the dilaton (Φ), the anti-symmetric tensor (H) and the tachyon (T). It is classically conformal invariant for $\Phi = T = 0$, and quantum mechanically, provided certain differential equations on G, H, T, Φ are satisfied.

C. Calculation of the Trace Anomaly

Important conformal field theories with bilinear actions are the scalar field and $b - c$ systems which we discuss later. This action is translation invariant in x^μ . We define the functional integral

$$e^{-W[g]} \equiv \int Dx^\mu e^{-S[x,g]} \tag{4C.1}$$

or more generally, some correlation functions. To study this integral and calculate it, we must find out what the measure is and how it can be calculated in a general

coordinate invariant way. To start, one introduces a $\text{Diff}(\Sigma)$ invariant $L^2(\Sigma)$ norm on the space of x -functions

$$\| \delta x^\mu \|_g^2 \equiv \int_\Sigma \sqrt{g} \delta x^\mu \delta x^\mu \quad (4C.2)$$

Notice that this norm is *not* Weyl invariant however. The measure is defined from this norm in the standard fashion.

To compute the functional integral, we diagonalize Δ_g

$$\Delta_g \psi_n = \lambda_n \psi_n \quad \| \psi_n \|_g^2 = 1 \quad (4C.3)$$

The Laplacian has a single zero mode so that $\lambda_0 = 0$, but is otherwise positive so that $\lambda_n > 0$ for $n \geq 1$. The field and the measure decompose as

$$x^\mu(\xi) = \sum_{n=0}^{\infty} x_n^\mu \psi_n(\xi) \quad Dx^\mu = \prod_{n,\mu} dx_n^\mu \quad (4C.4)$$

The functional integral is now easily performed

$$e^{-W[g]} = \int \prod_{n,\mu} dx_n^\mu \exp \left\{ - \sum_n x_n^\mu x_n^\mu \lambda_n \right\} \quad (4C.5)$$

From the integration over x_o^μ we get the volume of space-time, whereas from the remaining Gaussians, we get the product of the eigenvalues, which is related to the determinant of Δ_g .

$$\int dx^\mu = \int dx_o^\mu (\psi_o)^d = \Omega \quad \det' \Delta_g = \prod_{n \neq 0} \lambda_n \quad (4C.6)$$

Putting all together, we have

$$e^{-W[g]} = \Omega \left(\int_\Sigma \sqrt{g} \right)^{d/2} \left(\det' \Delta_g \right)^{-d/2} \quad (4C.7)$$

Now this determinant is infinite dimensional, so we need to regularize and renormalize it, preserving general coordinate invariance.

One particularly convenient prescription is the use of the heat-kernel method, where the determinant is defined as follows

$$\left(\frac{\partial}{\partial t} + \Delta_g \right)_\xi e^{-t\Delta_g}(\xi, \xi') \theta(t) = \delta_g(\xi, \xi') \delta(t) \quad (4C.8)$$

$$\ell n \det' \Delta_g = - \int_\epsilon^\infty \frac{dt}{t} \left(\text{tr} e^{-t\Delta_g} - 1 \right)$$

We have regularized the t -integral at short times, because there the heat-kernel is a δ -function, and the trace diverges. The 1 is subtracted since the zero mode should not contribute. This cutoff is completely $\text{Diff}(\Sigma)$ invariant.

In fact, one may obtain a simple expression in terms of the short time expansion of the heat-kernel as can be seen from the following manipulations, and using the fact that the Laplacian scales simply under Weyl transformations $\delta \Delta_g = -2\delta\sigma \Delta_g$

$$\begin{aligned} \delta \ln \det' \Delta_g &= \int_\epsilon^\infty dt \text{tr} \delta \Delta_g e^{-t\Delta_g} = 2 \int_\epsilon^\infty dt \frac{\partial}{\partial t} \text{tr} \delta \sigma e^{-t\Delta_g} \\ &= 2 \text{tr} \delta \sigma e^{-t\Delta_g} \Big|_\epsilon^\infty \end{aligned} \quad (4C.9)$$

As $t \rightarrow +\infty$ we have $e^{-t\Delta_g} \rightarrow 1$, so we get

$$\delta \ln \det' \Delta_g = 2 \text{tr} \delta \sigma - 2 \text{tr} \delta \sigma e^{-\epsilon \Delta_g} \quad (4C.10)$$

Furthermore we have

$$2 \text{tr} \delta \sigma = \frac{\int_\Sigma \sqrt{g} \, 2\delta\sigma}{\int_\Sigma \sqrt{g}} = \delta \ln \int_\Sigma \sqrt{g} \quad (4C.11)$$

Hence we find

$$\delta W[g] = -d \text{tr} \delta \sigma e^{-\epsilon \Delta_g} \quad (4C.12)$$

The last ingredient we need is the short time expression for the heat-kernel. For the Euclidean plane, we have

$$K_o^t(z, z') = \frac{1}{4\pi t} e^{\frac{-|z-z'|^2}{4t}} \theta(t) \quad K^t(z, z') = e^{-t\Delta_g} \theta(t) \quad (4C.13)$$

So for very small t , K_o^t is completely localized; i.e. sensitive to the metric at $z = z'$

only:

$$\begin{aligned}\delta W[g] &= -d \int_{\Sigma} \sqrt{g} \delta \sigma K^t(z, z) \\ K^t(z, z) &= \frac{1}{4\pi t} + \alpha R_g + \mathcal{O}(t)\end{aligned}\tag{4C.14}$$

It is easy to evaluate α by specializing to a metric $e^{2\sigma} \delta_{mn} : \alpha = \frac{1}{12\pi}$.

$$\delta W[g] = -d \left[\frac{1}{4\pi\epsilon} \int_{\Sigma} \sqrt{g} \delta \sigma + \frac{1}{12\pi} \int_{\Sigma} \sqrt{g} R_g \delta \sigma \right] \tag{4C.15}$$

Comparison with the definition of the stress tensor yields the trace of T

$$T^m_m = 4\pi \frac{\delta W}{\delta \sigma} = -\frac{d}{\epsilon} - \frac{d}{3} R_g \tag{4C.16}$$

We see that the central charge equals the dimension $c = d$ or $c = 1$ for each scalar field, as announced previously.

Another class of bilinear actions that are very important are *b-c systems* whose action is given by

$$S[b, c; g] = \frac{1}{2\pi} \int [b \nabla_{\bar{z}}^{(n)} c + \text{complex conjugate}] \tag{4C.17}$$

with c a tensor of conformal weight (n, o) and b a conformal tensor of weight $(1 - n, o)$ and both may be taken Grassmann valued. Formally, the functional integral is defined by

$$e^{-W[g]} = \int Db D\bar{b} Dc D\bar{c} e^{-S} \tag{4C.18}$$

Now, similarly as Δ_g had a zero mode, here too, in general, we may have zero modes

$$\begin{aligned}\nabla_{\bar{z}}^{(n)} \psi_p &= 0 & p = 1, \dots, \dim \text{Ker} \nabla_{\bar{z}}^{(n)} \\ \nabla_{\bar{z}}^{(1-n)} \phi_q &= 0 & q = 1, \dots, \dim \text{Ker} \nabla_{\bar{z}}^{(1-n)}\end{aligned}\tag{4C.19}$$

The presence of zero modes in the fields b, c renders the functional integral 0, due to its Grassmann character. The only non-zero functional integrals are those where some specific number of fields are inserted as well. Similarly as before, one must isolate these modes, since they do not contribute to the action and treat them separately.

We shall just state the result here

$$e^{-W[g]} = \frac{\det' \Delta_-^{(n)}}{\det \langle \phi_q | \phi_{q'} \rangle^{1/2} \det \langle \psi_p | \psi_{p'} \rangle^{1/2}} \quad (4C.20)$$

Notice that for $n = 0$, this answer reduces to the case of the scalar Laplacian, discussed earlier. For $n \neq 0$ we have

$$\begin{aligned} \Delta_+^{(n)} &\equiv 2(\nabla_z^{(n)})^\dagger \nabla_z^{(n)} = -2\nabla_{(n+1)}^z \nabla_z^{(n)} \\ \Delta_-^{(n)} &\equiv 2(\nabla_{(n)}^z)^\dagger \nabla_z^{(n)} = -2\nabla_z^{(n-1)} \nabla_{(n)}^z \end{aligned} \quad (4C.21)$$

The spectrum on compact, Riemann surfaces is *discrete* and ≥ 0 . In fact, the eigenfunctions and spectrum of $\Delta_\pm^{(n)}$ are related:

$$\Delta_+^{(n)} \psi_p = \lambda_p \psi_p \quad \text{or} \quad 2(\nabla_z^{(n)})^\dagger \nabla_z^{(n)} \psi_p = \lambda_p \psi_p \quad (4C.22)$$

now apply $\nabla_z^{(n)}$ to this equation and we find :

$$\Delta_-^{(n+1)} \nabla_z^{(n)} \psi_p = \lambda_p \nabla_z^{(n)} \psi_p \quad (4C.23)$$

So the non-zero eigenvalue spectra of $\Delta_+^{(n)}$ and $\Delta_-^{(n+1)}$ are identical. Of course, when the eigenvalue is zero this correspondence breaks down. As a result we have

$$\begin{aligned} \text{tr } e^{t\Delta_+^{(n)}} - \text{tr } e^{-t\Delta_-^{(n+1)}} &= \dim \text{Ker } \Delta_+^{(n)} - \dim \text{Ker } \Delta_-^{(n+1)} \\ &= \dim \text{Ker } \nabla_z^{(n)} - \dim \text{Ker } \nabla_{\bar{z}}^{(n+1)} \end{aligned} \quad (4C.24)$$

Using the expressions for the heat-kernel, one easily finds the *Riemann-Roch-Atiyah-Singer* theorem

$$\dim \text{Ker } \nabla_z^{(n)} - \dim \text{Ker } (\nabla_z^{(n)})^\dagger = (2n+1)\chi(\Sigma). \quad (4C.25)$$

The short-time expansion of the heat-kernel is something that will reoccur

constantly. A general heat-kernel is defined by

$$\left(\frac{\partial}{\partial t} + \Delta_{\pm}^{(n)} \right) e^{-t\Delta_{\pm}^{(n)}}(\xi, \xi') \theta(t) = \delta_g(\xi, \xi') \delta(t) \quad (4C.26)$$

These heat-kernels are tensors themselves, of the same dimension as $\delta_g(\xi, \xi')$ and in the limit of vanishing t , they indeed tend towards this quantity.

$$e^{-t\Delta_{\pm}^{(n)}}(\xi, \xi) \longrightarrow \delta_g(\xi, \xi) \quad \text{as} \quad t \rightarrow 0^+ \quad (4C.27)$$

The diagonal of the heat-kernel admits a Laurent expansion in powers of t , with coefficients that are local functionals of the metric. These are the *Seeley* coefficients and the first few are given as follows

$$e^{-t\Delta_{\pm}^{(n)}}(\xi, \xi) = \frac{1}{4\pi t} + \frac{1 \pm 3n}{12\pi} R_g + \mathcal{O}(t) \quad (4C.28)$$

The trace of the heatkernel is readily evaluated and we have

$$\text{tr} e^{-t\Delta_{\pm}^{(n)}} = \frac{1}{4\pi t} \int_{\Sigma} \sqrt{g} + \frac{1 \pm 3n}{12\pi} \int_{\Sigma} \sqrt{g} R_g + \mathcal{O}(t) \quad (4C.29)$$

Now one easily recovers the Riemann-Roch theorem as $t \rightarrow 0$:

$$\begin{aligned} \text{tr} e^{-t\Delta_+^{(n)}} - \text{tr} e^{-t\Delta_-^{(n+1)}} &= \frac{1 + 3n - (1 - 3(n+1))}{12\pi} \int_{\Sigma} \sqrt{g} R_g \\ &= \frac{1}{2} (2n+1) \chi(\Sigma). \end{aligned} \quad (4C.30)$$

Weyl transformation properties of these determinants may now be derived using heat-kernel methods. The Weyl transformation of the differential operators are given by

$$\begin{aligned} \delta \nabla_{(n)}^z &= -2\delta\sigma \nabla_{(n)}^z \\ \delta \nabla_z^{(n)} &= 2n\delta\sigma \nabla_z^{(n)} - 2n \nabla_z^{(n)} \delta\sigma \\ \delta \Delta_+^{(n)} &= -2(n+1)\delta\sigma \Delta_+^{(n)} - 4n \nabla_{(n+1)}^z \delta\sigma \nabla_z^{(n)} \\ \delta \Delta_-^{(n)} &= -2(1-n)\delta\sigma \Delta_-^{(n)} + 4n \nabla_z^{(n-1)} \delta\sigma \nabla_{(n)}^z \end{aligned} \quad (4C.31)$$

Determinants are defined by

$$\begin{aligned}\det' \Delta_{\pm}^{(n)} &\equiv - \int_{\epsilon}^{\infty} \frac{dt}{t} \left(\text{tr} e^{-t\Delta_{\pm}^{(n)}} - N_{\pm}^{(n)} \right) \\ N_{+}^{(n)} &= \dim \text{Ker } \Delta_{+}^{(n)} = \dim \text{Ker } \nabla_z^{(n)} \\ N_{-}^{(n)} &= \dim \text{Ker } \Delta_{-}^{(n)} = \dim \text{Ker } \nabla_{(n)}^z\end{aligned}\tag{4C.32}$$

Using the mapping between the spectra of $+$ and $-$ operators exhibited earlier, one finds

$$\text{tr} e^{-t\Delta_{+}^{(n)}} - N_{+}^{(n)} = \text{tr} e^{-t\Delta_{-}^{(n)}} - N_{-}^{(n+1)}\tag{4C.33}$$

and

$$\det' \Delta_{+}^{(n)} = \det' \Delta_{-}^{(n+1)}\tag{4C.34}$$

So, we may concentrate on $-$ only. We now evaluate explicitly the Weyl anomalies.

$$\delta \ell n \det' \Delta_{-}^{(n)} = \int_{\epsilon}^{\infty} dt \left[2(n-1) \text{tr} \delta\sigma \Delta_{-}^{(n)} e^{-t\Delta_{-}^{(n)}} + 4n \text{tr} \nabla_z^{(n-1)} \delta\sigma \nabla_{(n)}^z e^{t\Delta_{-}^{(n)}} \right]\tag{4C.35}$$

by rearranging the differential operators in the second line, one gets

$$\begin{aligned}&= \int_{\epsilon}^{\infty} dt \left[2(n-1) \text{tr} \delta\sigma \Delta_{-}^{(n)} e^{-t\Delta_{-}^{(n)}} - 2n \text{tr} \delta\sigma \Delta_{+}^{(n-1)} e^{-t\Delta_{+}^{(n-1)}} \right] \\&= \int_{\epsilon}^{\infty} dt \frac{\partial}{\partial t} \left[-2n(n-1) \text{tr} \delta\sigma e^{-t\Delta_{-}^{(n-1)}} + 2n \text{tr} \delta\sigma e^{-t\Delta_{+}^{(n-1)}} \right] \\&= \left[-2(n-1) \text{tr} \delta\sigma e^{-t\Delta_{-}^{(n)}} + 2n \text{tr} \delta\sigma e^{-t\Delta_{+}^{(n-1)}} \right]_{\epsilon}^{\infty}\end{aligned}\tag{4C.36}$$

as $t \rightarrow \infty$, heat-kernels do not tend to zero, when there are zero modes to the operators $\Delta_{\pm}$ so that

$$\begin{aligned}\delta \ell n \det' \Delta_{-}^{(n)} &= -2(n-1) \sum_j \langle \phi_j | \delta\sigma | \phi_j \rangle + 2n \sum_{\alpha} \langle \psi_{\alpha} | \delta\sigma | \psi_{\alpha} \rangle \\&\quad + 2(n-1) \text{tr} \delta\sigma e^{-\epsilon\Delta_{-}^{(n)}} - 2n \text{tr} \delta\sigma e^{-\epsilon\Delta_{+}^{(n-1)}}\end{aligned}\tag{4C.37}$$

The first two terms can be recast as follows:

$$\begin{aligned}
\langle \phi_j | \phi_k \rangle &= \int dz^2 (g^{z\bar{z}})^{n-1} \phi_j^* \phi_k \\
\delta \langle \phi_j | \phi_k \rangle &= 2(1-n) \langle \phi_j | \delta \sigma | \phi_k \rangle \\
\delta \ell n \det \langle \phi_j | \phi_k \rangle &= \delta \operatorname{tr} \ell n \langle \phi_j | \phi_k \rangle \\
&= \operatorname{tr} \delta \langle \phi_j | \phi_k \rangle \\
&= 2(1-n) \sum_j \langle \phi_j | \delta \sigma | \phi_j \rangle
\end{aligned} \tag{4C.38}$$

Putting this together with the short-time heat-kernel expressions we finally obtain

$$\delta \ell n \frac{\det' \Delta_-^{(n)}}{\det \langle \phi_j | \phi_k \rangle \det \langle \psi_\alpha | \psi_\beta \rangle} = -\frac{1}{2\pi\epsilon} \int_\Sigma \sqrt{g} \delta \sigma - \frac{6n^2 - 6n + 1}{6\pi} \int_\Sigma \sqrt{g} R_g \delta \sigma \tag{4C.39}$$

Hence the central charge of the b, c system is

$$c_n = \mp 2(6n^2 - 6n + 1) \tag{4C.40}$$

where the $-$ sign occurs for b, c Grassmanian and the $+$ sign for b, c commuting.

These anomaly calculations were performed with the use of the heat-kernel methods in Alvarez (1983); the above presentation follows D'Hoker and Phong (1988).

D. Conformal Ward Identities

From the operator identities

$$T_{mn} = T_{nm} \quad \nabla^m T_{mn} = 0 \quad T^m{}_m = -\frac{c}{3} R_g \tag{4D.1}$$

we can derive Ward identities in the usual way. Throughout, we shall choose a system of local conformal complex coordinates $z, \bar{z}$. The fields we consider here will be only *conformal tensor fields* or so-called *primary fields*, defined by

$$\phi'(z') = \left(\frac{\partial z}{\partial z'} \right)^h \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right)^{\bar{h}} \phi(z) \tag{4D.2}$$

$$\delta \phi = v^z \partial_z \phi + \hbar \partial_z v_z \phi + v^{\bar{z}} \partial_{\bar{z}} \phi + \hbar \partial_{\bar{z}} v_{\bar{z}} \phi$$

To make contact with standard terminology, the field ϕ is of conformal dimension

$(h, \bar{h})$ and

$$\begin{cases} \Delta = h + \bar{h} & \text{conformal scaling dimension} \\ s = h - \bar{h} & \text{conformal spin} \end{cases}$$

Classically, $h, \bar{h}$ would typically be integer or half integer, but quantum mechanically, we have anomalous dimensions to these operators and they are more generally real numbers.

We assume that the vacuum is invariant under analytic reparametrizations ($SL(2, \mathbf{R})$ -invariant vacuum)

$$L_n|0\rangle = \tilde{L}_n|0\rangle = 0 \quad n \geq -1 \quad (4D.3)$$

Notice that $L_n, n \geq -1$ form a subalgebra of Virasoro with zero central extension.

Consider a Green function for n primary fields ϕ_i of conformal dimension $(h_i, \bar{h}_i)$

$$Z(\xi_i) = \langle 0|T\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \quad (4D.4)$$

Hence we have the Weyl (anomalous) Ward identity

$$\langle 0|T T_{z\bar{z}}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle = -\frac{c}{6}Rg_{z\bar{z}}(z)Z(\xi_i) \quad (4D.5)$$

[Note in conformal coordinates $T_{z\bar{z}} = -\frac{c}{6}Rg_{z\bar{z}}$.] Or, upon introducing $T_{z\bar{z}}$ as a derivative with respect to $g^{z\bar{z}}$:

$$\frac{4\pi}{\sqrt{g}} \frac{\delta Z}{\delta g^{z\bar{z}}} = -\frac{c}{6}Rg_{z\bar{z}}(z)Z \quad (4D.6)$$

Now we also have a *reparametrization* Ward identity (for reparametrization vector fields v^z)

$$\delta_v Z = \sum_i \langle 0|T\phi_1(\xi_1)\cdots\delta\phi_i(\xi_i)\cdots\phi_n(\xi_n)|0\rangle \quad (4D.7)$$

where $\delta\phi_i$ is given by the transformation law of the primary fields ϕ_i under reparametrizations in (4D.2).

On the other hand, Z is a function of the metric, and the same quantity must be given by the change in the metric due to a reparametrization.

$$\delta_v Z = \int \sqrt{g} \delta g^{zz} \frac{1}{\sqrt{g}} \frac{\delta Z}{\delta g^{zz}} + \int \sqrt{g} \delta g^{z\bar{z}} \frac{1}{\sqrt{g}} \frac{\delta Z}{\delta g^{z\bar{z}}} \quad (4D.8)$$

with

$$\begin{aligned} \delta g^{zz} &= -(g^{z\bar{z}})^2 \delta g_{\bar{z}\bar{z}} = -2\nabla^z v^z \\ \delta g^{z\bar{z}} &= -(g^{z\bar{z}})^2 \delta g_{z\bar{z}} = -\nabla^{\bar{z}} v^z \end{aligned} \quad (4D.9)$$

Using the definition of the stress tensor insertion into Green functions and integrating the derivatives by part, we get

$$\delta_v Z = \frac{1}{2\pi} \int \sqrt{g} v^z \nabla^z \langle 0 | T T_{zz} \phi \cdots \phi | 0 \rangle - \frac{c}{24\pi} Z \int \sqrt{g} v^z \nabla^{\bar{z}} R g_{z\bar{z}} \quad (4D.10)$$

Identification of (4D.6) and (4D.10) for all vector fields v^z gives the reparametrization Ward identity

$$\begin{aligned} &\sum_i (h_i \nabla_{\xi_i} \delta(z, \xi_i) + \delta(z, \xi_i) \nabla_{\xi_i}) Z \\ &= \frac{1}{2\pi} \nabla^z \langle 0 | T T_{zz} \phi_1(\xi_1) \cdots \phi_n(\xi_n) | 0 \rangle - \frac{c}{24\pi} (\nabla_z R) Z \end{aligned} \quad (4D.11)$$

This equation may be solved by introducing the Green function

$$\nabla^z G_{zz}(z, w) = 2\pi \delta(z, w) \quad (4D.12)$$

[We ignore possible zero modes for the operator $(\nabla^z)^\dagger$].

$$\begin{aligned} \langle 0 | T T_{zz} \phi_1(\xi_1) \cdots \phi_n(\xi_n) | 0 \rangle &= \sum_{i=1}^n (h_i \nabla_{\xi_i} G_{zz}(z, \xi_i) + G_{zz}(z, \xi_i) \nabla_{\xi_i} Z \\ &\quad - \frac{c}{24\pi} Z \int d^2 y \sqrt{g} G_{zz}(z, y) \nabla_y R(y) \end{aligned} \quad (4D.13)$$

Whereas these complete expressions depend on the detailed geometry of the surface and the precise nature of the conformal field theory, the singularities as z approach

any one of the ξ_i 's is universal:

$$\langle 0|TT_{zz}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle = \sum_{i=1}^n \left(h_i \frac{1}{(z-\xi_i)^2} + \frac{1}{z-\xi_i} \partial_{\xi_i} \right) \langle 0|T\phi_1\cdots\phi_n|0\rangle \quad (4D.14)$$

up to additive regular terms as $z \rightarrow \xi_i$. This expression is in fact exact for the plane. One extracts the so-called operator product expansion (OPE)

$$T_{zz}\phi_i(\xi_i) = \frac{h_i}{(z-\xi_i)^2} \phi(\xi_i) + \frac{1}{(z-\xi_i)} \partial_{\xi_i} \phi_i(\xi_i) + \text{reg} \quad (4D.15)$$

One can now proceed to the Ward identity for two T 's; by differentiating with respect to g^{ww} :

$$\begin{aligned} \frac{1}{\sqrt{g}} \frac{\delta \nabla_z^{(n)}}{\delta g^{ww}} &= 0; & \frac{1}{\sqrt{g}} \frac{\delta \nabla_z^{(n)}}{\delta g^{ww}} &= \frac{1}{2} \delta(z, w) \nabla_z^{(n)} + \frac{n}{2} (\nabla_z \delta(z, w)) \\ \frac{1}{\sqrt{g}} \frac{\delta R(z)}{\delta g^{ww}} &= -\frac{1}{2} \nabla_z \nabla_z \delta(z, w). \end{aligned} \quad (4D.16)$$

Hence we find by differentiating the entire formula

$$\begin{aligned} & \sum_i (h_i \nabla_{\xi_i} \delta(z, \xi_i) + \delta(z, \xi_i) \nabla_{\xi_i}) \langle 0|TT_{ww}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \\ &= \frac{1}{2\pi} \nabla^z \langle 0|TT_{zz}T_{ww}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \\ &+ \left(\delta(z, w) \nabla_z + 2(\nabla_z \delta(z, w)) \right) \langle 0|TT_{zz}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \\ &- \frac{c}{24\pi} (\nabla_z R) \langle 0|TT_{ww}\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \\ &+ \frac{c}{12} (\nabla_z)^3 \delta(z, w) \langle 0|T\phi_1(\xi_1)\cdots\phi_n(\xi_n)|0\rangle \end{aligned} \quad (4D.17)$$

Again extracting the most singular piece (exact for the plane) as we let $z \rightarrow w$

$$\begin{aligned} \langle T_{zz}T_{ww}\phi_1(\xi_1)\cdots\phi_n(\xi_n) \rangle &= \left(\frac{1}{z-w} \partial_w + 2 \frac{1}{(z-w)^2} \right) \langle T_{ww}\phi_1(\xi_1)\cdots\phi_n(\xi_n) \rangle \\ &+ \frac{c}{12} \partial_w^3 \frac{1}{(z-w)} \langle \phi_1(\xi_1)\cdots\phi_n(\xi_n) \rangle \end{aligned} \quad (4D.18)$$

From this, one derives the OPE for two stress tensors

$$T_{zz}T_{ww} \sim \frac{c/2}{(z-w)^4} + \frac{2T_{ww}}{(z-w)^2} + \frac{\partial_w T_{ww}}{z-w} + \text{regular.} \quad (4D.19)$$

Notice that the most singular term has several interpretations

- (i) Weyl anomaly for non-flat metrics
- (ii) Schwinger term or central extension in Virasoro
- (iii) T_{zz} is *not* a primary conformal field when ($c \neq 0$)
- (iv) T_{zz} transforms as a connection.

Only the last two statements require clarification. Under infinitesimal reparametrizations:

$$\begin{aligned} Q_v &= \frac{1}{2\pi i} \oint dz v^z T_{zz}; L_n = \oint \frac{dz}{2\pi i} z^{n+1} T_{zz} \\ Q_v T_{ww} &= v^w \partial_w T_{ww} + 2\partial_w v^w T_{ww} + \frac{c}{12} \partial_w^3 v^w \end{aligned} \quad (4D.18)$$

This equation may be translated into an integral form

$$\begin{aligned} T_{zz}(z) &= \left(\frac{\delta w}{dz} \right)^2 T_{ww}(w) + \frac{c}{12} S\{z, w\} \\ S\{z, w\} &= \frac{d^3 w}{cz^3} \left(\frac{dw}{dz} \right)^{-1} = \frac{3}{2} \left(\frac{d^2 w}{dz^2} \right)^2 \left(\frac{dw}{dz} \right)^{-2} \end{aligned} \quad (4D.19)$$

S is the “Schwarzian derivative”. Notice that

$$S\left(z, \frac{az+b}{cz+d}\right) = 0. \quad (4D.20)$$

The approach of replacing equal time commutators with radial quantization goes back to Fubini, Hanson and Jackiw (1973). Conformal Ward identities in the operator product form are exploited in Friedan (1984), Belavin, Polyakov and Zamolodchikov (1984). The above derivation follows Eguchi and Ooguri (1987) and D’Hoker and Phong (1988).

E. Descendant Conformal Fields, Conformal Families

Descendant fields of a field ϕ are defined by

$$L_n \phi(w) = \oint \frac{dz}{2\pi i} (z-w)^{n+1} T_{zz} \phi(w). \quad (4E.1)$$

Here ϕ may be a primary conformal field or a descendant. From the OPE of T_{zz} with primary fields, we see that

$$L_n \phi(w) = 0 \quad n > 0. \quad (4E.2)$$

Further descendants may all be reduced to $n < 0$. Also from the OPE:

$$\begin{aligned} L_0 \phi(w) &= h \phi(w) \\ L_{-1} \phi(w) &= \partial_w \phi(w) \end{aligned} \quad (4E.3)$$

The set of

$$\phi^{\{k\}\{\bar{k}\}} \equiv L_{-k_1} \cdots L_{-k_N} \tilde{L}_{-\bar{k}_1} \cdots \tilde{L}_{-\bar{k}_N} \phi \quad k_i \geq 1, \bar{k}_i \geq 1 \quad (4E.4)$$

containing all descendants of a primary field is called a *conformal family*

$$[\phi] \equiv \left\{ \phi^{\{k\}\{\bar{k}\}} \quad \{k\}\{\bar{k}\} \right\} \quad (4E.5)$$

The Virasoro algebra maps $[\phi]$ into $[\phi]$ and in the language of group theory, this would be analogous to a representation of the Virasoro algebra with the descendants being the various weights. This is a standard thing in group theory: the primary field is like a *highest weight vector* and the descendants are like the *states* or *weights* of the representation $[\phi]$.

However, null vectors may be present beyond which the representation should really be truncated. Thus, this is more general than a representation, *i.e.* a so-called Verma Module. These null vectors are very special, in that they allow for considerable truncation of the Verma Module.

Definition. The conformal family $[\phi]$ admits a degeneracy at level $k \geq 1$, if there is a null state $|\chi\rangle \in [\phi]$ such that

$$\begin{aligned} L_n|\chi\rangle &= 0 \quad n \geq 1 \\ L_0|\chi\rangle &= (h+k)|\chi\rangle \\ L_0|\phi\rangle &= h|\phi\rangle \end{aligned} \tag{4E.6}$$

At level $k = 1$:

$$\begin{aligned} |\chi\rangle &= L_{-1}|\phi\rangle \\ L_1|\chi\rangle &= 2h|\phi\rangle \end{aligned} \tag{4E.7}$$

so we must have $h = 0$. (recall $L_{-1} \sim \partial_z$). At level $k = 2$.

$$\begin{aligned} |\chi\rangle &= \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) |\phi\rangle \\ L_1|\chi\rangle = 0 \quad L_2|\chi\rangle = 0 &\iff 4h + \frac{c}{2} - \frac{9}{2h+1}h = 0. \end{aligned} \tag{4E.8}$$

or

$$h = \frac{1}{16} \left[(5-c) \pm \sqrt{(1-c)(25-c)} \right] \tag{4E.9}$$

In terms of conformal fields, this means that the descendant

$$\chi(z) = L_{-2}\phi_h^{(z)} - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \phi_h(z) \tag{4E.10}$$

is null, and hence produces zero whenever inserted into a correlation function of primary fields:

$$\langle 0 | T\chi(z) \phi_1(\xi_1) \cdots \phi_n(\xi_n) | 0 \rangle = 0. \tag{4E.11}$$

Now using the conformal Ward identities, we have on the one hand

$$\begin{aligned} &\langle 0 | L_{-2} \phi_h(z) \phi_1(\xi_1) \cdots \phi_n(\xi_n) | 0 \rangle \\ &= \sum_{i=1}^n \left(\frac{h_i}{(z - \xi_i)^2} + \frac{1}{z - \xi_i} \partial_{\xi_i} \right) \langle 0 | T \phi_h(z) \phi_1 \cdots \phi_n | 0 \rangle \end{aligned} \tag{4E.12}$$

On the other hand, we use the vanishing of χ , so we get a differential equation on

the Green functions:

$$0 = \left[\frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^n \left(\frac{hi}{(z-\xi_i)^2} + \frac{1}{z-\xi_i} \frac{\partial}{\partial \xi_i} \right) \right] \langle 0 | T \phi_h(z) \phi_1(\xi_1) \cdots \phi_n(\xi_n) | 0 \rangle \quad (4E.13)$$

The solutions of these equations are hypergeometric functions in general. Degeneracies were analyzed by Kac (1983) in a more general context and given by the so-called Kac determinant formula.

$$h_{m,n} = \frac{c-1}{24} + \frac{1}{96} \left((m+1)\sqrt{25-c} \pm (n+1)\sqrt{1-c} \right)^2. \quad (4E.14)$$

for a conformal field theory with central charge c .

Other lecturers will expand considerably upon integrable conformal field theories or so-called minimal models, which were introduced by Belavin, Polyakov and Zamolodchikov (1984). For a reprint volume on conformal field theory, we refer to Itzykson, Saleur and Zuber (1990), and the review article by Ginsparg (1990).

V. SUMMING OVER SURFACES

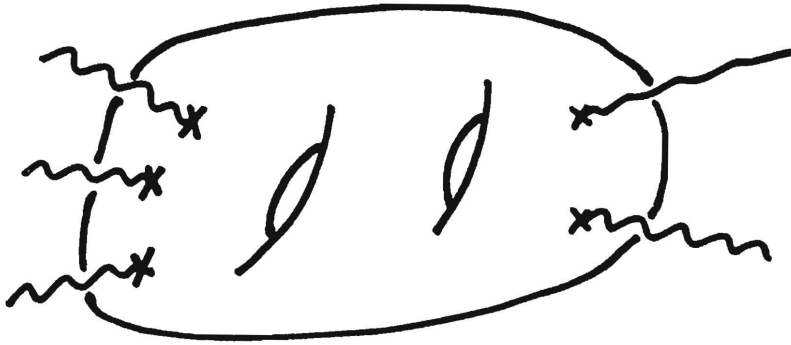
At the end of §II, we introduced Polyakov's proposal for formulating quantum string theory in terms of a sum over random surfaces. Three formulations are physically natural :

- (1) The incoming and outgoing states are full string configurations, represented by disc cut out of the surface.
- (2) In the infinite time limit, the full string states are represented by punctures. One may think of this limit as the infinite time limit in the LSZ formalism in quantum field theory where the field converges to a free field.
- (3) Punctures are replaced by vertex operators, inserted on a compact Riemann surface. In the LSZ equivalent in quantum field theory, these vertex operators play the role of an interpolating field.

It was shown long ago that these formulations are simply related to one another and as such equivalent, see D'Hoker and Giddings (1987). In these lectures, we shall only make use of the formulation in terms of vertex operators. Here, the amplitude is given in terms of a sum over random compact Riemann surfaces, with vertex operators inserted.

$$\text{Amplitude} = \sum_{\text{topologies}} \frac{1}{\mathcal{N}} \int Dg_{mn} \int Dx^\mu V_1 \cdots V_n e^{-I[x,g]} \quad (5.1)$$

Pictorially, this amplitude may be represented as in the figure below to two loop order for example.



The V_k 's are dimension (1,1) primary fields, integrated over the surface. The surfaces to be summed over are *compact*, *without boundaries* and *oriented* for the case of closed oriented strings, as we shall always assume. Topology is completely specified by the number of handles h . What remains is to properly define the integral over metrics and the normalization factor.

A. Structure of the Space of Metrics

On the space of all *positive metrics* $\{g_{mn}, g_{mn} > 0\}$ there is a $\text{Diff}(\Sigma)$ -invariant distance function

$$\|\delta g\|^2 = \int_{\Sigma} d^2\xi \sqrt{g} \delta g_{mn} \delta g_{pq} (g^{mp} g^{nq} + c g^{mn} g^{pq}) \quad (5A.1)$$

We have two natural actions: $\text{Diff}(\Sigma)$ and $\text{Weyl}(\Sigma)$, generated by a vector field v^m and a scalar $\delta\sigma$ respectively.

$$\begin{cases} \delta g_{mn} = (P_1 v)_{mn} + (2\delta\sigma + \nabla^p v_p) g_{mn} \\ (P_1 v)_{mn} = \nabla_m v_n + \nabla_n v_m - g_{mn} \nabla^p v_p \end{cases} \quad (5A.2)$$

where $(P_1 v)$ is traceless by construction.

We shall now quote a number of basic results in the form of theorems, concerning the structure of the space of all metrics and its tangent space, and how it is reduced to Moduli space.

Theorem 1. *Decomposition of the tangent space*

$$\{\delta g_{mn}\} = \{\delta\sigma g_{mn}\} \oplus (\text{Range } P_1) \oplus (\text{Ker } P_1^\dagger). \quad (5A.3)$$

The decomposition into trace part and traceless part is clear. From the fact that $P_1 v$ is traceless, it follows that we have the orthogonal decomposition

$$\delta g_{mn} = 2\delta\sigma g_{mn} + (P_1 v)_{mn} + \delta\hat{g}_{mn} \quad (5A.4)$$

where $\delta\hat{g}_{mn}$ is traceless and orthogonal to $P_1 v$ for all v :

$$\langle \delta\hat{g} | P_1 v \rangle = \langle P_1^\dagger \delta\hat{g} | v \rangle = 0 \quad \text{for all } v. \quad (5A.5)$$

Hence

$$\delta g \in \text{Ker } P_1^\dagger \quad (P_1^\dagger \delta g)_m = -2\nabla^n \delta\hat{g}_{mn}. \quad (5A.6)$$

Theorem 2. *Riemann-Roch Atiyah-Singer*

$$\dim_{\mathbf{R}} \text{Ker } P_n - \dim_{\mathbf{R}} \text{Ker } P_n^\dagger = (2n+1)\chi(\Sigma) = (2n+1)(2-2h). \quad (5A.7)$$

This fundamental theorem can be proven for example by using heat kernels for $P_1^\dagger P_1$ and $P_1 P_1^\dagger$ and comparing their short time limits as was done in (4C.30). One

has the following standard terminology.

Here $\begin{cases} \text{Ker } P_1 & \text{is the space of } \textit{conformal Killing vectors} \\ \text{Ker } P_1^\dagger & \text{is the space of } \textit{Teichmuüller or moduli deformations} \end{cases}$

Theorem 3. Uniformization

Any metric g_{mn} can be brought back by a Weyl rescaling to a constant curvature metric $\hat{g}$ such that

$$\begin{array}{lll} h = 0 & R_{\hat{g}} = 1 & \Rightarrow \int \sqrt{\hat{g}} = 4\pi \\ h = 1 & R_{\hat{g}} = 0 & \int \sqrt{\hat{g}} = 1 \\ h \geq 2 & R_{\hat{g}} = -1 & \Rightarrow \int \sqrt{\hat{g}} = 4\pi(h - 1) \end{array} \quad (5A.8)$$

This theorem is proven using properties of Liouville's equation.

Theorem 4. Vanishing Theorems

1. $h \geq 2$ $\text{Ker } P_1 = 0$
2. $h = 0$ $\text{Ker } P_1^\dagger = 0$.

This theorem is proven as follows. It is sufficient to prove it for constant curvature metrics, which would be $R = -1$ for case (1). This is because we have $P_1 v = 0 \iff \hat{P}_1 \hat{v} = 0$. Now, using the identities

$$\begin{aligned} [\nabla_m, \nabla_n]v_p &= (g_{mp}v_n - g_{np}v_m)R \\ \nabla_m v_n + \nabla_n v_m - g_{mn} \nabla^p v_p &= 0 \\ \nabla^m \nabla_m v_n + \nabla^m \nabla_n v_m - \nabla_n \nabla^m v_m &= 0 \\ \nabla^m \nabla_m v_n + R v_n = 0 &\implies \|\nabla_m v_n\|^2 - R \|v_n\|^2 = 0 \end{aligned} \quad (5A.9)$$

we see that this implies $v_n = 0$. As a result of the vanishing theorems, we have by using the Riemann Roch theorem

$$\dim_{\mathbf{R}} \text{Ker } P_1 \begin{cases} 6 & h = 0 \\ 2 & h = 1 \\ 0 & h \geq 2 \end{cases} \quad \dim_{\mathbf{R}} \text{Ker } P_1^\dagger \begin{cases} 0 & h = 0 \\ 2 & h = 1 \\ 6h - 6 & h \geq 2. \end{cases} \quad (5A.10)$$

Notice for the sphere $h = 0$; this is the same as Gauss' theorem: all metrics are

conformally flat. In stereographic projection, the conformal isometries are

$$z \longrightarrow \frac{az + b}{cz + d} \quad ad - bc = 1 \quad SL(2, \mathbb{C}). \quad (5A.11)$$

B. Moduli Space, Modular Invariance

There are various ways of properly defining the space of metrics modulo $\text{Weyl}(\Sigma)$ and $\text{Diff}(\Sigma)$ transformations.

1. The real definition.

Recall that any metric can be scaled down to constant curvature (defined previously) in a *unique* fashion. $\text{Diff}(\Sigma)$ leaves $R = \text{constant}$ *invariant*, because R is scalar. So we can properly define

$$\mathcal{M}_h \equiv \{\hat{g}_{mn}, R_{\hat{g}} = \text{constant}\} / \text{Diff}(\Sigma). \quad (5B.1)$$

This space is called the *moduli space* of compact, closed oriented Riemann surfaces. Now the $\text{Diff}(\Sigma)$ group is *not connected* :

$$\begin{aligned} \pi_o(\text{Diff}(\Sigma)) &= \mathcal{M} && \text{“mapping class group”} \\ \dim_{\mathbb{R}} \mathcal{M}_h &= \dim \text{Ker } P_1^\dagger \end{aligned}$$

The last identification shows that the space is of finite dimension in view of the Riemann Roch theorem. An important subgroup of the mapping class group is the modular group, i.e. the group of automorphisms of the homology cycles on the Riemann surface.

2. The complex definition

Any orientable surface with metric g_{mn} has automatically a *complex structure*, which is $\text{Weyl}(\Sigma)$ invariant.

$$J_m{}^n = \sqrt{g} \epsilon_{mp} g^{pn} \quad J^2 = -1 \quad (5B.2)$$

Hence we may define moduli space as the equivalence classes of complex structures

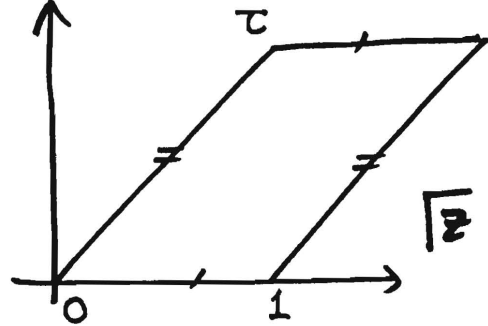
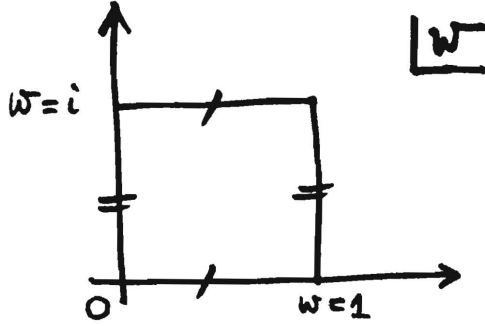
$$\mathcal{M}_h = \{J_m{}^n \quad J^2 = -1\} / \text{Diff}(\Sigma). \quad (5B.3)$$

This makes $\mathcal{M}_h$ itself a *complex orbifold* which is manifold with additional conical type singularities.

The example of the torus; $h = 1$.

For the torus, we have $R = 0$: any metric g_{mn} can be decomposed in Fourier modes and *non-zero* modes can be gauged away using $\text{Diff}_o(\Sigma)$. Hence moduli space can be parametrized by constant metrics modulo constant Weyl rescalings.

$$ds^2 = \lambda |dw + \mu d\bar{w}|^2 \quad \lambda > 0; \quad \mu \in \mathbb{C}, \quad |\mu| < 1. \quad (5B.4)$$



With the help of the following transformation, the square domain may be mapped onto a parallelogram domain.

$$z = \frac{w + \mu \bar{w}}{1 + \mu} \quad (5B.5)$$

This transformation is not conformal, but is *quasi-conformal*. It maps the constant metric (5B.3) into the Euclidean flat metric up to a constant Weyl rescaling.

$$ds^2 = \lambda' |dz|^2 \quad \tau = i \frac{1 - \mu}{1 + \mu} \quad (5B.6)$$

Hence Teichmüller space is the space of all complex τ with $\text{Im } \tau > 0$. But the torus also had diffeomorphisms not connected to identity: but then we get geometrically the same torus by redefining the fundamental periods of the torus ω^1, ω^2 :

$$\begin{aligned} \omega^1 &\longrightarrow d\omega^1 + c\omega^2 & a, b, c, d, \in \mathbb{Z} \\ \omega^2 &\longrightarrow b\omega^1 + a\omega^2 & ad - bc = 1. \end{aligned}$$

This forms a group $SL(2, \mathbb{Z})$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

One can translate this action into an action on τ :

$$\tau = \frac{\omega^2}{\omega^1} \longrightarrow \frac{a\omega^2 + b\omega^1}{c\omega^2 + d\omega^1} = \frac{a\tau + b}{c\tau + d}.$$

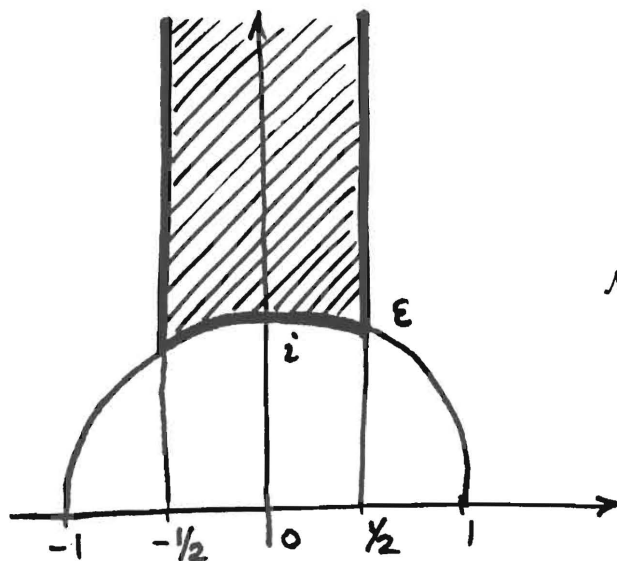
Hence moduli space is given by

$$\mathcal{M}_1 = \{\tau \in \mathbb{C}; \text{Im } \tau > 0\} / SL(2, \mathbb{Z}) \quad (5B.7)$$

This group is generated by

$$\begin{aligned} T : \tau &\rightarrow \tau + 1 \\ S : \tau &\rightarrow -1/\tau \quad S^2 = 1 \end{aligned}$$

Moduli space may be represented by a region in the upper half plane



$$\mathcal{M}_1 = \{\tau \in \mathbb{C}, |\text{Re } \tau| \leq \frac{1}{2}; |\tau| \geq 1; \text{Im } \tau \geq 0\}$$

Modular transformations have the following fixed points

$$(1) T : \infty \mapsto \infty$$

$$(2) S : i \mapsto i$$

$$(3) ST^{-1} : \epsilon \mapsto \epsilon \quad \epsilon = \frac{1}{2} + i\sqrt{\frac{3}{2}}$$

These are the only fixed points, and it is easy to see that topologically, $\mathcal{M}_1$ is equivalent to a sphere with three punctures.



$$\mathcal{M}_1 = S^2 - \{\infty, i, \epsilon\}. \quad (5B.8)$$

C. The Measure of Integration Over Metrics

The full scattering amplitude at h loops is given by

$$A_h = \int Dg_{mn} \frac{1}{\mathcal{N}} e^{-W_m[g] - \lambda\chi - \mu \int \sqrt{g}} \quad (5C.1)$$

The couplings λ and μ must be included since they multiply operators that arise from additive renormalizations of the matter conformal field theory coupled to the metric g_{mn} . Here W_m is the (conformal) matter effective action which in the case of the Polyakov string in flat space-time is given by

$$e^{-W_m[g]} = \int dx^\mu \quad V_1 \cdots V_n \quad e^{-I[x,g]} \quad (5C.2)$$

We assume that the matter theory is conformal: $g = \hat{g} e^{2\sigma}$ so that W_m has the Weyl transformation law given in (4B.3-4). In this section, we decompose and carry out the integration over g_{mn} . Amazingly enough, we will see that it reduces to a finite dimensional integral over moduli, plus an integral over σ .

The starting point is the parametrization of a general g_{mn} :

$$\begin{array}{ccccc} g_{mn} = (\exp v) & e^{2\sigma} & \hat{g}_{mn} \\ \nearrow & \uparrow & \nwarrow \\ \text{Diff}(\Sigma) & \text{Weyl}(\Sigma) & \text{parametrized by } \mathcal{M}_h. \end{array}$$

In particular, $\hat{g}_{mn}$ is generated by *Beltrami differentials*.

$$\begin{array}{ccc} \delta \hat{g}_{mn} = \sum_j \delta m_j (\mu_j)_m^p \hat{g}_{pn} & & \\ \nearrow & & \nwarrow \\ \text{changes in moduli} & & \text{Beltrami differentials.} \end{array} \quad (5C.3)$$

From the orthogonal decomposition of any metric in (5A.3-4), we easily de-

compose the norm on g_{mn} :

$$\| \delta g_{mn} \|^2 = \| \delta \sigma \|^2 + \| P_1 v \|^2 + \sum_{j,k} \frac{|\langle \mu_j | \phi_k \rangle_g \delta m_j|^2}{\langle \phi_k | \phi_k \rangle_g} \quad (5C.4)$$

where ϕ_k is assumed to be an *orthogonal* basis of $\text{Ker } P_1^\dagger$. As a result, the measure decomposes:

$$Dg_{mn} = D_g \sigma \left(\det' P_1^\dagger P_1 \right)^{1/2} D'_g v^m \frac{\det \langle \mu_j | \phi_k \rangle_g}{\det \langle \phi_j | \phi_k \rangle_g^{1/2}} \prod_j dw_j \quad (5C.5)$$

which is now valid for an arbitrary basis of ϕ_k 's in $\text{Ker } P_1^\dagger$. Conformal reparametrizations ($P_1 v = 0$) do not act, and hence must be excluded from Dv which is what $D'v$ means.

Under Weyl transformations, $\mu_j = e^{2\sigma} \hat{\mu}_j$, and as a result,

$$\det \langle \mu_j | \phi_k \rangle_g = \det \langle \hat{\mu}_j | \phi_k \rangle_{\hat{g}} \quad (5C.6)$$

Recalling the calculation of the Weyl scaling properties of determinants

$$\frac{(\det' P_1^\dagger P_1)^{1/2}}{\det \langle \phi_j | \phi_k \rangle_g^{1/2} \det \langle \psi_\alpha | \psi_\beta \rangle_g^{1/2}} = \frac{(\det' \hat{P}_1^\dagger \hat{P}_1)^{1/2}}{\det \langle \phi_j | \phi_k \rangle_{\hat{g}}^{1/2} \det \langle \psi_\alpha | \psi_\beta \rangle_{\hat{g}}^{1/2}} e^{-c_P S_L} \quad (5C.7)$$

$$c_P = -2(6n^2 + 6n + 1) \Big|_{n=1} = -26.$$

Hence, the full scattering amplitude for given number of loops becomes

$$A_h = \int_{\mathcal{M}_h} dm_j \int D\sigma \int Dv^n e^{-W_m[\hat{g}, o]} \left(\frac{\det' \hat{P}_1^\dagger \hat{P}_1}{\det \langle \phi_j | \phi_k \rangle_{\hat{g}}} \right)^{1/2} \frac{|\langle \mu_j | \phi_h \rangle|}{\text{Vol } C} \epsilon^{(26-c)S_L} \quad (5C.8)$$

where the normalization factor is taken $\mathcal{N} = \text{Vol Diff}(\Sigma) \times \text{Vol}(\text{Weyl}(\Sigma))$. C here stands for the conformal reparametrization group (globally) = $SL((2, \mathbb{C}))$ for the sphere, $\mathbb{R}^2/1^2$ for the torus.

We see that since everything is reparametrization invariant, we may simply omit the Dv^m integration. We also see that there is a *critical string theory* in which the Weyl-Liouville mode decouples provided

$$c = 26. \quad (5C.9)$$

The significance of the non-critical string /critical string is as follows:

1. In the critical string $c = 26, \alpha = 1$ we have a massless graviton and a massless dilaton. This is a real theory of gravity with massless gravitons. Technically, the infinite-dimensional integral over g_{mn} collapses to just an integral over a finite dimensional moduli space analogous to Feynman parameters in *QFT*.
2. In the non-critical string ($c < 25$) [$c = 25$ is equivalent to $c = 26$] we generically have massive spin 1 and spin 2-states, if Lorentz invariant at all. These could be theories of spontaneously broken gravity (massive gravitons) or of just massive hadrons (cfr. *QCD*). Technically, one now has to quantize this Liouville mode which is a difficult problem.

D. Faddeev-Popov Ghosts; BRST Invariance

As in non-Abelian gauge theories, we have produced an answer in terms of determinants that depend on the fields in the theory. In Yang-Mills theory, typically, one must do gauge fixed perturbation theory, and determinants are very inconvenient to handle. In string theory, this is not so, and determinants can often be evaluated completely explicitly.

Yet, many string theorists are passionately in love with ghosts, so I shall discuss them here. We had previously shown that the $b - c$ systems produce determinants of Laplace operators with corresponding spin. In general, things are a little bit complicated by the fact that b and c may have zero modes in the integral, and these have to be isolated. Doing so, one finds

$$\begin{aligned} & \int D(bc\bar{b}\bar{c}) e^{-\int (b\nabla_x c + \bar{b}\nabla_x \bar{c})} \prod_j \int b\mu_j \int \bar{b}\bar{\mu}_j \prod_\alpha \int cv_\alpha \int \bar{c}\bar{v}_\alpha \\ &= \left(\frac{\det' P_1^\dagger P_1}{\det \langle \phi_j | \phi_k \rangle} \right)^{1/2} \det \langle \mu_j | \phi_k \rangle \end{aligned} \quad (5D.1)$$

where μ_j are *complex* Beltrami differentials and v_α are generators of conformal Killing vectors, properly normalized. The final answer for the h -loop amplitude is

then

$$A_h = \int_{\mathcal{M}_h} \int Dx D(b\bar{c}\bar{c}) e^{-I[\hat{x},g] - \int (b\nabla_z c + \bar{b}\nabla_z \bar{c})} V_1 \dots V_n \prod_j \int b_{\mu_j} \int \bar{b}_{\bar{\mu}_j} \int c v_\alpha \int \bar{c} \bar{v}_\alpha \quad (5D.2)$$

Generally, when a gauge has been fixed, and Faddeev-Popov ghosts have been introduced, the resulting formulation has a new symmetry: BRST (Becchi Rouet Stora (1976) Tuytin (1975)) invariance. On ordinary fields, BRST acts simply by a ghost-dependent reparametrization

$$\delta x^\mu = \lambda c^z \partial_z x^\mu \quad \lambda \text{ Grassmann constant.} \quad (5D.3)$$

On the ghost fields, we have

$$\begin{aligned} \delta c^z &= -\lambda c^z \nabla_z c^z \\ \delta b_{zz} &= -\lambda \left[\underbrace{-\frac{1}{2} \partial_z x^\mu \partial_z x^\mu}_{T_{zz}^{\text{matter}}} + \underbrace{c^z \nabla_z b_{zz} + 2(\nabla_z c^z) b_{zz}}_{T_{zz}^{\text{ghost}}} \right] \end{aligned} \quad (5D.4)$$

The BRST charge (classically) is defined by the current

$$\begin{aligned} j_z^{\text{BRST}} &= c^z T_{zz}^{\text{matter}} + \frac{1}{2} c^z T_{zz}^{\text{ghost}} + \frac{3}{2} (\nabla_z)^2 c^z \\ Q_{\text{BRST}} &= \oint \frac{dz}{2\pi i} j_z^{\text{BRST}} \end{aligned} \quad (5D.5)$$

By working out the operator product expansion of the (renormalized) j_z^{BRST} with itself, one may show that

$$Q_{\text{BRST}}^2 = 0 \quad \text{provided } c = d = 26. \quad (5D.6)$$

The Virasoro conditions are now replaced with the BRST conditions. Reparametrization invariance is now cast in the form that all physical states should be annihilated by the BRST charge.

$$Q_{\text{BRST}} |\text{physical}\rangle = 0$$

However, any state that is already the BRST transform of another state is spurious (or longitudinal in the language of gauge theories), and should not be counted

separately as a physical state.

$$|\text{spurious}\rangle = Q_{BRST}/|\text{anything}\rangle$$

Hence the space of physical states is the set of equivalence classes (also called cohomology classes) as follows

$$\{\text{physical states}\} \sim \text{Ker } Q_{BRST}/\text{Im } Q_{BRST}$$

Many formal arguments can be constructed that exploit the mathematical structure of cohomology theory for this operator. It would however be beyond the scope of these lectures to go into these issues.

E. Tree-level Amplitudes

For simplicity, we only consider the scattering amplitude of tachyons $M^2 = -2$. The vertex operator for the tachyon is

$$V^T(k) = \epsilon \int d^2\xi \sqrt{g} e^{ikx} \quad k^2 = 2. \quad (5E.1)$$

Now on the sphere, we pick the round metric, stereographically projected onto the plane:

$$\hat{g}_{mn} = \frac{2\delta_{mn}}{(1 + |z|^2)^2} \quad (5E.2)$$

The Green function on scalar fields is defined by

$$\Delta_z G(z, z') = 4\pi\delta(z, z') - \frac{4\pi}{\int_{\Sigma} \sqrt{\hat{g}}} \quad (5E.3)$$

and we find

$$\hat{G}(z, z') = -\ell n \frac{|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)} \quad (5E.4)$$

We also need the renormalized Green function at coincident points. For the plane, this has a simple Weyl scaling law

$$G_R(z, z) = \hat{G}_R(z, z) + 2\sigma(z) \quad \text{if} \quad g = e^{2\sigma} \hat{g}. \quad (5E.5)$$

Clearly, this implies that the quantum V^T operator is scale invariant:

$$\sqrt{g} e^{ikx} = \sqrt{g} : e^{ikx} : e^{-\frac{1}{2}k^2 G_R(z, z)} = \sqrt{\hat{g}} : e^{ikx} : e^{-\frac{1}{2}k^2 \hat{G}_R} \quad (5E.6)$$

provided $k^2 = 2$.

All the ingredients in the scattering amplitudes are Weyl invariant, and recall that there are no moduli, so all determinants with respect to $\hat{g}$ are just constants. We shall denote the product of all these determinants by c . The amplitude then reads

$$\begin{aligned}
A_o &= e^{-2\lambda} \frac{1}{\text{Vol}C} c \langle V^T(k_1) \cdots V^T(k_n) \rangle \\
&= \frac{e^{-2\lambda} \epsilon^n}{\text{Vol}C} c \int d_2 z_1 \sqrt{\hat{g}_1} \cdots \int d_2 z_n \sqrt{\hat{g}_n} \langle e^{i k_1 x}(z_1) \cdots e^{i k_n x}(z_n) \rangle \\
&= (2\pi)^{26} \delta\left(\sum_i k_i\right) \frac{e^{-2\lambda} c \epsilon^n}{\text{Vol}C} \prod_i \int d^2 z_i \sqrt{\hat{g}} \exp\left\{-\frac{1}{2} \sum_{i \neq j} k_i \cdot k_j G(z_i, z_j) - \frac{1}{2} \sum_{i \neq j} k_i^2 \hat{G}_R(z_i, z_j)\right\}
\end{aligned} \tag{5E.7}$$

The renormalized Green function at coincident points is constant by conformal invariance, so we may define

$$\epsilon_R = \epsilon \exp\left\{-\hat{G}_R(z_i, z_i)\right\} \tag{5E.8}$$

Furthermore, the pieces in $\hat{G}$ that only depend on one variable, say z , add up:

$$\begin{aligned}
e.g. \quad \sum_{i \neq j} k_i \cdot k_j \left\{ -\ell n(1 + |z_i|^2) \right\} &= -k_i(-k_i) \ell n(1 + |z_i|^2) \\
&= 2\ell n(1 + |z_i|^2).
\end{aligned} \tag{5E.9}$$

these pieces cancel the $\sqrt{\hat{g}}$ factor in front, and we get

$$A_o = (2\pi)^{26} \delta\left(\sum_i k_i\right) e^{\frac{-2\lambda \epsilon^n c}{\text{Vol}C}} \prod_i \int d^2 z_i \prod_{i \neq j} |z_i - z_j|^{k_i \cdot k_j}. \tag{5E.10}$$

We still need to divide by the volume $\text{Vol}(C)$. Notice that the integral expression is invariant under transformation (5B.6). Under these transformations, we may always choose a, b, c, d in such a way that

$$z_n \longrightarrow \infty, \quad z_{n-1} \longrightarrow 1, \quad z_{n-2} \longrightarrow 0. \tag{5E.11}$$

Hence we get out precisely a factor of $\text{Vol} C$, which cancels the previous one. So,

we are left with

$$A_o = (2\pi)^{26} \delta\left(\sum_i k_i\right) e^{-2\lambda} \epsilon_R^n c \prod_{i=1}^{n-3} \int d^2 z_i \prod_{1 \leq i < j \leq n-1} |z_i - z_j|^{2k_i \cdot k_j} \quad (5E.12)$$

The zero point function (cosmological constant), one and two point functions vanish. For the three point function

$$A_o^{(3)} = (2\pi)^{26} \delta(k_1 + k_2 + k_3) e^{-2\lambda} \epsilon_R^3 c \quad (5E.13)$$

Four point function

$$A_o^{(4)} = (2\pi)^{26} \delta\left(\sum_i k_i\right) e^{-2\lambda} \epsilon_R^4 c \int dz^2 |z|^{2k_1 \cdot k_2} |z-1|^{2k_1 \cdot k_3} \quad (5E.14)$$

Or introducing the standard Mandelstam variables with $s + t + u = -8$

$$\begin{aligned} s &= -(k_1 + k_2)^2 = -4 - 2k_1 \cdot k_2 \\ t &= -(k_2 + k_3)^2 = -4 - 2k_2 \cdot k_3 \\ u &= -(k_1 + k_3)^2 = -4 - 2k_1 \cdot k_3 \end{aligned} \quad (5E.15)$$

$$A_o^{(4)} = (2\pi)^{26} \delta\left(\sum_i k_i\right) e^{-2\lambda} \epsilon_R^4 c \int dz^4 |z|^{s-4} |z-1|^{-u-4}$$

To perform this integral, we introduce Schwinger parameters

$$\frac{1}{A^s} = \int_0^\infty d\lambda e^{-e\lambda A} \lambda^{s-1} \quad (5E.16)$$

these integrals can be performed and we find

$$A_o^{(4)} = (2\pi)^{26} \delta\left(\sum_i k_i\right) e^{-2\lambda} \epsilon_R^4 \pi \frac{\Gamma(-1 - \frac{s}{2}) \Gamma(-1 - \frac{t}{2}) \Gamma(-1 - \frac{u}{2})}{\Gamma(2 + \frac{s}{2}) \Gamma(2 + \frac{t}{2}) \Gamma(2 + \frac{u}{2})} \quad (5E.17)$$

This expression was already encountered previously, and is called the *Virasoro-Shapiro amplitude* (1969). Notice that this amplitude is completely symmetric under interchange of s, t and u in particular, it is dual $s \leftrightarrow t$.

The pole structure is very simple: since Γ^{-1} is holomorphic, the only poles occur at

$$-1 - \frac{s}{2} = -k \quad k = 0, 1, 2, \dots$$

or

$$s = -2 + 2k \quad t = -2 + 2k' \quad u = -2 + 2k''$$

and we recover

$k = 0$	$s = -2$	tachyon
$k = 1$	$s = 0$	massless graviton etc.
$k \geq 2$	$s = 2, 4, \dots$	massive string states.

Factorization of the 4-point function on the tachyon reproduces the 3-point function, and fixes the normalization

$$c\epsilon_R^4 e^{-2\lambda} = (c\epsilon_R^3 e^{-2\lambda})^2 \quad (5E.18)$$

Recall that c is a known quantity, basically a product of the renormalized determinants.

F. One Loop Amplitudes.

The Riemann surface is a torus, which we may represent by a parallelogram as in (5B.3). The moduli space was described in (5B.6). Now we need to evaluate propagators and determinants. First it is easy to see that

$$\left(\det' P_1^\dagger P_1 \right)^{1/2} = \frac{1}{2} \det' \Delta \quad (5F.1)$$

The eigenvalues of the ordinary Laplacian with the above metric can be gotten directly from spectral Fourier analysis :

$$\lambda_{n_1 n_2} = 4\pi^2 \left(n_1^2 + \frac{1}{\tau_2^2} (n_2 - n_1 \tau_1)^2 \right). \quad (5F.2)$$

The determinant can be calculated by ζ -function regularization

$$\begin{aligned}
\zeta(s) &= \sum_{(n_1, n_2) \neq (0,0)} \left\{ n_1^2 \tau_2^2 + (n_2 - n_1 \tau_1)^2 \right\}^{-s} \\
\ell n \det' \Delta &= -\zeta'(0) + \ell n \frac{4\pi^2}{\tau_2^2} \zeta(0) \\
\det' \Delta &= \tau_2^2 |\eta(\tau)|^4 \\
\eta(\tau) &= e^{i\pi\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})
\end{aligned} \tag{5F.3}$$

By renormalizations analogous to the ones we carried out at tree level, we find

$$\begin{aligned}
A_1 &= \int_{\mathcal{M}_1} \frac{d^2 \tau}{\tau_2^2} \frac{1}{(4\pi^2 \tau_2)^{12}} |\eta(\tau)|^{-48} \prod_i \int dz_i^2 \exp \left[- \sum_{i < j} k_i \cdot k_j G(z_i, z_j, \tau) \right] \\
G(z_i, z_j; \tau) &= -\ell n \left| \frac{\vartheta_1(z_i - z_j | \tau)}{\vartheta_1'(0 | \tau)} \right|^2 - \frac{\pi}{2\tau_2} (z_i - \bar{z}_i - z_j + \bar{z}_j)^2
\end{aligned} \tag{5F.4}$$

One can show that the integrand is completely invariant under modular transformations of τ and z_i .

$$\begin{aligned}
e.g. \quad \tau &\rightarrow \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{Z}) \\
\frac{d^2 \tau}{\tau_2^2} &\text{ invariant} \quad d^2 \tau \rightarrow \frac{d^2 \tau}{|c\tau + d|^4} \quad ; \quad Im \tau \rightarrow \frac{Im \tau}{|c\tau + d|^2} \\
\eta(\tau) &\rightarrow \eta \left(\frac{a\tau + b}{c\tau + d} \right) = \epsilon (c\tau + d)^{1/2} \eta(\tau) \\
\tau_2^2 |\eta(\tau)|^2 &\text{ is invariant.}
\end{aligned} \tag{5F.6}$$

Notice that

$$\eta(\tau) \sim e^{-\frac{\pi}{12} \tau_2} \quad \text{as } \tau_2 \rightarrow \infty. \tag{5F.7}$$

Hence the bosonic string diverges:

$$A_1 \sim \int_{\mathcal{M}_1} \frac{d^2 \tau}{\tau_2^2} \frac{1}{\tau_2^{12}} e^{4\pi \tau_2} \quad (\text{Green functions}) \tag{5F.8}$$

This type of divergence is associated with the fact that the bosonic string spectrum contains a tachyon. Also, the massless dilaton produces extra divergences, because the dilaton tadpole graph is non-zero.

Physically, the reason for the divergence of the bosonic string is associated with the fact that we have expanded around flat Minkowskian space-time. This expansion really only makes sense provided this background is a stable state in the full string theory. The fact that we find a tachyonic excitation as well as a non-vanishing dilaton tadpole are two different symptoms of the same illness : Minkowski flat space-time is not a stable solution to bosonic string theory. One would like to look for other backgrounds where this expansion might not exhibit these problems. This is difficult technically, because we have no analogue to date of a fully satisfactory field theory for strings. So the best we can do is try out a variety of different backgrounds about which to expand and check whether the tachyon and tadpole divergences are absent. Such an approach was taken by Fischler and Susskind (1986). This question will be addressed more fully for the case of the superstring.

The summation over surfaces approach was initiated by Polyakov, see for example his book Polyakov (1987), and the reformulation in terms of integrals over moduli is extensively reviewed in D'Hoker and Phong (1988), which also has references to the original papers.

For mathematical references on theta functions, also on higher genus surfaces, we refer to Mumford (1983) and Fay (1973).

VI. SUPERSTRINGS

Models discussed thus far only describe space-time bosons. This is the case because the building blocks used are commuting creation and annihilation operators, transforming in the vector representation of the Lorentz group, and a Fock space of string states built on a bosonic vacuum.

To build a string theory that also contains space-time fermions, there are two fundamental approaches possible, which are almost diametrically opposite to one another. We shall now describe these first in words and then provide the associated equations.

1. The single string ground state is a bosonic state, as was the case for the bosonic string. In addition to the bosonic field that represents the position of the string x^μ , there is also a Grassmann-valued, space-time spinor field θ^A . Out of the bosonic one-string ground state, this fermionic field creates space-time spinor or fermionic states now, and it will turn out that these fermion fields are scalars on the worldsheet. This is called the Green-Schwarz (1984) covariant formulation. It has the advantage of being very geometrical, and space-time supersymmetric by construction; however, due to the intricate geometry of the system, very complicated constraints arise, and so far no one has been able to successfully quantize it beyond tree level.
2. In the second formulation, we have the bosonic position field x^μ , as in the bosonic string, but we also have a Grassmann-valued worldsheet spinor field ψ^μ that is a space-time Lorentz vector. There is the ordinary bosonic ground state, but there is also another ground state that is a space-time spinor. So even though all field operators in the theory transform under the vector representation, there are two sectors to the theory, one built on the bosonic ground state (this is the so-called Neveu-Schwarz sector) and another one built on the space-time spinor or fermionic ground state. All states built on the bosonic ground state are bosonic, and all states built on the fermionic ground state are fermionic. The remainder of this lecture will be devoted to the understanding of this Ramond Neveu-Schwarz theory and its truncation to the superstring.

A. Operator Formulation of the Neveu-Schwarz-Ramond String

One needs a new sector of the theory, that will produce a fermionic ground state. How can this be done? We start by putting an additional field on the worldsheet, transforming under the vector representation of the Lorentz group (in flat space-time) but under the spinor representation on the worldsheet. Thus the

fields on the world-sheet are

$$x^\mu(\tau, \sigma) \quad \psi_\alpha^\mu(\tau, \sigma) \sim (\psi_+^\mu, \psi_-^\mu) \quad (6A.1)$$

and we take ψ^μ to be *worldsheet Majorana spinors*. So ψ_+^μ and ψ_-^μ are worldsheet Majorana Weyl spinors.

When we attempt to put spinors on a non-trivial manifold, a new complication arises : can we do it ? For two dimensional surfaces, the answer is yes, provided they are orientable. Technically, the condition is that the second Stiefel-Whitney characteristic class vanishes, which is the case for oriented Riemann surfaces.

However, there are in general several different ways to put spinors on a manifold, and these are called *spin structures*. Technically, the different spin structures are represented by the first Stiefel-Whitney class of the surface. How do we see that there are several different ways of putting a spinor on a manifold ? We start with the vector representation. Under parallel transport around a closed cycle, the vector representation is transported in a unique way given by integrating the Christoffel connection around the closed cycle. The spinor representation corresponds to a double cover, or essentially transforms as the square root of the vector representation, and this is only defined up to a sign ambiguity.

For example on a flat cylinder, a vector comes back to itself, but a spinor may have periodic or antiperiodic boundary conditions. In fact, the cylinder is precisely the worldsheet with the correct geometry for a freely propagating single string state. The above sign ambiguity, as we shall show below is precisely in this case what distinguishes the Ramond (1971) and the Neveu-Schwarz (1971) sectors.

$$\begin{aligned} \psi^\mu(\sigma + 2\pi, \tau) &= +\psi^\mu(\sigma, \tau) & \text{Ramond} \\ \psi^\mu(\sigma + 2\pi, \tau) &= -\psi^\mu(\sigma, \tau) & \text{Neveu-Schwarz} \end{aligned} \quad (6A.2)$$

Let us now show how a spinor ground state is produced in the Ramond sector upon canonical quantization. We postulate canonical anti-commutation relations for ψ

$$\{\psi_\pm^\mu(\sigma, \tau), \psi_\pm^\nu(\sigma', \tau)\} = 4\pi\delta(\sigma - \sigma')\eta^{\mu\nu} \quad (6A.3)$$

Naturally, ψ obeys the (free) Dirac equation, which reads

$$\begin{aligned} \gamma^m \partial_m \psi &= 0 \\ (\partial_\tau \mp \partial_\sigma) \psi_\pm^\mu &= 0 \end{aligned} \quad (6A.4)$$

Taking into account the boundary periodic boundary conditions as given in (6A.2),

we get

$$\psi_{\pm}^{\mu}(\tau, \sigma) = \sum_n \psi_{\pm n}^{\mu} e^{-in(\tau \pm \sigma)} \quad (6A.5)$$

Canonical commutation relations imply

$$\{\psi_{\pm m}^{\mu}, \psi_{\pm n}^{\nu}\} = 2\eta^{\mu\nu} \delta_{m+n,0} \quad (6A.6)$$

In particular, ψ_{+0}^{μ} obeys the Clifford Algebra : $\psi_{+0}^{\mu} = \gamma^{\mu}$, where γ^{μ} are the space-time Dirac matrices. To find out the space-time spin of various states, we construct the angular momentum operator in this sector

$$J^{\mu\nu} = \frac{1}{2} \Sigma^{\mu\nu} + \sum_{n=1}^{\infty} (\psi_{-n}^{\mu} \psi_n^{\nu} - \psi_{-n}^{\nu} \psi_n^{\mu}) \quad (6A.7)$$

We see that due to the presence of the zero mode contribution, that yields the first term, all states in this sector have half-odd integer spin. For example on the ground state, the angular momentum operator reduces to

$$\psi_n^{\mu}|0\rangle = 0 \quad n > 0 \quad J^{\mu\nu} = \frac{1}{2} \Sigma^{\mu\nu} \quad (6A.8)$$

For the Neveu-Schwarz sector on the other hand,

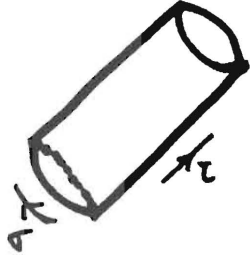
$$\psi_{\pm}^{\mu}(\tau, \sigma) = \sum_{r=\frac{1}{2}+\mathbb{Z}} \psi_{\pm r}^{\mu} e^{-ir(\tau \pm \sigma)} \quad (6A.9)$$

and

$$\{\psi_{\pm r}^{\mu}, \psi_{\pm s}^{\nu}\} = 2\eta^{\mu\nu} \delta_{r+s,0} \quad (6A.10)$$

Since neither r nor s are ever zero, there is no zero mode in this sector and hence all states are bosons. Notice that the existence of a fermionic ground state for the Ramond sector is an example of the so-called spin-isospin transmutation, as occurs *e.g.* for particles in the presence of magnetic monopoles or also in the molecular system of Polyacethylene.

It is also important to record how the conditions for periodicity and anti-periodicity on the cylinder translate into conditions onto the plane.



$$z = e^{\tau + i\sigma}$$

(6A.11)

$$\psi_+(\tau + i\sigma) = \left(\frac{\partial z}{\partial(\tau + i\sigma)} \right)^{\frac{1}{2}} \psi_+(z) = e^{\frac{1}{2}(\tau + i\sigma)} \psi_+(z)$$

The transformation coefficient is double valued :

Ramond

$$\psi_+(e^{2\pi i} z) = -\psi_+(z)$$



quadratic cut.

Neveu - Schwarz

$$\psi_+(e^{2\pi i} z) = +\psi_+(z)$$



However, we now face a new problem: ψ^μ whether in the Ramond or Neveu-Schwarz sector, generates new negative norm states, since it carries a Lorentz vector index. Conformal reparametrization invariance was responsible for eliminating only the negative norm states of x^μ . We need a new local symmetry : local superreparametrizations or local supersymmetry. We shall capture this symmetry in a variety of ways as we proceed. For the time being, we shall exhibit it as an extension of the Virasoro algebra, produced by the additional field ψ

$$\begin{aligned} L_n &= : \sum_p \frac{1}{2} x_{p-n}^\mu x_p^\mu + \sum_\nu \frac{1}{2} \nu \psi_{n-\nu}^\mu \psi_\nu : \\ F_\nu &= \sum_n \psi_{\nu-n}^\mu x_n^\mu \end{aligned} \quad (6A.12)$$

Here ν takes integer values for the Ramond sector and integer $+\frac{1}{2}$ values for the Neveu-Schwarz sector.

$$\begin{aligned} [L_m, L_n] &= i(m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, F_\nu] &= \nu F_{m+\nu} \\ \{F_\mu, F_\nu\} &= 2L_{\mu+\nu} + \begin{cases} R \\ NS \end{cases} \end{aligned} \quad (6A.13)$$

Physical states are defined by

$$L_m |\psi\rangle = 0 \quad m > 0 \quad F_\nu |\psi\rangle = 0 \quad \nu > 0 \quad (6A.14)$$

and there is a no-ghost theorem for this string model as well.

B. Component Formulation of the RNS String

To produce locally supersymmetric theories, one needs to introduce a new field, the Rarita-Schwinger spin 3/2 field :

$$\chi_m^\alpha = e_m^a \chi_a^\alpha \quad (6B.1)$$

In fact, we shall use the frame (zweibein) formalism, since worldsheet spinors have been introduced : e_m^a provides a locally inertial Lorentz frame

$$g_{mn} = e_m^a e_n^b \delta_{ab} \quad (\text{or } \eta_{ab}) \quad (6B.2)$$

By introducing e_m^a , we now have 4 fields (instead of 3 for g_{mn}), but we also have *local* $U(1)$ *frame rotation* invariance which was not present before, and which kills one physical degree of freedom.

The appropriate action for the dynamics of a two-dimensional theory with local supersymmetry invariance is supergravity. $N = 1$ supergravity in 2-dimensions for flat target space-time is

$$I = \frac{1}{4\pi} \int_\Sigma \sqrt{g} \left[\frac{1}{2} g^{mn} \partial_m x^\mu \partial_n x^\mu + \psi^\mu \gamma^m \partial_m \psi^\mu - \psi^\mu \gamma^a \gamma^m \chi_a \partial_m x^\mu - \frac{1}{4} \psi^\mu \gamma^a \gamma^b \chi_a (\chi_b \psi^\mu) \right] \quad (6B.3)$$

Local invariances of this action are

- | | | |
|-------------------------------|---|--|
| 1. Diff(Σ) | | |
| 2. Weyl (Σ) | $\delta e_m^a = \delta \sigma e_m^a$ | $\delta \chi_m = \frac{1}{2} \delta \sigma \chi_m$ |
| 3. Supersymmetry | $\delta e_m^a = \xi \gamma^a \chi_m$ | $\delta \chi_m = -2 \nabla_m \xi$ |
| $\delta x^\mu = \xi \psi^\mu$ | $\delta \psi^\mu = \gamma^m \nabla_m \xi$ | |
| 4. Super-Weyl | $\delta e_m^a = 0$ | $\delta \chi_m = \gamma_m \lambda$ |

Here, ξ and λ are spin $\frac{1}{2}$ fields on the worldsheet.

Amplitudes are defined by additional functional integrations over ψ^μ and χ :

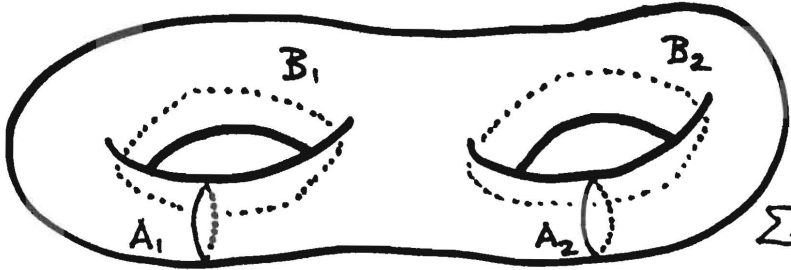
$$A = \sum_{\text{topologies}} \int De_m^a \sum_{\substack{\text{spin} \\ \text{structures}}} \int d\chi_m^\alpha \int D\psi^\mu \int D_n^\mu V_1 \cdots V_n e^{-I} \quad (6B.4)$$

The summation over spin structures only affects the spinor fields, and ψ^μ and χ_m^α have (in principle) the same spin structure. We shall see shortly though that their holomorphic and antiholomorphic components have different spin structures, which must however be correlated as follows:

$$\begin{array}{lll} \psi_+^\mu & \text{same spin structure as} & \chi_{\bar{z}}^+ \\ \psi_-^\mu & \text{same spin structure as} & \chi_z^- \end{array} \quad (6B.5)$$

to make the Lagrangian density single valued.

Spin structures were defined only on the cylinder so far. On general Riemann surfaces, one looks for the first homology group $H^1(\Sigma)$, or the Abelian group of independent 1-cycles on Σ .



1-cycles = $2h$.

For every cycle, there is a 2-fold choice and we have 2^{2h} independent spin structures.

As we gauged away reparametrizations in the bosonic case, we can also gauge away local supersymmetry. However, exactly as some moduli parameters remained in the bosonic case, so here there are supermoduli.

$$\{\chi_m^\alpha\} = \{\gamma_m \lambda\} \oplus \{P_{1/2} \xi\} \oplus \hat{\chi}_m^\alpha \quad (6B.6)$$

Where $P_{1/2}$ is the γ - traceless covariant derivative on ξ ; and $\hat{\chi}_m^\alpha$ may be viewed as dual to $\text{Ker } P_{1/2}^\dagger$

$$\begin{array}{ll} \dim_{\mathbf{R}} \text{Ker } P_{1/2}^\dagger = 2 & h \leq 2 \\ = 0 & h = 1 \\ = 0 & h = 0 \end{array} \quad (6B.7)$$

From the change of integration from χ_m^α to λ, ξ and $\hat{\chi}_m^\alpha$, we retain the fol-

lowing infinite-dimensional determinants

$$\left(\det' P_1^\dagger P_1\right)^{1/2} \left(\det' P_{1/2}^\dagger P_{1/2}\right)^{-1/2} \quad (6B.8)$$

It is now easy to deduce the critical dimension. Recall the Weyl anomaly for each

$$c_n = \pm 2(6n^2 + 6n + 1) \quad (6B.9)$$

Hence we have the following central charges

$$\begin{array}{ll} P_1 & -26 \\ P_{1/2} & +11 \\ \psi^\mu & +d/2 \\ x^\mu & +d \end{array} \quad (6B.10)$$

Adding up these contributions, we get $\frac{3}{2}(d-10)$, which vanishes for the critical dimension $d=10$. So, this dimension plays the same role as 26 did in the bosonic string. We shall see however that for the fermionic string, there are a large number of additional special features that occur when the space-time dimension is 10, and they are related to the nature of spinors in ten dimensions.

In addition of course, there are these finite dimensional determinants etc. For these and for the infinite-dimensional determinants, one may introduce ghosts

$$\begin{array}{llll} b, c & \text{anticommuting} & c(-1, 0) & b(1, 0) \\ \beta, \gamma & \text{commuting} & \gamma\left(\frac{1}{2}, 0\right) & \beta\left(\frac{3}{2}, 0\right). \end{array} \quad (6B.11)$$

The associated supersymmetric action is

$$I_{gh} = \frac{1}{2\hbar} \int_{\Sigma} (b\nabla_{\bar{z}}c + \bar{b}\nabla_{\bar{z}}\bar{c} + \beta\nabla_{\bar{z}}\gamma + \bar{\beta}\nabla_z\bar{\gamma} + \chi_{\bar{z}}^+ S_z^- + \chi_z^- S_{\bar{z}}^+) \quad (6B.12)$$

where the ghost supercurrent is

$$S_z^- = \frac{1}{2}b\gamma - \frac{3}{2}\beta D_z c - (D_z \beta)c. \quad (6B.13)$$

In principle, one may now use this action and functional integrals to work out scattering amplitudes as one has done for the case of the bosonic string. This will

work fine for tree level amplitudes and some simple one-loop amplitudes as well. However, if one wants to develop a systematic loop expansion to all orders, fix a gauge for all the local symmetries and define appropriate moduli and supermoduli in a global fashion, it will turn out that the component formulation is awkward and confusing. Things are much more natural from a geometrical point of view in the superfield formulation, which we shall now briefly discuss.

C. Superfield formulation of N=1 supergravity

Locally, $N = 1$ superspace is parametrized by two real $\xi^m = (\xi^1, \xi^2)$ or one complex coordinate $\xi = \frac{1}{\sqrt{2}}(\xi^1 + i\xi^2)$ and two real odd coordinates $\theta^\mu = (\theta^1, \theta^2)$ or one complex odd coordinate $\theta = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2)$ and its complex conjugate $\bar{\theta}$. These coordinates are collected into one supercoordinate $z^M = (\xi, \bar{\xi}; \theta, \bar{\theta})$, where the index M is a coordinate or Einstein index. We will also use a local Lorentz frame with indices $A = (z, \bar{z}; +, -)$ where z and $\bar{z}$ refer to the vector representation of the $U(1)$ frame group and $+$ and $-$ refer to the spinor representation. The corresponding lower case latin and greek letters correspond to the even and odd parts of these coordinates respectively.

The $N = 1$ supergravity multiplet consists of the superzweibein E_M^A , and the $U(1)$ superconnection Ω_M , from which a $U(1)$ covariant super-derivative $\mathcal{D}_M$ may be constructed. When this derivative acts on $U(1)$ tensors V of weight n , it is given by

$$\mathcal{D}_M^n V = \partial_M V + in\Omega_M V \quad (6C.1)$$

We shall mostly be using the covariant derivatives with $U(1)$ indices $\mathcal{D}_A^n = E_A^M \mathcal{D}_M^n$, because they are manifestly super-reparametrization invariant. Torsion T_{BC}^A and curvature R_{AB} tensors are defined by

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}^C \mathcal{D}_C + inR_{AB} \quad (6C.2)$$

The super-geometry may be specified by imposing the standard torsion constraints

$$T_{ab}^c = T_{\alpha\beta}^\gamma = 0 \quad T_{\alpha\beta}^c = 2(\gamma^c)_{\alpha\beta} \quad (6C.3)$$

Another way of looking at the latter constraints is that they entirely specify the commutation relations between $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$, so that with the help of (6C.2) all components of torsion and curvature can be computed in terms of the single scalar superfield R_{+-} .

The super-geometry is invariant under transformations which preserve the torsion constraints. We list them below, in terms of infinitesimal changes in the fields $H_A{}^B$

$$H_A{}^B = E_A{}^M \delta E_M{}^B \quad (6C.4)$$

The symmetries of the super-geometry are

(i) Local U(1) transformations, forming a group $sU(1)$ are generated by a real superfield L acting by

$$\delta E_M{}^A = -E_M{}^B E_B{}^A \delta L \quad H_A{}^B = -\delta L E_A{}^B \quad (6C.5)$$

(ii) Super-reparametrizations, forming a group $sDiff(M)$. The infinitesimal ones are generated by super vector fields δV^M and are given by

$$H_A{}^B = \mathcal{D}_A \delta V^B - \delta V^C T_{CA}{}^B + \delta V^C \Omega_C E_A{}^B \quad (6C.6)$$

(iii) Super Weyl transformations, forming a group $sWeyl(M)$. These are generated by a real scalar superfield Σ

$$\begin{aligned} H_a{}^b &= \delta \Sigma \delta_a{}^b & H_\alpha{}^\beta &= \frac{1}{2} \delta \Sigma \delta_\alpha{}^\beta \\ H_\alpha{}^a &= 0 & H_a{}^\alpha &= (\gamma_a)^\alpha{}^\beta \mathcal{D}_\beta \delta \Sigma . \end{aligned}$$

By analogy with two dimensional geometry we introduce a supercomplex structure

$$J_M{}^N = E_M{}^a \epsilon_a{}^b E_b{}^N + E_M{}^\alpha (\gamma_5)_\alpha{}^\beta E_\beta{}^N \quad (6C.7)$$

which is a super-reparametrization tensor, and a local U(1) scalar. The main properties of $J_M{}^N$ are

$$J_M{}^N J_N{}^P = -\delta_M{}^P \quad (6C.8)$$

and the fact that it depends only on the *super-conformal class* of $E_A{}^M$, i.e., it is invariant under the super-Weyl transformations of (iii). When the torsion constraints are satisfied, this almost complex structure becomes an integrable complex structure.

Flat $N=1$ superspace is given by the following superzweibein

$$E_m^a = \delta_m^a \quad E_m^\alpha = 0 \quad E_\mu^a = (\gamma^a)_\mu{}^\beta \theta_\beta \quad E_\mu^\alpha = \delta_\mu^\alpha \quad (6C.9)$$

and the superderivatives take the simple form

$$\mathcal{D}_+ = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \quad \mathcal{D}_- = \frac{\partial}{\partial \bar{\theta}} + \bar{\theta} \frac{\partial}{\partial \bar{z}} \quad (6C.10)$$

Locally every supergeometry is locally superconformal (i.e. equivalent under a super-Weyl and local $U(1)$ transformation) to flat supergeometry. Globally however, there may be topological obstructions, and it will be necessary to introduce supermoduli space, i.e. the space of inequivalent superconformal structures.

To obtain a better understanding of super-geometry and supergravity, it should be useful to immediately discuss its associated component formulation. The passage from the superfield to the component language requires the elimination of the auxiliary fields required by the superfields. This is usually accomplished by fixing the Wess-Zumino gauge for the superzweiben. The results for their full θ expansions are for example

$$\begin{aligned} E_m^a &= e_m^a + \theta \gamma^a \chi_m - \frac{i}{2} \theta \bar{\theta} e_m^a A \\ E_\mu^\alpha &= \delta_\mu^\alpha (1 + i \theta \bar{\theta} A/4) \end{aligned} \quad (6C.11)$$

where e_m^a is the frame, χ_m^α the Rarita-Schwinger field introduced previously and A an auxiliary field which turn out to be the leading component of the curvature superfield R_{+-} . Notice that since E_μ^α is basically the Kronecker symbol between a μ and an α index, the distinction between $U(1)$ and Einstein spinor indices is lost in Wess-Zumino gauge, and θ may be written either with α or μ indices.

It is now straightforward to translate the symmetries of the superzweibein into component language as well. The super-reparametrizations relevant to the component language are the ones that preserve the Wess-Zumino gauge up to local $U(1)$ and super-Weyl transformations. They decompose into reparametrization invariance and an $N = 1$ supersymmetry. Super Weyl transformations will take us out of this gauge, so the component transformations written below are obtained only after compensation by a super-reparametrization and a local $U(1)$ transformation taking us back to Wess-Zumino gauge.

A good review on supergravity in four dimensions is Wess and Bagger (1983), besides the original papers by Freedman, Van Nieuwenhuizen and Ferrara (1976) and Deser and Zumino (1976). Two dimensional supergravity is discussed by Brink, di Vecchia and Howe (1976) and by Howe (1979).

D. The GSO Projections - Summation Over Spin Structures.

So far, we have constructed two complete sectors of the Ramond Neveu-Schwarz string : the Ramond sector with space-time spinor fermion states only, and the Neveu-Schwarz sector with space-time bosons only. In particular, the latter sector contains all the states of the bosonic closed string.

Now by construction, the bosonic closed critical string contained the graviton, antisymmetric tensor field and dilator, all massless, and these states are fine. However recall that it also contains the scalar tachyon. This tachyon gave rise for example to non-causal propagation and produced divergences already in the simplest one-loop amplitudes. In any physical, stable theory, a tachyon should not be present. One needs to find a mechanism or prescription for eliminating this excitation from the spectrum.

In a remarkable paper, Gliozzi, Scherk and Olive, (1976) proposed to put both sectors together, and to truncate the states:

- i. in the NS sector : keep states with odd G-parity (*i.e.* odd $\#\psi_r$'s)
- ii. in the R sector : restrict the ground state (fermion) to be Majorana-Weyl.

Notice that the truncation in the Ramond sector only makes sense in certain dimensions of space-time in which the Majorana and Weyl conditions can be enforced simultaneously. This is the case for dimensions of the type $d = 8k + 2$ for positive or zero integers k . We see from this point of view how the critical dimension for the RNS string is precisely a dimension in which the GSO truncation can be performed.

The restriction on the NS sector automatically eliminates the tachyon, whereas the full truncation produces equal numbers of bosonic and fermionic physical states at each mass level. GSO conjectured that this string theory has 10-dimensional supersymmetry. This was proven by Green and Schwarz (1981). Note that the above restrictions have to be made on the ψ_+ 's and ψ_- 's separately. When the chiralities of both Ramond sectors are the same, (opposite) we have Type II B (Type IIA) strings.

In the Riemann-surface formulation, we had seen for the cylinder that

Ramond	$\leftrightarrow$	periodic boundary conditions
Neveu-Schwarz	$\leftrightarrow$	antiperiodic boundary conditions

which are precisely the two spin structures on the cylinder. More generally, they are associated with $\pm$ assignments in $H^1(\Sigma)$. However, spin structures transform non-trivially under the group of large diffeomorphisms that transforms $H^1(\Sigma) \rightarrow H^1(\Sigma)$: the so-called modular group $Sp(2h, \mathbf{Z})$.

There are two types of spin structures:

even spin structure	$\gamma^a D_a \psi_+ = 0$	admits 0 (mod 2) solution
odd spin structure	$\gamma^a D_a \psi_+ = 0$	admits 1 (mod 2) solution

Clearly the number of solutions is modular invariant, so the modular group transforms even spin structures into even spin structures and odd ones into odd spin structures.

Since a consistent string theory must be modular invariant (a special case of $\text{Diff}(\Sigma)$ invariance), as soon as one spin structure is included, necessarily all the ones obtained by modular transformation must also be included. Thus, a modular invariant theory requires a summation over all spin structures.

But recall that in the GSO projection, we needed to sum independently over the ψ_+ 's and ψ_- 's, so one must do the same in the RNS formulation: sum independently over the spin structures of both worldsheet spinor chiralities. How this can be done in general is discussed in the next section.

E. Chiral Splitting and Holomorphicity on Moduli Space.

We need to be able to assign different spin structures to different world-sheet spinor chiralities. That means we have to be able to quantize opposite chiralities separately. But in Euclidean signature, two opposite chiralities are complex conjugates of one another, so should carry the same spin structure.

In fact, this issue is much more general, and results directly from the general structure of conformal field theory. Recall that for conformal field theory on closed oriented surfaces, we had duplication of Virasoro generators L_n and $\tilde{L}_n$, which mutually commute. So holomorphic and anti-holomorphic behavior almost entirely decouples, except for the constraint

$$L_0 - \tilde{L}_0 = 0$$

Unfortunately, there was not enough time to discuss these issues when we were dealing with conformal field theory in these lectures.

The key to the understanding of this issue is that the Cauchy-Riemann operators on the Riemann surface depend holomorphically on moduli. Recall that moduli space was a complex (orbifold), so we may split the Beltrami differentials

into analytic and anti-analytic directions

$$\begin{aligned}\mu_{\bar{z}}^z(m_j) &= \mu; & \mu_z^{\bar{z}}(\bar{m}_j) &= \bar{\mu} \\ \delta \nabla_{\bar{z}}^{(n)} &= \mu_{\bar{z}}^z \nabla_z^{(n)} + n(\nabla_z \mu_{\bar{z}}^z)\end{aligned}\tag{6E.1}$$

This provides essentially a factorized form to functional integrals:

$$\begin{aligned}I_S &= \frac{1}{4\pi} \int (\psi_+ \nabla_{\bar{z}}^{(\frac{1}{2})} \psi_+ + \psi_- \nabla_z^{(-\frac{1}{2})} \psi_-) \\ \int D\psi_+ D\psi_- e^{-I_S} &= \det \nabla_{\bar{z}}^\dagger \nabla_{\bar{z}} \sim \underbrace{\det \nabla_{\bar{z}}^\dagger}_{\bar{\mu}} \underbrace{\det \nabla_{\bar{z}}}_{\mu}\end{aligned}\tag{6E.2}$$

up to issues of zero modes. So one expects holomorphic factorization. Actually, Belavin and Knizhnik (1986) showed that if you want to do this splitting in a reparametrization invariant way, there is the so-called “holomorphic anomaly”

$$\delta_\mu \delta_{\bar{\mu}} \ell n \frac{\det' \nabla_{\bar{z}}^\dagger \nabla_{\bar{z}}^{(n)}}{(\text{zero modes})} = \frac{6n^2 - 6n + 1}{12\pi} \int \left(\nabla_z \bar{\mu} \nabla_{\bar{z}} \mu + 2R\mu\bar{\mu} \right)\tag{6E.3}$$

and we notice that the overall coefficient is precisely proportional to the central charge, as was the case for the Weyl anomaly. Actually, another way of looking at this problem is to investigate when the Lorentz anomaly cancels separately for left and right chiralities.

So, individually, one cannot quite factorize these determinants, but if we put together the $\beta - \gamma$ ghost and $b - c$ ghost ψ fields and x^μ determinants as well, then we can split in a reparametrization invariant way:

$$(\det P_1^\dagger P_1)^{1/2} (\det P_{1/2})_\nu^{-1/2} (\det' \not{D})_\nu^5 (\det' \Delta)^{-10} = |Z_\nu^o(m_j)|^2\tag{6E.4}$$

and we now have a concept of *chirally split* amplitude in a (local) reparametrization invariant way. The chiral quantity

$$Z_\nu^o(m_j)\tag{6E.5}$$

is defined up to a phase and this phase is independent of moduli, but dependent on on the spin structure in general in view of the modular transformation properties.

Now, there is more than determinants; the action reads

$$I_m = \frac{1}{4\pi} \int \left[\partial_z x^\mu \partial_{\bar{z}} x^\mu + \psi_+^\mu \partial_{\bar{z}} \psi_+^\mu + \psi_-^\mu \partial_z \psi_-^\mu + \psi_+^\mu \chi_{\bar{z}}^+ \partial_z x^\mu + \psi_-^\mu \chi_z^- \partial_{\bar{z}} x^\mu - \frac{1}{2} \chi_+^\mu \psi_-^\mu \chi_{\bar{z}}^+ \chi_z^- \right] \quad (6E.6)$$

and we see that the last term for example is not chirally split. It would seem that the nice chiral splitting properties exhibited at the cost of great pain for determinants would be lost for the full Green function contributions. However, contraction of other chiral terms yield contact terms that cancel the above obstruction to chiral splitting :

$$\langle \psi_+^\mu \chi_{\bar{z}}^+ \partial_z x^\mu \psi_-^\mu \chi_z^- \partial_{\bar{z}} x^\mu \rangle_x \partial_z \partial_{\bar{z}} \langle x^\mu(z) x^\mu(z') \rangle = 2\pi \delta(z - z') - \frac{1}{\int_\Sigma \sqrt{g}} \quad (6E.7)$$

This type of cancellation was first noticed by D'Hoker and Phong (1986). Further mixings arise due to zero modes as well, but proper account of all contact terms shows that such intermediate contributions can be put in a chirally split form for any given set of fixed loop momenta. The full amplitude is then an integral over all loop momenta as well.

The approach based on chiral splitting was initiated in D'Hoker and Phong (1988) and presented in full in D'Hoker and Phong (1990).

F. Heterotic Strings

For *oriented* strings, like Type II, A, B, or the bosonic string, we have essentially independent degrees of freedom associated with holomorphic and antiholomorphic coordinates on the Riemann surface. (left-right movers) Already for Type II Strings, we saw that GSO projection requires independent spin structures for both chiralities. The heterotic string is constructed as a cross bread between these two strings in the following way

$$\text{Heterotic String} \left\{ \begin{array}{l} \text{Type II for left movers} \\ x^\mu, \psi_+^\mu, \mu = 1, \dots, 10 \\ \text{Bosonic for right movers} \\ x^\mu, x^I, I = 16, \dots, 26. \\ (x^I \text{ "are chiral bosons"}). \end{array} \right. \quad (6F.1)$$

The internal degrees x^I are compactified on a 16-dimensional torus and by fermionization are equivalent to ψ_-^A $A = 1, \dots, 32$, Majorana-Weyl internal fermions. In

each sector, we need Weyl invariance which is equivalent to combined Weyl and $U(1)$ local frame rotation invariance. As a result we obtain the following critical dimensions :

$$\begin{cases} \text{left : like Type II in 10 dimension.} \\ \text{right : like the bosonic in 26 dimension.} \end{cases} \quad (6F.2)$$

However, since we have introduced chiral fermions, there is an additional requirement that is analogous to the summation over spin structures that we had in the Type II strings. Indeed, chiral fermions exhibit global anomalies under modular transformations. For the case of the torus and the chiral Dirac operator, we have

$$\begin{aligned} \det \partial_{\bar{z}} \Big|_{\nu} &= \frac{\vartheta_{\nu}[o, \tau]}{\eta(\tau)} & \nu \text{ even} \\ \det {}'\partial_{\bar{z}} \Big|_{\nu \text{ odd}} &= \eta(\tau)^2 \end{aligned} \quad (6F.3)$$

Combining these contributions with those for $b, c, \beta, \gamma, x^{\mu}, \psi_{+}^{\mu}$ fields, in the left-moving sector, we find that

$$\eta(\tau)^2 \eta^{-10}(\tau) \left(\frac{\vartheta_{\nu}(o|\tau)}{\eta(\tau)} \right)^{-1} \left(\frac{\vartheta_{\nu}(o|\tau)}{\eta(\tau)} \right)^5 = \frac{\vartheta_{\nu}(o|\tau)}{\eta(\tau)^{12}} \quad (6F.4)$$

$$\vartheta_{\nu}(o|\tau + 1) = \vartheta_{\nu'}(o|\tau) \epsilon \quad \epsilon^8 = 1.$$

On the other hand, the contributions of b, c, x^{μ}, ψ^A of the right-moving sector give rise to

$$\eta(\tau)^2 \eta^{-10}(\tau) \left(\frac{\vartheta_{\nu_1}(o, \tau)}{\eta(\tau)} \right)^{A_1} \left(\frac{\vartheta_{\nu_2}(o|\tau)}{\eta(\tau)} \right)^{A_2} \left(\frac{\vartheta_{\nu_3}(o|\tau)}{\eta(\tau)} \right)^{A_3} \quad (6F.5)$$

with $A_1 + A_2 + A_3 = 16$. where, under modular transformations, φ transforms with 8-th root of unity so that each of the A_i must be divisible by 8 to obtain a modular invariant answer.

This gives rise to two possible choices : The first where one $A_i = 16$, the others being zero; in this case, all the internal fermions have always the same spin structure, and hence the internal sector exhibits a $O(32)$ symmetry. (Actually, the representation content is that of a slightly different group : $\text{Spin}(32)/Z_2$.) The second choice is where two of the $A = 8$, and then the manifest symmetry group of the internal fermions is $O(16) \times O(16)$, by the same reasoning as before. However, there are a number of non-linearly realized hidden symmetries, which enhance the full symmetry to $E_8 \times E_8$.

From the point of view of space-time physics, the Type II superstrings reduce to $N=2$, $d=10$ supergravity in the low energy limit, as was established by Green, Schwarz and Brink (1984). For Type IIA, this supergravity is non-chiral, whereas for Type IIB, it is chiral. Both heterotic strings give rise to $N=1$, $d=10$ supergravity-super Yang-Mills theories.

Heterotic strings were discovered and discussed by Gross, Harvey, Martinec and Rohm (1985-86).

G. Tree Level Amplitudes

To tree level, the worldsheet is the super-sphere, or by super-conformal mapping the super plane. Since super conformal invariants are perhaps not so familiar, we shall start by examining them first.

The superconformal invariance group is isomorphic to complexified $OSp(1,1)$ — the superconformal extension of $SL(2, \mathbb{C})$. This group of transformations is defined as follows; consider the matrices

$$K = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & A \end{pmatrix} \quad (6G.1)$$

$$z \longrightarrow \frac{az + b + \alpha\theta}{cz + d + \beta\theta} \quad \theta \longrightarrow \frac{\gamma z + \delta + A\theta}{cz + d + \beta\theta} \quad (6G.2)$$

where latin entries are commuting and greek entries are anticommuting. To obtain a *superconformal* transformation T , the line element $d\mathbf{z} = dz + \theta d\theta$ must be transformed into itself up to a conformal scaling. Equivalently “the quadratic form”

$$z_{12} = z_1 - z_2 - \theta_1\theta_2 = \frac{v_1w_2 - v_2w_1 - \psi_1\psi_2}{w_1w_2} \quad (6G.3)$$

should transform into itself up to a conformal scaling. This is uniquely achieved when the orthosymplectic form is left invariant under T :

$$T^T K T = K \quad (6G.4)$$

The weight under which the difference transforms is easily derived, and we have

$$T : z_{12} \longrightarrow \tilde{z}_{12} = \frac{z_{12}}{(cz_1 + d + \beta\theta_1)(cz_2 + d + \beta\theta_2)} \quad (6G.5)$$

Similarly the line element transforms as

$$dz \longrightarrow d\tilde{z} = \frac{dz}{(cz + d + \beta\theta)^2} \quad (6G.6)$$

and the volume element as

$$dz \wedge d\theta \longrightarrow \frac{dz \wedge d\theta}{(cz + d + \beta\theta)} \quad (6G.7)$$

Elements in $OSp(1,1)$ are in unique correspondence with a triplet of points in the superplane (z_1, θ_1) , (z_2, θ_2) , (z_3, θ_3) obeying one single (Grassmann valued) constraint. The counting works out because $OSp(1,1)$ has 3 commuting and 2 anticommuting parameters. The constraint is an $OSp(1,1)$ invariant Grassmann valued function dependent on three points, given by

$$\Delta = \frac{z_{12}\theta_3 + z_{31}\theta_2 + z_{23}\theta_1 + \theta_1\theta_2\theta_3}{(z_{12}z_{23}z_{31})^{1/2}} \quad (6G.8)$$

The natural value for Δ is of course 0, which implies that one θ is dependent. With this value for Δ , it is easy to see that there is a unique correspondence between triplets of points satisfying $\Delta = 0$, and elements of $OSp(1,1)$, so that the latter may be accordingly parametrized. The invariant volume element induced on $OSp(1,1)$ is then

$$d\mu = \frac{dz_1 dz_2 dz_3 d\theta_1 d\theta_2 d\theta_3}{(z_{12} z_{23} z_{31})^{1/2}} \Delta \quad (6G.9)$$

As for the bosonic string, here the volume of the super conformal transformation group must be extracted from the amplitudes.

The propagator on the super plane is very simple :

$$G(\mathbf{z}, \mathbf{z}') = -\ln(|z - z' - \theta\theta'|^2 + \epsilon^2) \quad (6G.10)$$

and ϵ is understood to be infinitesimal. The simplest possible physical vertex operator is the one for bosonic particles at zero mass level $k^2 = 0$

$$V(\epsilon, k) = g \epsilon_{\mu; \bar{\mu}} \int d^2\mathbf{z} E \mathcal{D}_+ X^\mu \mathcal{D}_- X^{\bar{\mu}} e^{i\mathbf{k} \cdot \mathbf{X}} \quad (6G.11)$$

describing the graviton, antisymmetric tensor field and the dilaton. The polarization tensor $\epsilon_{\mu; \bar{\mu}}$ is understood to be transverse in k , and the vertex is effectively

normal ordered. To compute correlation functions of several of these vertices, it is useful to recall a trick known from the bosonic string. It consists of introducing a source for both X and its derivatives, and then isolating the correct expansion coefficient when developing in powers of the source. The key observation is that we may *formally* write $\epsilon_{\mu;\bar{\mu}} = \zeta_{\mu}\bar{\zeta}_{\bar{\mu}}$, where ζ_{μ} and $\bar{\zeta}_{\bar{\mu}}$ are Grassmann-valued vectors. By linearity of any amplitude in the $\epsilon_{\mu;\bar{\mu}}$'s, clearly any $\epsilon_{\mu;\bar{\mu}}$ can be written as a formal sum, but we shall not explicitly need this construction. Once this has been done, we may introduce a generalized vertex

$$V^*(\zeta, \bar{\zeta}, k) = g \int d^2z E e^{ik \cdot X + \zeta \cdot \mathcal{D}_+ X + \bar{\zeta} \cdot \mathcal{D}_- X} \quad (6G.12)$$

whose $\zeta\bar{\zeta}$ coefficient is precisely $V(\epsilon, k)$. Thus we shall perform our calculations on V^* , introducing a different set of $\zeta\bar{\zeta}$'s for every ϵ of V , and selecting the correct term in the expansion in ζ 's.

We thus calculate the n vertex correlation function starting from the V^* operators

$$\langle V^*(\zeta_1, \bar{\zeta}_1, k_1) \cdots V^*(\zeta_n, \bar{\zeta}_n, k_n) \rangle = g^n \int d^2z_1 \cdots d^2z_n \langle \exp \int d^2z J^{\mu}(z) X_{\mu}(z) \rangle \quad (6G.13)$$

where the source can be read off from the definition of V^* :

$$J^{\mu}(z) = \sum_{i=1}^n (ik_i^{\mu} + \zeta_i^{\mu} \mathcal{D}_+^i + \bar{\zeta}_i^{\mu} \mathcal{D}_-^i) \delta^2(z, z_i) \quad (6G.14)$$

By completing the square in the expectation value, we get

$$\langle \exp \int d^2z J^{\mu}(z) X_{\mu}(z) \rangle = (2\pi)^{10} \delta(k) \exp \left[\mathcal{G}_n - \frac{1}{2} \sum_{i \neq j=1}^n k_i \cdot k_j G(z_i, z_j) \right] \quad (6G.15)$$

Here the terms with $i = j$ are independent of momenta k and of the coordinates z_i . Their contribution is absorbed into an overall normalization factor for each vertex, which will be omitted here.

$$\begin{aligned} \mathcal{G}_n = \sum_{i \neq j=1}^n \{ & -ik_i \cdot \zeta_j \mathcal{D}_+^j - ik_i \cdot \bar{\zeta}_j \mathcal{D}_-^j - \frac{1}{2} \zeta_i \cdot \zeta_j \mathcal{D}_+^i \mathcal{D}_+^j \\ & - \frac{1}{2} \bar{\zeta}_i \cdot \bar{\zeta}_j \mathcal{D}_-^i \mathcal{D}_-^j - \frac{1}{2} \zeta_i \cdot \bar{\zeta}_j \mathcal{D}_+^i \mathcal{D}_-^j - \frac{1}{2} \bar{\zeta}_i \cdot \zeta_j \mathcal{D}_-^i \mathcal{D}_+^j \} G(z_i, z_j) \end{aligned} \quad (6G.16)$$

For tree level amplitudes, we work on the superplane and we use the Green function

of (6G.10). Thus we have (with $\theta_{ij} = \theta_i - \theta_j$)

$$\mathcal{G}_n = \sum_{ij=1}^n \left\{ +ik_i \cdot \zeta_j \frac{\theta_{ij}}{z_{ij}} + ik_i \cdot \bar{\zeta}_j \frac{\bar{\theta}_{ij}}{\bar{z}_{ij}} + \frac{1}{2} \zeta_i \cdot \zeta_j \frac{1}{z_{ij}} + \frac{1}{2} \bar{\zeta}_i \cdot \bar{\zeta}_j \frac{1}{\bar{z}_{ij}} \right\} \quad (6G.17)$$

We now work out the three point amplitude first and separate $\mathcal{G}_n$ as a function of ζ 's and $\bar{\zeta}$'s: $\mathcal{G}_n = \mathcal{G}_n^\zeta + \mathcal{G}_n^{\bar{\zeta}}$. In evaluating $\exp \mathcal{G}_n^\zeta$, one retains terms proportional to $\zeta_1 \zeta_2 \zeta_3$; however the term with three θ 's vanishes because $\theta_{12} \theta_{23} \theta_{31} = 0$. Defining the tensors

$$K_{\mu_1 \mu_2 \mu_3} = \eta_{\mu_1 \mu_2} k_{1\mu_3} + \eta_{\mu_2 \mu_3} k_{2\mu_1} + \eta_{\mu_3 \mu_1} k_{3\mu_2}$$

we find for the three point function

$$\langle V(\epsilon_1, k_1) V(\epsilon_2, k_2) V(\epsilon_3, k_3) \rangle = 4(2\pi)^{10} \delta(k) \epsilon_1^{\mu_1 \bar{\mu}_1} \epsilon_2^{\mu_2 \bar{\mu}_2} \epsilon_3^{\mu_3 \bar{\mu}_3} K_{\mu_1 \mu_2 \mu_3} K_{\bar{\mu}_1 \bar{\mu}_2 \bar{\mu}_3} \quad (6G.18)$$

To compute the four point amplitude, we shall fix superconformal invariance as follows : $z = z_1, z_2 = 0, z_3 = 1, z_4 = \infty, \theta_1, \theta_2, \theta_3 = \theta_4 = 0$, and then have

$$\begin{aligned} \exp \mathcal{G}_4^\zeta = & \left\{ \zeta_1 \cdot \zeta_2 \zeta_3 \cdot \zeta_4 \frac{1}{z_{12} z_{34}} + \zeta_1 \cdot \zeta_3 \zeta_2 \cdot \zeta_4 \frac{1}{z_{13} z_{24}} + \zeta_1 \cdot \zeta_4 \zeta_2 \cdot \zeta_3 \frac{1}{z_{14} z_{23}} \right\} \\ & + \left\{ \zeta_1 \cdot \zeta_2 \left(k_1 \cdot \zeta_3 k_2 \cdot \zeta_4 \frac{\theta_2 \theta_1}{z_{12} z_{13} z_{24}} + k_1 \cdot \zeta_4 k_2 \cdot \zeta_3 \frac{\theta_2 \theta_1}{z_{12} z_{14} z_{23}} \right) + \text{perm.} \right\} \end{aligned} \quad (6G.19)$$

In principle, one should now multiply this whole expression by the one involving the $\bar{\zeta}$'s, perform the integrals over z and θ and regroup terms, clearly a feudal task. The calculation is enormously simplified by the factorization properties of the Veneziano-integrals. Recall that we have the ordinary integrals

$$\int \frac{d^2 z}{\pi} z^A \bar{z}^{\tilde{A}} (1-z)^B (1-\bar{z})^{\tilde{B}} = \frac{\Gamma(-1-\tilde{A}-\tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})} \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \quad (6G.20)$$

provided $A - \tilde{A}$ and $B - \tilde{B}$ are integers, which is always the case in string theory. Using the reciprocity formula for Γ -functions and the fact that $A - \tilde{A}$ and $B - \tilde{B}$ are integers, this expression is actually symmetric under $(A, B) \leftrightarrow (\tilde{A}, \tilde{B})$, as one might expect from complex conjugation. More importantly, the answer factorizes into a product of factors each only dependent either on the parameters for the z or $\bar{z}$ coordinates. This product property implies that one must only consider say the z coordinates to find the full amplitude, which by the same token will also

completely factorize as a function of ζ 's and $\bar{\zeta}$'s. An analogous formula is derived for the super-integrals we need:

$$\begin{aligned} \int \frac{d^2 \mathbf{z}_1}{\pi} d^2 \theta_2 [\theta_1 \theta_2]^a [\bar{\theta}_1 \bar{\theta}_2]^{\tilde{a}} z_{12}^A \bar{z}_{12}^{\tilde{A}} (1 - z_1)^B (1 - \bar{z}_1)^{\tilde{B}} \\ = (-2i)^{1-a} (+2i)^{1-\tilde{a}} \frac{\Gamma(-\tilde{a} - \tilde{A} - \tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})} \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+1+a)} \end{aligned} \quad (6G.22)$$

Here a and $\tilde{a}$ are either 0 or 1, and the integrals are symmetric under $(aAB) \leftrightarrow (\tilde{a}\tilde{A}\tilde{B})$. With the help of (6G.22), it is now straightforward to evaluate the four point function

$$\begin{aligned} \langle V(\epsilon_1, k_1) V(\epsilon_2, k_2) V(\epsilon_3, k_3) V(\epsilon_4, k_4) \rangle \\ = (2\pi)^{10} \delta(k) g^4 \int d^2 \mathbf{z}_1 d^2 \theta_2 |z_{12}|^{-s} |z_1 - 1|^{-u} e^{g_4^\zeta + g_4^{\bar{\zeta}}} \\ = \pi (2\pi)^{10} \delta(k) g^4 \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma(1+\frac{s}{2})\Gamma(1+\frac{t}{2})\Gamma(1+\frac{u}{2})} \epsilon^{1\bar{1}} \epsilon^{2\bar{2}} \epsilon^{3\bar{3}} \epsilon^{4\bar{4}} K_{1234} K_{\bar{1}\bar{2}\bar{3}\bar{4}} \end{aligned} \quad (6G.23)$$

Using the abbreviation i for μ_i to save some writing we have $K_{\mu_1 \mu_2 \mu_3 \mu_4} = K_{1234}$ and K is then given by

$$\begin{aligned} K_{1234} = (st \eta_{13} \eta_{24} - su \eta_{14} \eta_{23} - tu \eta_{12} \eta_{34}) \\ - s (k_1^4 k_3^2 \eta_{24} + k_2^3 k_4^1 \eta_{13} - k_1^3 k_4^2 \eta_{23} - k_2^4 k_3^1 \eta_{14}) \\ + t (k_2^1 k_4^3 \eta_{13} + k_3^4 k_1^2 \eta_{24} - k_2^4 k_1^3 \eta_{34} - k_3^1 k_4^2 \eta_{12}) \\ - u (k_1^2 k_4^3 \eta_{23} + k_3^4 k_2^1 \eta_{14} - k_1^4 k_2^3 \eta_{34} - k_3^2 k_4^1 \eta_{12}) \end{aligned} \quad (6G.24)$$

We conclude this subsection by remarking that by superconformal invariance, the zero, one and two point functions of the superstring all vanish. The fastest way of obtaining this result is by remarking that $SL(2, C)$ is a subgroup of the superconformal group, and that the respective subgroups leaving 0, 1 or 2 points fixed all have infinite volume, so that the amplitudes vanish.

H. One loop amplitudes for the type II superstring

On the torus, there are 4 spin structures, one odd corresponding to periodic $\times$ periodic boundary conditions for all worldsheet spinors, and three even spin structures, containing at least one anti-periodic boundary condition. For even spin structure, there is one complex modulus and one complex conformal Killing vector. For odd spin structure, there is in addition an odd modulus and a complex

conformal Killing spinor. It will be convenient to represent a spin structure by its corresponding characteristics $\nu = (a, b)$. Here a and b take the value 0 or 1 according to whether the boundary conditions are anti-periodic or periodic respectively about A and B cycles. Left and right chiralities will be endowed with separate spin structures ν and $\bar{\nu}$. Thus, it is appropriate to decompose the one loop amplitude as follows

$$\langle V_1 \dots V_n \rangle = \sum_{\nu \bar{\nu}} C_{\nu \bar{\nu}} \langle V_1 \dots V_n \rangle_{\nu \bar{\nu}} \quad (6H.1)$$

The presence of conformal Killing vectors and spinors requires the insertion of the ghost c and the superghost $\delta(\gamma_0)$ where γ_0 is the zero mode (for odd spin structure).

We shall now evaluate this expression for the case of bosonic vertex operators. Then, the vertex operators are independent of the ghosts and this integral may be performed separately. Both ghost chiralities may be integrated over independently, and one recovers the formulas derived earlier. For even spin structure ν

$$\mathcal{A}_{sgb} = \int_{\nu} D(bc\beta\gamma) bce^{-I_{sgb}} = \left(\frac{1}{2\tau_2^2} \det' \nabla_{-1}^z \right) (\det \nabla_{-\frac{1}{2}}^z)^{-1}_{\nu} \quad (6H.2)$$

whereas for odd spin structure this expression is just $\mathcal{A}_{sgb} = 1$

It is straightforward to evaluate

$$\frac{1}{2\tau_2^2} \det' \nabla_{-1}^z = \frac{1}{2} \eta(\tau)^2 \quad (6H.3)$$

and for even spin structure ν

$$(\det \nabla_{-\frac{1}{2}}^z)_{\nu} = \frac{\vartheta[\nu](0, \tau)}{\eta(\tau)} \quad (6H.4)$$

Notice that the superghost part of the amplitude is independent of the supermodulus χ .

Next, we evaluate the matter contribution, and again use the results of chiral splitting, as in D'Hoker and Phong (1988). Recall that in principle all vertex insertions for bosonic external particles could be obtained from the insertion of (unintegrated) exponential factors. Thus, it is best to evaluate these first since

they are simplest. Consider the amplitude

$$\mathcal{A}_m = \int D(x\psi) \prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i, \theta_i)} e^{-I_m} \quad (6H.5)$$

where $X^\mu = x^\mu + \theta\psi_+^\mu + \bar{\theta}\psi_-^\mu$ and I_m is the matter action in components. It is implicit that left and right spin structures are fixed to be ν and $\bar{\nu}$. We have

$$\begin{aligned} \mathcal{A}_m = (2\pi)^{10} \delta(k) & \left(\frac{4\pi^2 \det' \Delta}{(\text{Im}\tau)^2} \right)^{-5} \left(\frac{\det' \not{D}_+}{\langle h|h \rangle} \right)_\nu \left(\frac{\det' \not{D}_-}{\langle h|h \rangle} \right)_{\bar{\nu}}^5 \\ & \times \int_{\text{Im}} d^5 p^\mu \mathcal{F}'_\nu(z_i, \theta_i, \tau; p^\mu) \overline{\mathcal{F}'_{\bar{\nu}}(z_i, \theta_i, \tau; p^\mu)} \end{aligned} \quad (6H.6)$$

where the reduced chiral amplitude $\mathcal{F}_\nu$ is given by

$$\mathcal{F}_\nu = \exp\{ip^\mu \tau p^\mu + i2\pi p^\mu \sum_i k_i^\mu z_i - \sum_{i < j} k_i \cdot k_j G_\nu(z_i, z_j)\} \quad (6H.7)$$

The full propagator for the X -field is simply related to G_ν :

$$\begin{aligned} G_{\nu\bar{\nu}}(z, w) &= \langle X(z) X(w) \rangle \\ &= G_\nu(z, w) + \overline{G_{\bar{\nu}}(z, w)} - \frac{\pi}{2\tau_2} (z - w - \bar{z} + \bar{w})^2 \end{aligned} \quad (6H.8)$$

where the “chiral X -propagator” is defined by

$$\begin{aligned} G_\nu(z, w) &= G_1(z, w) + \theta_z \theta_w S_\nu(z, w) \\ G_1(z, w) &= -\ln E(z, w) \end{aligned} \quad (6H.9)$$

It may be useful to also recall that the prime form E takes on a simple form for the torus

$$E(z, w) = \frac{\vartheta_1(z - w, \tau)}{\vartheta_1'(0, \tau)} \quad (6H.10)$$

On the other hand, $S_\nu(z, w)$ is the Dirac propagator. For even spin structure, this is given by the Szegő kernel

$$S_\nu(z, w) = \frac{\vartheta[\nu](z - w, \tau) \vartheta_1'(0, \tau)}{\vartheta[\nu](0, \tau) \vartheta_1(z - w, \tau)} \quad (6H.11)$$

For odd spin structure, the Dirac zero mode contributes, however since they are linear in χ , each of them can only be contracted with the exponential insertion.

The ψ_+ propagator $\tilde{S}_\nu$, orthogonal to the constant zero mode of $\mathcal{D}$ is given by

$$\tilde{S}_0(z, w) = \frac{\vartheta'_1(z - w, \tau)}{\vartheta_1(z - w, \tau)} - \frac{\pi}{\tau_2}(z - w - \bar{z} + \bar{w}) \quad (6H.12)$$

It is easy to see that this is a well defined function on the torus.

Putting all together, considerable simplifications occur, and we have for the full amplitude $\mathcal{A} = \mathcal{A}_m \times \mathcal{A}_{sgb} \times \overline{\mathcal{A}_{sgb}}$:

$$\begin{aligned} \mathcal{A} &= \int D(x\psi bc\beta\gamma) \prod_{i=1}^n e^{ik_i^\mu X^\mu(z_i, \theta_i)} \mathcal{I}_\nu \bar{\mathcal{I}}_{\bar{\nu}} e^{-I} \\ &= (2\pi)^{10} \delta(k) M_\nu \bar{M}_{\bar{\nu}} \int_{\text{Im}} dp^\mu \mathcal{F}_\nu \bar{\mathcal{F}}_{\bar{\nu}} \end{aligned} \quad (6H.13)$$

For even spin structure, we have

$$M_\nu = \frac{\vartheta[\nu](0, \tau)^4}{\eta(\tau)^{12}} \quad (6H.14)$$

whereas for odd spin structure $M_{(1,1)} = 1$.

Modular invariance

We now discuss the coefficients $C_{\nu\bar{\nu}}$ occurring in the summation over spin structures. $G_{\nu\bar{\nu}}$ is manifestly modular covariant. The only effect of a modular transformation on $G_{\nu\bar{\nu}}$ is to permute the spin structure according to the modular group.

$$\begin{aligned} G_{\nu\bar{\nu}}(z - z', \theta\theta'; \tau + 1) &= G_{\nu_1\bar{\nu}_1}(z - z', \theta\theta'; \tau) \\ G_{\nu\bar{\nu}}\left(\frac{z - z'}{\tau}, \frac{\theta\theta'}{\tau}, -\frac{1}{\tau}\right) &= G_{\nu_\tau\bar{\nu}_\tau}(z - z', \theta\theta'; \tau) \end{aligned} \quad (6H.15)$$

where

$$\begin{aligned} \nu_1 &= (a, b + a + 1) & \bar{\nu}_1 &= (\bar{a}, \bar{b} + \bar{a} + 1) \\ \nu_\tau &= (b, a) & \bar{\nu}_\tau &= (\bar{b}, \bar{a}). \end{aligned} \quad (\text{mod } 2)$$

Note that the odd spin structure is transformed into itself. This at once implies that the vertex operator contractions $\langle\langle V_1(k_1) \cdots V_n(k_n) \rangle\rangle_{\nu\bar{\nu}}$ are also modular invariant

in this sense. Modular invariance of the full amplitude $\langle V_1(k_1) \cdots V_n(k_n) \rangle$ will be achieved provided a choice for $C_{\nu\bar{\nu}}$ is made that is consistent with modular invariance. It is easily checked that the measure in (3M.12) transforms correctly under modular transformations, except may be for a constant phase:

$$\begin{aligned} \langle V_1(k_1) \cdots V_n(k_n) \rangle_{\nu\bar{\nu}}(\tau+1) &= (-1)^{a+\bar{a}} \langle V_1(k_1) \cdots V_n(k_n) \rangle_{\nu_1\bar{\nu}_1}(\tau) \\ \langle V_1(k_1) \cdots V_n(k_n) \rangle_{\nu\bar{\nu}}(-\frac{1}{\tau}) &= \langle V_1(k_1) \cdots V_n(k_n) \rangle_{\nu\tau\nu\tau}(\tau). \end{aligned} \quad (6H.16)$$

Hence modular invariance of the full amplitude requires the following choice for the constants $C_{\nu\bar{\nu}}$:

$$\begin{aligned} \tau \rightarrow -\frac{1}{\tau} \quad & C_{(1,0)\bar{\nu}} = C_{(0,1)\bar{\nu}} \\ & C_{\nu(1,0)} = C_{\nu(0,1)} \\ \tau \rightarrow \tau+1 \quad & C_{(0,1)\bar{\nu}} = -C_{(0,0)\bar{\nu}} \\ & C_{\nu(0,1)} = -C_{\nu(0,0)} \end{aligned} \quad (6H.17)$$

and this should hold for all ν and $\bar{\nu}$. Note that since the odd spin structure $(1,1)$ transforms as a singlet under the modular group, the relative magnitude with even spin structures is not fixed by modular invariance. It should be determined by factorization, in the limit where the torus degenerates to the sphere.

Three and four point amplitudes for massless bosons

Let us consider scattering amplitudes with massless external particles only (the graviton, dilaton and anti-symmetric tensor field). Such operators are produced by the generating vertex $V^*(\zeta, \bar{\zeta}; k)$ introduced in (6G.). As in the case of tree level, the amplitude (3L.18) is expressed through (3L.20) and (3L.21), but the propagator is now understood to be $G_{\nu\bar{\nu}}$, of (3M.19).

For even spin structure, we consider the chirality conserving form first and then split it to obtain the chiral amplitude. Thus, we may separate $\mathcal{G}_n$ of (3L.21) into two chiral parts expressed only in terms of the chiral propagator G_ν where the chiral part is given by

$$\mathcal{G}_n^\zeta = \sum_{i \neq j} \{ -ik_i \cdot \zeta_j \mathcal{D}_+^j G_\nu(\mathbf{z}_i, \mathbf{z}_j) - \frac{1}{2} \zeta_i \cdot \zeta_j \mathcal{D}_+^i \mathcal{D}_+^j G_\nu(\mathbf{z}_i, \mathbf{z}_j) \} \quad (6H.17)$$

Thus, the full amplitude (still for even spin structure) may be recast in a familiar

form

$$\langle V_1^* \dots V_n^* \rangle_{\nu\bar{\nu}} = (2\pi)^{10} \delta(k) M_\nu \bar{M}_{\bar{\nu}} \int_{\text{Im}} dp^\mu \mathcal{F}_\nu \bar{\mathcal{F}}_{\bar{\nu}} \quad (6H.18)$$

where

$$\mathcal{F}_\nu(z_i, k, \zeta, \tau; p^\mu) = \exp\{i\pi p^2 \tau + 2\pi p^\mu (-\zeta_i^\mu \theta_i + i k_i^\mu z_i) - \frac{1}{2} \sum_{i \neq j} k_i \cdot k_j G_\nu(z_i, z_j) + \mathcal{G}_n^\zeta\} \quad (3M.29)$$

Of course this amplitude should now be integrated over moduli space.

To evaluate the 0, 1, 2, 3, 4 and 5 point amplitudes, the above is in fact enough, for only the even spin structures contribute to their amplitudes. Indeed, for the odd spin structure the Dirac operator has one (chiral) zero mode for each dimension of space-time. In $d = 10$, there is thus a total of 10 zero modes. Inserting e.g. a massless vertex eats up 2 zero modes. However, one fermion mode is also eaten up by fixing a conformal spinor gauge for the supersymmetry operator. One more is produced by the presence of the supermoduli parameters. All zero modes must of course be killed, so naively the lowest number of vertex operator insertions necessary to make the amplitude non-zero is 5. However, overall momentum conservation implies that this amplitude also vanishes, and one has to go to six external particles to obtain a non-zero contribution from the odd spin structure.

We first show that the 0,1,2 and 3 point functions vanish identically. This fact is based upon two fundamental observations. For 3 or less external massless particles, one always has $k_i \cdot k_j = 0$ for all i and j , so that $\mathcal{F}_\nu$ only involves $\mathcal{G}_n^\zeta$, which depends on the derivatives of G_ν only. These derivatives are given by

$$\begin{aligned} \mathcal{D}_+^i G_\nu(z_i, z_j; \tau) &= \theta_j S_\nu(z_i, z_j) + \theta_i \partial_i G_1(z_i, z_j) \\ \mathcal{D}_+^i \mathcal{D}_+^j G_\nu(z_i, z_j; \tau) &= -S_\nu(z_i, z_j) + \theta_i \theta_j \partial_i \partial_j G_1(z_i, z_j) \end{aligned} \quad (6H.18)$$

The partition function, the 1 and 2 particle amplitudes all vanish simply by the use of the famous Jacobi identity of (E.11) and the assignments of the coefficients $C_{\nu\bar{\nu}}$:

$$\sum_\nu C_{\nu\bar{\nu}} \vartheta_{ab}(0, \tau)^4 = 0 \quad \nu = (a, b) \quad \bar{\nu} = (\bar{a}, \bar{b}) \quad (6H.19)$$

For the 3-point function, one uses in addition to the above, the facts that

$$\begin{aligned} \sum_\nu C_{\nu\bar{\nu}} \vartheta_{ab}(o, \tau)^4 \mathcal{D}_+^1 G_\nu(1, 2) \mathcal{D}_+^2 G_\nu(2, 3) \mathcal{D}_+^3 G_\nu(3, 1) &= 0 \\ \sum_\nu C_{\nu\bar{\nu}} \vartheta_{ab}(o, \tau)^4 \mathcal{D}_+^1 G_\nu(1, 2) \mathcal{D}_+^2 G_\nu(2, 1) \mathcal{D}_+^3 G_\nu(3, 1) &= 0 \end{aligned} \quad (6H.20)$$

All these identities are easily proven with the help of (6H.18).

The calculation of the 4-point funtion is more involved, and ϑ -function identities are heavily used. There are three types of terms: those with 4 factors of k , with 2 factors of k , and without explicit k 's at all contracted onto the polarization tensors. Our first task is to show that the terms with 4 factors of k cancel after summation over all spin structures. One needs the following Riemann-type identity

$$\begin{aligned} \sum_{\nu} C_{\nu\bar{\nu}} \vartheta_{ab}(0, \tau)^4 \mathcal{D}_+^1 G_{\nu}(1, i_1) \mathcal{D}_+^2 G_{\nu}(2, i_2) \mathcal{D}_+^3 G_{\nu}(3, i_3) \mathcal{D}_+^4 G_{\nu}(4, i_4) \\ = \theta_{i_1} \theta_{i_2} \theta_{i_3} \theta_{i_4} \sum_{\nu} C_{\nu\bar{\nu}} \vartheta_{ab}(0, \tau)^4 S_{\nu}(1, i_1) S_{\nu}(2, i_2) S_{\nu}(3, i_3) S_{\nu}(4, i_4) \end{aligned} \quad (6H.21)$$

Permutations $(1, 2, 3, 4) \mapsto (i_1, i_2, i_3, i_4)$ which leave one or more points fixed need not be considered, as their contribution cancels due to transversality of the polarization tensors. The remaining 9 permutations cancel in view of the Riemann identity.

Now we calculate the terms with two momenta k ; it is useful to take an example. Consider terms arising as the coefficient of

$$\zeta_1 \cdot \zeta_2 \zeta_3 \cdot k_{i_3} \zeta_4 \cdot k_{i_4}$$

where i_3 and i_4 are different from 3 and 4 respectively. The spin structure sum is then again simplified, with the help of the Riemann identities and one finds

$$\begin{aligned} \sum_{\nu} C_{\nu\bar{\nu}} \vartheta_{\nu}^4(0, \tau) \mathcal{D}_+^1 \mathcal{D}_+^2 G_{\nu}(1, 2) \mathcal{D}_+^3 G_{\nu}(3, i_3) \mathcal{D}_+^4 G_{\nu}(4, i_4) \prod_{i < j} (1 + k_i \cdot k_j \theta_i \theta_j S_{\nu}(i, j)) \\ = \theta_{i_3} \theta_{i_4} \sum_{\nu} C_{\nu\bar{\nu}} \vartheta_{\nu}(0, \tau)^4 S_{\nu}(1, 2) S_{\nu}(3, i_3) S_{\nu}(4, i_4) \prod_{i < j} () \end{aligned} \quad (6H.22)$$

where the last factor arises from the expansion of the superspace Green function. Since we must end up with 4 θ 's the product $\prod_{i < j}$ only produces terms with two θ 's, so that the answer will be a linear function of s, t and u . With some further use of the Riemann identities, one can evaluate it rather easily and one finds

$$\zeta_1 \cdot \zeta_2 [t \zeta_3 \cdot k_1 \zeta_4 \cdot k_2 + u \zeta_3 \cdot k_2 \zeta_4 \cdot k_1] \quad (6H.23)$$

Upon inspection, one notices that this result is reminiscent of the tree level answer obtained in (3L.37). One can now easily complete the analysis by examining that

the other terms also have the same form as the tree-level answer. Thus our final expression for the one loop 4-pt function in the Type II superstring is

$$\langle V(\epsilon_1, k_1) \cdots V(\epsilon_4, k_4) \rangle = g^4 \delta(k) \mathcal{A}_1 \epsilon^{1\bar{1}} \epsilon^{2\bar{2}} \epsilon^{3\bar{3}} \epsilon^{4\bar{4}} K_{1234} K_{\bar{1}\bar{2}\bar{3}\bar{4}} \quad (6H.24)$$

where the reduced amplitude is given by

$$\mathcal{A}_1 = \int_{\mathcal{M}_1} \frac{d^2 \tau}{2\tau_2^2} \frac{1}{(\tau_2)^4} \int d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4 |F_{12} F_{34}|^{-s/2} |F_{23} F_{14}|^{-t/2} |F_{13} F_{24}|^{-u/2} \quad (6H.25)$$

Overall translation invariance on the torus allows us to integrate over one of the four positions, so that we may set $z_4 = 0$ and

$$\mathcal{A}_1 = \frac{1}{2} \int_{\mathcal{M}_1} \frac{d^2 \tau}{(\tau_2)^5} \int d^2 z_1 d^2 z_2 d^2 z_3 \left| \frac{F_{12} F_{34}}{F_{13} F_{24}} \right|^{-s/2} \left| \frac{F_{23} F_{14}}{F_{13} F_{24}} \right|^{-t/2} \quad (6H.26)$$

which agrees with the classic formula derived in the operator formalism.

Several remarks are in order here. First, it is remarkable that the kinematical form for the one loop amplitude coincides with that for the tree level amplitude. Second, our calculation of the one-loop four point amplitude is perhaps more involved than when it is performed in the light-cone operator formalism. However it has to be recalled that the corresponding calculation in the light-cone formulation was simple only for graphs with very few external legs, ultimately becoming unwildy for graphs with more than 6 legs. In our covariant RNS formulation, the difficulty increases, but only slightly so.

For lack of space we shall not discuss the one loop amplitudes for general numbers of external states. The Type II superstring one-loop amplitude was first calculated in Green and Schwarz (1982), where also its finiteness was suggested. For heterotic strings, the one loop amplitudes are calculated in Gross, Harvey, Martinec and Rohm (1986). General one loop amplitudes, including the case of odd spin structures are calculated in D'Hoker and Phong (1988), which is also the approach we followed here. References to prior work can also be found in this review.

Some Concluding Remarks.

To conclude, we collect a few remarks on the present status of string theory as considered from the point of view of grand unification. First, we have found that the theory is remarkably unique in flat Minkowski space-time. The dimension is fixed and the gauge group for heterotic or Type I strings is determined almost uniquely.

However, we do not live in ten dimensions. Our long-distance world is four-dimensional, and if string theory is to be a good description of the real world as we see it then we must find a mechanism by which the dynamics of string theory itself can account for this *compactification*. In addition, the ten-dimensional theory exhibits supersymmetry, which at some stage must also be broken. Perhaps the mechanism responsible for compactification is the same as the one that drives supersymmetry breaking, at least down to $N=1$ supergravity in four dimensions. This is what was originally proposed by Candelas, Horowitz, Strominger and Witten (1985).

The mechanism for driving these compactifications has so far not been identified though. In fact, the outstanding problem in superstring theory today is the identification and understanding of this type of mechanism. It is a hard, and non-perturbative problem.

Actually, it may be a good idea to pause here, and attempt to describe what we understand by a good theory of quantum gravity. Einstein gravity coupled to matter is a non-renormalizable theory, as it stands. This means that upon quantization, an infinite number of counterterms must be supplied to render all amplitudes finite. At the same time an infinite number of renormalization conditions must be supplied to define the theory, and this makes the non-renormalizable theory non-predictable.

String theory as it stands is uniquely defined in flat space-time, but the number of compactifications is very large. Presumably this number is not infinite. Yet, just for Calabi-Yau type spaces, there are probably more than one million such spaces, and most have many *moduli* i.e. continuously variable parameters. Let's assume that there is just a million parameters, no more than that. In principle, since the number of parameters is finite, we have found a renormalizable quantum theory of gravity, coupled to Yang-Mills theory and fermions. Yet, the number of parameters is so large that one would have to supply an unreasonable number of input parameters. The theory is perhaps renormalizable, but in practice pretty unpredictable. To obtain a predictive theory, these parameters must somehow be fixed or understood dynamically.

One thing one can really get at, with the present technology, is to find out whether a certain compactification is perturbatively a solution or not. When space-time supersymmetry is unbroken, it is perhaps not that difficult to investigate such issues.

The other thing one can get at is the behavior of the string that is pretty much independent of compactification, i.e. the high energy behavior, where high energy is compared with the compactification scale. This information can be obtained by applying perturbation theory around flat Minkowski space-time. These questions were investigated *e.g.* in Gross and Mende (1988).

So an enormous amount of work remains to be done. Let us not be too modest either though. String theory is beautiful, and *obviously* correct. This quantum theory of gravity is expected to be perturbatively finite, causal, and as was shown by Aoki, D'Hoker and Phong (1990) also unitary. From that point of view it is much better than any other candidate for quantum gravity proposed so far.

The next question we should address is whether it is a good theory. A good scientific theory must be falsifiable, *i.e.* one must be able to carry out some experiment that can conceivably show it to be incorrect. It is very unclear at this point whether string theory is indeed falsifiable. Recall that all our energy scales compared to the Planck scale are very very small, and for any quantum theory of gravity, one must look for some sort of coherent effects. Can this be done? It is an important question whether any experiment can ever be done in our life-time that could falsify the theory.

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