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POLARIZATION DECOMPOSITION OF FLUXES AND KINEMATICS IN *ep* REACTIONS

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ABSTRACT

Flux matrices for γ, Z^0, and W^- particles in *ep* scattering are calculated using polarization decomposition. These matrices are useful for estimating cross sections especially when the interaction matrix element is only partially known (there is no need to deal with polarization decomposition when the interaction matrix element is known exactly). These matrices contain complete Q^2 dependence and the transverse photon flux reduces to the Weizsäcker-Williams Approximation (WWA) at low-Q^2 and also at large-y and hence the WWA is valid when the scattered electron is co-linear with the beam electron. Kinematical effects are important especially if the outgoing lepton can be scattered at arbitrary angles. Lepton beam polarization effects are also included.

1. Kinematics and Polarization Vectors

Consider a vector boson (i.e., a γ, Z^0, or W^-) emitted from a charged lepton as shown in Figure 1. Let P_1 be the incoming lepton four-momentum and let P_2

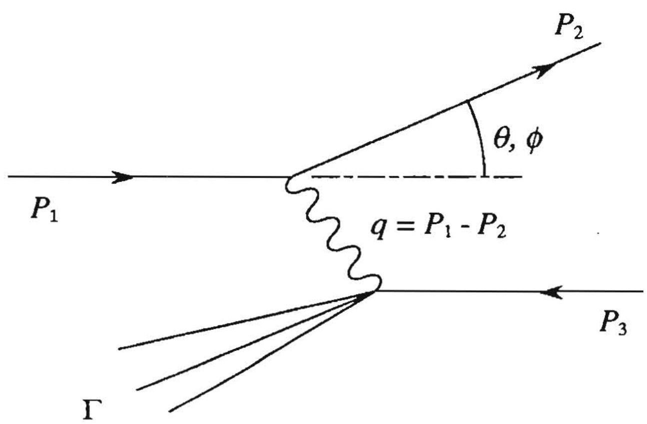

Figure 1: Definition of Kinematic Variables.

be the outgoing four-momentum of the scattered lepton. The four-vector of the emitted vector boson is $q = P_1 - P_2$ which is absorbed by a proton, P_3, assumed to be

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traveling in the $+z$ direction. We take the incoming lepton P_1 to be traveling in the $-z$ direction and consider it to be ultra-relativistic (i.e., $m_1 \ll E_1$). The direction of P_2 is given by the angles θ and ϕ with respect to the initial electron direction (scattering angles). These basic definitions provide us with

$$
\begin{aligned}
P_1 &= (E_1, 0, 0, -E_1), && (1)\\
P_2 &= (E_2, E_2 \sin\theta\cos\phi, E_2 \sin\theta \sin\phi, -E_2\cos\theta), && (2)\\
P_3 &= (E_3, 0, 0, \beta E_3), && (3)\\
q &= (E_1 - E_2, -E_2\sin\theta\cos\phi, -E_2\sin\theta\sin\phi, E_2\cos\theta - E_1). && (4)
\end{aligned}
$$

The usual invariants used to describe the ep interaction are $Q^2 = -q^2$, $x = Q^2/2p_3 \cdot q$, and $y = (q \cdot P_3)/(P_1 \cdot P_3)$. These can be approximated as

$$
\begin{aligned}
y &\approx 1 - \frac{E_2}{E_1}\frac{(1+\cos\theta)}{2}, && (5)\\
\frac{Q^2}{4E_1^2} &\approx \frac{E_2}{E_1}\frac{(1-\cos\theta)}{2}, && (6)\\
x &\approx \frac{Q^2}{sy}. && (7)
\end{aligned}
$$

The coordinates of the vector q in terms of these variables are given by

$$
\begin{aligned}
q_0 &= E_1(y - \frac{Q^2}{4E_1^2}), && (8)\\
q_1 &= -\sqrt{1-y}Q\cos\phi, && (9)\\
q_2 &= -\sqrt{1-y}Q\sin\phi, && (10)\\
q_3 &= -E_1(y + \frac{Q^2}{4E_1^2}). && (11)
\end{aligned}
$$

We define a parameter τ, to measure the difference between the initial and final lepton energy

$$2\tau \equiv y - \frac{Q^2}{4E_1^2}. \tag{12}$$

We have the following simple relations between the above kinematical quantities

$$-q_3 = q_0 + \frac{Q^2}{2E_1}, \quad \text{and} \tag{13}$$

$$E_1 + E_2 = 2E_1(1-\tau). \tag{14}$$

We choose the laboratory frame to determine the following polarization vectors

$$
\begin{aligned}
\epsilon^1 &= (0, -\sin\phi, \cos\phi, 0), && (15)\\
\epsilon^2 &= (0, (E_1 - E_2\cos\theta)\cos\phi, (E_1 - E_2\cos\theta)\sin\phi, -E_2\sin\theta)/\sqrt{q_0^2 + Q^2}, && (16)\\
\epsilon^3 &= (|\vec{q}|, q_0\vec{e}_q)/\sqrt{Q^2}, && (17)\\
\epsilon^4 &= q/\sqrt{|Q^2|}. && (18)
\end{aligned}
$$

where $\vec{e}_q = \vec{q}/|\vec{q}|$. In general one has the following Reduction of Unity:

$$-g_{\mu\nu} = \sum_{\lambda=1}^{4} \epsilon_\mu^{*\lambda} \eta_\lambda \epsilon_\nu^\lambda, \tag{19}$$

where $\eta_1 = 1$, $\eta_2 = 1$, $\eta_3 = -1$, $\eta_4 = 1$. Identity (19) can be written in the case of spacelike four-vector q as

$$-g_{\mu\nu} = \frac{q_\mu q_\nu}{Q^2} + \epsilon_\mu^{*1}\epsilon_\nu^1 + \epsilon_\mu^{*2}\epsilon_\nu^2 - \epsilon_\mu^{*3}\epsilon_\nu^3. = \epsilon_\mu^4\epsilon_\nu^4 + \epsilon_\mu^{*1}\epsilon_\nu^1 + \epsilon_\mu^{*2}\epsilon_\nu^2 - \epsilon_\mu^{*3}\epsilon_\nu^3. \tag{20}$$

For notational convenience we denote the contraction of two Lorentz four-vectors a and b (dot product) as

$$(ab) \equiv g_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu. \tag{21}$$

2. Polarized Lepton Beams

Consider a positive energy fermion whose momentum four-vector is denoted by p and spin λ. The corresponding spinor is $u_\lambda(p)$. In the fermion rest frame the spin four-vector is given by $s = (0, \vec{s})$, with $\vec{s}$ being a unit vector in the direction of the spin quantization. We have $p \cdot s = 0$ in the rest frame in particular and by Lorentz invariance this dot product vanishes in all frames. The ultra-relativistic spinor product can be represented as

$$u_\lambda(p)\overline{u}_\lambda(p) \rightarrow (\not{p} + m)\left[\frac{1 - \lambda\gamma_5}{2}\right], \tag{22}$$

where λ is the helicity (right-handed beams have $\lambda = 1$, left-handed beams have $\lambda = -1$).

3. Definition of Cross Section and Fluxes

Consider the matrix-element for the lepton-proton interaction. Imagine P_1 to be the source of the exchanged vector boson. Let us consider the ep reaction given by $P_1 + P_3 \rightarrow P_2 + \Gamma$ proceeding via vector boson exchange ($P_1 = P_2 + q$). Also consider the sub-reactions: a) $i = 1$ for γ exchange, b) $i = 2$ for Z^0 exchange, c) and $i = 3$ for W^- exchange. Consider first γ and Z^0 exchange given by $q + P_3 \rightarrow \Gamma$. The cross section for the ep reaction has the general form

$$d\sigma_{ep} = \frac{(2\pi)^4 \delta^4(P_1 - P_2 + P_3 - \Gamma)}{4\sqrt{(P_1 \cdot P_3)^2 - m_1^2 m_3^2}} \overline{|M_{ep}|^2} \frac{d^3 P_2}{(2\pi)^3 2E_2} d\Gamma. \tag{23}$$

For the above possible intermediate states, $i, j = 1, 2, 3$, the form of the cross section for interactions at the proton vertex can be represented as

$$d\sigma_{ij} = \frac{(2\pi)^4 \delta^4(q + P_3 - \Gamma)}{4\sqrt{(q \cdot P_3)^2 + Q^2 m_3^2}} \overline{|M_i M_j^\dagger|}\, d\Gamma, \tag{24}$$

where we allow for interference between different intermediate channels.

The final-state phase space factor at the proton vertex is represented by $d\Gamma$, where

$$d\Gamma = \prod_k \frac{d^3 p_k}{(2\pi)^2 2E_k}. \tag{25}$$

Forming ratios from the above definitions we have (using units where $e^2 = 4\pi\alpha$)

$$d\sigma_{ep} = 4\pi\alpha \sqrt{\frac{(q \cdot P_3)^2 + Q^2 m_3^2}{(P_1 \cdot P_3)^2 - m_1^2 m_3^2}} \sum_{ij} \sum_{\lambda\rho} \left[F_i(Q^2)\, F_j^\dagger(Q^2) \mathcal{D}^{ij}_{\lambda\rho} \right] (d\sigma_{ij}^{\lambda\rho}) \frac{d^3 P_2}{(2\pi)^3 2E_2}. \tag{26}$$

The final interaction cross section at the proton vertex is defined as

$$d\sigma_{ij}^{\lambda\rho} = \frac{(2\pi)^4 \delta^4(q + P_3 - \Gamma)}{4\sqrt{(q \cdot P_3)^2 + Q^2 m_3^2}} \overline{|M_i M_j^\dagger|}^{\lambda\rho}\, d\Gamma, \tag{27}$$

and the propagator factors are

$$F_1(Q^2) = \frac{1}{Q^2}, \tag{28}$$

$$F_2(Q^2) = \frac{1}{2 \sin\theta_W \cos\theta_W \left[Q^2 + (M_Z - i\Gamma_Z/2)^2\right]}, \tag{29}$$

$$F_3(Q^2) = \frac{1}{2\sqrt{2} \sin\theta_W \left[Q^2 + (M_W - i\Gamma_W/2)^2\right]}. \tag{30}$$

The $\mathcal{D}^{ij}_{\lambda\rho}$ are a set of matrix-valued functions with components over the polarization states $\lambda\rho$ of the exchanged vector boson. Each combination of ij refers to the possible exchanged vector bosons in the intermediate state and the interference effects between exchanged particles. We will derive expressions for the $\mathcal{D}$ matrices below. Since $\sqrt{s}$ is so much larger than the mass of the proton, (m_3), we make the following approximation (valid for most of the phase space)

$$\sqrt{\frac{(q \cdot P_3)^2 + Q^2 m_3^2}{(P_1 \cdot P_3)^2 - m_1^2 m_3^2}} \approx \sqrt{\frac{y^2 (s - m_3^2)^2 + 4Q^2 m_3^2}{(s - m_3^2)^2}} \approx y. \tag{31}$$

The flux expressions are the polarization dependent factors multiplying $d\sigma_{ij}^{\lambda\rho}$. Therefore, we define the polarization flux factors as

$$d^3 \mathcal{L}^{ij}_{\lambda\rho} = (4\pi\alpha) y \left[F_i(Q^2)\, F_j^\dagger(Q^2) \mathcal{D}^{ij}_{\lambda\rho} \right] \frac{d^3 P_2}{(2\pi)^3 2E_2}. \tag{32}$$

If the matrix elements are azimuthally symmetric, then integration over the azimuthal angle of P_2 yields

$$\frac{d^2 \mathcal{L}^{ij}_{\lambda\rho}}{dy\, dQ^2} = \left(\frac{\alpha}{4\pi}\right) y \left[F_i(Q^2)\, F_j^\dagger(Q^2) \mathcal{D}^{ij}_{\lambda\rho} \right]. \tag{33}$$

4. Definition of Amplitudes for Electroweak Flux Matrices

We now use Standard Model Feynman rules to write down the matrix element for the ep reaction in terms of the coupling of the lepton to the intermediate state

$i = 1$ for γ, and $i = 2$ for Z^0. The matrix element for positive energy leptons interacting via neutral currents becomes

$$M_{ep} = e[\overline{u}(P_2)\gamma^\mu u(P_1)]\frac{-g_{\mu\nu}}{q^2}M_1^\nu \tag{34}$$

$$+ \; e\frac{1}{2\sin\theta_W\cos\theta_W}[\overline{u}(P_2)\gamma^\mu(g_V - g_A\gamma_5)u(P_1)]\frac{-g_{\mu\nu}+q_\mu q_\nu/M_Z^2}{[q^2-(M_Z-i\Gamma_Z/2)^2]}M_2^\nu. \tag{35}$$

Using the Reduction of Unity we obtain

$$-M_{ep} = eF_1(Q^2)\sum_{\lambda=1,2,3}[\overline{u}(P_2)\gamma^\mu u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda\,[\epsilon_\nu^\lambda M_1^\nu] \tag{36}$$

$$+ \; eF_2(Q^2)\sum_{\lambda=1,2,3}[\overline{u}(P_2)\gamma^\mu(g_V - g_A\gamma_5)u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda[\epsilon_\nu^\lambda M_2^\nu] \tag{37}$$

$$+ \; eF_2(Q^2)g_A\frac{2m_1}{Q}[\overline{u}(P_2)\gamma_5 u(P_1)][1+Q^2/M_Z^2][\epsilon_\nu^4 M_2^\nu]. \tag{38}$$

There is clearly a separation of factors into vector boson emission terms and vector boson interaction terms in the above formula. We can take as a definition of the polarization amplitudes for i the expression

$$M_i^\lambda = [\epsilon_\nu^\lambda M_i^\nu] = (\epsilon^\lambda M_i). \tag{39}$$

The above formula can then be decomposed into

$$-M_{ep} = eF_1(Q^2)\sum_{\lambda=1,2,3}[\overline{u}(P_2)\gamma^\mu u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda\,M_1^\lambda \tag{40}$$

$$+ \; eF_2(Q^2)\sum_{\lambda=1,2,3}[\overline{u}(P_2)\gamma^\mu[g_V - g_A\gamma_5]u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda M_2^\lambda \tag{41}$$

$$+ \; eF_2(Q^2)g_A\frac{2m_1}{Q}[\overline{u}(P_2)\gamma_5(P_1)][1+Q^2/M_Z^2]M_2^4. \tag{42}$$

Using the ultra-relativistic spin-projection operator above we can take into account lepton beam polarization defined as

$$f_P = f_R - f_L, \tag{43}$$

where the fractions of the lepton beam that are right- and left-handed are given by f_R and f_L (i.e., $f_R + f_L = 1$).

Projecting out the polarization dependences we express this above amplitude as a sum over the terms A_1^λ and A_2^λ

$$-M_{ep} = e\sum_{i=1}^{2}\sum_{\lambda=1}^{4}F_i(Q^2)A_i^\lambda M_i^\lambda, \tag{44}$$

where

$$A_1^\lambda = [\overline{u}(P_2)\gamma^\mu u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda, \qquad \lambda = 1,2,3 \tag{45}$$

$$A_1^4 = 0, \tag{46}$$

$$A_2^\lambda = [\overline{u}(P_2)\gamma^\mu[g_V - g_A\gamma_5]u(P_1)]\epsilon_\mu^{*\lambda}\eta_\lambda, \qquad \lambda = 1,2,3 \tag{47}$$

$$A_2^4 = g_A \frac{2m_1}{Q}[\overline{u}(P_2)\gamma_5 u(P_1)][1 + Q^2/M_Z^2]. \tag{48}$$

Let us compute the parts that form the square of the overall matrix element. We sum over the final lepton spins and use the fact that the initial electron beam has an overall polarization f_P (i.e., we average the initial lepton spins with a weighted sum given by

$$< A_i^\lambda A_j^\rho > = \sum_{s_f=1,2} [f_R A_i^\lambda A_j^\rho + f_L A_i^\lambda A_j^\rho] \, . \tag{49}$$

We define the following useful tensors in the polarization space ($\lambda, \rho = 1,2,3$)

$$U^{\lambda\rho} = 2\left[(P_1\epsilon^{*\lambda})(P_2\epsilon^\rho) + (P_1\epsilon^\rho)(P_2\epsilon^{*\lambda}) - \frac{Q^2}{2}(\epsilon^{*\lambda}\epsilon^\rho)\right]\eta_\lambda\eta_\rho, \tag{50}$$

$$V^{\lambda\rho} = -2m_1^2(\epsilon^{*\lambda}\epsilon^\rho)\eta_\lambda\eta_\rho, \quad \text{and} \tag{51}$$

$$W^{\lambda\rho} = 2\left[\epsilon^{\alpha\mu\beta\nu}P_{1\alpha}\epsilon_\mu^{*\lambda}P_{2\beta}\epsilon_\nu^\rho\right]\eta_\lambda\eta_\rho. \tag{52}$$

For $\lambda, \rho = 1,2,3$ we have

$$A_1^\lambda A_1^{\dagger\rho} = U^{\lambda\rho} - if_P W^{\lambda\rho}, \tag{53}$$

$$A_1^\lambda A_2^{\dagger\rho} = (g_V - f_P g_A)U^{\lambda\rho} - i(f_P g_V - g_A)W^{\lambda\rho}, \quad \text{and} \tag{54}$$

$$A_2^\lambda A_2^{\dagger\rho} = (g_V^2 + g_A^2 - f_P g_V g_A)U^{\lambda\rho} - (g_V^2 - g_A^2)V^{\lambda\rho} + i[2g_V g_A - f_P(g_V^2 + g_A^2)]W^{\lambda\rho}. \tag{55}$$

In the above expressions the terms in the Levi-Civita tensor are expressible as determinants of the four-by-four matrix formed by taking the vectors contracted with the tensor as column or row elements.

The terms involving A_2^4 in the above amplitude are due to the axial coupling of the weak current to the fermions. There is a non-zero projection of the axial part onto q. These terms are proportional to the mass of the lepton and when squared or contracted with the other amplitudes, give rise to terms proportional to the square of the lepton mass and we will ignore all such contributions. Hence we will ignore the fourth component in all the polarization matrices derived below, because they are small compared to the longitudinal and transverse contributions.

The square of the matrix element can be summed over final lepton spins and represented as (ignoring the ϵ^4 component)

$$\overline{|M_{ep}|}^2 = e^2 \sum_{i,j=1}^{2} \sum_{\lambda,\rho=1}^{3} A_i^\lambda A_j^{\dagger\rho} F_i(Q^2) F_j^\dagger(Q^2) M_i^\lambda M_j^{\dagger\rho}. \tag{56}$$

The flux density matrices are defined as

$$D_{ij}^{\lambda\rho} \equiv A_i^\lambda A_j^{\dagger\rho}. \tag{57}$$

5. Polarization Decomposition of Electroweak Fluxes

As an aide in evaluating the density matrices in the laboratory frame we define some of the dot products that will enter into the calculation

$$
\begin{aligned}
&P_1\cdot\epsilon^1=0;\quad P_1\cdot\epsilon^2=-\tfrac{E_1Q\sqrt{1-y}}{|\vec{q}|};\quad P_1\cdot\epsilon^3=\tfrac{E_1Q(1-\tau)}{|\vec{q}|};\quad P_1\cdot\epsilon^4=-Q/2\\
&P_2\cdot\epsilon^1=0;\quad P_2\cdot\epsilon^2=-\tfrac{E_1Q\sqrt{1-y}}{|\vec{q}|};\quad P_2\cdot\epsilon^3=\tfrac{E_1Q(1-\tau)}{|\vec{q}|};\quad P_2\cdot\epsilon^4=Q/2,
\end{aligned}
\tag{58}
$$

where

$$
|\vec{q}|=\sqrt{q_0^2+Q^2}=2E_1\sqrt{\tau^2+\frac{Q^2}{4E_1^2}}. \tag{59}
$$

Let $|ABCD|$ stand for the determinant of the 4-by-4 matrix that is formed by taking the four-vectors A, B, C, and D as columns. Including the effects of beam polarization as well as axial currents requires that we compute in addition to the above dot products also the following determinants:

$$
|P_1\epsilon^1P_2\epsilon^2|=\frac{E_1E_2}{|\vec{q}|}\begin{vmatrix}1 & 0 & 1 & 0\\ 0 & -\sin\phi & \sin\theta\cos\phi & (E_1-E_2\cos\theta)\cos\phi\\ 0 & \cos\phi & \sin\theta\sin\phi & (E_1-E_2\cos\theta)\sin\phi\\ -1 & 0 & -\cos\theta & -E_2\sin\theta\end{vmatrix} \tag{60}
$$

$$
|P_1\epsilon^1P_2\epsilon^3|=\frac{E_1E_2}{Q|\vec{q}|}\begin{vmatrix}1 & 0 & 1 & q_0^2+Q^2\\ 0 & -\sin\phi & \sin\theta\cos\phi & (E_1-E_2)q_1\\ 0 & \cos\phi & \sin\theta\sin\phi & (E_1-E_2)q_2\\ -1 & 0 & -\cos\theta & (E_1-E_2)q_3\end{vmatrix} \tag{61}
$$

$$
|P_1\epsilon^2P_2\epsilon^3|=\frac{E_1E_2\sin\phi\cos\phi}{Q(q_0^2+Q^2)}\begin{vmatrix}1 & 0 & 1 & q_0^2+Q^2\\ 0 & E_1-E_2\cos\theta & \sin\theta & -(E_1-E_2)\sin\theta\\ 0 & (E_1-E_2\cos\theta) & \sin\theta & -(E_1-E_2)\sin\theta\\ -1 & -E_2\sin\theta & -\cos\theta & (E_1-E_2)q_3\end{vmatrix} \tag{62}
$$

Notice that the 2nd and 3rd rows of the last determinant are linearly dependent. Therefore,

$$
|P_1\epsilon^1P_2\epsilon^2| = \frac{E_1Q^2(1-\tau)}{|\vec{q}|}=\frac{Q^2(1-\tau)}{2\sqrt{\tau^2+\frac{Q^2}{4E_1^2}}}, \tag{63}
$$

$$
|P_1\epsilon^1P_2\epsilon^3| = -\frac{E_1Q^2\sqrt{1-y}}{|\vec{q}|}=-\frac{Q^2\sqrt{1-y}}{2\sqrt{\tau^2+\frac{Q^2}{4E_1^2}}}, \tag{64}
$$

$$
|P_1\epsilon^2P_2\epsilon^3| = 0. \tag{65}
$$

Using the above quantities, calculated in the lab frame, we now evaluate the matrices $U, V,$ and W for values of $\lambda, \rho = 1, 2, 3$ and ignore the contribution from the ϵ^4 polarization since it depends on the square of the electron mass and is not multiplied by comparatively large factors. We summarize the results:

$$U^{\lambda\rho} = \frac{4Q^2}{4\tau^2 + \frac{Q^2}{E_1^2}} \begin{pmatrix} \tau^2 + \frac{Q^2}{4E_1^2} & 0 & 0 \\ 0 & 1 - y + \tau^2 + \frac{Q^2}{4E_1^2} & (1-\tau)\sqrt{1-y} \\ 0 & (1-\tau)\sqrt{1-y} & 1-y \end{pmatrix}, \tag{66}$$

$$V^{\lambda\rho} = 2m_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{67}$$

$$W^{\lambda\rho} = \frac{2Q^2}{\sqrt{4\tau^2 + \frac{Q^2}{E_1^2}}} \begin{pmatrix} 0 & (1-\tau) & \sqrt{1-y} \\ -(1-\tau) & 0 & 0 \\ -\sqrt{1-y} & 0 & 0 \end{pmatrix}. \tag{68}$$

As can be seen the elements of $V^{\lambda\rho}$ are proportional to the square of the electron mass and will be ignored.

We obtain the following density matrices with rows and columns denoting $\lambda, \rho = 1, 2, 3$ respectively. The density matrix for pure photon exchange, $\mathcal{D}^{11}_{\lambda\rho}$, is given by

$$\mathcal{D}^{11}_{\lambda\rho} = A_1^\lambda A_1^{\dagger\rho} = U^{\lambda\rho} - if_P W^{\lambda\rho}. \tag{69}$$

The above matrices together with the other factors in the definition of the flux matrices, Eqs. (32) and (33), give the final result for emission of photons from Spin-1/2 fermions. In the unpolarized case $f_P = 0$ and the density matrix, $\mathcal{D}^{11}_{\lambda\rho}$, has the property that all the longitudinal components and off-diagonal elements vanish for backwards scattering $y = 1$. Re-writing the contributions from the two transverse diagonal elements we have

$$d^2\mathcal{L}^{11}\sigma_{11} + d^2\mathcal{L}^{22}\sigma_{22} = [d^2\mathcal{L}^{11} + d^2\mathcal{L}^{22}]\left[\frac{\sigma_{11}+\sigma_{22}}{2}\right] \tag{70}$$

$$+ [d^2\mathcal{L}^{22} - d^2\mathcal{L}^{11}]\left[\frac{\sigma_{22}-\sigma_{11}}{2}\right]. \tag{71}$$

The last term represents the interference between the two polarization states and the transverse flux is the term multiplying the average over the two transverse cross sections: $d^2\mathcal{L}^T = d^2\mathcal{L}^{11} + d^2\mathcal{L}^{22}$.

Then

$$d^2\mathcal{L}^T = \frac{\alpha}{2\pi}\frac{dy\,dQ^2}{Q^2} y\left[\frac{2(1-y) + (y - \frac{Q^2}{4E_1^2})^2 + \frac{Q^2}{E_1^2}}{(y - \frac{Q^2}{4E_1^2})^2 + \frac{Q^2}{E_1^2}}\right]. \tag{72}$$

In the limit $Q^2 \to 0$, this reduces to the usual WWA

$$d^2\mathcal{L}^T = \frac{\alpha}{2\pi}\frac{dy\,dQ^2}{yQ^2}[1 + (1-y)^2]. \tag{73}$$

Also note that in the case of backward scattered electrons the transverse flux and WWA also agree since $y \to 1$ for backwards scattered electrons and the formulas are in agreement. This means that the WWA is a reasonable approximation when the initial and final electrons are co-linear, regardless if they are parallel ($Q^2 \to 0$) or anti-parallel ($y \to 1$). Defining the longitudinal flux as $d^2\mathcal{L}^L = d^2\mathcal{L}^{33}$ we obtain

$$d^2\mathcal{L}^L = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{Q^2}y\frac{(1-y)}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}, \tag{74}$$

with the low-Q^2 limit of

$$d^2\mathcal{L}^L = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{yQ^2}(1-y). \tag{75}$$

Continuing again for the other contributions we obtain for γ and Z^0 interference

$$\mathcal{D}^{12}_{\lambda\rho} = [g_V - f_P g_A]U^{\lambda\rho} - i(f_P g_V - g_A)W^{\lambda\rho}, \quad \text{and} \tag{76}$$

$$\mathcal{D}^{21}_{\lambda\rho} = [g_V - f_P g_A]U^{\lambda\rho} + i(f_P g_V - g_A)W^{\lambda\rho}, \tag{77}$$

and for pure Z^0 exchange

$$\mathcal{D}^{22}_{\lambda\rho} = [g_V^2 + g_A^2 - f_P g_V g_A]U^{\lambda\rho} + i[2g_V g_A - f_P(g_V^2 + g_A^2)]W^{\lambda\rho}. \tag{78}$$

To conclude, the matrix $\mathcal{D}^{33}_{\lambda\rho}$ which describes the analogous matrix for the electron-neutrino-W^- vertex is

$$\mathcal{D}^{33}_{\lambda\rho} = 2(1-f_P)\left[U^{\lambda\rho} + iW^{\lambda\rho}\right]. \tag{79}$$

Notice that the density matrix for W^- production vanishes in the limit of a purely right-handed electron beam ($f_P \to 1$).

6. Comparisons with Weizsäcker-Williams Approximation at HERA Energy

The essence of the WWA lies in ignoring the Q^2 dependent terms with respect to y. This limiting procedure can be expressed as

$$\lim_{Q^2 \ll 4E_1^2 y} 4\tau^2 + \frac{Q^2}{4E_1^2} \to y^2. \tag{80}$$

The result of this approximation is that

$$y\frac{\left[2(1-y)+(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}\right]}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}} \to \frac{[1+(1-y)^2]}{y}. \tag{81}$$

Proceeding as with the transverse photon flux given above, we obtain the transverse flux for virtual Z^0 and W^- emission from polarized electron beams

$$\frac{d^2\mathcal{L}^T_{22}}{dy\,dQ^2} = \frac{\alpha}{2\pi}\frac{[g_V^2+g_A^2-f_P g_V g_A]}{4\sin^2\theta_W\cos^2\theta_W}\frac{Q^2}{(Q^2+M_Z^2)^2+(\Gamma_Z M_Z)^2}y\left[\frac{2(1-y)+(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}\right], \tag{82}$$

and

$$\frac{d^2\mathcal{L}_{33}^T}{dy\,dQ^2} = \frac{\alpha}{2\pi}\frac{[1-f_P]}{4\sin^2\theta_W}\frac{Q^2}{(Q^2+M_W^2)^2+(\Gamma_W M_W)^2}y[\frac{2(1-y)+(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}]. \tag{83}$$

The limiting cases with respect to the above approximation are

$$\frac{d^2\mathcal{L}_{22}^T}{dy\,dQ^2} = \frac{\alpha}{2\pi}\frac{[g_V^2+g_A^2-f_P g_V g_A]}{4\sin^2\theta_W\cos^2\theta_W}\frac{Q^2}{(Q^2+M_Z^2)^2+(\Gamma_Z M_Z)^2}\frac{[1+(1-y)^2]}{y}, \tag{84}$$

and

$$\frac{d^2\mathcal{L}_{33}^T}{dy\,dQ^2} = \frac{\alpha}{2\pi}\frac{[1-f_P]}{4\sin^2\theta_W}\frac{Q^2}{(Q^2+M_Z^2)^2+(\Gamma_W M_Z)^2}\frac{[1+(1-y)^2]}{y}. \tag{85}$$

For a comparison we look at γ, Z^0 and W^- emission from the lepton beam determined by the flux only (the proton vertex enters with a factor of unity). As an example we compute the integrated flux for invariant masses of the final state (Γ) above 20 GeV at HERA energies. We obtain 1.31 : 1 : 0.94 for the ratios of the integrals of the fluxes for WWA:Transverse:Longitudinal contributions for the photon. At $f_P = 0$ we obtain 3.6 : 1 : 0.0065 for the ratios of the same integrated fluxes for Z^0 emission and 3.57 : 1 : 0.0073 for the ratios of the integrated fluxes for W^- emission. As seen in Figure 2, the discrepancy between the WWA and

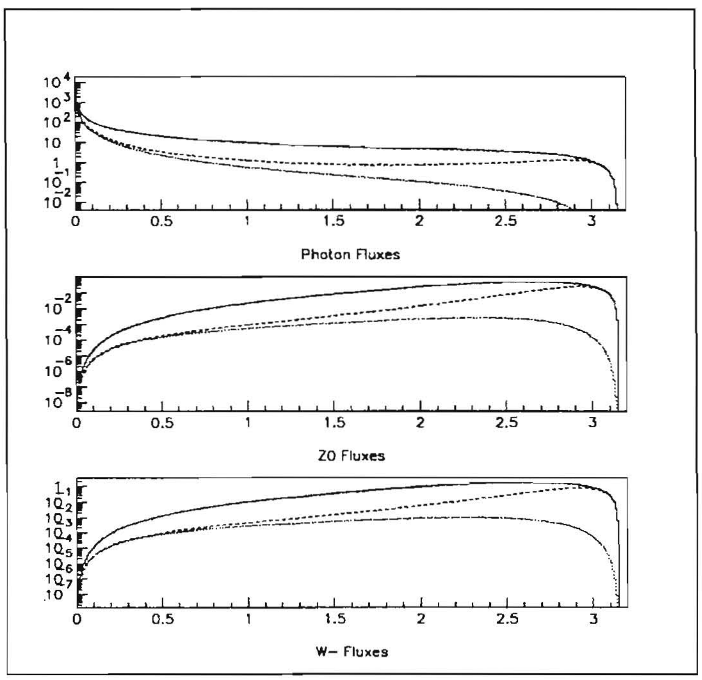

Figure 2: Electron Scattering Angle in Radians. Solid–WWA Flux, Dashed–Transverse Flux, Dotted–Longitudinal Flux.

the transverse flux reaches a maximum around $\frac{\pi}{2}$ radians and goes away at 0 and π radians for the scattered electron. Figure 3 shows the p_T spectrum for the exchanged particle. There is a large discrepancy for larger p_T values that must be taken into account in all Monte Carlo programs that use flux approximations. Setting $p_T = 0$ in a Monte Carlo program does not correctly describe the break in the p_T distribution of the produced final state, because it ignores contributions to p_T from the emission process of the exchanged vector boson. It is also incorrect to approximate the p_T distribution by the WWA over the complete phase space. In order to get the details of the p_T distribution put in correctly, attention to the exact flux formulas is essential.

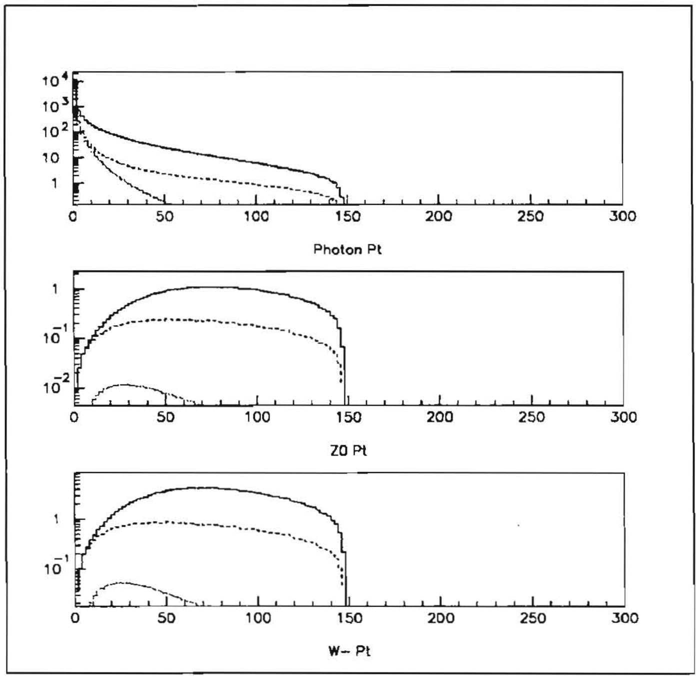

Figure 3: p_T in GeV/c at HERA ($\sqrt{s} \approx 300\ GeV/c^2$). Solid–WWA Flux, Dashed–Transverse Flux, Dotted–Longitudinal Flux.

7. Conclusions

Whenever the initial and scattered electrons are co-linear, (i.e., low-Q^2 or large-y), then the WWA works fine. The emission of photons involves a massless propagator which biases the events to low-Q^2 values and hence one typically sees small effects regarding integrals over Q^2 or electron scattering angle θ. However, in the case of Z^0 or W^- exchange, the scattered electron angle is biased towards larger angles and the discrepancy between WWA and the transverse flux is especially large. Therefore one should always check the relative sizes of y and $\frac{Q^2}{4E_1^2}$ in the region of application of the WWA.

8. Acknowledgements

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