UCD/IIRPA 92-24

AN EXPERIMENTALIST'S GUIDE TO PHOTON FLUX CALCULATIONS

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1 Introduction

The purpose of this report is to determine photon fluxes using polarization vectors. It expands the ideas presented in an earlier internal report [1] for the H1 Collaboration and is intended to be a guide for researchers and students who are interested in understanding the use of photon fluxes and the limitations and pitfalls inherent in this approach. One may use the Weizsäcker-Williams Approximation (WWA) many times in kinematic regions where it works well and discover that in other regions it does not provide enough detail to achieve the precision required. This leads to the unfortunate habit of stating a result based on the WWA and then concluding that a significant error may be present due to uncertainties in the flux integral. There may be theoretical uncertainties in the matrix element for various interesting physics processes, but given a matrix element it is always possible to remove that part of the uncertainty that comes from the flux. There are two reasons that the WWA becomes problematic: 1) The WWA flux formula is missing the required Q^2 dependence to the degree needed; and 2) There are other contributions coming from various polarizations of the virtual photon. We consider the implications of these points for the interesting example of Unpolarized Electron-Proton Scattering.

2 Kinematics and Photon Polarization Vectors

Consider a photon emitted from a charged particle as shown in Figure 1. Let P_1 be the initial four-vector of this particle and P_2 be the final four-vector of the scattered particle after photon emission. The photon has four-vector $q = P_1 - P_2$ and subsequently interacts with a particle P_3 assumed to be traveling in the $+z$ direction. We take the particle P_1 to be traveling in the $-z$ direction and consider it to be ultra-relativistic (i.e. $m_1 \ll E_1$). The direction of P_2 is given by the angles θ and ϕ with respect to the initial electron direction (scattering angles). Therefore the four-vectors in question have the following form

$$
\begin{aligned}
P_1 &= (E_1, 0, 0, -E_1), \\
P_2 &= (E_2, E_2 \sin\theta \cos\phi, E_2 \sin\theta \sin\phi, -E_2 \cos\theta), \\
P_3 &= (E_3, 0, 0, \beta E_3), \\
q &= (E_1 - E_2, -E_2 \sin\theta \cos\phi, -E_2 \sin\theta \sin\phi, E_2 \cos\theta - E_1).
\end{aligned}
$$

We consider cases where the target (particle 3) moves along the z-axis with velocity

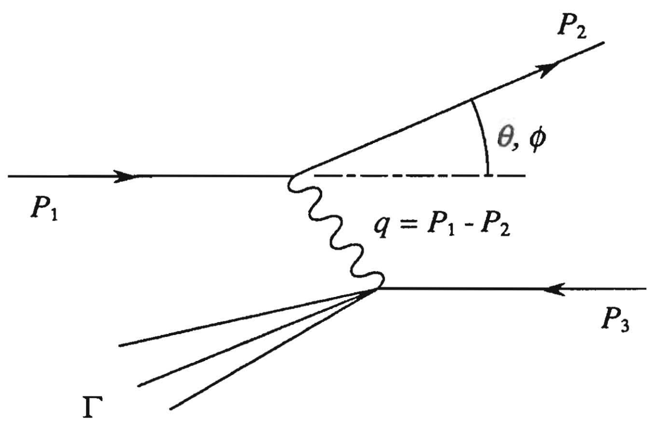

Figure 1: Definition of Kinematic Variables.

β. In the ultra-relativistic case $\beta \approx 1$, and in this case the square of the overall center-of-mass energy is given by $s = 4E_1E_3$. For the moment we consider the ultra-relativistic situation. For completeness we define the variables x, y and Q^2 which are used commonly to describe the scattering process. In terms of invariants we have $Q^2 = -q^2$, $x = Q^2/2p_3 \cdot q$, and $y = (q \cdot P_3)/(P_1 \cdot P_3)$. Therefore we have the following approximations

$$
\begin{aligned}
y &\approx 1 - \frac{E_2}{E_1}\frac{(1+\cos\theta)}{2}, \\
\frac{Q^2}{4E_1^2} &\approx \frac{E_2}{E_1}\frac{(1-\cos\theta)}{2}, \\
x &\approx \frac{Q^2}{sy}.
\end{aligned}
$$

Notice that in the elastic scattering limit (elastic limit), $E_2 \to E_1$ and therefore $y \to \hat{y}$, where $\hat{y} \equiv \frac{Q^2}{4E_1^2}$. It is useful to define a parameter τ, corresponding to inelasticity

$$2\tau \equiv y - \frac{Q^2}{4E_1^2}.$$

In the limit of small Q^2, $2\tau \to y$. In the center-of-mass system of $P_1 + P_3$, $s = 4E_1E_2$, and

$$E_1 \longrightarrow \frac{s + m_1^2 - m_3^2}{2\sqrt{s}},$$

and

$$E_3 \longrightarrow \frac{s - m_1^2 + m_3^2}{2\sqrt{s}}.$$

The absolute minimum value of Q^2 is $Q^2_{min} = (m_1 y)^2/(1-y)$. The invariant mass M_{qP_3} of the system formed by the photon and particle 3 satisfies the relation

$$M_{qP_3}^2 = m_3^2 + sy - Q^2,$$

where we have ignored the mass of particle 1 compared to the mass of particle 3 and the other scales in the interaction. These variables can be inverted to obtain the energy and scattering angle as follows:

$$\begin{aligned} \frac{E_2}{E_1} &= 1 - y + \frac{Q^2}{4E_1^2}, \\ \cos\theta &= \frac{1 - y - \frac{Q^2}{4E_1^2}}{1 - y + \frac{Q^2}{4E_1^2}}. \end{aligned}$$

The coordinates of the vector q in terms of these variables are given by

$$\begin{aligned} q_0 &= E_1(y - \frac{Q^2}{4E_1^2}), \\ q_1 &= -\sqrt{1-y}Q\cos\phi, \\ q_2 &= -\sqrt{1-y}Q\sin\phi, \\ q_3 &= -E_1(y + \frac{Q^2}{4E_1^2}). \end{aligned}$$

The Jacobians of the transformed phase space are defined by

$$dE_2\, d\cos\theta = \frac{dy\, dQ^2}{2E_2} = \left(\frac{Q^2}{sx^2}\right)\frac{dx\, dQ^2}{2E_2}.$$

There are three vectors that are orthogonal to q in the four-vector sense. Two of these vectors can be chosen to be transverse to the spatial direction of q (i.e., orthogonal to

$\vec{q}$). For an interesting example of this technique see [2]. Let ϵ^1 and ϵ^2 be unit vectors that have spatial parts transverse to $\vec{q}$. We take the remaining unit vector ϵ^3 to have its spatial part parallel to $\vec{q}$, but still orthogonal to q in the four-vector sense. Grouping these vectors together we obtain the following system

$$\begin{aligned}
q &= (E_1 - E_2, -E_2 \sin\theta\cos\phi, -E_2\sin\theta\sin\phi, E_2\cos\theta - E_1), \\
\epsilon^1 &= (0, -\sin\phi, \cos\phi, 0), \\
\epsilon^2 &= (0, (E_1 - E_2\cos\theta)\cos\phi, (E_1 - E_2\cos\theta)\sin\phi, -E_2\sin\theta)/\sqrt{q_0^2 + Q^2}\,, \\
\epsilon^3 &= (|\vec{q}|, q_0\vec{e}_q)/\sqrt{Q^2},
\end{aligned}$$

where $\vec{e}_q = \vec{q}/|\vec{q}|$. Observe that q, ϵ^1, and ϵ^2 are spacelike four-vectors and ϵ^3 is timelike. We also have $\vec{e}_q = \vec{\epsilon}^1 \times \vec{\epsilon}^2$. In general one has the following identity

$$-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} = \sum_{\lambda,\lambda'=1,2,3} \epsilon_\mu^{*\lambda'} \eta_{\lambda'\lambda} \epsilon_\nu^\lambda, \tag{1}$$

where $\eta_{11} = 1, \eta_{22} = 1, \eta_{33} = -1$, and $\eta_{ij} = 0$ for $i \neq j$ in the case of spacelike photons with polarization vectors defined as above. In the timelike case we have $\eta_{\lambda'\lambda} = \delta_{\lambda'\lambda}$ (Kronecker delta). Identity (1) can be simply written in the spacelike case as

$$-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} = \epsilon_\mu^{*1}\epsilon_\nu^1 + \epsilon_\mu^{*2}\epsilon_\nu^2 - \epsilon_\mu^{*3}\epsilon_\nu^3. \tag{2}$$

Equation (1) is known as a Reduction of Unity, or as a Completeness Relation for the metric in Lorentz space. There is an invariance of the matrix elements for all transformations that maintain the above identity. Looking at the right-hand side of Equation (1) as a quadratic form we see that it remains invariant under certain orthogonal transformations corresponding to the sign of the diagonal elements of η given above. An important example of this invariance is the case of circular polarization. Circular polarization states correspond to helicity states. We take the following definitions of the circular polarization vectors

$$\begin{aligned}
\epsilon^+ &= \frac{\epsilon^2 + i\epsilon^1}{\sqrt{2}}, \\
\epsilon_0 &= \epsilon^3, \\
\epsilon^- &= \frac{\epsilon^2 - i\epsilon^1}{\sqrt{2}}.
\end{aligned}$$

3 Relation to Gauge Invariance and Low-Q^2 Limit

The photon four-potential, denoted by A_μ, is related to the the polarization vector for plane-wave states by

$$A_\mu = \epsilon_\mu \exp(iq \cdot x).$$

Gauge transformations have the following effect on A_μ

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi.$$

If one considers the special function $\chi = f(\vec{q}) \exp(iq \cdot x)$, then the gauge transformation has the form

$$A_\mu \rightarrow [\epsilon_\mu + q_\mu f(\vec{q})] \exp(iq \cdot x).$$

Hence invariance of the matrix elements under gauge transformations implies vanishing of the matrix elements upon substitution of ϵ_μ by q_μ.

Let us examine the low-Q limit of the Reduction of Unity. At low-Q (given finite photon energy q_0) we have the following approximation based on the binomial theorem

$$\begin{aligned} q_0 &= \sqrt{|\vec{q}|^2 - Q^2} \\ &\approx |\vec{q}| - \frac{Q^2}{2|\vec{q}|} \\ &= \sqrt{Q^2}\epsilon_0^3 - \frac{Q^2}{2|\vec{q}|}. \end{aligned}$$

We see that $(q_0)^2/Q^2 \approx (\epsilon_0^3)^2 + \mathcal{O}(Q)$ and therefore

$$\lim_{Q^2 \rightarrow 0} \left[(\epsilon_0^3)^2 - \frac{(q_0)^2}{Q^2} \right] = 0.$$

Likewise, in the low-Q^2 limit of the spatial components,

$$q_i = \sqrt{Q^2}\frac{|\vec{q}|}{q_0}\epsilon_i^3 \approx \sqrt{Q^2}(1 + Q^2/2q_0^2)\, \epsilon_i^3,$$

hence

$$\lim_{Q^2 \rightarrow 0} \left[\epsilon_i^3 \epsilon_j^3 - \frac{q_i q_j}{Q^2} \right] = 0,$$

for $i, j = 1, 2, 3$. The remaining case involves the product of the time and space components. Using the above expansions we have for the product $q_0 q_i/Q^2$, $i = 1, 2, 3$ the approximation

$$\frac{q_0 q_i}{Q^2} = \epsilon_0^3 \epsilon_i^3 + \mathcal{O}(Q).$$

This implies that in the low-Q^2 limit, the longitudinal polarization vector term cancels the $q_\mu q_\nu$ term on the right-hand side of the identity

$$\lim_{Q^2\to 0}\left[\epsilon^3_\mu\epsilon^3_\nu - \frac{q_\mu q_\nu}{Q^2}\right] = 0.$$

The identity then becomes in the limit of zero Q^2 just a sum over the transverse terms

$$-g_{\mu\nu} = \sum_{\lambda,\lambda'=1,2} \epsilon^{*\lambda}_\mu \eta_{\lambda\lambda'}\epsilon^{\lambda'}_\nu .$$

4 A Simple Example of the Use of Reduction of Unity Identity

In working with Feynman diagrams one deals with currents which, being conserved, are orthogonal to q. Consider one such example of a charged, spinless, massless particle emitting a virtual photon. As before let the incoming particle have four-momentum P_1 and the outgoing particle P_2. The photon has four-momentum $q = P_1 - P_2$; therefore $q\cdot(P_1+P_2) = 0$. Suppose one wants to know $(P_1+P_2)^2$. We can compute this directly as

$$(P_1+P_2)^2 = 2P_1\cdot P_2 = -q^2 = Q^2.$$

Using the identity we obtain

$$(P_1+P_2)^2 = -\sum_{\lambda\lambda'}[(P_1+P_2)\cdot\epsilon^{\lambda'}][(P_1+P_2)\cdot\epsilon^{\lambda}]\eta_{\lambda'\lambda}.$$

Working through the expressions using the fact that $\epsilon\cdot P_1 = \epsilon\cdot P_2$ we obtain

$$(P_1+P_2)^2 = -4[(P_1\cdot\epsilon^2)^2 - (P_1\cdot\epsilon^3)^2].$$

Also we have from the definitions of the four-vectors for P_1 and P_2 that $Q^2 \approx 2E_1E_2(1-\cos\theta)$ and $q_0 = E_1 - E_2$. Therefore we have

$$\begin{aligned} P_1\cdot\epsilon^2 &= \frac{E_1E_2\sin\theta}{\sqrt{q_0^2+Q^2}}, \\ P_1\cdot\epsilon^3 &= \sqrt{\frac{Q^2}{q_0^2+Q^2}}\frac{(E_1+E_2)}{2}. \end{aligned}$$

The product reduces to

$$(P_1+P_2)^2 = -\frac{Q^2}{q_0^2+Q^2}[-Q^2-q_0^2] = Q^2.$$

So we obtain the same result as before.

5 Basic Definitions of Cross Sections and Expansions

Consider two charged particles P_1 and P_3 interacting via exchange of a photon. We imagine P_1 to be the source of the photon and endeavor to write down the cross section in terms of photon emission and photon interaction [1]. Let us consider the reaction R_1 given by $P_1 + P_3 \rightarrow P_2 + \Gamma$ proceeding via photon exchange ($P_1 = P_2 + q$). Also let us consider the sub-reaction R_2 given by $q + P_3 \rightarrow \Gamma$. The cross section for reaction R_1 has the general form

$$d\sigma_{R_1} = \frac{(2\pi)^4 \delta^4 (P_1 - P_2 + P_3 - \Gamma)}{4\sqrt{(P_1 \cdot P_3)^2 - m_1^2 m_3^2}} \overline{|M_{R_1}|^2} \frac{d^3 P_2}{(2\pi)^3 2E_2} d\Gamma,$$

and for R_2 the form

$$d\sigma_{R_2} = \frac{(2\pi)^4 \delta^4 (q + P_3 - \Gamma)}{4\sqrt{(q \cdot P_3)^2 + Q^2 m_3^2}} \overline{|M_{R_2}|^2} \, d\Gamma.$$

The final-state phase space factor is represented by $d\Gamma$,

$$d\Gamma = \prod_k \frac{d^3 p_k}{(2\pi)^3 2E_k},$$

where we represent all the final state particle phase space in one symbol.

We now use QED to write down the matrix element for reaction R_1 in terms of the coupling of P_1 to the photon and the reaction R_2. Consider two cases; (I) Spin-0 Bosons, and (II) Spin-1/2 Fermions. Case (I): Let P_1 be a Spin-0 Boson; then the matrix element becomes

$$M_{R_1} = ie(P_1 + P_2)^\mu \frac{-g_{\mu\nu}}{q^2} M^\nu_{R_2}.$$

Case (II): Let P_1 be a Spin-1/2 Fermion; then the matrix element becomes

$$M_{R_1} = ie\overline{u}(P_2)\gamma^\mu u(P_1) \frac{-g_{\mu\nu}}{q^2} M^\nu_{R_2}.$$

Using the Reduction of Unity and the gauge-invariance property of the matrix elements we obtain for case (I)

$$M_{R_1} = ie \sum_{\lambda'\lambda} [\epsilon^{*\lambda'}_\mu (P_1 + P_2)^\mu \frac{\eta_{\lambda'\lambda}}{Q^2}] [\epsilon^\lambda_\nu M^\nu_{R_2}].$$

For case (II) we obtain

$$M_{R_1} = ie \sum_{\lambda'\lambda} [\epsilon^{*\lambda'}_\mu \overline{u}(E_2)\gamma^\mu u(E_1) \frac{\eta_{\lambda'\lambda}}{Q^2}] [\epsilon^\lambda_\nu M^\nu_{R_2}].$$

There is clearly a separation of factors into photon emission terms and photon interaction terms in the above formula. We can take as a definition of the polarization amplitudes for R_2 the expression

$$M^{\lambda}_{R_2} = [\epsilon^{\lambda}_{\nu} M^{\nu}_{R_2}].$$

The above formula then become for case (I)

$$M_{R_1} = ie \sum_{\lambda'\lambda} [\epsilon^{*\lambda'}_{\mu} (P_1 + P_2)^{\mu} \frac{\eta_{\lambda'\lambda}}{Q^2}]\, M^{\lambda}_{R_2},$$

with matrix-element squared given by

$$|M_{R_1}|^2 = \frac{e^2}{(Q^2)^2} \sum_{\lambda'\lambda} \sum_{\rho'\rho} [\epsilon^{*\lambda'}_{\mu} (P_1 + P_2)^{\mu} \eta_{\lambda'\lambda}][\epsilon^{\rho'}_{\nu} (P_1 + P_2)^{\nu} \eta_{\rho'\rho}]\, M^{\lambda}_{R_2} M^{\rho}_{R_2}.$$

For case (II) we have

$$M_{R_1} = ie \sum_{\lambda'\lambda} [\epsilon^{*\lambda'}_{\mu} \overline{u}(E_2) \gamma^{\mu} u(E_1) \frac{\eta_{\lambda'\lambda}}{Q^2}]\, M^{\lambda}_{R_2},$$

with spin-average square of matrix element given by

$$\overline{|M_{R_1}|^2} = 2 \frac{e^2}{(Q^2)^2} \sum_{\lambda'\lambda} \sum_{\rho'\rho} [\epsilon^{*\lambda'}_{\mu}\, \epsilon^{\rho'}_{\nu} (P^{\mu}_1 P^{\nu}_2 + P^{\nu}_1 P^{\mu}_2 - \frac{Q^2}{2} g^{\mu\nu}) \eta_{\lambda'\lambda} \eta_{\rho'\rho}]\, M^{\lambda}_{R_2} M^{\rho}_{R_2}.$$

We now go on to examine the implications of these matrix elements. To simplify the following formula we define two matrices to describe the photon emission matrix elements and photon interaction matrices. We define a matrix called the photon density matrix $\mathcal{D}_{\lambda\rho}$. For case (I) $\mathcal{D}^{I}_{\lambda\rho}$ is given by

$$\mathcal{D}^{I}_{\lambda\rho} = \sum_{\lambda'\rho'} [\epsilon^{*\lambda'}_{\mu} (P_1 + P_2)^{\mu} \eta_{\lambda'\lambda}][\epsilon^{\rho'}_{\nu} (P_1 + P_2)^{\nu} \eta_{\rho'\rho}].$$

For case (II) $\mathcal{D}^{II}_{\lambda\rho}$ is given by

$$\mathcal{D}^{II}_{\lambda\rho} = 2 \sum_{\lambda'\rho'} \epsilon^{*\lambda'}_{\mu}\, \epsilon^{\rho'}_{\nu} (P^{\mu}_1 P^{\nu}_2 + P^{\nu}_1 P^{\mu}_2 - \frac{Q^2}{2} g^{\mu\nu}) \eta_{\lambda'\lambda} \eta_{\rho'\rho}.$$

The photon interaction matrix squared $\overline{|M_{R_2}|^2}^{\lambda\rho}$ is defined as

$$\overline{|M_{R_2}|^2}^{\lambda\rho} = M^{\lambda}_{R_2} M^{*\rho}_{R_2}.$$

We return to the definition of cross section and forming ratios from the above definitions we can write (using units where $e^2 = 4\pi\alpha$)

$$d\sigma_{R_1} = \frac{4\pi\alpha}{(Q^2)^2} \sqrt{\frac{(q \cdot P_3)^2 + Q^2 m_3^2}{(P_1 \cdot P_3)^2 - m_1^2 m_3^2}} \sum_{\lambda\rho} \mathcal{D}_{\lambda\rho} \frac{d^3 P_2}{(2\pi)^3 2E_2} (d\sigma^{\lambda\rho}_{R_2}),$$

where

$$d\sigma_{R_2}^{\lambda\rho} = \frac{(2\pi)^4\delta^4(q+P_3-\Gamma)}{4\sqrt{(q\cdot P_3)^2+Q^2m_3^2}}\overline{|M_{R_2}|^2}^{\lambda\rho}\,d\Gamma.$$

Note that for small values of Q^2 and ignoring the mass in P_1 we obtain the following approximation

$$\sqrt{\frac{(q\cdot P_3)^2+Q^2m_3^2}{(P_1\cdot P_3)^2-m_1^2m_3^2}} \approx \sqrt{\frac{y^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}} \approx y.$$

The flux expressions are the polarization dependent factors multiplying $d\sigma_{R_2}^{\lambda\rho}$. Therefore we can define the helicity flux factors as

$$d^3\mathcal{L}_{\lambda\rho} = \frac{4\pi\alpha}{(Q^2)^2}\sqrt{\frac{y^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}}\mathcal{D}_{\lambda\rho}\frac{d^3P_2}{(2\pi)^3 2E_2}.$$

If the matrix elements are azimuthally symmetric, then it is possible work with the integrated flux matrix. We assume azimuthal symmetry in this paper, but point out that this needs to be checked on a case by case basis. Integrating over the azimuthal angle of the outgoing particle P_2 and changing variables to (y, Q^2) we obtain

$$d^2\mathcal{L}_{\lambda\rho} = \frac{\alpha}{4\pi}\frac{dy\,dQ^2}{(Q^2)^2}\sqrt{\frac{y^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}}\mathcal{D}_{\lambda\rho}.$$

It is interesting also to see what happens in the elastic limit. In the elastic limit we can introduce a factor from the phase space for energy conservation. For example in the case of two-body elastic scattering $P_1+P_3 \to P_2+P_4$, we have a factor of $\delta(E_1-E_2+E_3-E_4)$ from the phase space. This can be written, assuming m_1 is negligible, as

$$\frac{2E_4}{(s-m_3^2)}\delta(y-\frac{Q^2}{4E_1^2}).$$

In the elastic limit the center-of-mass polarization vectors become

$$\begin{aligned}
\epsilon^1 &= (0,-\sin\phi,\cos\phi,0),\\
\epsilon^2 &= (0,(1-\cos\theta)\cos\phi,(1-\cos\theta)\sin\phi,-\sin\theta)/\sqrt{2(1-\cos\theta)}\,,\\
\epsilon^3 &= (1,0,0,0)\,.
\end{aligned}$$

The elastic limit is just one example of the Breit-Frame (Brick-Wall Frame) and in it the photon transfers zero energy. In this case ϵ^3 is always of the simple form $\epsilon^3 = (1,0,0,0)$. We can boost along z to the Breit-Frame by using the relativistic parameter $\beta = -q_0/(q_0+Q^2/2E_1)$. It is not surprising that many of the invariants involved in the contraction with polarization vectors will be expressed in the lab in terms of this boost parameter. The Breit-Frame is very natural to use because so many of the invariants in the matrix elements have very simple form in this frame. However, what is transverse and longitudinal depends on the observer and the contribution to the observed transverse spectrum in the lab is not the same in a different frame of reference. In general, one must take into account all the contributions from the complete set of polarizations, because only the complete set transforms as a tensor (i.e., covariantly). Boosting along the z-direction by β as given above to the Breit-Frame (indicated by a *) we find

$$
\begin{aligned}
P_1^* &= \frac{Q}{2\sqrt{y}}(1,0,0,-1), \\
P_3^* &= \frac{s\sqrt{y}}{2Q}(1,0,0,1).
\end{aligned}
$$

The photon four-vector is given by

$$q^* = Q(0, -\sqrt{1-y}\cos\phi, -\sqrt{1-y}\sin\phi, -\sqrt{y}).$$

In this frame of reference the polarization vectors can be chosen as follows

$$
\begin{aligned}
\epsilon^1 &= (0, -\sin\phi, \cos\phi, 0), \\
\epsilon^2 &= (0, \sqrt{y}\cos\phi, \sqrt{y}\sin\phi, -\sqrt{1-y}), \\
\epsilon^3 &= (1,0,0,0).
\end{aligned}
$$

The polarization four-vectors have simpler expressions, but the transverse/longitudinal components in the Breit-Frame are not the result of applying a Lorentz transformation to the laboratory polarization vectors. What is transverse and longitudinal is frame-dependent. All the polarization vectors taken together still combine as in Equation (1) to form a tensor expression and therefore are covariant taken as a set.

We now investigate the implications for case (I) and case (II).

6 Flux Matrix for Photons from Spin-0 Bosons

Using the definitions above we obtain the following matrix elements for the polarization flux factors for photon emission from Spin-0 particles of negligible mass in the presence of the field from particle 3.

$$\mathcal{D}^I_{\lambda\rho} = \sum_{\lambda'\rho'} [\epsilon^{*\lambda'}_{\mu}(P_1+P_2)^{\mu}][\epsilon^{\rho'}_{\nu}(P_1+P_2)^{\nu}]\eta_{\lambda'\lambda}\eta_{\rho'\rho}.$$

With this definition we obtain a density matrix with rows and columns corresponding to $\lambda, \rho = 1, 2, 3$ respectively

$$\mathcal{D}^I_{\lambda\rho} = \frac{[2E_1 Q]^2}{[2E_1\tau]^2 + Q^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-y & (1-\tau)\sqrt{1-y} \\ 0 & (1-\tau)\sqrt{1-y} & (1-\tau)^2 \end{pmatrix}.$$

We see that the density matrix has an empty first row and column indicating that there is no flux for photons polarized in the plane perpendicular to the scattering plane of the Spin-0 particle. The polarization in the transverse direction must be in the scattering plane itself. Considering that the other polarization available is in the $\vec{q}$ direction we conclude that the outgoing virtual photon must be polarized in the scattering plane. As one can see the density matrix above has components along the diagonal and also has non-zero off-diagonal components. The diagonal components can be interpreted as direct contributions from given polarization states and the off-diagonal components represent interference terms between different polarizations. If $y = 1$ all the terms involving transverse components vanish. This condition is the case whenever the boson is scattered directly backwards ($y = 1$ at $\cos\theta = -1$).

This also gives another way to examine the well-known phenomena that the coupling of Spin-0 particles to photons vanishes [3] in the Breit-Frame for the case of backwards scattering in deep-inelastic scattering. We conclude that there is no coupling of Spin-0 Bosons to transverse photons in the limit as $y \to 1$. In the Breit-Frame the density matrix is given by the simple expression

$$\mathcal{D}^I_{\lambda\rho} = \frac{Q^2}{y} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-y & \sqrt{1-y} \\ 0 & \sqrt{1-y} & 1 \end{pmatrix}.$$

Let us investigate the effect in the density matrix of converting to circular polarization vectors. The matrix $\mathcal{D}$ would now become a complex-valued matrix and we can obtain the elements from the array formed with linear polarizations. In terms of circular polarizations the indexes λ and ρ vary as $\lambda, \rho = +, 0, -$ and denote the respective matrix elements. With this convention we obtain the following transformed density matrix (denoted by $\mathcal{C}_{\lambda\rho}$ to avoid confusion with $\mathcal{D}$)

$$\mathcal{C}^I_{\lambda\rho} = \frac{1}{2}\frac{[2E_1Q]^2}{[2E_1\tau]^2+Q^2}\begin{pmatrix} 1-y & (1-\tau)\sqrt{1-y}/\sqrt{2} & 1-y \\ (1-\tau)\sqrt{1-y}/\sqrt{2} & (1-\tau)^2 & (1-\tau)\sqrt{1-y}/\sqrt{2} \\ 1-y & (1-\tau)\sqrt{1-y}/\sqrt{2} & 1-y \end{pmatrix},$$

where $\lambda, \rho = +, 0, -$ respectively. To see the relationship between linear and circular polarization matrices in more detail, consult the Appendix. Observe now that the $++$ term is equal to the $--$ and their sum is equal to the sum of the two diagonal elements corresponding to the two transverse polarizations in $\mathcal{D}$. We define the net transverse flux as the sum of the first two diagonal elements of $\mathcal{D}$. $d^2\mathcal{L}^T = d^2\mathcal{L}^{11} + d^2\mathcal{L}^{22} = d^2\mathcal{L}^{++} + d^2\mathcal{L}^{--}$, where we use whatever representation of polarization suits a given problem. The circularly polarized expressions are more balanced looking. They are also more closely related to virtual-photon helicity. Using

$$Q^2 + q_0^2 = Q^2 + E_1^2(y - \frac{Q^2}{4E_1^2})^2,$$

$$d^2\mathcal{L}^T = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{Q^2}\sqrt{\frac{y^2(s-m_3^2)^2 + 4Q^2m_3^2}{(s-m_3^2)^2}}\frac{(1-y)}{(y-\frac{Q^2}{4E_1^2})^2 + \frac{Q^2}{E_1^2}}.$$

In the limit $Q^2 \to 0$, this reduces to

$$d^2\mathcal{L}^T = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{yQ^2}(1-y).$$

Defining the longitudinal flux as $d^2\mathcal{L}^L = d^2\mathcal{L}^{33}$ we obtain

$$d^2\mathcal{L}^L = \frac{\alpha}{4\pi}\frac{dy\,dQ^2}{Q^2}\sqrt{\frac{y^2(s-m_3^2)^2 + 4Q^2m_3^2}{(s-m_3^2)^2}}\frac{(2-y+\frac{Q^2}{4E_1^2})^2}{(y-\frac{Q^2}{4E_1^2})^2 + \frac{Q^2}{E_1^2}},$$

with the low-Q^2 limit of

$$d^2\mathcal{L}^L = \frac{\alpha}{4\pi}\frac{dy\,dQ^2}{yQ^2}(2-y)^2.$$

In the elastic limit ($\tau = 0$), the matrix $\mathcal{D}$ becomes much simpler. At high energy we obtain obtain

$$\mathcal{D}^I_{\lambda\rho} = s\begin{pmatrix} 0 & 0 & 0 \\ 0 & (1-\hat{y}) & \sqrt{1-\hat{y}} \\ 0 & \sqrt{1-\hat{y}} & 1 \end{pmatrix}.$$

The expressions for the longitudinal and transverse fluxes in this limit become

$$d\mathcal{L}^L \to \frac{\alpha}{\pi}\frac{dQ^2}{Q^4}E_1^2\sqrt{\frac{\hat{y}^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}},$$

and

$$d\mathcal{L}^T \to \frac{\alpha}{\pi}\frac{dQ^2}{Q^4}E_1^2\sqrt{\frac{\hat{y}^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}}(1-\hat{y}).$$

7 Flux Matrix for Photons from Spin-1/2 Fermions

Using the definitions above we obtain the following matrix elements for the polarization flux factors for photon emission from Spin-1/2 particles of negligible mass in the presence of the field from particle 3.

$$\mathcal{D}^{II}_{\lambda\rho} = 2\sum_{\lambda'\rho'}\epsilon^{*\lambda'}_{\mu}\epsilon^{\rho'}_{\nu}[P_1^\mu P_2^\nu + P_2^\mu P_1^\nu - \frac{Q^2}{2}g^{\mu\nu}]\eta_{\lambda'\lambda}\eta_{\rho'\rho}.$$

We obtain the following density matrix with rows and columns denoting $\lambda,\rho = 1,2,3$ respectively

$$\mathcal{D}^{II}_{\lambda\rho} = \frac{[2E_1Q]^2}{[2E_1\tau]^2+Q^2}\begin{pmatrix} \tau^2+\frac{Q^2}{4E_1^2} & 0 & 0 \\ 0 & 1-y+\tau^2+\frac{Q^2}{4E_1^2} & (1-\tau)\sqrt{1-y} \\ 0 & (1-\tau)\sqrt{1-y} & 1-y \end{pmatrix}.$$

Using this matrix together with the other factors in the definition of the flux matrix gives the final result for emission of photons from Spin-1/2 Fermions. Notice that the above density matrix has the property that all the longitudinal components and off-diagonal elements vanish for backwards scattering $y = 1$. The leftover terms are the two diagonal components from the transverse polarization vectors which are both equal to each other and, taking into account the factors in front, proportional to Q^2. The limit $y = 1$ limit therefore corresponds to the overall flux having a $1/Q^2$ dependence. Another point is that the vanishing of the longitudinal elements implies that backwards scattered Spin-1/2 particles couple only to transverse photons [3]. This property is converse to the Spin-0 coupling illustrated above. In the Breit-Frame, the density matrix for Spin-1/2 particles coupling to photons becomes

$$\mathcal{D}^{II}_{\lambda\rho} = \frac{Q^2}{y}\begin{pmatrix} y & 0 & 0 \\ 0 & 1 & \sqrt{1-y} \\ 0 & \sqrt{1-y} & 1-y \end{pmatrix}.$$

Using circular polarizations the density matrix becomes in this case

$$\mathcal{C}^{II}_{\lambda\rho} = \frac{1}{2}\frac{[2E_1Q]^2}{[2E_1\tau]^2+Q^2]}\begin{pmatrix} 1-y+2\tau^2+\frac{Q^2}{2E_1^2} & \sqrt{2}(1-\tau)\sqrt{1-y} & 1-y \\ \sqrt{2}(1-\tau)\sqrt{1-y} & 1-y & \sqrt{2}(1-\tau)\sqrt{1-y} \\ 1-y & \sqrt{2}(1-\tau)\sqrt{1-y} & 1-y+2\tau^2+\frac{Q^2}{2E_1^2} \end{pmatrix}.$$

Let us consider the sum of the two transverse fluxes and define $d^2\mathcal{L}^T = d^2\mathcal{L}^{11} + d^2\mathcal{L}^{22}$. Making the same substitution as in the Spin-0 case for the sum $q_0^2 + Q^2$ we obtain

$$d^2\mathcal{L}^T = \frac{\alpha}{2\pi}\frac{dy\,dQ^2}{Q^2}\sqrt{\frac{y^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}}\left[\frac{2(1-y)+(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}}\right].$$

In the limit $Q^2 \to 0$, this reduces to the usual Weizsäcker-Williams Approximation.

$$d^2\mathcal{L}^T = \frac{\alpha}{2\pi}\frac{dy\,dQ^2}{yQ^2}[1+(1-y)^2].$$

Defining the longitudinal flux as $d^2\mathcal{L}^L = d^2\mathcal{L}^{33}$ we obtain

$$d^2\mathcal{L}^L = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{Q^2}\sqrt{\frac{y^2(s-m_3^2)^2+4Q^2m_3^2}{(s-m_3^2)^2}}\frac{(1-y)}{(y-\frac{Q^2}{4E_1^2})^2+\frac{Q^2}{E_1^2}},$$

with the low-Q^2 limit of

$$d^2\mathcal{L}^L = \frac{\alpha}{\pi}\frac{dy\,dQ^2}{yQ^2}(1-y).$$

In the elastic limit the density matrix for Spin-1/2 at high energy becomes

$$\mathcal{D}^{II}_{\lambda\rho} \to s\begin{pmatrix} \hat{y} & 0 & 0 \\ 0 & 1 & \sqrt{1-\hat{y}} \\ 0 & \sqrt{1-\hat{y}} & (1-\hat{y}) \end{pmatrix}.$$

8 Examples

We now present some examples of interactions and how the calculations using polarization vectors proceed. We make use the Mandelstam variables defined by

$$\begin{aligned} s &= (P_1+P_3)^2 = (P_2+P_4) \approx 2P_1\cdot P_3 \approx 2P_2\cdot P_4 \\ t &= (P_1-P_2)^2 = (P_3-P_4) \approx -2P_1\cdot P_2 \approx -2P_3\cdot P_4 \\ u &= (P_1-P_4)^2 = (P_2-P_3) \approx -2P_1\cdot P_4 \approx -2P_2\cdot P_4 \end{aligned}$$

The relationship between these variables and (y, Q^2) is given by

$$Q^2 = -t^2$$

$$
\begin{aligned}
y &= \frac{s+u}{s} \\
1-y &= -\frac{u}{s} \\
2-y &= \frac{s-u}{s}
\end{aligned}
$$

Expressing the relationship between the double sum over all the polarization contributions to the square of the matrix element we can express this sum in compact form using the definition of matrix product and trace. We have

$$\sum_{\lambda\rho} \mathcal{D}^{*i}_{\lambda\rho}\mathcal{D}^{f}_{\lambda\rho} = \mathrm{Trace}[\mathcal{D}^{i\dagger}\mathcal{D}^{f}],$$

where $\mathcal{D}^{i\dagger}$ is the complex conjugate transpose of $\mathcal{D}^{i}$.

The general form of the cross section applicable to the two body case is

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4st^2}\mathrm{Trace}[\mathcal{D}^{i\dagger}\mathcal{D}^{f}],$$

where i, f are used to denote the density matrix of the two vertices of photon interaction. It is interesting to note that if an orthogonal transformation was performed on the $\mathcal{D}$ matrices, then the resulting cross section would be invariant because the overall interaction involves taking a trace of the product of a matrix and its transpose-conjugate which is an invariant operation under orthogonal transformations. This is connected with rotational invariance of the sum in Equation (1).

8.1 Two Non-Identical Spin-0 Bosons

Now we use the above derived quantities to examine the contributions to well known cross sections coming from the different elements in the density matrix. Consider the simple case of two charged Spin-0 particles scattering via photon exchange in their center-of-mass system. In this case we can work with the elastic limit because this is the case of elastic scattering. In this case the square of the matrix element for the absorption of the photon at the vertex involving the photon and particles 3 and 4 is

$$\overline{|M_{R_2}|^2}^{\lambda\rho} = e^2[\epsilon^{\lambda}_{\mu}(P_3+P_4)^{\mu}][\epsilon^{\rho}_{\nu}(P_3+P_4)^{\nu}],$$

which at high energy is

$$\overline{|M_{R_2}|^2}^{\lambda\rho} = e^2\mathcal{D}^{I}_{\lambda\rho}.$$

For high-energy elastic scattering we obtain

$$\mathcal{D}^{I}_{\lambda\rho} = s \begin{pmatrix} 0 & 0 & 0 \\ 0 & (1-\hat{y}) & \sqrt{1-\hat{y}} \\ 0 & \sqrt{1-\hat{y}} & 1 \end{pmatrix},$$

where $\hat{y} = \frac{Q^2}{4E_1^2}$. Since we have azimuthal symmetry in the above we obtain in the limit of high energy

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4st^2}\text{Trace}[\mathcal{D}^{I\dagger}\mathcal{D}^{I}],$$

which reduces to

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2 s}{4t^2}[2-\hat{y}]^2.$$

In terms of the Mandelstam variables this has the form

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4s}\left[\frac{s-u}{t}\right]^2,$$

which can be written as

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4s}\left[\frac{3+\cos\theta}{1-\cos\theta}\right]^2.$$

This is the well known form of the cross section and if the transverse-longitudinal interference terms had been neglected the result would have been different. We would have obtained in that case

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2 s}{4t^2}[1+(1-\hat{y})^2],$$

which reduces to

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4s}\left[\frac{5+2\cos\theta+\cos^2\theta}{(1-\cos\theta)^2}\right].$$

We conclude that it is important to consider the terms involving interference between different polarizations.

8.2 One Spin-1/2 Fermion and One Spin-0 Boson

Again we can make use the above derived matrices to derive the elastic limit result for Spin-1/2 particles interacting with Spin-0 particles. In this case we have

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4st^2}\text{Trace}[\mathcal{D}^{I\dagger}\mathcal{D}^{II}].$$

In terms of the Mandelstam variables this becomes

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \alpha^2\frac{(s-u)}{t^2},$$

which can be written as

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{2\alpha^2}{s}\left[\frac{1+\cos\theta}{(1-\cos\theta)^2}\right].$$

Again this is a well known cross section and the interference between different polarization states is important.

8.3 Two Non-Identical Spin-1/2 Fermions

In this case we have

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{4st^2}\mathrm{Trace}[\mathcal{D}^{II\dagger}\mathcal{D}^{II}],$$

which produces

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2 s}{4t^2}\left[\hat{y}^2 + (2-\hat{y})^2\right],$$

and this can be further reduced to

$$\frac{d^2\sigma_{R_1}}{d\Omega} = \frac{\alpha^2}{2s}\left[\frac{(s^2+u^2)}{t^2}\right].$$

The interference between different polarization states plays an important role in this result as well.

8.4 Unpolarized Lepton-Nucleon Scattering

In the case where the matrix elements for the reaction R_2 have equal transverse terms, we can sum the transverse polarization contributions to the flux and average the final interaction terms together and form a correct total sum. Let us consider the standard form of the Lepton-Nucleon Interaction. The square of the proton current is usually represented by a general tensor of the form

$$W^{\mu\nu} = \frac{W_2}{m_3^2}\left[P_3^\mu - \frac{P_3\cdot q}{q^2}q^\mu\right]\left[P_3^\nu - \frac{P_3\cdot q}{q^2}q^\nu\right] + W_1\left[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right].$$

Defining the photon interaction elements in the usual way we have

$$\mathcal{W}^{\lambda\rho} = \epsilon_\mu^\lambda W^{\mu\nu}\epsilon_\nu^{*\rho}.$$

The polarization vectors are orthogonal to q and therefore the result of the above contraction is

$$\mathcal{W}^{\lambda\rho} = \frac{W_2}{m_3^2}[\epsilon^\lambda\cdot P_3][\epsilon^{*\rho}\cdot P_3] - W_1[\epsilon^\lambda\cdot\epsilon^{*\rho}].$$

The above tensor is related to the proton current by

$$4\pi e^2 m_3 W^{\mu\nu} = (2\pi)^2 \delta^2(P_3 + q - P_4) j_p^\mu j_p^\nu \prod_k \frac{d^3 p_k}{(2\pi)^3 2E_k},$$

where the square of the proton current is defined to be the square of the γp matrix element

$$j_p^\mu j_p^\nu = \overline{|M_{\gamma p}|^2}^{\mu\nu}.$$

The relation to the γP cross section is given in terms of the polarizations as

$$\sigma_{\gamma p}^{\lambda\rho} = \frac{8\pi^2 \alpha m_3}{\sqrt{y^2(s-m_3^2)^2 + 4Q^2 m_3^2}} \mathcal{W}^{\lambda\rho}.$$

Evaluating this matrix element as in the above cases except to allow the possibility that the target has a range of velocities (i.e. $P_3 = (E_3, 0, 0, \beta E_3)$, $\gamma = E_3/m_3$) we obtain

$$\mathcal{W}^{\lambda\rho} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & \beta^2\gamma^2 W_2 \frac{(1-y)Q^2}{q_0^2+Q^2} + W_1 & \beta\gamma^2 W_2 \sqrt{1-y}\left[1 + \frac{\beta q_0(q_0+Q^2/2E_1)}{q_0^2+Q^2}\right] \\ 0 & \beta\gamma^2 W_2\sqrt{1-y}\left[1+\frac{\beta q_0(q_0+Q^2/2E_1)}{q_0^2+Q^2}\right] & \frac{\gamma^2 W_2}{Q^2(q_0^2+Q^2)}\left[q_0^2 + Q^2 + \beta q_0(q_0 + \frac{Q^2}{2E_1})\right]^2 - W_1 \end{pmatrix}.$$

In the ultra-relativistic case $\beta \to 1$ and boosting to the Breit-Frame the above interaction matrix assumes a simpler form given by

$$\mathcal{W}^{\lambda\rho} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & \frac{W_2}{m_3^2}\frac{s^2 y(1-y)}{4Q^2} + W_1 & \frac{W_2}{m_3^2}\frac{s^2 y\sqrt{1-y}}{4Q^2} \\ 0 & \frac{W_2}{m_3^2}\frac{s^2 y\sqrt{1-y}}{4Q^2} & \frac{W_2}{m_3^2}\frac{s^2 y}{4Q^2} - W_1 \end{pmatrix},$$

but the mixture of transverse and longitudinal in the Breit-Frame is different than in the lab. One obtains the same results by summing over all the polarization states. Again, as one can see from the above the interference terms in matrix elements do not vanish and should be included in a correct description of Lepton-Nucleon Scattering. Taking the limit $\beta \to 0$ ($E_3 = m_3$) gives the well-known result [4] for a stationary target which no longer possesses interference terms,

$$\mathcal{W}^{\lambda\rho} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & W_2\left[\frac{q_0^2+Q^2}{Q^2}\right] - W_1 \end{pmatrix}.$$

In the $\beta = 0$ limit the photon interaction matrix is diagonal and the two transverse terms are equal to each other. There is no interference in this limit. The same is true for the $\mathcal{D}^I$ and $\mathcal{D}^{II}$ matrices defined before. They were derived in the ultrarelativistic region, but had they been evaluated in the rest frame the result would have been diagonal matrices with equal transverse elements. Therefore it is a general principle that there are no interference effects in the rest frame of a charged particle.

In the above limit the two diagonal terms for the transverse polarizations are equal and one may add up the flux terms together for each and use an effective summed transverse flux (i.e. $d^2\mathcal{L}^T$ mentioned above for the Spin-1/2 Photon Flux). As was indicated above this summed flux reduces to the WWA in the limit of small Q^2. The above example shows the conditions necessary in order to add flux terms together based on the terms in the interaction matrix element. There are, in general, interference effects in Lepton-Nucleon Scattering between photon polarization states that do not show up in the specific limiting case of $\beta = 0$. If one uses the limit $\beta = 0$, then the total ep interaction has the form

$$d^2\sigma_{ep} = d^2\mathcal{L}^T\sigma^T_{\gamma p} + d^2\mathcal{L}^T\sigma^L_{\gamma p},$$

with the γp cross sections given by their usual expressions in terms of the structure functions

$$\sigma^T_{\gamma p} = \frac{8\pi^2\alpha m_3}{\sqrt{y^2(s-m_3^2)^2 + 4Q^2m_3^2}} W_1,$$

and

$$\sigma^L_{\gamma p} = \frac{8\pi^2\alpha m_3}{\sqrt{y^2(s-m_3^2)^2 + 4Q^2m_3^2}} \left[\left(\frac{q_0^2+Q^2}{Q^2}\right) W_2 - W_1\right].$$

However, in the general case when the proton is moving, this form is not complete; and the full expression involves the total flux matrix for Spin-1/2 Leptons and the total photon interaction matrix for the Lepton-Nucleon current. We have the following ingredients

$$\begin{aligned}
d^2\mathcal{L}_{\lambda\rho} &= \frac{\alpha}{4\pi}\frac{dy\,dQ^2}{Q^4}\sqrt{\frac{y^2(s-m_3)^2+4Q^2m_3^2}{(s-m_3^2)^2}}\,\mathcal{D}_{\lambda\rho}, \\
\sigma^{\lambda\rho}_{\gamma p} &= \frac{8\pi^2\alpha m_3}{\sqrt{y^2(s-m_3)^2+4Q^2m_3^2}}\,\mathcal{W}^{\lambda\rho}, \\
d^2\sigma_{ep} &= \sum_{\lambda\rho} d^2\mathcal{L}_{\lambda\rho}\sigma^{\lambda\rho}_{\gamma p}.
\end{aligned}$$

We therefore obtain for the general case

$$\frac{d^2\sigma_{ep}}{dy\,dQ^2} = \frac{2\pi\alpha^2 m_3}{(s-m_3^2)Q^2}\Bigg(2W_1 + \frac{4E_1^2(1-y)}{q_0^2+Q^2}\left[\frac{\gamma^2 W_2}{Q^2(q_0^2+Q^2)}[q_0^2+Q^2+\beta q_0(q_0+Q^2/2E_1)]^2\right]$$
$$+ \left[\frac{4E_1^2(1-y)}{q_0^2+Q^2}+1\right]\frac{\beta^2\gamma^2 W_2 Q^2(1-y)}{q_0^2+Q^2}$$
$$+ \frac{4E_1(E_1+E_2)\beta\gamma^2 W_2(1-y)}{q_0^2+Q^2}\left[1+\frac{\beta q_0(q_0+Q^2/2E_1)}{q_0^2+Q^2}\right]\Bigg).$$

The factors of the form $q_0 + Q^2/2E_1$ contain the quantities that define the Breit-Frame boosts. Assuming that P_3 is ultra-relativistic this general form reduces, using the Breit-Frame representation, to

$$\frac{d^2\sigma_{ep}}{dy\,dQ^2} = \frac{2\pi\alpha^2 m_3}{sQ^4}\left[s^2(1-y)\frac{W_2}{m_3^2}+2Q^2W_1\right].$$

We compare this general form with the approximation to the total ep cross section based on the two terms often used in the literature [4], $d^2\sigma_{ep} \approx d^2\mathcal{L}^T\sigma^T_{\gamma p} + d^2\mathcal{L}^L\sigma^L_{\gamma p}$. We have at high energy

$$\frac{d^2\sigma_{ep}}{dy\,dQ^2} \approx \frac{2\pi\alpha^2 m_3}{sQ^2}\left(2W_1 + \frac{4E_1^2(1-y)}{q_0^2+Q^2}\left[\frac{E_3^2}{Q^2(q_0^2+Q^2)}[q_0^2+Q^2+q_0(q_0+Q^2/2E_1)]^2\frac{W_2}{m_3^2}\right]\right).$$

This expression reduces in the low-Q^2 limit to

$$\frac{d^2\sigma_{ep}}{dy\,dQ^2} = \frac{2\pi\alpha^2 m_3}{sQ^4}\left[s^2(1-y)\frac{W_2}{m_3^2}+2Q^2W_1\right].$$

As can be seen, in the ultrarelativistic region and at low-Q^2, there is no difference between the exact formulation and the approximation resulting from adding the two terms $d^2\mathcal{L}^2\sigma^T_{\gamma p}+d^2\mathcal{L}^L\sigma^L_{\gamma p}$. We can express the above results in terms of the usual structure functions F_1 and F_2 by substitution of the forms of these structure functions given by

$$W_1 = \frac{F_1}{m_3},$$
$$W_2 = \frac{2m_3x}{Q^2}F_2.$$

At this point one can make the substitutions and investigate the implications of various models. For example the Callen-Gross relation specifies that $2xF_1 = F_2$. Then F_1 could be eliminated and one would have a relationship between the total ep cross section and F_2 of the form (valid at high energy and low-Q^2) given by

$$\frac{d^2\sigma_{ep}}{dy\,dQ^2} = \frac{2\pi\alpha^2}{sQ^4}\left[2(1-y)\frac{s^2x}{Q^2}+\frac{Q^2}{x}\right]F_2.$$

Transforming to the variables (x, Q^2) this becomes

$$\frac{d^2\sigma_{ep}}{dx\, dQ^2} = \frac{2\pi\alpha^2}{xQ^4}\left[2(1-y)+y^2\right] F_2.$$

Another interesting limit to consider for Lepton-Nucleon Scattering is the elastic limit defined by equal energies for the incoming and outgoing lepton: $E_1 = E_2$, $y = \frac{Q^2}{4E_1^2}$, $q_0 = 0$. Putting these values into the photon interaction matrix and using the notation $\hat{y} = \frac{Q^2}{4E_1^2}$ we obtain

$$\mathcal{W}^{\lambda\rho} = \begin{pmatrix} W_1 & 0 & 0 \\ 0 & \beta^2\gamma^2 W_2(1-\hat{y}) + W_1 & \beta\gamma^2 W_2\sqrt{1-\hat{y}} \\ 0 & \beta\gamma^2 W_2\sqrt{1-\hat{y}} & W_2\gamma^2 - W_1 \end{pmatrix}.$$

The elastic limit is interesting because it indicates how the interference dies as $\beta \to 0$ or $\hat{y} \to 1$. By examining the relative magnitudes of the matrix elements we see that the interference terms are important in the interpretation of the cross section.

9 Conclusions

We have illustrated working with photon polarization vectors to obtain matrices of flux factors for photon-mediated interactions. In certain cases one can sum over the various polarization states. The Weizsäcker-Williams Approximation results from summing the transverse flux components and taking the high energy and low-Q^2 limit. The off-diagonal terms in the photon density matrices represent magnitudes of interferences among the three different polarizations possible for the virtual photon. Whether or not interference is important depends on the nature of the final photon interaction matrix element, but is potentially sizeable. Also, if the photon interaction matrix element has inhomogeneities between the different polarization states, then one must multiply each term by the appropriate flux and form the sum over all the contributions. If the matrix element is homogeneous then one can sum the fluxes and then multiply the sum by the photon interaction cross section. The implication – to be careful about lumping together different polarization contributions in cross sections – is demonstrated with well known examples. In studying cross sections for Lepton-Nucleon scattering with arbitrary kinematics it is important to consider the magnitude of the contributions coming from interference among polarization states.

10 Acknowledgements

I would like to thank colleagues at the University of California, Davis, Physics Department especially Richard L. Lander, Iris Abt, and Sean Willard for helpful discussions and encouragement. Thanks especially to Sean for many suggestions. This work was supported in part by the Intercampus Institute for Research at Particle Accelerators (IIRPA) and USDOE grant DE FG03 91ER40674.

11 Appendix

In this section we describe a simple method to translate from the linear polarization representation matrix, $\mathcal{D}^{\lambda\rho}$, to the circular polarization representation matrix, $\mathcal{C}^{mn}$. Using the definitions we have that the representation of a tensor $M^{\mu\nu}$ in terms of the polarization states is

$$\mathcal{M}^{\lambda\rho} = \epsilon^{*\lambda}_{\mu} M^{\mu\nu} \epsilon^{\rho}_{\nu}.$$

This holds for either set of polarization vectors. Hence we can use the definition of the circular polarization vectors to determine the relationship among the matrix elements. The relationship that we use between the vectors is

$$\begin{aligned} \epsilon^{+} &= \frac{\epsilon^2 + i\epsilon^1}{\sqrt{2}}, \\ \epsilon_0 &= \epsilon^3, \\ \epsilon^{-} &= \frac{\epsilon^2 - i\epsilon^1}{\sqrt{2}}. \end{aligned}$$

The circular polarization vectors satisfy the relation $\epsilon^{\pm *} = \epsilon^{\mp}$, so we have

$$\begin{array}{lll} \mathcal{M}^{++} = (\mathcal{M}^{22} + \mathcal{M}^{11})/2, & \mathcal{M}^{+0} = (\mathcal{M}^{23} - i\mathcal{M}^{13})/\sqrt{2}, & \mathcal{M}^{+-} = (\mathcal{M}^{22} - \mathcal{M}^{11})/2, \\ \mathcal{M}^{0+} = (\mathcal{M}^{32} + i\mathcal{M}^{31})/\sqrt{2}, & \mathcal{M}^{00} = \mathcal{M}^{33}, & \mathcal{M}^{0-} = (\mathcal{M}^{32} - i\mathcal{M}^{31})/\sqrt{2}, \\ \mathcal{M}^{-+} = (\mathcal{M}^{22} - \mathcal{M}^{11})/2, & \mathcal{M}^{-0} = (\mathcal{M}^{23} + i\mathcal{M}^{13})/\sqrt{2}, & \mathcal{M}^{--} = (\mathcal{M}^{22} + \mathcal{M}^{11})/2. \end{array}$$

In the case where all the cross terms with ϵ^1 vanish we have the particular case that is satisfied by our system of basis vectors

$$\begin{array}{lll} \mathcal{M}^{++} = (\mathcal{M}^{22} + \mathcal{M}^{11})/2, & \mathcal{M}^{+0} = \mathcal{M}^{23}/\sqrt{2}, & \mathcal{M}^{+-} = (\mathcal{M}^{22} - \mathcal{M}^{11})/2, \\ \mathcal{M}^{0+} = \mathcal{M}^{32}/\sqrt{2}, & \mathcal{M}^{00} = \mathcal{M}^{33}, & \mathcal{M}^{0-} = \mathcal{M}^{32}/\sqrt{2}, \\ \mathcal{M}^{-+} = (\mathcal{M}^{22} - \mathcal{M}^{11})/2, & \mathcal{M}^{-0} = \mathcal{M}^{23}/\sqrt{2}, & \mathcal{M}^{--} = (\mathcal{M}^{22} + \mathcal{M}^{11})/2. \end{array}$$

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