THIRD ORDER DIFFERENTIAL EQUATION POSSESSING THREE SYMMETRIES

(The Two Homogeneous Ones plus the Time Translation)

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Abstract

The solvability of second order differential equations invariant under time translation and rescaling and third order differential equations invariant under time translation and two homogeneity symmetries is studied. We give in both cases analytic results for the solution of the equations up to a single quadrature instead of the usual two (resp three).

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1 Introduction

A differential equation

$$E\left(t, x, \dot{x}, \ldots, x^{(n)}\right) = 0, \tag{1}$$

where $x^{(n)}$ represents $\mathrm{d}^n x/\mathrm{d}t^n$, possesses a Lie symmetry

$$G = \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}, \tag{2}$$

where τ and η are C^n functions, if

$$G^{[n]} E_n|_{E_n=0} = 0, \tag{3}$$

where $G^{[n]}$ is the nth extension of G (required to determine the effect of the infinitesimal transformation induced by G on derivatives up to the nth) given by [1]

$$\begin{aligned} G^{[n]} &= G \\ &+ \sum_{i=1}^{n} \left\{\eta^{(i)} - \sum_{j=1}^{i} \begin{pmatrix} i \\ j \end{pmatrix} x^{(i+1-j)}\tau^{(j)}\right\} \frac{\partial}{\partial x^{(i)}}. \end{aligned} \tag{4}$$

The symmetry (2) is point if τ and η are functions of t and x [2], generalised if they contain derivatives of x with respect to t [3] and nonlocal if they contain integrals which cannot be evaluated without a knowledge of $x(t)$ [4]. In this paper we confine our attention to point symmetries.

The existence of a point symmetry enables the order of an equation to be reduced by one through the introduction of new variables based on the two invariants of $G^{[1]}$ which are obtained from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}x}{\eta} = \frac{\mathrm{d}\dot{x}}{\dot{\eta} - \dot{x}\dot{\tau}}. \tag{5}$$

If the reduced equation has a point symmetry, the order may again be reduced and, if there are sufficient point symmetries, the solution of the original nth order equation becomes the performance of n successive quadratures. A sufficient condition for the complete reduction of a nth order ordinary differential equation is that the equation possesses n point symmetries with a solvable Lie algebra [5]. However, as Type

II hidden symmetries [6] may arise in the reduction process, the condition is not necessary.

In the modelling of natural phenomena two symmetries, representing invariance under time translation and recycling, frequently occur. We write the generators as

$$G_1 = \frac{\partial}{\partial t} \qquad G_4 = -qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{6}$$

Since $[G_1, G_4] = -qG_1$ and $G_1 \neq \rho(t,x)G_4$, G_1 and G_4 constitute a representation of Lie's Type III two-dimensional algebra [2] and a study of second order ordinary differential equations possessing these two symmetries is a study of the class of second order ordinary differential equations possessing the Type III symmetry. We recall that second order ordinary differential equations with the Types II and IV algebras have linear canonical representations. This is not the case for second order ordinary differential equations with Type III symmetry. They are inherently nonlinear.

A related ordinary differential equation is the class of third order ordinary differential equations possessing the three symmetries associated with the generators

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also interesting to observe that a second order differential equation possessing the symmetries G_1 and G_4 is transformed into an equation of type (8) when we use the

Riccati transformation

$$x = \alpha \left(\frac{\dot{u}}{u}\right)^{1/q}, \quad \alpha = \left(1 + \frac{2}{q}\right)^{1/q} \tag{9}$$

To the symmetry of rescaling is associated G_4 with $xt^{1/q}$ as invariant.

The central purpose of this paper is to push the analytic calculations as far as possible in the solutions of these second order ordinary differential equations and third order ordinary differential equations. For instance, as was shown in [11], the most general form of a second order ordinary differential equation possessing the symmetries G_1 and G_4 is

$$\ddot{x} + x^{2q+1} f(\xi) = 0, \quad \xi = \frac{\dot{x}}{x^{q+1}}. \tag{10}$$

Equation (10) has a first integral $[[11], Eq.(8)]$ corresponding to the integration of

$$\frac{\mathrm{d}x}{x} + \frac{\xi \mathrm{d}\xi}{(q+1)\xi^2 + f(\xi)} = 0. \tag{11}$$

Two steps still remain for the solution process. Firstly we must explicitly obtain $\dot{x}$ as function of x using the first integral obtained from (11). Secondly

$$t = \int \mathrm{d}t = \int \frac{\mathrm{d}x}{\dot{x}(x)} \tag{12}$$

which solves the problem. Even in the case of function as simple as $f(\xi) = \xi + k$, an equation extensively studied in [12, 13, 14, 15], these two steps involve numerical solution of the algebraic relation connecting $\dot{x}$ and x and a numerical final quadrature. We shall see that for some functions $f(\xi)$ in (10) the first step can be performed analytically. Things are worse with the third order ordinary differential equation (8). We introduce the variable

$$\rho = \frac{x\ddot{x}}{\dot{x}^2} \tag{13}$$

which is the second order differential invariant common to G_1, G_2 and G_3. Writting $\ddot{x} = \rho \dot{x}^2/x$, computing $\dddot{x}$ and introducing (8) we obtain the first integral (in analogy with (11)) from the integration of

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}\rho}{F(\rho) + 2\rho^2 - \rho} = 0. \tag{14}$$

According to (13) this first integral contains a second derivative and so can be rewritten as a second order ordinary differential equation with the value of the integral as a parameter, the solution of which is not obvious. This paper deals with two problems. Firstly the solution of (8) pushing the analytical computation as far as possible for a certain class of the function $F(\rho)$ and secondly the solution of (10) going first to a third order ordinary differential equation after the use of the Riccati transformation as given by (9). The trick is to recognise that the possession of G_1, G_2 and G_3 is equivalent to having G_1 and G_4 with arbitrary values of q (in the first problem) and to take advantage of the arbitrariness of q. In the second problem we shall see that the third order ordinary differential equation obtained after the Riccati transformation has the three symmetries G_1, G_2 and G_3. This game of exchange between second order ordinary differential equation and third order ordinary differential equation brings quite useful results.

2 Third order ordinary differential equations of the type (8)

It was shown [[11], *Eq* (6)] that a third order ordinary differential equation invariant under time translation and a rescaling symmetry can be written as

$$\dddot{x} + x^{3q+1} f(\xi, \eta) = 0 \tag{15}$$

with

$$\xi = \frac{\dot{x}}{x^{q+1}}, \qquad \eta = \frac{\ddot{x}}{x^{2q+1}}, \tag{16}$$

where f is an arbitrary function of its arguments. The connection between the two forms as given by (8) and (15) (*ie* the relation between f and F) is easy to establish. Firstly the argument ρ as given by (13) is equal to η/ξ^2 for all q. The identification of the two terms in front of f and F gives

$$f(\xi, \eta) = \xi^3 F\left(\frac{\eta}{\xi^2}\right). \tag{17}$$

Moreover it was shown [[11], *Eq* (89)] that by applying the two symmetries G_1 and G_4 to (15) we obtain

$$\frac{d\eta}{d\xi} = \frac{f(\xi,\eta) + (2q+1)\xi\eta}{(q+1)\xi^2 - \eta}. \tag{18}$$

We introduce (17) and (18), change the variable ξ with $z = \frac{1}{2}\xi^2$ and reformulate (18) with z and ρ to obtain separation of variables and the following first integral

$$\frac{dz}{z} = \frac{2(q+1-\rho)}{F(\rho) + 2\rho^2 - \rho} d\rho. \tag{19}$$

Equation (19) is reminescent of (14) except that now q is at our disposal. An interesting possibility exists for $F(\rho)$ which are first or second order degree polynomials in ρ. It consists in selecting $q = A - 1$, where A is a root of the denominator of (19), *ie*

$$F(A) + 2A^2 - A = 0. \tag{20}$$

If we do not select $q = A - 1$, calling B the other root of (20) we obtain a first integral I as written

$$I = \xi^2 \left(\frac{\eta}{\xi^2} - A\right)^a \left(\frac{\eta}{\xi^2} - B\right)^b, \tag{21}$$

where a and b depend upon q and the coefficients of the polynomial $F(\rho)$ while the initial conditions are introduced *via* I as given by (15). If $F(\rho)$ is a first degree polynomial then $a + b = 1$. Reverting to $x, \dot{x}$ and $\ddot{x}$ we rewrite (21) in the following form

$$\left(\ddot{x} - A\frac{\dot{x}^2}{x}\right)^a \left(\ddot{x} - B\frac{\dot{x}^2}{x}\right)^b = I\dot{x}^{2(a+b-1)} x^{2(q+1)-(a+b)}. \tag{22}$$

We obtain a SODE similar to the one we could have obtained with (14) but certainly not a friendly one. Now we take q as explained above, *ie* $q + 1 = A$, where A is the root of (20). In that case we can take $a = 0$ and (22) becomes

$$\ddot{x} - B\frac{\dot{x}^2}{x} = J\dot{x}^\alpha x^\beta, \tag{23}$$

where

$$J = I^{1/b}, \quad \alpha = \frac{2(b-1)}{b}, \quad \beta = \frac{(2+2q-b)}{b}. \tag{24}$$

Now we take x as a new independent variable and

$$y = \frac{1}{p}\dot{x}^p$$

as a new function. We have

$$\ddot{x} = \frac{dy}{dx}(py)^{\frac{2}{p}-1} \tag{25}$$

and (23) is now written

$$\frac{dy}{dx}(py)^{\frac{2}{p}-1} - \frac{B}{x}(py)^{\frac{2}{p}} = Jx^\beta(py)^{\frac{\alpha}{p}}. \tag{26}$$

As p is at our disposal, we select it ao that $\alpha = 2 - p$. Equation (26) is now written as

$$\frac{dy}{dx} - Bp\frac{y}{x} = Jx^\beta. \tag{27}$$

Equation (27) is an inhomogeneous linear Euler type equation the solution of which is

$$y = \frac{Kx^{Bp} + Jx^{\beta+1}}{(\beta + 1 - Bp)}, \tag{28}$$

where initial conditions enter through K and J. The only numerical computation left is eventually the time scale with the simple quadrature

$$t = \int \mathrm{d}t = \int \frac{\mathrm{d}x}{\dot{x}} = \int \frac{\mathrm{d}x}{(py)^{\frac{1}{p}}}, \tag{29}$$

where y is given by (28).

It is interesting to observe a connection between the Painlevé analysis and the class of equations (15) into which we have introduced (17), *viz.*(8). When we apply to (8) the usual Painlevé analysis, *ie* we look for the behaviour around the time singularity, t_0, with

$$x \sim A_0(t - t_0)^s \tag{30}$$

and on balancing the terms in (15) we find that

$$s - 3 = s - 3. \tag{31}$$

So as one expects for a third order ordinary differential equation with the symmetries G_1, G_2 and G_3, we obtain an arbitrary value for s. We just have here the first part of the Painlevé analysis.

Now the dominant coefficient, A_0, is determined by the following functional equation obtained by introducing (30) into (15), where F is given by (17), *viz.*

$$A_0\left[(s-1)(s-2)+s^2F\left(\frac{s(s-1)}{s^2}\right)\right]=0. \tag{32}$$

A nontrivial solution for A_0 requires that the term in crochets be zero. If we select $s=-1/q$ and $q=\rho-1$, we obtain

$$A_0\left[F(\rho)+2\rho^2-\rho\right]=0. \tag{33}$$

We have already seen that the right hand member is equal to zero. Consequently the coefficient A_0 is arbitrary and the power $-1/q$ is a fixed value. This result corresponds to the obtaining of the resonance $r=0$ in the Painlevé analysis.

3 Transformation of a second order ordinary differential equation into a third order ordinary differential equation

We consider firstly (10) with $q=1$ and the Riccati transformation $x=\alpha\dot{u}/u$. A little calculation gives

$$\dddot{u}+\frac{\dot{u}^3}{u^2}\left[-3\rho+2+\alpha^2\mathrm{f}\left(\frac{\rho}{\alpha}-\frac{1}{\alpha}\right)\right]=0, \tag{34}$$

where ρ is given by (13) after we have replaced x by u. Indeed (34) is of type (8) with

$$F(\rho)=-3\rho+2+\alpha^2\mathrm{f}\left(\frac{\rho}{\alpha}-\frac{1}{\alpha}\right). \tag{35}$$

Since the analysis given in the section above shows that $F(\rho)$ must be a polynomial at most of second degree, we see that f must also be polynomial of degree one or two. We consequently consider

$$f(\xi)=\lambda\xi^2+\mu\xi+\nu. \tag{36}$$

The general equation with q arbitrary is

$$\ddot{x} + \lambda \frac{\dot{x}^2}{x} + \mu x^q \dot{x} + \nu x^{2q+1} = 0. \tag{37}$$

The essential parameters obtained by invariance of (37) under homothetic transformations are λ and ν/μ^2. Without loss of generality we take $\mu = 1$ and $\nu = k$ (which allows, by taking $\lambda = 0$, one to have the notation of the much studied case $f(\xi) = \xi + k$). With the Riccati transformation given by (9) we obtain for u after a little algebra

$$\dddot{u} + \left(\frac{\lambda + 1}{q} - 1\right)\frac{\ddot{u}^2}{\dot{u}} - \left(\frac{2\lambda}{q}\right)\frac{\dot{u}\ddot{u}}{u} + \left[\frac{k(q+2)^2 - 1 + \lambda}{q}\right]\frac{\dot{u}^3}{u^2} = 0\cdot \tag{38}$$

We introduce

$$y = \frac{1}{p}\dot{x}^p \tag{39}$$

and take u as new independent variable and y as the new function; we obtain

$$\dot{u} = py^{1/p}, \quad \ddot{u} = \frac{\mathrm{d}y}{\mathrm{d}u}(py)^{2/p-1} \tag{40}$$

$$\dddot{u} = \frac{\mathrm{d}^2 y}{\mathrm{d}u^2}(py)^{3/p-1} + (2-p)\left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2 (py)^{3/p-2}. \tag{41}$$

Introducing (40) and (41) into (38) we select p in order to cancel the term in $(\mathrm{d}y/\mathrm{d}u)^2$ which appears in the first and second terms on the right hand side of (41). We have

$$p = 1 + \frac{1}{q} + \frac{\lambda}{q} \tag{42}$$

and (38) is now

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} - 2\frac{\lambda}{qu}\frac{\mathrm{d}y}{\mathrm{d}u} + \left[\frac{k(q+2)^2 + \lambda - 1}{q}\right] p\frac{y}{u^2} = 0. \tag{43}$$

Equation (43) is an Euler type linear equation the solution of which is

$$y = A_1 u^{\mu_1} + A_2 u^{\mu_2}, \tag{44}$$

where the exponents μ_i $(i = 1, 2)$ are the roots of the characteristic equation

$$\mu^2 - \left(1 + \frac{2\lambda}{q}\right)\mu + \frac{q + 1 + \lambda}{q^2}\left[k(q+2)^2 - 1 + \lambda\right] = 0. \tag{45}$$

The derivatives $\dot{u}$ and $\dot{x}$ are given by

$$\dot{u} = (py)^{1/p} \quad \text{and} \quad \dot{x} = \left(1 + \frac{2}{q}\right)^{1/q} \frac{\dot{u}}{u}. \tag{46}$$

The only numerical computation is that of the time scale with

$$\begin{aligned} t &= \int \mathrm{d}t = \int \frac{\mathrm{d}u}{\dot{u}} = \int \frac{\mathrm{d}u}{(py)^{1/p}} \\ &= p^{-1/p} \int \frac{\mathrm{d}u}{(A_1 u^{\mu_1} + A_2 u^{\mu_2})^{1/p}}. \end{aligned} \tag{47}$$

The last integration is usually a numerical one, but in the case of $\lambda = 0$ and $k = 1/(q+2)^2$ we have

$$y = A_1 + A_2 u, \tag{48}$$

where A_1 and A_2 are some arbitrary constants and the final quadrature can be obtained analytically. This makes contact with an already obtained result for this case [[11], *Eq* (85)].

Finally (39) gives the critical value of k, for which we pass from an oscillating (periodic) solution to a nonperiodic one. The link is given by the appearance of complex roots in (45). This value is

$$k_c = \frac{1}{4(q+1+\lambda)}, \tag{49}$$

in agreement with the result $k = 1/8$ obtained in (49) for $\lambda = 0$ and $q = 1$ [13]. We finally state the following result. Third order ordinary differential equations possessing the three symmmetries (associated to G_i ($i = 1, 2, 3$) and second order ordinary differential equations possessing the two associated to G_1 and G_4, where F (for the third order ordinary differential equations) and f (for the second order ordinary differential equations), are polynomial of degree two or lower can be analytically solved up to a final quadrature which provides the time scale. In general a second order ordinary differential equation (third order ordinary differential equation) with two (three) point symmetries requires two (three) quadratures to obtain the solution.

4 Conclusion

This paper deals with the possibility of exhibiting explicit solutions for second order ordinary differential equations and third order ordinary differential equations possessing the necessary number of symmetries to be formally integrable. Knowledge of the associated invariant provides in the best case (second order ordinary differential equation) an algebraic formula connecting $\dot{x}$ and x. Two numerical steps are needed, the first to solve this equation and the second to compute the time scale. A change of initial conditions implies doing again this second step. We have shown that for functions f and F polynomials of at most second degree, the first step can be explicitly written (including the initial conditions) as an integrand with sometimes the possibility to evaluate the final integral. This property can be connected to the passing of the Painlevé test.

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