Fermilab

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To: J. Butteris, K. Coulter, S. Delchamps, W. Kinney, and M. Lamm
From: Jim Strait
Subj: Notes on Tangential Probes: Basic Equations

Attached are some of my notes on tangential probes. I have a moderate amount of material in my file on these probes, including an analysis of the sensitivity to various manufacturing errors. Most of it is not entirely self-explanatory, so it may be more productive for me to give you only that part which I think I can explain just now. The attached notes are various calculations which I will try to explain here.

Page 1 shows the assumed harmonic expansion given in terms of the radial component of the field. This is useful because the radial component is the component perpendicular to both the probe wires and their velocity. The voltage across a single wire segment parallel to the axis is

$$V(\text{1-wire}) = l \; r \; \omega \; B_r$$

where l is the length of the probe, r is its radius and ω is its angular velocity. In this expansion n = 0 is the dipole component, n = 1 is the quadrupole etc. I also define $\bar{n}$ = n+1 which appears in many places. Therefore $\bar{n}$ = 1 is the dipole, $\bar{n}$ = 2 is the quadrupole etc. The reference radius for the harmonic coefficients is ρ, which is 1 cm for SSC magnets and 1 inch for Tevatron magnets.

The voltage across a full probe winding is gotten by summing the voltages across the wire segments that make up the full winding, keeping appropriate track of signs. The 1-wire voltage is shown on page 1. Also shown is a tangential probe with 1 M-wire tangential winding and two 1-wire dipole bucking coils. The table at the bottom specifies the phase angles of each of the windings and the sign to be used in summing over the wires in each coil. The "bucking factor" 'd' is a quantity by which the signals from the dipole windings are multiplied so that when they are added to the tangential signal the dipole component is cancelled. This bucking can be done in an analog fashion by summing the probe signals through a set of resistor dividers before digitization or digitally by taking appropriate sums or differences between the probe signals after digitization. The former is the method used by magnetic measuring systems built at MTF and the latter is used by the mole.

For further calculations it is convenient to work with harmonic coefficients as magnitude and phase rather than as normal and skew components. On page 2 I work out the relation between the magnitude and phase (h_n and α_n) and the skew and normal components (a_n and b_n). On Page 3 I sum over the six axial wire segments in the figure on page 1. The quantity V_n is the bucked voltage from the probe for harmonic 'n'. I define the symbols C_n, S_n, F_n and γ_n in terms of which V_n looks particularly simple. Note that these four quantities (actually two independent quantities) are characteristics of the probe only and not of the field. F_n gives the sensitivity of the probe to harmonic 'n' and γ_n is a phase angle characteristic of the probe.

On page 4 I compute the integral Φ_n^T of the voltage, which is what HAL2 measures. I then write down the amplitude A_n^T and phase δ_n^T, which would be the output from the FFT, in terms of the the harmonic coefficients and probe related quantities. (I am not sure what the signficanct of the superscript "T" is supposed to be. It does not appear elsewhere in my notes.) I have explicitly included a phase ϕ which represents the phase at which the integral starts, that is, the angle between the zero of the encoder and the x-axis, where the dipole field is defined to lie along the y-axis.

On page 5.0 I calculate the bucking factor 'd' assuming that the two dipole coils have precisely the same geometry and the probe is a perfect representation of the figure on page 1. The bucking factor is calculated such that the dipole signal is exactly 0, that is $\Phi_0 = 0$. On page 5.1 I calculate separate bucking factors for the two dipole coils to allow for the case that they have different lengths and radii than each other and than the tangential winding. On pp. 5.2 and 5.3 I relax the assumption of perfect angular placement of each winding and let each winding have its own independent phase angle. This is the most general case calculated here and only makes the assumption that all the windings are perfectly parallel to the rotation axis.

On pp. 6 and 7 are some computations of C_n and S_n with the sums over the 6 windings explicitly carried out. Here each of the windings is allowed to have its own length and radius as is often the case by design in a real probe. The angles of the windings, however, are assumed to be ideal. That is, these equations are appropriate for a "real" probe with no manufacturing errors. These equations show that C_n and S_n have different forms for n = even and n = odd. Note that $C_n = 0$ for all n = odd and that if the two dipole bucking coils are the same length then $S_n = 0$ for all n = even also. Page 8 has some helpful relations about sin's and cos's that are used on pp. 6 and 7. On page 9 is a comparison between some C_n's and S_n's, but it is not immediately clear to me what the significance of this is. (Occasionally there are numbers plugged in for some of the parameters. These are obviously taken from some real probe, but I do not remember which one it is.)

Pages 10.0 and 10.1 are computations of similar quantities for a Morgan coil. These formulas could be used to extract harmonic coefficients from the HAL2 Morgan coil data just taken on DS0311. Page 11 shows how one would extract harmonic coefficients from a tangential coil which was analyzed by a program that only knew about Morgan coils. (This, presumably is of little interest to us and was done only because at some point at MTF the first tangential probe was ready before the corresponding software was ready. This computation applies to the case in which the Morgan coil output is analyzed with an FFT and it does not obviously apply to the Magnetometer's algorithm.)

The the figure on page 1 represents the geometry of the mole probe. I believe that Peter Mazur's "DDL" probes have an additional dipole winding which is parallel to the tangential winding, that is at $\phi = 0$ and 180°. This winding is termed the "belly band." In this case, by judicious choices of the number of turns in the tangential winding and its opening angle β the bucking factor can be set to 1. Then the tangential winding and the belly band can be wired in series and out of phase and the combination will have no sensitivity to the dipole field except for manufacturing errors. The other two dipole coils are needed only to buck out the dipole signal that remains due to these errors and their bucking factors will be very small. In this case digital bucking is easier or, if analog bucking is used the results are less sensitive to thermal drifts in the resistor divider network. If the probe is well enough made the other bucking coils may not need to be used. It is left as an exercise to the reader to work out the sensitivity factors F_n and γ_n for the DDL design.

In a future note (which I will try to write "soon") I will show output from a FORTRAN program (which is mentioned at the top of p. 5.2) which computes the probe sensitivity functions C_n, S_n, F_n and γ_n as a function of n and the effect of various probe manufacturing errors. This note will include examples of sensivity functions and plausible manufacturing errors for a particular probe geometry and may be useful for "getting a feel" for how tangential probes work.

cc: S. Gourlay
W. Koska

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^{n} (a_n \cos\bar{n}\theta + b_n \sin\bar{n}\theta) \qquad \bar{n} \equiv n+1$$

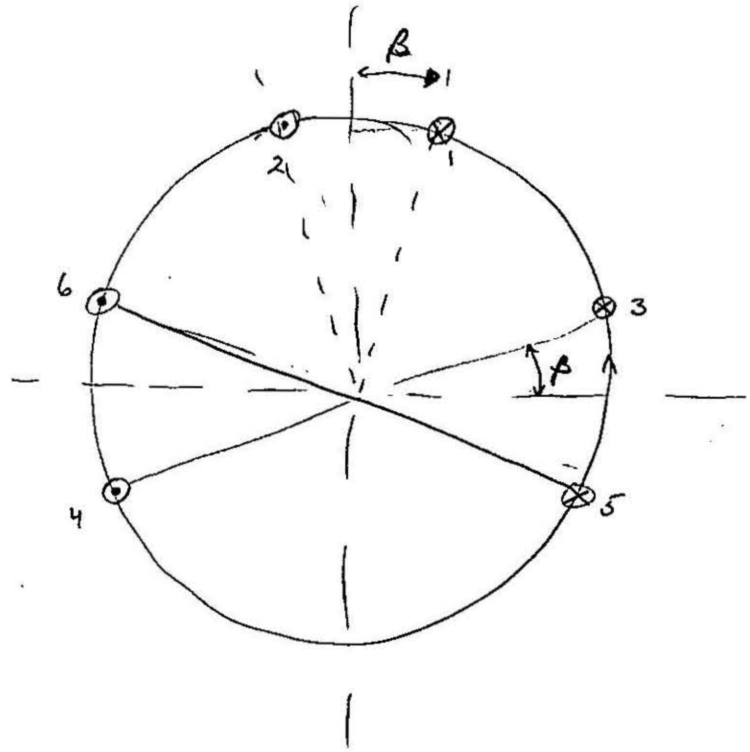
$$\theta = (\omega t + \phi)$$

$$V^{(t)}_{1\,WIRE} = \omega \ell B_0 \sum_{n=0}^{\infty} \frac{r^{\bar{n}}}{\rho^{n}} \left[a_n \cos(\bar{n}\omega t + \bar{n}\phi) + b_n \sin(\bar{n}\omega t + \bar{n}\phi)\right]$$

for the i-th winding

wire sense x bucking factor

$$v_i = B_0 \omega f_i N_i \ell_i \sum_{n=0}^{\infty} \frac{r_i^{\bar{n}}}{\rho^{n}} \left[a_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_i)\right]$$



$\ell\, v \times B$

i	ϕ_i	N_i	f_i	ℓ_i	r_i
1	$90-\beta$	M	$+1$	$\ell_1 = 42.028$	$r_1 =$ ~~.585~~ .586
2	$90+\beta$	M	-1	ℓ_1	r_1
3	β	1	$-d$	$\ell_3 = 42.563$	$r_3 =$ ~~.595~~ .605
4	$180+\beta$	1	$+d$	ℓ_3	r_3
5	$-\beta$	1	$-d$	$\ell_5 = 42.417$	r_3
6	$180-\beta$	1	$+d$	ℓ_5	r_3

$\beta_{nominal} = 8.2$

or L_{mag} if $L_{mag} < \ell_i$

$$a_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_i)$$

$$= h_n \left[\sin\alpha_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + \cos\alpha_n \sin(\bar{n}\omega t + \bar{n}\phi_i)\right]$$

$$= h_n \sin(\bar{n}\omega t + \alpha_n + \bar{n}\phi_i)$$

$$= h_n \left[\sin(\bar{n}\omega t + \alpha_n)\cos\bar{n}\phi_i + \cos(\bar{n}\omega t + \alpha_n)\sin\bar{n}\phi_i\right]$$

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n h_n \sin(\bar{n}\theta + \alpha_n)$$

$$= B_0 \sum_{n=0}^{\infty} h_n (\sin\alpha_n \cos\bar{n}\theta + \cos\alpha_n \sin\bar{n}\theta)$$

$$\Rightarrow \quad a_n = h_n \sin\alpha_n$$

$$b_n = h_n \cos\alpha_n$$

$$V^{(t)} = \sum_{i=1}^{6} v_i = B_0 \bar{\omega} \sum_{n=0}^{\infty} \frac{1}{\rho^n} \sum_{i=1}^{6} f_i^{\bar{n}} N_i l_i r_i^{\bar{n}} h_n \, \cancel{c_n}\, e^{i(\bar{n}\omega t + \alpha_n)} e^{i\bar{n}\phi_i}$$

$$\Phi_n = \int_0^t V_n(t)\,dt$$

$$= -B_0 \frac{1}{\bar{n}\rho^n} \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \, Re\left(e^{i(\bar{n}\omega t + \alpha_n)} e^{i\bar{n}\phi_j}\right)\Big|_0^t$$

$$= -B_0 \frac{1}{\bar{n}\rho^n} Re\left\{\left(e^{i(\bar{n}\omega t + \alpha_n)} - e^{i\alpha_n}\right) \underbrace{\sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} e^{i\bar{n}\phi_j}}_{F e^{i\gamma}}\right\}$$

$$\cancel{V_n = \frac{B_0\omega}{\rho^n} h_n \sum_{j=1}^{6} f_j N_j l_j r_j^{n} \Big[}$$

$$V_n = \frac{B_0 \omega}{\rho^n} h_n \Big[\sin(\bar{n}\omega t + \alpha_n) \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \cos\bar{n}\phi_j + \cos(\bar{n}\omega t + \alpha_n) \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \sin\bar{n}\phi_j \Big]$$

$$\equiv \frac{B_0 \omega}{\rho^n} h_n \left[\sin(\bar{n}\omega t + \alpha_n) F \cos\gamma + \cos(\bar{n}\omega t + \alpha_n) F \sin\gamma\right]$$

$$F_n \cos\gamma_n \equiv \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \cos\bar{n}\phi_j \equiv C_n$$

$$F_n \sin\gamma_n \equiv \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \sin\bar{n}\phi_j \equiv S_n$$

$$F_n = \left(C_n^2 + S^2\right)^{1/2} \qquad \gamma_n = \tan^{-1}\left(\frac{S_n}{C_n}\right)$$

$$V_n = \frac{B_0 \omega}{\rho^n} h_n F_n \sin(\bar{n}\omega t + \alpha_n + \gamma_n)$$

$$\cancel{= \frac{B_0 \omega}{\rho^n} h_n F_n \cos(\bar{n}\omega t + \alpha_n + \gamma_n - \tfrac{\pi}{2})}$$

Integrate the voltage:

$$\int_0^{\theta/\omega} V(t)\,dt = \frac{B_0}{\bar{n}\rho^n} h_n F_n \left[\cos(\alpha_n + \gamma) - \cos(\bar{n}\theta + \alpha_n + \gamma_n)\right]$$

Drop the constant term (FFT "sums" the constant terms from all Φ_n and computes a "monopole term") absorb the (−) sign with $-\cos(\theta) = +\cos(\theta - \pi)$

$$\boxed{\Phi_n^{TC} = \frac{B_0}{\bar{n}\rho^n} F_n h_n \cos(\bar{n}\theta + \alpha_n + \gamma_n - \pi)}$$

$$\boxed{\text{Amplitude} \quad A_n^T = \frac{B_0}{\bar{n}\rho^n} F_n h_n} \rightarrow$$

Phase $\quad \tilde{\delta}_n^T = \alpha_n + \gamma_n - \pi$

If coil is rotated by an angle Ψ at $t=0$, this advances phase of signal from harmonic n by $\bar{n}\Psi$

Measured phase $\quad \delta_n^T = \alpha_n + \gamma_n + \bar{n}\Psi - \pi$

~~We measure Ψ from $\alpha_0 = 0 \Rightarrow \Psi = \delta_0^T - \gamma_0 + \pi$~~

$$\Rightarrow \alpha_n = \delta_n^T - \gamma_n - \bar{n}\Psi + \pi$$

$$\cancel{\boxed{\alpha_n = (\delta_n^T - \bar{n}\delta_0^T) - (\gamma_n - \bar{n}\gamma_0) + \pi(\bar{n} - 1)}}$$

Bucking factor "d" chosen to make $\Phi_0 = 0$

$\Rightarrow F_0 = 0 \Rightarrow C_0 = S_0 = 0$

$$C_0 = \sum_{j=1}^{6} f_j \, N_j \, l_j \, r_j \cos\phi_j$$

$$= M l_1 r_1 [\cos(90^\circ - \beta) - \cos(90 + \beta)] + l_3 r_3 d [\cos(180 + \beta) - \cos\beta] + l_5 r_3 d [\cos(180 - \beta) - \cos(-\beta)]$$

$$= M l_1 r_1 \, 2\sin\beta + d\, r_3 [-2l_3 \cos\beta - 2l_5 \cos\beta]$$

$$= M l_1 r_1 \, 2\sin\beta - 4 d\, r_3 l_3 \cos\beta \left(\frac{l_3 + l_5}{2 l_3}\right)$$

for $C_0 = 0$ $\Rightarrow$ $d = \frac{M}{2} \frac{r_1}{r_3} \frac{l_1}{l_3} \left(\frac{2l_3}{l_3 + l_5}\right) \tan\beta \approx \frac{M}{2} \tan\beta$

$$S_0 = \sum_{j=1}^{6} f_j \, N_j \, l_j \, r_j \sin\phi_j$$

$$= M l_1 r_1 [\sin(90 - \beta) - \sin(90 + \beta)] + l_3 r_3 d [\sin(180 + \beta) - \sin\beta] + l_5 r_3 d [\sin(180 - \beta) - \sin(-\beta)]$$

$$= M l_1 r_1 [\cos\beta - \cos\beta] - 2 l_3 r_3 d \sin\beta + 2 l_5 r_3 d \sin\beta$$

$$= 2 \cancel{l_3} r_3 d \sin\beta \left[\frac{l_5 - l_3}{\cancel{l_3}}\right] \approx 0$$

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Allow 2 bucking factors, one for each dipole coil

$$C_0 = M_1 l_1 r_1 2\sin\beta + 2r_3\cos\beta\left[d_3 l_3 + d_5 l_5\right] = 0$$

$$S_0 = 2r_3\sin\beta\left[l_5 d_5 - l_3 d_3\right] = 0$$

form: $A + B d_3 + C d_5 = 0$ $\quad A = M_1 l_1 r_1 2\sin\beta$

$B = 2r_3\cos\beta\, l_3$

$-D d_3 + E d_5 = 0$ $\quad C = 2r_3\cos\beta\, l_5$

$D = 2r_3\sin\beta\, l_3$

$AE - (BE + CD)d_3 = 0$ $\quad E = 2r_3\sin\beta\, l_5$

$$d_3 = \frac{AE}{BE+CD} =$$

$$AD - (CD - BE)d_5 = 0$$

$$d_5 = \frac{AD}{CD+BE} = \frac{D}{E}d_3 = \frac{l_3}{l_5}d_3$$

$$d_3 = \frac{(M_1 l_1 r_1 \cancel{2}\sin\beta)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5})}{(\cancel{2}r_3\cos\beta\, l_3)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5}) + (\cancel{2}r_3\cos\beta\, l_3)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5})}$$

$$\boxed{d_3 = \frac{M_1 l_1 r_1}{2 \cdot l_3 r_3}\tan\beta} = .491$$

$$\boxed{d_5 = \frac{M_1 l_1 r_1}{2 l_5 r_3}\tan\beta} = .492$$

single bucking factor given before $\frac{1}{d} = \frac{1}{2d_3} + \frac{1}{2d_5}$

calc bucking factors with angular symmetry relaxed $\quad f_i = + - + - + -$ on this page as in program TANGENTIAL OFOR

$$C_0 = M l_1 r_1 [\cos\phi_1 - \cos\phi_2]$$
$$+ l_3 r_3 d_3 [\cos\phi_3 - \cos\phi_4]$$
$$+ l_5 r_3 d_5 [\cos\phi_5 - \cos\phi_6] \equiv A + B d_3 + C d_5 = 0 \quad E$$

$$S_0 = M l_1 r_1 [\sin\phi_1 - \sin\phi_2]$$
$$+ l_3 r_3 d_3 [\sin\phi_3 - \sin\phi_4]$$
$$+ l_5 r_3 d_5 [\sin\phi_5 - \cos\phi_6] \equiv D + E d_3 + F d_5 = 0 \quad B$$

$$(AF - CD) + (BF - CE) d_3 = 0$$

$$d_3 = \frac{AF - CD}{CE - BF}$$

$$(AE - BD) + (CE - BF) d_5 = 0$$

$$d_5 = -\frac{AE - BD}{CE - BF}$$

$$d_3 = \frac{(M l_1 r_1 c_{12})(\cancel{l_5} \cancel{r_3} s_{56}) - (\cancel{l_5} \cancel{r_3} c_{56})(M l_1 r_1 s_{12})}{(\cancel{l_5} \cancel{r_3} c_{56})(l_3 r_3 s_{34}) - (l_3 r_3 c_{34})(\cancel{l_5} \cancel{r_3} s_{56})}$$

$$\boxed{d_3 = \frac{M l_1 r_1 (c_{12} s_{56} - s_{12} c_{56})}{l_3 r_3 (c_{56} s_{34} - s_{56} c_{34})}}$$

$$d_5 = -\frac{(M l_1 r_1 C_{12})(\cancel{l_5} r_3 S_{34}) - (\cancel{l_5} r_3 C_{34})(M l_1 r_1 S_{12})}{(l_5 r_3 C_{56})(\cancel{l_5} r_3 S_{34}) - (\cancel{l_5} r_3 C_{34})(l_5 r_3 S_{56})}$$

$$\boxed{d_5 = -\frac{M l_1 r_1 (C_{12} S_{34} - S_{12} C_{34})}{l_5 r_3 (C_{56} S_{34} - S_{56} C_{34})}}$$

for harmonic "n"

$$C_n = \sum_{j=1}^{6} S_j N_j l_j r_j^{\bar{n}} \cos \bar{n} \phi_j$$
$$S_n = \sum \qquad \sin \bar{n} \phi_j$$

$$C_n = M l_1 r_1^{\bar{n}} \left[\cos \bar{n}(90-\beta) - \cos \bar{n}(90+\beta) \right]$$
$$+ l_3 r_3^{\bar{n}} d \left[\cos \bar{n}(180+\beta) - \cos \bar{n}\beta \right]$$
$$+ l_5 r_3^{\bar{n}} d \left[\cos \bar{n}(180-\beta) - \cos(-\bar{n}\beta) \right]$$

for odd $\bar{n} = 1, 3, 5, \ldots$ or even $n = 0, 2, 4, \ldots$

$$C_{n=even} = 2M l_1 r_1^{\bar{n}} (-1)^{\frac{\bar{n}-1}{2}} \sin(\bar{n}\beta)$$
$$+ l_3 r_3^{\bar{n}} d \left(1 - (-1)^{\bar{n}}\right) \cos(\bar{n}\beta)$$
$$- l_5 r_3^{\bar{n}} d \left(1 - (-1)^{\bar{n}}\right) \cos(\bar{n}\beta)$$

$$\boxed{C_{\substack{n=even \\ \bar{n}=odd}} = 2M l_1 r_1^{\bar{n}} (-1)^{\frac{\bar{n}-1}{2}} \sin \bar{n}\beta - 2d\left(1 - (-1)^{\bar{n}}\right) r_3^{\bar{n}} l_3 \left(\frac{l_5 + l_3}{2 l_3}\right) \cos \bar{n}\beta}$$

for even $\bar{n} = 2, 4, 6, \ldots$ or odd $n = 1, 3, 5, \ldots$

$$C_{n=odd} = 2M l_1 r_1^{\bar{n}} (-1)^{\bar{n}/2} \left(\cos \bar{n}\beta - \cos \bar{n}\beta \right)$$
$$+ l_3 r_3^{\bar{n}} d \left[\cos \bar{n}\beta - \cos \bar{n}\beta \right] + l_5 r_3^{\bar{n}} d \left[\cos \bar{n}\beta - \cos \bar{n}\beta \right] = 0$$

$$\boxed{C_{\substack{n=odd \\ \bar{n}=even}} = 0}$$

$$S_n = M l_1 r_1^{\bar{n}} \left[\sin(\bar{n}90 - \bar{n}\beta) - \sin(\bar{n}90 + \bar{n}\beta) \right]$$
$$+ l_3 r_3^{\bar{n}} d \left[\sin(\bar{n}180 + \bar{n}\beta) - \sin \bar{n}\beta \right]$$
$$+ l_5 r_3^{\bar{n}} d \left[\sin(\bar{n}180 - \bar{n}\beta) - \sin(-\bar{n}\beta) \right]$$

for <u>odd $\bar{n}$</u> $= 1, 3, \cdots$ or <u>even n</u> $= 0, 2, \cdots$

$$S_n = M l_1 r_1^{\bar{n}} (-1)^{\frac{\bar{n}-1}{2}} \left[\cos \bar{n}\beta - \cos \bar{n}\beta \right]$$
$$+ l_3 r_3^{\bar{n}} d \left[(-1)^{\frac{\bar{n}+1}{2}} - 1 \right] \sin \bar{n}\beta$$
$$+ l_5 r_3^{\bar{n}} d \left[1 - (-1)^{\frac{\bar{n}+1}{2}} \right] \sin \bar{n}\beta$$

$$\boxed{\begin{aligned} S_{\substack{n=\text{even}\\ \bar{n}=\text{odd}}} &= l_3 r_3^{\bar{n}} d \left[1 - (-1)^{\frac{\bar{n}+1}{2}} \right] \left[\frac{l_5 - l_3}{l_3} \right] \sin \bar{n}\beta \\ &= 0 \text{ for } \bar{n} = 3, 7, 11, \ldots \\ &= 2 l_3 r_3^{\bar{n}} d \left[\frac{l_5 - l_3}{3} \right] \sin \bar{n}\beta \approx 0 \text{ for } \bar{n} = 1, 5, 7, \ldots \end{aligned}}$$

for <u>even $\bar{n}$</u> $= 2, 4, 6, \ldots$ or <u>odd n</u> $= 1, 3, 5, \ldots$

$$S_{n=\text{odd}} = -2 M l_1 r_1^{\bar{n}} (-1)^{\bar{n}/2} \sin(\bar{n}\beta)$$
$$+ l_3 r_3^{\bar{n}} d \left[\sin \bar{n}\beta - \sin \bar{n}\beta \right]$$
$$+ l_5 r_3^{\bar{n}} d \left[- \sin \bar{n}\beta + \sin \bar{n}\beta \right]$$

$$\boxed{S_{\substack{n=\text{odd}\\ \bar{n}=\text{even}}} = -2 M l_1 r_1^{\bar{n}} (-1)^{\bar{n}/2} \sin \bar{n}\beta}$$

$$\left.\begin{aligned}\cos(\bar{n}\,90 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}+1}{2}}\sin(\bar{n}\beta)\\ \cos(\bar{n}\,180 \pm \bar{n}\beta) &= (-1)^{\bar{n}}\cos(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{odd}$$

$$\left.\begin{aligned}\cos(\bar{n}\,90 \pm \bar{n}\beta) &= (-1)^{\bar{n}/2}\cos(\bar{n}\beta)\\ \cos(\bar{n}\,180 \pm \bar{n}\beta) &= \cos(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{even}$$

$$\left.\begin{aligned}\sin(\bar{n}\,90 \pm \bar{n}\beta) &= (-1)^{\frac{\bar{n}-1}{2}}\cos(\bar{n}\beta)\\ \sin(\bar{n}\,180 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}+1}{2}}\sin(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{odd}$$

$$\left.\begin{aligned}\sin(\bar{n}\,90 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}}{2}}\sin(\bar{n}\beta)\\ \sin(\bar{n}\,180 \pm \bar{n}\beta) &= \pm\sin(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{even}$$

to compare order of magnitude of $C_{n=\text{even}}$ with non-zero $S_{n=\text{even}}$ compare

$$\frac{l_5 - l_3}{2(l_3+l_5)} = -8.6\times 10^{-4}$$

$$C_0 = 0 \Rightarrow d \approx \frac{7}{2}\tan 8.2^\circ = .504$$

$$d_{exact} = .504 \times \frac{r_1}{r_3} \quad \left(\frac{2l_1}{l_3+l_5}\right)$$

$$= .504 \times .990 \times .989 = .504 \times .979$$

$$= .494$$

$$S_0 = -.012$$

harmonic → n, coil → m

$$V_{nm}(t) = B_0 2\bar{m}\,\omega\, l_m \frac{r_m^{\bar{n}}}{\rho^n}\left[a_n \cos(\bar{n}\omega t + \bar{n}\phi_m) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_m)\right]$$

$$= B_0 2\bar{m}\,\omega\, l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \sin(\bar{n}\omega t + \alpha_n + \bar{n}\phi_m)$$

$$\int_0^{\theta/\omega} V_{nm}(t)\,dt = B_0 2\frac{\bar{m}}{\bar{n}} l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \left[\cos(\alpha_n + \bar{n}\phi_m) - \cos(\bar{n}\theta + \alpha_n + \bar{n}\phi_m)\right]$$

drop the "monopole" term and absorb the minus sign with $-\cos(\theta) = \cos(\theta - \pi)$

$$\boxed{\Phi_{nm}^{M} = 2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \cos(\bar{n}\theta + \alpha_n + \bar{n}\phi_m - \pi)}$$

$$\boxed{\text{Amplitude} \quad A_{nm}^{M} = 2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n} \rightarrow h_n^M = \frac{A^M}{2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n}}$$

Phase $\quad \tilde{\delta}_{nm}^{M} = \alpha_n + \bar{n}\phi_m - \pi$

if coil is rotated by an angle Ψ at $t=0$, this advances the phase for harmonic n by $\bar{n}\Psi$

Measured phase $\quad \delta_{nm}^{M} = \alpha_n + \bar{n}\phi_m + \bar{n}\Psi - \pi$

measure Ψ by requirement $\alpha_n = 0 \Rightarrow \Psi = \delta_{00}^{M} - \phi_0 + \pi$

$$\Rightarrow \alpha_{nm} = \delta_{nm}^{M} - \bar{n}\phi_m - \bar{n}\Psi + \pi$$

$$\boxed{\alpha_{nm} = \left(\delta_{nm}^{M} - \bar{n}\delta_{00}^{M}\right) - \bar{n}(\phi_m - \phi_0) - \pi(\bar{n} - 1)}$$

these match Mike Gormley, exactly for α_n and hopefully for h_n

$$\alpha_{nm}^{M} = (\delta_{nm} - \bar{n}\,\delta_{00}) - \bar{n}\,(\phi_m - \phi_0) + \pi(\bar{n} - 1)$$

$$\alpha_{nm}^{T} = (\delta_{nm} - \bar{n}\,\delta_{00}) - (\gamma_n - \bar{n}\,\gamma_0) + \pi(\bar{n} - 1)$$

$$h_n^{T} = A_n \frac{\bar{n}\,\rho^n}{B_0 F_n}$$

$$A_{nm}^{M} = 2\frac{\bar{m}}{\bar{n}} B_0 \ell_m \frac{r_m^{\bar{n}}}{\rho^n} h_n$$

by comparing expressions for Amplitudes and phases between Tangential (p4) and Morgan (p10) coils we find that if Tangential coil is analysed assuming it is a Morgan coil then

$$h_n = h^M_{nm} \times \frac{2\bar{m}}{\cancel{\bar{n}}} \cancel{B_0} l_m \frac{r_m^{\bar{n}}}{\cancel{\rho^n}} \times \frac{\cancel{\bar{n}}\cancel{\rho^n}}{\cancel{B_0} F_n} =$$

$$\boxed{h_n = h^M_{nm} \left[\frac{2\bar{m}\, l_m r_m^{\bar{n}}}{F_n} \right]}$$

$$\boxed{\alpha_n = \alpha^M_{nm} + \bar{n}(\phi_m - \phi_0) - (\gamma_n - \bar{n}\gamma_0)}$$

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n \left(a_n \cos\bar{n}\theta + b_n \sin\bar{n}\theta\right) \qquad \bar{n} \equiv n+1$$

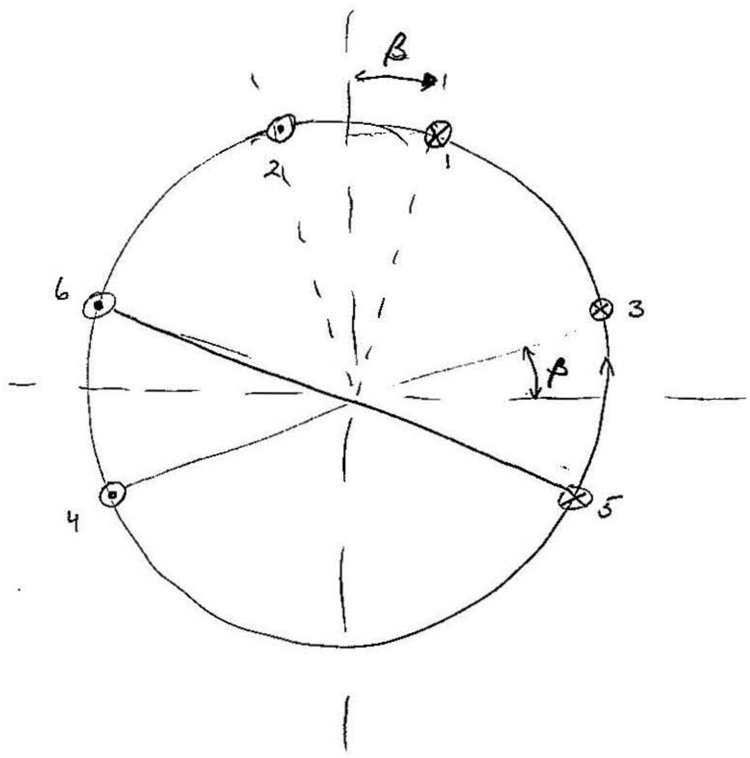
$$\theta = (\omega t + \phi)$$

$$V_{1\,WIRE}(t) = \omega\, l\, B_0 \sum_{n=0}^{\infty} \frac{r^{\bar{n}}}{\rho^n}\left[a_n \cos(\bar{n}\omega t + \bar{n}\phi) + b_n \sin(\bar{n}\omega t + \bar{n}\phi)\right]$$

for the i-th winding

wire sense × bucking factor

$$v_i = B_0\, \omega\, f_i\, N_i\, l_i \sum_{n=0}^{\infty} \frac{r_i^{\bar{n}}}{\rho^n}\left[a_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_i)\right]$$



$l\; v \times B$

i	ϕ_i	N_i	f_i	l_i	r_i
1	$90-\beta$	M	+1	$l_1 = 42.028$	$r_1 = \cancel{.587}\ .596$
2	$90+\beta$	M	−1	l_1	r_1
3	β	1	$-d$	$l_3 = 42.563$	$r_3 = \cancel{.595}\ .605$
4	$180+\beta$	1	$+d$	l_3	r_3
5	$-\beta$	1	$-d$	$l_5 = 42.417$	r_3
6	$180-\beta$	1	$+d$	l_5	r_3

$\beta_{nominal} = 8.2$

↑ or L_{mag} if $L_{mag} < l_i$

$$a_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_i)$$

$$= h_n \left[\sin\alpha_n \cos(\bar{n}\omega t + \bar{n}\phi_i) + \cos\alpha_n \sin(\bar{n}\omega t + \bar{n}\phi_i)\right]$$

$$= h_n \sin(\bar{n}\omega t + \alpha_n + \bar{n}\phi_i)$$

$$= h_n \left[\sin(\bar{n}\omega t + \alpha_n)\cos\bar{n}\phi_i + \cos(\bar{n}\omega t + \alpha_n)\sin\bar{n}\phi_i\right]$$

$$B_r = B_0 \sum_{n=0}^{\infty} \left(\frac{r}{\rho}\right)^n h_n \sin(\bar{n}\theta + \alpha_n)$$

$$= B_0 \sum_{n=0}^{\infty} h_n (\sin\alpha_n \cos\bar{n}\theta + \cos\alpha_n \sin\bar{n}\theta)$$

$$\Rightarrow a_n = h_n \sin\alpha_n$$

$$b_n = h_n \cos\alpha_n$$

$$V^{(t)} = \sum_{i=1}^{6} v_i = B_0 \omega \sum_{n=0}^{\infty} \frac{1}{\rho^n} \sum_{i=1}^{6} f_i N_i l_i r_i^{\bar n} h_n d_n e^{i(\bar n \omega t + \alpha_n)} e^{i\bar n \phi_i}$$

$$\Phi_n = \int_0^t V_n(t)\,dt$$

$$= -B_0 \frac{1}{\bar n \rho^n} \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n} \,\mathrm{Re}\left(e^{i(\bar n\omega t+\alpha_n)} e^{i\bar n\phi_j}\right)\Big|_0^t$$

$$= -B_0 \frac{1}{\bar n\rho^n} \mathrm{Re}\left\{\left(e^{i(\bar n\omega t+\alpha_n)} - e^{i\alpha_n}\right)\underbrace{\sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n} e^{i\bar n\phi_j}}_{F e^{i\gamma}}\right\}$$

$$\cancel{V_n = \frac{B_0\omega}{\rho^n} h_n \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n}}\Big[$$

$$V_n = \frac{B_0\omega}{\rho^n} h_n \left[\sin(\bar n\omega t+\alpha_n)\sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n}\cos\bar n\phi_j + \cos(\bar n\omega t+\alpha_n)\sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n}\sin\bar n\phi_j\right]$$

$$\equiv \frac{B_0\omega}{\rho^n} h_n \left[\sin(\omega t+\alpha_n) F\cos\gamma + \cos(\bar n\omega t+\alpha_n) F\sin\gamma\right]$$

$$F_n \cos\gamma_n \equiv \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n}\cos\bar n\phi_j \equiv C_n$$

$$F_n \sin\gamma_n \equiv \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar n}\sin\bar n\phi_j \equiv S_n$$

$$F_n = \left(C_n^2 + S^2\right)^{1/2} \qquad \gamma_n = \tan^{-1}\left(\frac{S_n}{C_n}\right)$$

$$V_n = \frac{B_0\omega}{\rho^n} h_n F_n \sin(\bar n\omega t+\alpha_n+\gamma_n)$$

$$= \cancel{\frac{B_0\omega}{\rho^n} h_n F_n \cos\left(\bar n\omega t+\alpha_n+\gamma_n-\frac{\pi}{2}\right)}$$

Integrate the voltage:

$$\int_0^{\theta/\omega} V(t)\,dt = \frac{B_0}{\bar{n}\rho^n} h_n F_n \left[\cos(\alpha_n+\gamma) - \cos(\bar{n}\theta+\alpha_n+\gamma_n)\right]$$

Drop the constant term (FFT "sums" the constant terms from all Φ_n and computes a "monopole term")
absorb the (−) sign with $-\cos(\theta) = +\cos(\theta-\pi)$

$$\boxed{\Phi_n^{TC} = \frac{B_0}{\bar{n}\rho^n} F_n h_n \cos(\bar{n}\theta+\alpha_n+\gamma_n-\pi)}$$

$$\boxed{\text{Amplitude} \quad A_n^T = \frac{B_0}{\bar{n}\rho^n} F_n h_n} \rightarrow$$

Phase $\quad \tilde{\delta}_n^T = \alpha_n+\gamma_n-\pi$

If coil is rotated by an angle Ψ at $t=0$, this advances phase of signal from harmonic n by $\bar{n}\Psi$

Measured phase $\quad \delta_n^T = \alpha_n+\gamma_n+\bar{n}\Psi-\pi$

~~We measure Ψ from $\alpha_0 = 0 \rightarrow \Psi = \delta_0^T - \gamma_0 + \pi$~~

$$\Rightarrow \alpha_n = \delta_n^T - \gamma_n - \bar{n}\Psi + \pi$$

$$\cancel{\boxed{\alpha_n = (\delta_n^T - \bar{n}\delta_0^T) - (\gamma_n - \bar{n}\gamma_0) + \pi(\bar{n}-1)}}$$

Bucking factor "d" chosen to make $\Phi_0 = 0$

$$\Rightarrow F_0 = 0 \quad \Rightarrow C_0 = S_0 = 0$$

$$C_0 = \sum_{j=1}^{6} f_j \, N_j \, l_j \, r_j \cos\phi_j$$

$$= M l_1 r_1 \left[\cos(90^\circ - \beta) - \cos(90 + \beta)\right]$$

$$+ l_3 r_3 d \left[\cos(180 + \beta) - \cos\beta\right]$$

$$+ l_5 r_3 d \left[\cos(180 - \beta) - \cos(-\beta)\right]$$

$$= M l_1 r_1 \, 2\sin\beta$$

$$+ d \, r_3 \left[-2 l_3 \cos\beta - 2 l_5 \cos\beta\right]$$

$$= M l_1 r_1 2\sin\beta - 4 d r_3 l_3 \cos\beta \left(\frac{l_3 + l_5}{2 l_3}\right)$$

for $C_0 = 0 \Rightarrow d = \frac{M}{2} \frac{r_1}{r_3} \frac{l_1}{l_3} \left(\frac{2 l_3}{l_3 + l_5}\right) \tan\beta \approx \frac{M}{2} \tan\beta$

$$S_0 = \sum_{j=1}^{6} f_j \, N_j \, l_j \, r_j \sin\phi_j$$

$$= M l_1 r_1 \left[\sin(90 - \beta) - \sin(90 + \beta)\right]$$

$$+ l_3 r_3 d \left[\sin(180 + \beta) - \sin\beta\right]$$

$$+ l_5 r_3 d \left[\sin(180 - \beta) - \sin(-\beta)\right]$$

$$= M l_1 r_1 \left[\cos\beta - \cos\beta\right] - 2 l_3 r_3 d \sin\beta + 2 l_5 r_3 d \sin\beta$$

$$= 2 \cancel{l_3} r_3 d \sin\beta \left[\frac{l_5 - l_3}{\cancel{l_3}}\right] \approx 0$$

DT 3/4/9

Allow 2 bucking factors, one for each dipole coil

$$C_0 = M\, l_1 r_1\, 2\sin\beta - 2 r_3 \cos\beta\left[d_3 l_3 + d_5 l_5\right] = 0$$

$$S_0 = 2 r_3 \sin\beta\left[l_5 d_5 - l_3 d_3\right] = 0$$

form: $A - B d_3 - C d_5 = 0$ $A = M_1 l_1 r_1 2\sin\beta$

$- D d_3 + E d_5 = 0$ $B = 2 r_3 \cos\beta\, l_3$

$C = 2 r_3 \cos\beta\, l_5$

$D = 2 r_3 \sin\beta\, l_3$

$E = 2 r_3 \sin\beta\, l_5$

$$AE - (BE + CD) d_3 = 0$$

$$d_3 = \frac{AE}{BE + CD} =$$

$$AD - (CD - BE) d_5 = 0$$

$$d_5 = \frac{AD}{CD + BE} = \frac{D}{E} d_3 = \frac{l_3}{l_5} d_3$$

$$d_3 = \frac{(M_1 l_1 r_1 \cancel{2} \sin\beta)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5})}{(\cancel{2} r_3 \cos\beta\, l_3)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5}) + (\cancel{2} r_3 \cos\beta\, l_3)(\cancel{2}\cancel{r_3}\cancel{\sin\beta}\cancel{l_5})}$$

$$\boxed{d_3 = \frac{M\, l_1 r_1}{2\, l_3 r_3} \tan\beta} = .491$$

$$\boxed{d_5 = \frac{M\, l_1 r_1}{2\, l_5 r_3} \tan\beta} = .492$$

single bucking factor given before $\frac{1}{d} = \frac{1}{2 d_3} + \frac{1}{2 d_5}$

calc bucking factors with angular symmetry relaxed

$f_i = + \ - \ + \ - \ + \ -$ on this page as in program TANGENTIAL.FOR

$$C_0 = M l_1 r_1 [\cos\phi_1 - \cos\phi_2] + l_3 r_3 d_3 [\cos\phi_3 - \cos\phi_4] + l_5 r_3 d_5 [\cos\phi_5 - \cos\phi_6] \equiv A + B d_3 + C d_5 = 0 \quad E$$

$$S_0 = M l_1 r_1 [\sin\phi_1 - \sin\phi_2] + l_3 r_3 d_3 [\sin\phi_3 - \sin\phi_4] + l_5 r_3 d_5 [\sin\phi_5 - \cos\phi_6] \equiv D + E d_3 + F d_5 = 0 \quad B$$

$$(AF - CD) + (BF - CE) d_3 = 0$$

$$d_3 = \frac{AF - CD}{CE - BF}$$

$$(AE - BD) + (CE - BF) d_5 = 0$$

$$d_5 = -\frac{AE - BD}{CE - BF}$$

$$d_3 = \frac{(M l_1 r_1 c_{12})(\cancel{l_5} \cancel{r_3} s_{56}) - (\cancel{l_5} \cancel{r_3} c_{56})(M l_1 r_1 s_{12})}{(\cancel{l_5} \cancel{r_3} c_{56})(l_3 r_3 s_{34}) - (l_3 r_3 c_{34})(\cancel{l_5} \cancel{r_3} s_{56})}$$

$$\boxed{d_3 = \frac{M l_1 r_1 (c_{12} s_{56} - s_{12} c_{56})}{l_3 r_3 (c_{56} s_{34} - s_{56} c_{34})}}$$

$$d_5 = -\frac{(M\ell_1 r_1 c_{12})(\cancel{\ell_3} r_3 s_{34}) - (\cancel{\ell_5 r_3}\, c_{34})(M\ell_1 r_1 s_{12})}{(\ell_5 r_3 c_{56})(\cancel{\ell_3} r_3 s_{34}) - (\cancel{\ell_5 r_3}\, c_{34})(\ell_5 r_3 s_{56})}$$

$$\boxed{d_5 = -\frac{M\ell_1 r_1 (c_{12} s_{34} - s_{12} c_{34})}{\ell_5 r_3 (c_{56} s_{34} - s_{56} c_{34})}}$$

for harmonic "n"

$$C_n = \sum_{j=1}^{6} f_j N_j l_j r_j^{\bar{n}} \cos \bar{n} \phi_j$$
$$S_n = \sum \qquad\qquad \sin \bar{n} \phi_j$$

$$C_n = M l_1 r_1^{\bar{n}} \left[\cos \bar{n}(90-\beta) - \cos \bar{n}(90+\beta) \right]$$
$$+ l_3 r_3^{\bar{n}} d \left[\cos \bar{n}(180+\beta) - \cos \bar{n}\beta \right]$$
$$+ l_5 r_3^{\bar{n}} d \left[\cos \bar{n}(180-\beta) - \cos(-\bar{n}\beta) \right]$$

for <u>odd</u> $\bar{n} = 1, 3, 5, \ldots$ or <u>even</u> $n = 0, 2, 4, \ldots$

$$C_{n=even} = 2M l_1 r_1^{\bar{n}} (-1)^{\frac{\bar{n}-1}{2}} \sin(\bar{n}\beta)$$
$$+ l_3 r_3^{\bar{n}} d \left(1 - (-1)^{\bar{n}}\right) \cos(\bar{n}\beta)$$
$$- l_5 r_3^{\bar{n}} d \left(1 - (-1)^{\bar{n}}\right) \cos(\bar{n}\beta)$$

$$\boxed{C_{\substack{n=even \\ \bar{n}=odd}} = 2M l_1 r_1^{\bar{n}} (-1)^{\frac{\bar{n}-1}{2}} \sin \bar{n}\beta - 2d\left(1 - (-1)^{\bar{n}}\right) r_3^{\bar{n}} l_3 \left(\frac{l_5 + l_3}{2 l_3}\right) \cos \bar{n}\beta}$$

for <u>even</u> $\bar{n} = 2, 4, 6, \ldots$ or <u>odd</u> $n = 1, 3, 5, \ldots$

$$C_{n=odd} = 2M l_1 r_1^{\bar{n}} (-1)^{\bar{n}/2} \left(\cos \bar{n}\beta - \cos \bar{n}\beta \right)$$
$$+ l_3 r_3^{\bar{n}} d \left[\cos \bar{n}\beta - \cos \bar{n}\beta \right] + l_5 r_3^{\bar{n}} d \left[\cos \bar{n}\beta - \cos \bar{n}\beta \right] = 0$$

$$\boxed{C_{\substack{n=odd \\ \bar{n}=even}} = 0}$$

$$S_n = M l_1 r_1^{\bar n}\left[\sin(\bar n 90 - \bar n\beta) - \sin(\bar n 90 + \bar n\beta)\right]$$
$$+ l_3 r_3^{\bar n} d\left[\sin(\bar n 180 + \bar n\beta) - \sin \bar n\beta\right]$$
$$+ l_5 r_3^{\bar n} d\left[\sin(\bar n 180 - \bar n\beta) - \sin(-\bar n\beta)\right]$$

for <u>odd $\bar n$</u> $= 1, 3, \cdots$ or <u>even n</u> $= 0, 2, \cdots$

$$S_n = M l_1 r_1^{\bar n}(-1)^{\frac{\bar n - 1}{2}}\left[\cos \bar n\beta - \cos \bar n\beta\right]$$
$$+ l_3 r_3^{\bar n} d\left[(-1)^{\frac{\bar n + 1}{2}} - 1\right]\sin \bar n\beta$$
$$+ l_5 r_3^{\bar n} d\left[1 - (-1)^{\frac{\bar n + 1}{2}}\right]\sin \bar n\beta$$

$$\boxed{\begin{aligned} S_{\substack{n = \text{even} \\ \bar n = \text{odd}}} &= l_3 r_3^{\bar n} d\left[1 - (-1)^{\frac{\bar n + 1}{2}}\right]\left[\frac{l_5 - l_3}{l_3}\right]\sin \bar n\beta \\ &= 0 \text{ for } \bar n = 3, 7, 11, \ldots \\ &= 2 l_3 r_3^{\bar n} d\left[\frac{l_5 - l_3}{3}\right]\sin \bar n\beta \approx 0 \text{ for } \bar n = 1, 5, 7, \ldots \end{aligned}}$$

for <u>even $\bar n$</u> $= 2, 4, 6, \ldots$ or <u>odd n</u> $= 1, 3, 5, \ldots$

$$S_{n = \text{odd}} = -2 M l_1 r_1^{\bar n}(-1)^{\bar n/2}\sin(\bar n\beta)$$
$$+ l_3 r_3^{\bar n} d\left[\sin \bar n\beta - \sin \bar n\beta\right]$$
$$+ l_5 r_3^{\bar n} d\left[-\sin \bar n\beta + \sin \bar n\beta\right]$$

$$\boxed{S_{\substack{n = \text{odd} \\ \bar n = \text{even}}} = -2 M l_1 r_1^{\bar n}(-1)^{\bar n/2}\sin \bar n\beta}$$

$$\left.\begin{aligned}\cos(\bar{n}90 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}+1}{2}}\sin(\bar{n}\beta)\\ \cos(\bar{n}180 \pm \bar{n}\beta) &= (-1)^{\bar{n}}\cos(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{odd}$$

$$\left.\begin{aligned}\cos(\bar{n}90 \pm \bar{n}\beta) &= (-1)^{\bar{n}/2}\cos(\bar{n}\beta)\\ \cos(\bar{n}180 \pm \bar{n}\beta) &= \cos(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{even}$$

$$\left.\begin{aligned}\sin(\bar{n}90 \pm \bar{n}\beta) &= (-1)^{\frac{\bar{n}-1}{2}}\cos(\bar{n}\beta)\\ \sin(\bar{n}180 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}\pm1}{2}}\sin(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{odd}$$

$$\left.\begin{aligned}\sin(\bar{n}90 \pm \bar{n}\beta) &= \pm(-1)^{\frac{\bar{n}}{2}}\sin(\bar{n}\beta)\\ \sin(\bar{n}180 \pm \bar{n}\beta) &= \pm\sin(\bar{n}\beta)\end{aligned}\right\}\quad \bar{n} = \text{even}$$

to compare, order of magnitude of $C_{n=even}$ with non-zero $S_{n=even}$ compare

$$\frac{l_5 - l_3}{2(l_3 + l_5)} = -8.6 \times 10^{-4}$$

$$C_0 = 0 \Rightarrow d \approx \frac{7}{2} \tan 8.2° = .504$$

$$d_{exact} = .504 \times \frac{r_1}{r_3} \quad \left(\frac{2 l_1}{l_3 + l_5}\right)$$

$$= .504 \times .990 \times .989 = .504 \times .979$$

$$= .494$$

$$S_0 = -.012$$

Mogan Coil Harmonics - Integrator + FFT

$$V_{nm}(t) = B_0 2\bar{m}\,\omega\, l_m \frac{r_m^{\bar{n}}}{\rho^n}\left[a_n \cos(\bar{n}\omega t + \bar{n}\phi_m) + b_n \sin(\bar{n}\omega t + \bar{n}\phi_m)\right]$$

harmonic (n), coil (m)

$$= B_0 2\bar{m}\,\omega\, l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \sin(\bar{n}\omega t + \alpha_n + \bar{n}\phi_m)$$

$$\int_0^{\theta/\omega} V_{nm}(t)\,dt = B_0 2\frac{\bar{m}}{\bar{n}} l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \left[\cos(\alpha_n + \bar{n}\phi_m) - \cos(\bar{n}\theta + \alpha_n + \bar{n}\phi_m)\right]$$

drop the "monopole" term and absorb the minus sign with $-\cos(\theta) = \cos(\theta - \pi)$

$$\boxed{\bar{\Phi}^M_{nm} = 2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n \cos(\bar{n}\theta + \alpha_n + \bar{n}\phi_m - \pi)}$$

$$\boxed{\text{Amplitude} \quad A^M_{nm} = 2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n} h_n} \rightarrow h^M_n = \frac{A^M}{2\frac{\bar{m}}{\bar{n}} B_0 l_m \frac{r_m^{\bar{n}}}{\rho^n}}$$

Phase $\quad \tilde{\delta}^M_{nm} = \alpha_n + \bar{n}\phi_m - \pi$

if coil is rotated by an angle Ψ at $t=0$, this advances the phase for harmonic n by $\bar{n}\Psi$

Measured phase $\delta^M_{nm} = \alpha_n + \bar{n}\phi_m + \bar{n}\Psi - \pi$

measure Ψ by requirement $\alpha_n = 0 \Rightarrow \Psi = \delta^M_{00} - \phi_0 + \pi$

$$\Rightarrow \alpha_{nm} = \delta^M_{nm} - \bar{n}\phi_m - \bar{n}\Psi + \pi$$

$$\boxed{\alpha_{nm} = \left(\delta^M_{nm} - \bar{n}\delta^M_{00}\right) - \bar{n}(\phi_m - \phi_0) - \pi(\bar{n} - 1)}$$

these match Mike Gormley, exactly for α_n and hopefully for h_n

$$\alpha^{M}_{nm} = (\delta_{nm} - \bar{n}\,\delta_{00}) - \bar{n}(\phi_m - \phi_0) + \pi(\bar{n} - 1)$$

$$\alpha^{T}_{nm} = (\delta_{nm} - \bar{n}\,\delta_{00}) - (\gamma_n - \bar{n}\,\gamma_0) + \pi(\bar{n} - 1)$$

$$h^{T}_{n} = A_n \frac{\bar{n}\,\rho^{n}}{B_0 F_n}$$

$$A^{M}_{nm} = 2\frac{\bar{m}}{\bar{n}} B_0 \ell_m \frac{r_m^{\bar{n}}}{\rho^{n}} h_n$$

by comparing expressions for Amplitudes and phases between Tangential (p4) and Morgan (p10) coils we find That if Tangential coil is analysed assuming it is a Morgan coil Then

$$h_n = h_{nm}^M \times \frac{2\bar{m}}{\cancel{\bar{n}}} \cancel{B_0} l_m \frac{r_m^{\bar{n}}}{\cancel{\rho^n}} \times \frac{\cancel{\bar{n}} \cancel{\rho^n}}{\cancel{B_0} F_n}$$

$$\boxed{h_n = h_{nm}^M \left[\frac{2\bar{m}\, l_m\, r_m^{\bar{n}}}{F_n} \right]}$$

$$\boxed{\alpha_n = \alpha_{nm}^M + \bar{n}(\phi_m - \phi_0) - (\gamma_n - \bar{n}\gamma_0)}$$