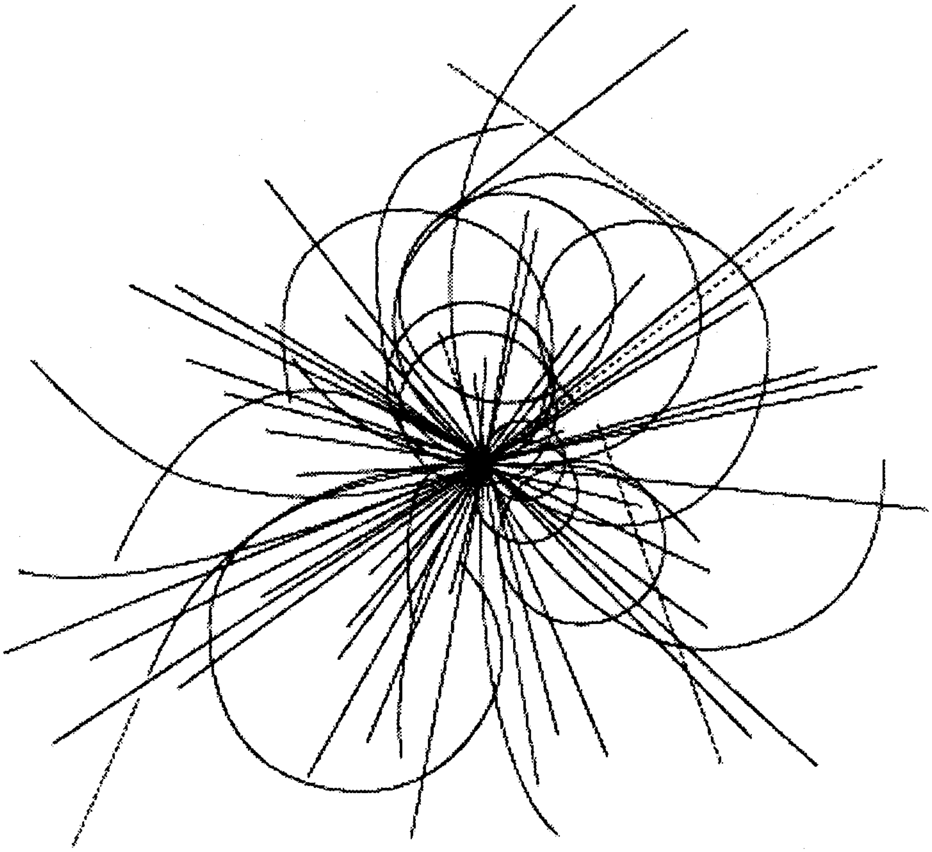
SSCL-606

SSCL-606
April 1993
Distribution Category: 414

B. Chen
A. Chao

Longitudinal Head-Tail Instability in a Non-Harmonic Potential Well



Superconducting Super Collider Laboratory

SSCL-606

Longitudinal Head-Tail Instability in a Non-Harmonic Potential Well

B. Chen and A. Chao

Superconducting Super Collider Laboratory*
2550 Beckleymeade Avenue
Dallas, TX 75237

April 1993

* Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

Longitudinal Head-Tail Instability in a Non-Harmonic Potential Well

B. Chen and A. Chao

Abstract

A perturbation technique is developed that can be applied to study the collective instability problem when the unperturbed system is not described by a simple harmonic oscillator. The Longitudinal Head-Tail instability effect is well studied as applications of this technique. Applications of the longitudinal head-tail instability effects to the CERN SPS and the SSC are included.

1.0 INTRODUCTION

The ideal motion of a single particle in a circular accelerator is that of a simple harmonic oscillator. In reality, the accelerator contains various perturbation effects that cause deviations from this simple-harmonic environment. To study the problem of beam instability, therefore, it is customary to consider the various perturbation effects to be imposed on the idealized simple-harmonic environment. In particular, the conventional theory of collective instabilities is developed by imposing the perturbation of collective wake forces on the simple harmonic system.

However, when the new collective longitudinal instability was observed in the CERN Super Proton Synchrotron (SPS),[1] the analyses suggested the "longitudinal chromaticity" playing a role. Drawing analogy to the transverse case where the betatron chromaticity causes the head-tail instability, this new instability was named Longitudinal Head-Tail (LHT) instability. The theoretical existence of the LHT instability was pointed out by Hereward;[2] it results from the non-simple-harmonic nature of the system when the longitudinal chromaticity effect is considered. When the longitudinal chromaticity vanishes — and therefore the system is simple-harmonic — there would not be a LHT instability. To study the LHT instability, the conventional theory does not suffice because it treats only the simple-harmonic case.

In this paper we develop a new formalism that extends the conventional approach to the non-simple-harmonic Hamiltonian system. The LHT instability is studied as an application to demonstrate the technique. By using the water-bag particle distribution model, it is possible to solve the problem exactly and obtain the growth rates for the various collective modes (the dipole, quadrupole, sextupole modes, *etc.*). Although not discussed below, the potential-well distortion, as well as its effects on collective instabilities, can also be studied with this technique.

In Section 2, we follow Reference 1 to illustrate the basic physical mechanism of the LHT instability effect. In Section 3, a perturbation formalism for the non-simple-harmonic Hamiltonian system is developed. For one distribution, the water-bag model, the problem is analytically solved. In Section 4, the results of Section 3 are applied to the CERN SPS, and the SSC collider and boosters.

2.0 MECHANISM OF THE LONGITUDINAL HEAD-TAIL INSTABILITY

The LHT instability, like its well-known transverse counterpart, the transverse head-tail instability (or simply the head-tail instability), is a single bunch effect. The mechanisms of these instabilities are quite similar. In the transverse head-tail instability, the betatron frequency of a particle depends on its momentum deviation $\delta = \Delta E/E$. As a consequence, the accumulated betatron phase of the particle depends on its longitudinal location z in the bunch as it executes a synchrotron oscillation. If the particle motion is then perturbed by the collective wake forces, this betatron phase difference between the bunch head and bunch tail can lead to the transverse head-tail instability.[3,4] A similar situation happens for the LHT instability. This instability is caused by a dependence of the accumulated synchrotron phase on the longitudinal position of the particle, coupled with a perturbation due to the collective wake forces.

However, the situation is more subtle in the longitudinal case for the following reason. In the transverse case, the betatron motion of a particle is modulated by δ and z, which, in the description of the transverse effects, can be regarded as *external* parameters. In the longitudinal case, the synchrotron motion is also modulated by δ and z, but in this case, δ and z are the dynamic variables which describe the particle motion. The analysis of this problem is therefore more involved.

We will postpone the analysis till Section 3. In this section, we will illustrate the basic mechanism of the LHT instability, at least for the collective dipole mode. To do so, consider a circular accelerator whose slippage factor η contains a higher order term in δ, *i.e.*,

$$\eta = \eta_0(1 + \frac{3}{2}\epsilon\delta), \tag{1}$$

where η_0 is the leading contribution of the momentum slippage factor, and ϵ is a parameter that specifies the strength of the higher order contribution. The unperturbed equations of motion of a single particle are given by

$$\begin{aligned} \frac{dz}{ds} &= -\eta_0\delta(1 + \frac{3}{2}\epsilon\delta), \\ \frac{d\delta}{ds} &= \frac{\omega_s^2}{\eta_0 c^2} z, \end{aligned} \tag{2}$$

where s is the longitudinal coordinate along the accelerator circumference, and ω_s is the unperturbed synchrotron oscillation frequency for small amplitudes.

The Hamiltonian of the system is given by

$$H_0 = \frac{\eta_0^2}{2}\delta^2(1 + \epsilon\delta) + \frac{\omega_s^2}{2c^2}z^2. \tag{3}$$

Equation (2) follows from the Hamiltonian (3) if we take the canonical variables to be

$$q = z \quad \text{and} \quad p = -\eta_0\delta. \tag{4}$$

The coefficient ϵ describes the deviation from the simple harmonicity of the system. We consider small ϵ so that $|\epsilon\delta| \ll 1$. The motion of a single particle in the z-δ space follows a constant Hamiltonian contour. One such contour is shown in Figure 1. The contour would be elliptical if $\epsilon = 0$. When $\epsilon \neq 0$, the contour is deformed. The contour in Figure 1 shows the deformation when $\epsilon > 0$.

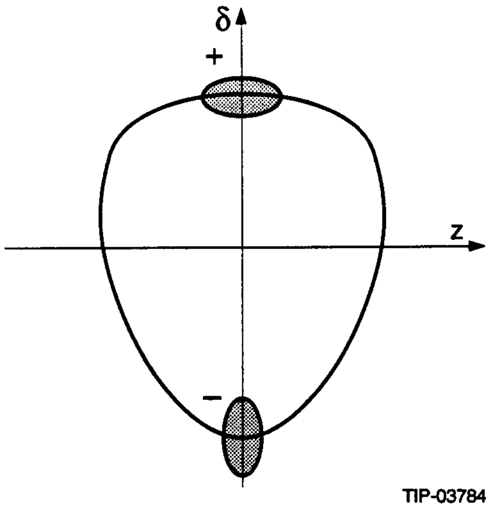


Figure 1. The Phase Space Trajectory due to the Non-simple-harmonic Hamiltonian, Eq. (3). The case shown is for $\epsilon > 0$..

Also shown in Figure 1 is the motion of a beam bunch. The center of the bunch is considered to move along the constant Hamiltonian contour shown. The other particles in the bunch move along neighboring contours which are not shown. The bunch is considered to be executing a longitudinal dipole oscillation, the amplitude of which has been exaggerated in Figure 1. The main effect of a non-vanishing ϵ is that it has introduced an asymmetry between the upper and the lower halves of the phase plane.

As the beam bunch executes its dipole oscillation in this deformed phase space, the shape of the phase space area occupied by the bunch varies, although its area is conserved. The bunch shape at two instances (marked by + and −) are shown as shaded areas in Figure 1. In particular, the bunch lengths $\hat{z}_+$ and $\hat{z}_-$ at the two instances are related by the Liouville theorem according to

$$\frac{\hat{z}_-}{\hat{z}_+} = \frac{|\frac{dz}{ds}_-|}{|\frac{dz}{ds}_+|} = \frac{-\delta_-(1+\frac{3}{2}\epsilon\delta_-)}{\delta_+(1+\frac{3}{2}\epsilon\delta_+)} \approx \frac{1-\epsilon\delta_0}{1+\epsilon\delta_0} \approx \frac{1+\epsilon\delta_-}{1+\epsilon\delta_+}, \tag{5}$$

where $\delta_0 = \sqrt{2H_0}/|\eta_0|$, and $\delta_\pm \approx \pm\delta_0 - \frac{1}{2}\epsilon\delta_0^2$ are the values of δ at the + and − locations. We conclude from Eq.(4) that, to first order in $|\epsilon\delta_0|$, the bunch length is modulated according to

$$\hat{z} \propto 1 + \epsilon\delta \tag{6}$$

as the bunch executes the dipole oscillation in the phase space.

Next we introduce the effect of the collective wake fields. The bunch will lose energy due to interacting with the surroundings through the wake fields. Since the energy loss of the beam bunch depends on the bunch length, the bunch energy loss is also modulated by the same factor of Eq. (6). Adding the energy loss term to Eq.(2), we obtain the equations of motion

$$\begin{aligned} \frac{dz}{ds} &= -\eta_0\delta(1+\frac{3}{2}\epsilon\delta), \\ \frac{d\delta}{ds} &= \frac{\omega_s^2}{\eta_0 c^2}z + \frac{1}{NEC}[\Delta\varepsilon\mid_{\hat{z}(1+\epsilon\delta)} - \Delta\varepsilon\mid_{\hat{z}}] \\ &\approx \frac{\omega_s^2}{\eta_0 c^2}z + \epsilon\frac{\hat{z}}{NEC}\frac{d\Delta\varepsilon}{d\hat{z}}\delta, \end{aligned} \tag{7}$$

where N is number of particles per bunch, E is the particle energy, C is machine circumference, and $\Delta\varepsilon$ is the bunch energy loss per turn. Compared with Eq.(2), Eq.(7) contains an extra term which is proportional to $\Delta\varepsilon\mid_{\hat{z}(1+\epsilon\delta)} - \Delta\varepsilon\mid_{\hat{z}}$, which is the portion of $\Delta\varepsilon$ which is modulated by the instantaneous beam energy δ, and we have kept only its leading contribution to first order in δ.

To first order in δ, the two equations in (7) can be combined to give

$$\frac{d^2\delta}{ds^2} - \epsilon\frac{\hat{z}}{NEC}\frac{d\Delta\varepsilon}{d\hat{z}}\frac{d\delta}{ds} + \frac{\omega_s^2}{c^2}\delta = 0. \tag{8}$$

Equation (8) represents a system with growth (or damping if negative) rate:

$$\tau^{-1} = \epsilon\frac{c\hat{z}}{2NEC}\frac{d\Delta\varepsilon}{d\hat{z}}. \tag{9}$$

The result (9) was first obtained in Reference 1.

The dipole LHT instability growth rate is proportional to the nonlinear slippage factor ϵ and the dependence of the beam energy loss on the bunch length. Whether the system is stable or unstable is determined by the sign of $\epsilon(d\Delta\varepsilon/d\hat{z})$. Usually the quantity $d\Delta\varepsilon/d\hat{z}$ is positive (a short bunch loses more energy than a long one. In our convention, $\Delta\varepsilon < 0$). This means the bunch oscillation is unstable if $\epsilon > 0$ and stable if $\epsilon < 0$.

Consider a bunch executing a longitudinal dipole oscillation relative to a synchronous particle. Due to the nonlinear slippage factor, as the bunch executes a dipole oscillation, its length is modulated by the factors $1+\epsilon\delta$, where δ is the instantaneous relative energy of the bunch. If $\epsilon > 0$, the length of the bunch is going to be shorter when its energy is lower than the synchronous energy ($\delta < 0$), and is longer when its energy is higher than the synchronous energy ($\delta > 0$). If a short bunch loses more energy in the wake field than a long bunch, *i.e.*, if $d\Delta\varepsilon/d\hat{z} > 0$. the bunch will lose energy when $\delta < 0$ and gains energy when $\delta > 0$. This means an ever-increasing amplitude of the dipole motion of the bunch, leading to an instability. If $\epsilon < 0$, the opposite happens, and the beam dipole motion is damped.

We have so far studied the effect of a non-harmonic term in the dz/ds equation which is nonlinear in δ. Naturally one could ask the counterpart problem when the non-harmonicity is contained in the $d\delta/ds$ equation due to a term nonlinear in z. This system describes a potential-well distortion effect. The analysis to be described in the next section, as well as the physical picture described in the present section, can be extended to that system as well. We have not pursued it in the present report. Suffice it to say here that the collective wake forces do not cause an instability of this system; they cause only collective mode frequency shifts.

3.0 PERTURBATION APPROACH

The conventional approach to treat the longitudinal collective instabilities is as follows. One starts with a certain stationary bunch distribution (usually ignoring the potential well distortion effects). One then assumes that on top of this stationary distribution, there is a time-dependent perturbation which oscillates with a certain coherent frequency Ω that is to be determined. The equation that governs the perturbation distribution is the Vlasov equation. By solving the Vlasov equation, one obtains solutions for Ω. The imaginary part of Ω then gives the stability growth rate.[5,6]

If we adopt the simple harmonic oscillation as the unperturbed model (unperturbed is used here to refer to the case when wake field effects are neglected), the action-angle variables form a pair of canonical variables. In fact, the action is proportional to the Hamiltonian and the canonical transformation from the (z, δ) to the action-angle variables is the transformation from the Cartesian coordinates to the polar coordinates.

From the previous section we knew that to study the LHT instability, we will have to consider an unperturbed system which is described by a non-simple-harmonic Hamiltonian. For such a system, the conventional method of Cartesian-to-polar transformation no longer applies. The technique we develop in this paper is to introduce a new pair of dynamical variables: the Hamiltonian H itself and another variable Q which assumes the role of the time variable. The advantage of using the new variables is we only need to deal with one complicated variable Q. This point will become clear in the later derivation. Having introduced the new dynamical variables, the procedure we use to solve the Vlasov equation then follows basically the conventional treatment.

We start with a general situation when the accelerator is described by a Hamiltonian $H(q,p;s)$. The beam distribution $\psi(q,p;s)$ in the phase space (q,p) behaves in this environment according to the Vlasov equation

$$\frac{\partial\psi}{\partial s} + \{\psi, H\} = \frac{\partial\psi}{\partial s} + \frac{\partial\psi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial\psi}{\partial p}\frac{\partial H}{\partial q} = 0. \tag{10}$$

Here we have introduced the Poisson bracket notation. Later we will relate the canonical variables q and p to z and δ according to Eq.(4), but we leave them general for now.

The unperturbed part of the Hamiltonian H is

$$H_0(q,p) = \frac{1}{2}p^2[1 + f(p)] + \frac{\omega_s^2}{2c^2}q^2, \tag{11}$$

where the function $f(p)$ represents a small deviation of the system from simple harmonicity. In the following we will study how $f(p)$ contributes to the LHT instability, particularly for the system described by Eq.(3) for which

$$f(p) = -\frac{\epsilon}{\eta_0}p. \tag{12}$$

Consider a beam with an unperturbed distribution ψ_0 which is executing a collective oscillation due to the interaction of wake fields. Let the collective oscillation be described by a small distribution perturbation ψ_1 and let the oscillation frequency be Ω. The total beam distribution is then given by

$$\psi = \psi_0 + \psi_1 e^{-i\Omega s/c}. \tag{13}$$

The normalization is chosen such that

$$\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\ \psi_0 = N. \tag{14}$$

To describe the stationary distribution of the unperturbed system, ψ_0 must be a function of H_0 alone.

The distribution perturbation induces a collective wake force which affects the motion of the beam particles. This additional wake force is described by the perturbed Hamiltonian

$$H(q,p;s) = H_0(q,p) + H_1(q)e^{-i\Omega s/c}, \tag{15}$$

where the unperturbed term H_0 is given by Eq. (11), and the wake induced term H_1 is given by

$$H_1 = -\frac{\eta_0 e}{EC}\int_{-\infty}^{q} V_1(q')dq', \tag{16}$$

where $V_1(q)$ is the retarding wake voltage per turn induced by ψ_1 and is related to the longitudinal wake function $W(q)$ and impedance $Z_0^{\|}(\omega)$ according to

$$
\begin{aligned}
V_1(q) &= \epsilon \int_{-\infty}^{\infty} dq' \int_{-\infty}^{\infty} dp \ \psi_1(q',p) W(q-q') \\
&= \frac{\epsilon}{2\pi} \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega) e^{i\omega q/c} \times \\
& \int_{-\infty}^{\infty} dq' e^{-i\omega q'/c} \int_{-\infty}^{\infty} dp \ \psi_1(q',p).
\end{aligned}
\tag{17}
$$

The wake term H_1 contains the effect due to the perturbation distribution ψ_1; the wake force induced by the unperturbed distribution ψ_0 has been ignored. This amounts to ignoring the potential-well distortion effect, which is not of interest in the present study. In writing down Eq.(17), we have also ignored multi-turn wake effects.

Considering ψ_1 is a small quantity, the Vlasov equation (10) can be linearized by keeping the first order terms in ψ_1,

$$
-i\frac{\Omega}{c}\psi_1 + \{\psi_1, H_0\} + \{\psi_0, H_1\} = 0. \tag{18}
$$

We now introduce a canonical transformation from the old variables (q,p) to the new variables (Q,P),

$$
q = -\frac{\partial F}{\partial p} \quad \text{and} \quad Q = \frac{\partial F}{\partial P}, \tag{19}
$$

with the generating function

$$
F(p,P) = -\int_0^p q(P,p')dp'. \tag{20}
$$

The basic idea is to choose the unperturbed Hamiltonian H_0 as the new canonical momentum, *i.e.*, $P = H_0$. Then the other canonical variable is given by

$$
Q = -\int_0^p \frac{\partial q(H_0,p')}{\partial H_0} dp'. \tag{21}
$$

The advantage of having $P = H_0$ lies in the fact that ψ_0 depends on H_0 only, and we have

$$
\psi(Q,H_0;s) = \psi_0(H_0) + \psi_1(Q,H_0)e^{-i\Omega s/c}. \tag{22}
$$

Notice the period of the motion of a particle is

$$
\Phi = \oint \frac{\partial q(H_0,p')}{\partial H_0} dp'. \tag{23}
$$

This period depends on the value of H_0 for the particle under consideration. In the simple harmonic case, we have $\Phi = 2\pi c/\omega_s$.

Since the Poisson bracket is invariant under canonical transformations, we can express it in either the new or the old variables. In particular, we use the new variables to obtain

$$\{\psi_1, H_0\} = \frac{\partial\psi_1}{\partial Q}\frac{\partial H_0}{\partial H_0} - \frac{\partial\psi_1}{\partial H_0}\frac{\partial H_0}{\partial Q} = \frac{\partial\psi_1}{\partial Q}, \tag{24}$$

and, noting H_1 is independent of p, use the old variables to obtain

$$\begin{aligned} \psi_0, H_1\} &= \frac{\partial\psi_0}{\partial q}\frac{\partial H_1}{\partial p} - \frac{\partial\psi_0}{\partial p}\frac{\partial H_1}{\partial q} \\ &= -\frac{\partial\psi_0}{\partial H_0}\frac{\partial H_0}{\partial p}\frac{\partial H_1}{\partial q}. \end{aligned} \tag{25}$$

Following the definition of H_1 in Eq.(16), we have

$$\frac{\partial H_1}{\partial q} = -\frac{\eta_0 e}{EC}V_1(q). \tag{26}$$

The linearized Vlasov equation (18) becomes

$$-i\frac{\Omega}{c}\psi_1 + \frac{\partial\psi_1}{\partial Q} + \frac{\eta_0 e}{EC}V_1(q)\frac{\partial\psi_0}{\partial H_0}\frac{\partial H_0}{\partial p} = 0. \tag{27}$$

To solve the Vlasov equation, we first Fourier expand ψ_1 as

$$\psi_1 = \sum_{l=-\infty}^{\infty} R_l(H_0)e^{i2\pi lQ/\Phi(H_0)}, \tag{28}$$

where the $l = 0$ term in the summation is to be excluded because it violates the total charge conservation for a given H_0. The Fourier expansion is possible because the motion is periodic in Q with period Φ. Note that Φ depends on H_0.

Substituting Eq.(28) into Eq.(17), we find

$$\begin{aligned} V_1 = \frac{e}{2\pi}\int_{-\infty}^{\infty} d\omega Z_0^{\parallel}(\omega)\exp\left[i\omega\frac{q(Q,H_0)}{c}\right] \\ \times\int_{-\infty}^{\infty} dH_0' \int_0^{\Phi(H_0')} dQ' \exp\left[-i\omega\frac{q(Q',H_0')}{c}\right] \\ \times\sum_{l=-\infty}^{\infty} R_l(H_0')\exp\left[i2\pi l\frac{Q'}{\Phi(H_0')}\right]. \end{aligned} \tag{29}$$

Multiplying both side of Eq.(27) by $\exp(-i2\pi lQ/\Phi)$ and integrating over Q from 0 to Φ, we obtain

$$\left[\Omega - \frac{2\pi lc}{\Phi(H_0)}\right] R_l(H_0) + i\frac{\eta_0 e^2 c}{2\pi EC\Phi(H_0)}$$
$$\times \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega) \int_0^{\Phi(H_0)} dQ\, \frac{\partial\psi_0}{\partial H_0}\frac{\partial H_0}{\partial p} \exp\left[i\omega\frac{q(Q,H_0)}{c}\right]$$
$$\times \int_{-\infty}^{\infty} dH_0' \int_0^{\Phi(H_0')} dQ' \exp\left[-i\omega\frac{q(Q',H_0')}{c}\right]$$
$$\times \sum_{l'=-\infty}^{\infty} R_{l'}(H_0') \exp\left[i2\pi l'\frac{Q'}{\Phi(H_0')} - i2\pi l\frac{Q}{\Phi(H_0)}\right] = 0, \quad l = \pm \text{ integers.} \tag{30}$$

For a general equilibrium distribution ψ_0 (Gaussian, for example), the analysis to solve Eq.(30) is involved. Pursuing along this line would yield the radial modes of the collective oscillation. For one simple beam distribution, the water-bag model, however, the radial modes degenerate and the equation can be solved analytically. In the following, we will assume that the unperturbed beam has a water-bag distribution

$$\psi_0(H_0) = \begin{cases} N/\int_0^{\hat{H}} dH_0\Phi(H_0) & \text{if } 0 < H_0 < \hat{H}, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

The normalization is given by Eq.(14), together with the condition $dQdH_0 = dqdp$. For small ϵ, the overall normalization of (31) can be approximated to give

$$\psi_0(H_0) \approx \frac{\omega_s N}{2\pi c\hat{H}}\Theta(\hat{H} - H_0), \tag{32}$$

where $\Theta(x)$ is the step function.

Since any perturbation of a water-bag distribution has to occur around the edge of the bag, we have

$$R_l(H_0) \propto \delta(H_0 - \hat{H}). \tag{33}$$

After adopting the water-bag model, Eq.(30) simplifies considerably to yield, for the l-th mode (for example, $l = 1, 2, 3$ correspond to the dipole, quadrupole, and sextupole modes),

$$\Omega^{(l)} - \frac{2\pi lc}{\Phi} - i\frac{\eta_0 N e^2 \omega_s}{4\pi^2 EC\hat{H}\Phi} \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)$$
$$\times \int_0^{\Phi} dQ \frac{\partial H_0}{\partial p} \exp\left[i\omega\frac{q(Q,\hat{H})}{c} - i2\pi l\frac{Q}{\Phi}\right] \tag{34}$$
$$\times \int_0^{\Phi} dQ' \exp\left[-i\omega\frac{q(Q',\hat{H})}{c} + i2\pi l\frac{Q'}{\Phi}\right] = 0,$$

where, and form this point on, Φ is evaluated at $H_0 = \hat{H}$. In obtaining Eq.(34), the coupling among the different modes with $l' \neq l$ are neglected. The validity of this approximation assumes the mode frequency shifts are small compared with $2\pi c\Phi \approx \omega_s$.

For the longitudinal head-tail instability problem, we now substitute Eq.(11) for H_0 and Eq.(12) for $f(p)$. We further define an angular variable θ according to

$$q = \frac{c}{\omega_s}\sqrt{2H_0}\cos\theta,$$
$$p\sqrt{1-\frac{\epsilon}{\eta_0}p} = \sqrt{2H_0}\sin\theta. \tag{35}$$

We then have, from Eqs.(21) and (23),

$$Q = -\frac{c}{\omega_s}\int_0^{\theta} G(\theta')d\theta',$$
$$\Phi = \frac{c}{\omega_s}\int_0^{2\pi} G(\theta)d\theta \equiv \frac{2\pi c}{\omega_s}\langle G\rangle, \tag{36}$$

with

$$G(p) = \frac{\sqrt{1-\frac{\epsilon}{\eta_0}p}}{1-\frac{3}{2}\frac{\epsilon}{\eta_0}p}. \tag{37}$$

The size of the water-bag $\hat{H}$ is related to the bunch length $\hat{z}$ through $\hat{H} = \omega_s^2\hat{z}^2/8c^2$. Below, we will introduce another convenient parameter

$$\hat{\tau} = \frac{\hat{z}}{c} = \frac{2\sqrt{2\hat{H}}}{\omega_s}. \tag{38}$$

In terms of the new variable θ, Eq.(34) can be written as

$$\Omega^{(l)} - \frac{l\omega_s}{\langle G\rangle} - i\frac{\eta_0 N e^2 c}{2\pi^3 E C\omega_s\hat{\tau}\langle G\rangle}\int_{-\infty}^{\infty} d\omega Z_0^{\parallel}(\omega)$$
$$\times \int_0^{2\pi} d\theta \sin\theta \exp\left[i\frac{\omega\hat{\tau}}{2}\cos\theta + il\frac{\int_0^{\theta} G(\theta'')d\theta''}{\langle G\rangle}\right]$$
$$\times \int_0^{2\pi} d\theta' G(\theta')\exp\left[-i\frac{\omega\hat{\tau}}{2}\cos\theta' - il\frac{\int_0^{\theta'} G(\theta'')d\theta''}{\langle G\rangle}\right] = 0. \tag{39}$$

If the non-harmonicity is weak, we assume $|\epsilon\sqrt{2\hat{H}}/\eta_0| \ll 1$ and keep the first order terms in ϵ to obtain

$$G(\theta) \approx 1 + \frac{\epsilon}{\eta_0}\sqrt{2\hat{H}}\sin\theta,$$
$$\langle G\rangle = 1 + O(\epsilon^2) \approx 1. \quad (40)$$

Substituting the above expression into Eq.(39), we finally find the mode frequency

$$\Omega^{(l)} = l\omega_s + i\frac{\eta_0 N e^2 c}{2\pi^3 E C \omega_s \hat{\tau}}(A + \frac{\epsilon}{2\eta_0}\omega_s\hat{\tau}B), \quad (41)$$

where

$$A = \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)\int_0^{2\pi} d\theta\ \sin\theta e^{i\omega\hat{\tau}\cos\theta/2+il\theta}$$
$$\int_0^{2\pi} d\theta' e^{-i\omega\hat{\tau}\cos\theta/2-il\theta'},$$
$$B = \int_{-\infty}^{\infty} d\omega Z_0^{\|}(\omega)\Big\{il\int_0^{2\pi} d\theta\ \sin\theta(1-\cos\theta)e^{i\omega\hat{\tau}\cos\theta/2+il\theta}$$
$$\int_0^{2\pi} d\theta' e^{-i\omega\hat{\tau}\cos\theta/2'-il\theta'} + \int_0^{2\pi} d\theta\sin\theta e^{i\omega\hat{\tau}\cos\theta/2+il\theta}$$
$$\int_0^{2\pi} d\theta'[\sin\theta' - il(1-\cos\theta')]e^{-i\omega\hat{\tau}\cos\theta/2-il\theta'}\Big\}. \quad (42)$$

The quantities A and B can be simplified as

$$A = \frac{8\pi^2}{\hat{\tau}}l\int_{-\infty}^{\infty} d\omega\frac{Z_0^{\|}(\omega)}{\omega}J_l^2(\frac{\omega\hat{\tau}}{2}),$$
$$B = -\frac{32\pi^2}{\hat{\tau}^2}l^2\int_{-\infty}^{\infty} d\omega\frac{Z_0^{\|}(\omega)}{\omega^2}[\frac{\omega\hat{\tau}}{2}J_l(\frac{\omega\hat{\tau}}{2})J_{l+1}(\frac{\omega\hat{\tau}}{2}) +$$
$$(1-l)J_l^2(\frac{\omega\hat{\tau}}{2})]. \quad (43)$$

The longitudinal impedance satisfies

$$Z_0^{\|}(\omega) = Z_0^{\|*}(-\omega). \quad (44)$$

It follows that A is a purely imaginary and B is real. If $\epsilon = 0$, only the A coefficient plays a role; the result describes the solution of the conventional longitudinal instability problem. In particular, the fact that A is purely imaginary means the mode frequency Ω is real, and the beam is always stable. This is a well-known result [5,6] when mode coupling and multi-turn effects are ignored, as is presently assumed. If $\epsilon \neq 0$, the B term also contributes to the mode frequency Ω. This contribution, being imaginary, is the cause of the LHT instability. The instability growth rate is

$$\tau_l^{-1} = \mathrm{Im}\Omega = -\epsilon \frac{Ne^2c}{4\pi^3 EC} B$$
$$= \epsilon \frac{8Ne^2c^3}{\pi EC\hat{z}^2} l^2 \int_{-\infty}^{\infty} d\omega \frac{\Re Z_0^{\|}(\omega)}{\omega^2} \left[\frac{\omega\hat{z}}{2c} J_l(\frac{\omega\hat{z}}{2c}) J_{l+1}(\frac{\omega\hat{z}}{2c}) + \right.$$
$$\left. (1-l) J_l^2(\frac{\omega\hat{z}}{2c}) \right]. \tag{45}$$

Equation (45) is our main result of the LHT instability growth rate for mode l.

We will next establish Eq.(9) as follows. The energy loss in one turn is given by

$$\Delta\varepsilon = -\frac{e^2}{2\pi} \int_{-\infty}^{\infty} d\omega |\tilde{\rho}(\omega)|^2 \Re Z_0^{\|}(\omega). \tag{46}$$

For the water-bag distribution, we have

$$\rho(z) = \frac{8N}{\pi\hat{z}^2} \sqrt{\frac{\hat{z}^2}{4} - z^2}, \tag{47}$$

and the corresponding Fourier spectrum is

$$\tilde{\rho}(\omega) = \frac{4Nc}{\omega\hat{z}} J_1(\frac{\omega\hat{z}}{2c}). \tag{48}$$

Substituting Eqs.(46) and (48) into Eq.(9), we find

$$\tau^{-1} = \epsilon \frac{4Ne^2c^2}{\pi EC\hat{z}} \int_{-\infty}^{\infty} d\omega \frac{\Re Z_0^{\|}(\omega)}{\omega} J_1(\frac{\omega\hat{z}}{2c}) J_2(\frac{\omega\hat{z}}{2c}). \tag{49}$$

This is the same result as Eq.(45) for the case $l = 1$. The simple physical picture and the self-consistent calculation thus give identical result for dipole motion.

4.0 NUMERICAL RESULTS

We may apply the result of the last section to specific models of the impedance. For example, the diffraction model of cavity structures gives an impedance [7]

$$Z_0^{\parallel}(\omega) = \frac{Z_0}{2\pi b}\sqrt{\frac{cg}{\pi|\omega|}}[1+\mathrm{sgn}(\omega)i], \tag{50}$$

where $Z_0 = 377\Omega$ is the impedance of free space, b is the radius of the beam pipe at the location of the cavity structure, and g is the longitudinal length of the gap of the cavity. The corresponding growth rate is found to be

$$\tau_l^{-1} = \frac{3l^2\Gamma(l-\frac{3}{4})}{32\pi^2\Gamma^2(\frac{7}{4})\Gamma(l+\frac{7}{4})}\epsilon\frac{Ne^2c^2Z_0}{ECb}\sqrt{\frac{g}{\hat{z}}}. \tag{51}$$

For the $l = 1$ dipole mode, this gives

$$\tau^{-1} = \frac{2\Gamma^4(\frac{1}{4})}{63\pi^5}\epsilon\frac{Ne^2c^2Z_0}{ECb}\sqrt{\frac{2g}{\hat{z}}}\ . \tag{52}$$

In case the dominating cavity structures are the accelerating rf cavities, one would have $g \approx c/2f_{\mathrm{rf}}$ where f_{rf} is the rf frequency.

As a second impedance model, we consider the resonator model

$$Z_0^{\parallel}(\omega) = \frac{R}{1+iQ(\frac{\omega_r}{\omega}-\frac{\omega}{\omega_r})}. \tag{53}$$

The growth rate for dipole mode is found to be

$$\tau^{-1} = \frac{4Ne^2c^2R}{EC\hat{z}\sqrt{4Q^2-1}}\epsilon\Re(I_1-I_2), \tag{54}$$

where

$$I_{1,2} = J_1(x_{1,2})J_2(x_{1,2}) - \frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!!(x_{1,2})^{2n}}{(2n+1)!!\Gamma(n-\frac{1}{2})\Gamma(n+\frac{5}{2})} \tag{55}$$

and

$$x_{1,2} \equiv \frac{\omega_r\hat{z}}{4Qc}\left(-i\pm\sqrt{4Q^2-1}\right). \tag{56}$$

For the CERN SPS collider, we take $E = 26$ GeV, $N = 10^{11}$, $C = 6.9\times10^3$ m, $\hat{z}/c = 5$ ns (Reference 1) which gives a bunch length 7 ns for the cosine squared distribution. The $\hat{z}$ is obtained by equaling rms bunch length of two distribution, the water-bag and the cosine squared), and $\epsilon = 1$. We also assume there are two sets of rf cavities. The first set contains 198 rf cells with $f_{\mathrm{rf}} = 200.222$ MHz, $b = 6.5$ cm, $R/Q = 114.5\ \Omega$. The second set contains 32 rf cells with $f_{\mathrm{rf}} = 200.3982$ MHz, $b = 7.8$ cm, $R/Q = 216\ \Omega$.[1,8] The growth time τ is

5.4 s for the diffraction model and 23 s for the resonator model. The observed growth rate is 5 $\sim$ 6 s.[1]

For SSCL machines the parameters are as follows: For the Collider, we assume E = 2000 GeV, N = 8.1×10^9, C = 8.712×10^4 m, $\hat{z}$ = 5.4 cm and there are 40 rf cells with $f_{\rm rf}$ = 359.75901 MHz. For the HEB, we assume E = 199.1 GeV, N = 8.1×10^9, $C = 1.08 \times 10^4$ m, $\hat{z}$ = 30.7 cm and there are 10 rf cells with $f_{\rm rf}$ = 59.957832 MHz. For the MEB, we assume E = 11.1 GeV, $N = 8.3 \times 10^9$, $C = 3.96 \times 10^3$ m, $\hat{z}$ = 22.0 cm and there are 18 rf cells with $f_{\rm rf}$ = 59.776 MHz. For the LEB, we assume E = 0.6 GeV, $N = 8.7 \times 10^9$, $C = 5.7 \times 10^2$ m, $\hat{z}$ = 143.0 cm and there are 8 rf cells with $f_{\rm rf}$ = 47.514 MHz, Reference 9. We also assume $b \approx 5$ cm and $\epsilon \approx 1$ for all machines. The growth time are 7.0×10^4 s, 3.4×10^3 s, 32 s and 1.2 s, respectively.

The LHT instabilities tend to play a more important role in the lower energy accelerators, particularly those operated close to transition. In all cases studied, however, the LHT instability does not constitute a serious limit on beam intensities.

We may also apply the result to the resistive wall impedance

$$Z_0^{\|}(\omega) = \frac{C}{bc}\sqrt{\frac{|\omega|}{2\pi\sigma}}[1 - \mathrm{sgn}(\omega)i], \tag{57}$$

where σ is metal conductivity. We obtain

$$\tau_l^{-1} = \frac{3l^2\Gamma(l-\frac{1}{4})}{4\pi\Gamma^2(\frac{5}{4})\Gamma(l+\frac{5}{4})}\epsilon\frac{Ne^2c}{E\hat{z}b}\sqrt{\frac{2c}{\sigma\hat{z}}}. \tag{58}$$

For the resistive wall, we find growth rate is negligible. For the LHT instability, resistive walls do not play an important role.

5.0 SUMMARY

If the unperturbed beam motion is distorted from that of a simple harmonic motion, the non-harmonic distortion will create a new collective instability. We have developed a formalism based on the Vlasov equation to analyze this instability. The technique is then applied to the longitudinal head-tail instability effect. Explicit expressions of growth rates are obtained for the water-bag distribution model for various collective modes. The analytical result for the dipole mode seems to agree with the observation made at the SPS. Application to the SSC Collider and Boosters show that the longitudinal head-tail instability is not a serious limit on the SSC beam intensities.

REFERENCES

1. D. Boussard and T. Linnecar, *Proc. 2nd Euro. Part. Accel. Conf.*, edited by P. Marin and P. Mandrillon (Editions Frontières, Gif-sur-Yvette Cedex, France, 1990), pp. 1560–1562 and CERN Report SPS/89-38(ARF) (1989).
2. G. Hereward, Rutherford Lab. Report RL-74-062 (1974) and RL-75-021 (1975).
3. C. Pellegrini, *Nuovo Cimento* **64A**, 447 (1969).
4. M. Sands, SLAC Report TN-69-8 and TN-69-10 (1969).
5. F. Sacherer, CERN Report SI-BR/72-5 (1972) and *Proc. 9th Int. Conf. High Energy Accel.* (SLAC, 1974) p. 347.
6. A. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (John Wiley & Sons, New York, 1993).
7. J. Lawson, *Part. Accel.* **25**, 107 (1990).
8. Some of this information is extracted from L. Vos, CERN Report SPS/86-21(MS) (1986).
9. SSC Central Design Group, SSCL Report SSC-SR-2020 (1986).