SISSA ISAS

NOTE ON A FREE PARTICLE IN NONCOMMUTATIVE SPACE-TIME

Ludwik Dąbrowski and Preeti Parashar

SISSA, 34014 Trieste, Italy

Abstract

By examining some known wave equations it is shown that the dynamics of a free quantum particle in a slightly noncommutative space-time is equivalent to that of a charged particle moving in a self-generated (weak) electromagnetic field.

Ref. SISSA 134/95/FM

Via Beirut 2-4 - 34014 Trieste - ITALY

Recently several noncommutative deformations of space-time coordinates have been considered [1], [2], [3], [4] in relation to the development of quantum groups. It is well known cf. [5] p.188, that the classical mechanics of a point particle moving in an electromagnetic field generated by the vector potential A_μ, can be equivalently described as a free particle with a modified Poisson bracket

$$\{x_\mu, x_\nu\} = 0 \ , \ \{x_\mu, p_\nu\} = \eta_{\mu,\nu} \ , \ \{p_\mu, p_\nu\} = e\ F_{\mu,\nu} \ , \tag{1}$$

where $F_{\mu,\nu}$ is the field strength of A_μ.

Note that such a deformation occurs only in the sector of momenta, the coordinates x remaining 'undeformed'.

It is interesting to study a related question on the quantum level. A particle in slightly noncommutative space has been recently considered by Chaturvedi et al. [6]. It turns out that for the space-like coordinates as well as for the conjugate momenta being noncommutative, similarly to the classical case, the net effect amounts to introducing a weak magnetic field (upto first order in the deformation parameter). In this note we extend the results of the paper [6] to the four dimensional space-time and study some known wave equations. As we shall demonstrate, a free particle moving in such a SNC space-time can be described as a charged particle moving in the ordinary space-time coupled to a suitable magnetic and electric field.

Consider a noncommutative space-time spanned by four real coordinates $X^i, i = 0, 1, 2, 3$; satisfying the commutation relations

$$X^i X^j = q X^j X^i, \ \ i < j, \ \ q = e^{i\theta}. \tag{2}$$

The q-commutation relations between these NC variables and their corresponding partial derivatives are:

$$\begin{aligned} X^i X^j &= q X^j X^i, \ \ i < j, \\ \mathcal{D}_i X^j &= q X^j \mathcal{D}_i \ , \\ \mathcal{D}_i X^i &= 1 + q^2 X^i \mathcal{D}_i + (q^2 - 1) \sum_{j>i} X^j \mathcal{D}_j \ , \\ \mathcal{D}_i \mathcal{D}_j &= q^{-1} \mathcal{D}_j \mathcal{D}_i. \end{aligned} \tag{3}$$

This differential calculus is covariant under the coaction of the quantum group $GL_q(4)$. Following [7], [9] one can explicitly realize the above noncommutative 'quantum' space-time coordinates and their derivatives in terms of commutative 'classical' space-time coordinates x^i and derivatives ∂_i as:

$$\begin{aligned} X^i &= x^i \left\{ \frac{q^{2(N_i+1)} - 1}{(q^2 - 1)(N_i + 1)} \right\}^{\frac{1}{2}} q^{\sum_{j>i} N_j} \ , \\ \mathcal{D}_i &= q^{\sum_{j>i} N_j} \left\{ \frac{q^{2(N_i+1)} - 1}{(q^2 - 1)(N_i + 1)} \right\}^{\frac{1}{2}} \partial_i \ , \end{aligned} \tag{4}$$

where $N_i = x^i \partial_i$.

This realization is not unique as there are other equivalent realizations [7], [8]. Also there exists an inverse map by which (x^i, ∂_i) can be written in terms of $(X^i, \mathcal{D}_i)$, see [6], [9].

We now make an expansion in θ and assuming θ to be small retain terms upto only first order in θ. In this small θ approximation the above realization can be written as

$$
\begin{aligned}
X^i &= x^i + i\theta(x^i \sum_{j>i} \left\{x^j\partial_j\right\}_s + \frac{1}{2}\left\{(x^i)^2\partial_i\right\}_s) , \\
\mathcal{D}_i &= \partial_i + i\theta(\partial_i \sum_{j>i} \left\{x^j\partial_j\right\}_s + \frac{1}{2}\left\{x^i{\partial_i}^2\right\}_s) ,
\end{aligned} \tag{5}
$$

where

$$\left\{x^i\partial_i\right\}_s = \frac{1}{2}(x^i\partial_i + \partial_i x^i)$$

Let us now consider a particle moving in a slightly noncommutative space-time ($q \approx 1 + i\theta$). To describe the quantum mechanical behaviour of this particle we call:
X^0 : Time coordinate operator ,
$X^i, i = 1,2,3$: Position coordinate operators ,
$P_0 = i\hbar\mathcal{D}_0$: The Energy operator , and
$P_i = -i\hbar\mathcal{D}_i$:The Momentum operators.
Then the above expressions become

$$
\begin{aligned}
X^0 &= x^0 - \frac{\theta}{\hbar}(x^0 \sum_{i=1}^{3} \left\{x^i p_i\right\}_s - \frac{1}{2}\left\{(x^0)^2 p_0\right\}_s) , \\
X^i &= x^i - \frac{\theta}{\hbar}(x^i \sum_{j>i} \left\{x^j p_j\right\}_s + \frac{1}{2}\left\{(x^i)^2 p_i\right\}_s) ,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
P_0 &= p_0 - \frac{\theta}{\hbar}(p_0 \sum_{i=1}^{3} \left\{x^i p_i\right\}_s - \frac{1}{2}\left\{(x^0 p_0^2\right\}_s) , \\
P_i &= p_i - \frac{\theta}{\hbar}(p_i \sum_{j>i} \left\{x^j p_j\right\}_s + \frac{1}{2}\left\{x^i p_i^2\right\}_s) ,
\end{aligned} \tag{7}
$$

where

$$p_0 = i\hbar\partial_0 \ , \ p_i = -i\hbar\partial_i \ .$$

The deformed Heisenberg commutation relations in this approximation (infinitesimally small θ) are:

$$
\begin{gathered}
[X^0 \,, X^i] = i\theta X^i X^0 \ , \ [X^i \,, X^j] = i\theta X^j X^i \ , \ [P_0 \,, X^i] = i\theta X^i P_0 \ , \\
[P_i \,, X^j] = i\theta X^j P_i \ , \ [P_0 \,, P_i] = -i\theta P_i P_0 \ , \ [P_i \,, P_j] = -i\theta P_j P_i \ , \\
[X^0 \,, P_0] = -i\hbar + 2i\theta(-X^0 P_0 + \sum_{i=1}^{3} X^i P_i) \ , \ [X^i \,, P_i] = i\hbar - 2i\theta(\sum_{j\geq i} X^j P_j).
\end{gathered} \tag{8}
$$

For $\theta = 0$ they reduce to the usual Heisenberg relations.The state of a quantum mechanical system can be represented by the wavefunction $\psi(x^0, x^i)$ .The effect of noncommutativity of the underlying space-time is manifested in the deformation of the hamiltonian H to $\mathcal{H}$. With the above setup in mind , we now study some known wave equations in slightly non-commutative space-time.

1. SCHRÖDINGER EQUATION:

The non-relativistic Schrödinger equation for a free particle of mass m moving in a SNCST can be written as

$$i\hbar\mathcal{D}_0\psi = \frac{1}{2m}\sum_{i=1}^{3} P_i^2\psi \tag{9}$$

Let us assume a solution of the form

$$\psi(r, x^0) = \phi_k(r)e^{i(k.r-\omega x^0)} \tag{10}$$

Following Chaturvedi et al.[6] $\phi_k(r)$ is considered to be a slowly varying function of r such that

$$|\frac{-i}{\phi_k}\partial_i\phi_k| \ll |k_i| \tag{11}$$

Then the above equation takes the form

$$\mathcal{P}_0\psi \approx \frac{1}{2m}\sum_{i=1}^{3}\mathcal{P}_i{}^2\psi \tag{12}$$

where

$$\begin{aligned}
\mathcal{P}_0 &= p_0 - 3i\theta p_0 - \theta\hbar\omega(-\frac{1}{2}\omega x^0 + \sum_{i=1}^{3} k_i x^i)\ , \\
\mathcal{P}_1 &= p_1 - \frac{3}{2}i\theta p_1 - \theta\hbar k_1(\frac{1}{2}k_1 x^1 + k_2 x^2 + k_3 x^3)\ , \\
\mathcal{P}_2 &= p_2 - i\theta p_2 - \theta\hbar k_2(\frac{1}{2}k_2 x^2 + k_3 x^3)\ , \\
\mathcal{P}_3 &= p_3 - \frac{1}{2}i\theta p_3 - \theta\hbar k_3(\frac{1}{2}k_3 x^3)\ ,
\end{aligned} \tag{13}$$

Writing the above energy and momentum operators in the form

$$\mathcal{P}_0 = p_0 - e\phi\ ,\ \mathcal{P}_i = p_i - eA_i\ \ , c = 1\ ,$$

we infer that the behaviour of a particle in a SNCST is described by the Schrödinger equation for a charged particle moving in a self generated electro-magnetic field given by

$$E_i = -\frac{\theta}{e}\hbar\omega k_i\ ,\ B_i = \frac{\theta}{e}\hbar\epsilon_{ijk}k_j k_l \tag{14}$$

Remarks:
1. if ψ depends on only one coordinate then the particle feels only electric field and not magnetic field.
2. Since the commutation relations of NCST coordinates does not possess the usu al $SO(3,1)$ covariance of the commutative space time, it implies that there ex ist preferred axes corresponding to a given (X^0, X^i) frame.
3. In the special case of the time coordinate remaining undeformed,we have only magnetic field and hence obtain the results of Chaturvedi et al.
4.In [6] the second terms on rhs in the last three equations of (13) were neglected since θ is assumed to be very small. But since we are considering the case of infinitesimally small θ it

would not be proper to neglect them. Howevere, it may be noted that since E_i and B_i are gauge invariant, the presence of these terms in the gauge potential has no net effect on the electromagnetic field components.

2. KLEIN-GORDON EQUATION:

The quantum mechanical relativistic Schrödinger equation or the Klein Gordon equation in a SNCST where the coordinates are noncommutative,is written as

$$-\hbar^2 \mathcal{D}_0 \psi = \sum_{i=1}^{3} P_i^2 \psi + m^2 \psi \tag{15}$$

The solution of this equation is of the type

$$\psi(r, x^0) = \phi_k(r) e^{i(k.r \mp \omega x^0)} , \quad p_0 = \mp \hbar\omega$$

Taking the first solution ($p_0 = +\hbar\omega$) and using the previous approximations we obtain the Klein Gordon eq. in the form

$$\mathcal{P}_0{}^2 \psi \approx \sum_{i=1}^{3} \mathcal{P}_i{}^2 \psi + m^2 \psi \tag{16}$$

where $\mathcal{P}_0$ and $\mathcal{P}_i$ are given by the (13). Also the field components E_i and B_i are the same as in the Schrödinger case.

On the other hand, the negative energy solution ($p_0 = -\hbar\omega$) yields

$$\mathcal{P}_0 = p_0 + \theta\hbar\omega(\frac{1}{2}\omega x^0 + \sum_{i=1}^{3} k_i x^i) , \tag{17}$$

and $\mathcal{P}_i$ are the same as given by the first solution. As a consequence, the Magnetic field B remains the same whereas the electric field changes sign

$$E_i = \frac{\theta}{e}\hbar\omega k_i \tag{18}$$

This can be thought of as two particles of opposite charges $(e, -e)$ moving in the same direction in an ordinary space-time coupled to an electromagnetic field.This situation is similar to the particle-antiparticle description.

3. DIRAC EQUATION:

The four component Dirac equation with noncommuting coordinates can be written as

$$(i\hbar\gamma^\mu \mathcal{D}_\mu - m)\psi = 0 \tag{19}$$

or

$$\gamma^0 P_0 \psi = \sum_{i=1}^{3} \gamma^i P_i \psi + m\psi \tag{20}$$

We consider a plane wave solution:

$$\psi(r, x^0) = U(k) e^{i(k.r \mp \omega x^0)} , \quad p_0 = \mp \hbar\omega$$

where $U(k)$ is a four-component spinor and γ^μ are the usual gamma matrices. Performing the same analysis, we obtain same results as in the Klein Gordon case. This is due to the fact that the effect of spinors is neglected.

We would like to conclude by making the following remarks. By studying some known wave equations we have shown that the dynamics of a free quantum particle in a slightly noncommutative space-time is, in the first approximation, equivalent to that of a charged particle moving in a self-generated electromagnetic field. The explicit form of such a (weak) electromagnetic field is essentially universal as it turns out to be independent of the kind of wave equation chosen , upto some numerical coefficients.
The wave equations we have considered here are not covariant under the quantum group $GL_q(4)$. (Only the coordinates are covariant). For the internal consistency it would be interesting to obtain and discuss a q-deformation of these equations which are covariant under some q-deformed Lorentz group. However, we expect that also in this case the results of our study should be valid (possibly upto another numerical constant), just due to the 'universality' mentioned above.

References

[1] W. Pusz and S. Woronowicz, Rep. Math. Phys. 27 (1989) 231

[2] J. Wess and B. Zumino, Nucl. Phys. (Proc. Suppl.) B 18 (1990) 302 : B. Zumino, Mod. Phys. Lett. A 6 (1991) 1225

[3] S. Majid and U. Meyer, Z. Phys. C, 63, (1994) 357

[4] S. Doplicher, Talk at the XI International Congress of Math. Phys., Paris, July 1994

[5] W. Thirring, A Course on Mathematical Physics, vol I, Springer Verlag, New York 1978

[6] S. Chaturvedi, R. Jagannathan, R. Sridhar and V. Srinivasan, J. Phys. A: Math. Gen. 26 (1993) L105

[7] R. Jagannathan, R. Sridhar, R. Vasudevan, S. Chaturvedi, M. Krishnakumari, P. Shanta, and V. Srinivasan, J. Phys. A : Math. Gen. 25 (1992) 6429

[8] M. Arik, Z. Phys. C, 51, (1991) 627

[9] D. Fairlie and C. Zachos, Phys. Lett. 256B (1991) 43