ON THREE-QUARK ANOMALOUS DIMENSIONS

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ABSTRACT
The anomalous dimensions of three-quark operators are calculated by diagonalizing the one-gluon exchange kernel of the evolution equation of nucleon distribution amplitudes. This is done within a symmetrized basis of Appell polynomials for polynomial degrees $\geq 1$. Technically, this is accomplished by a combination of analytical and numerical algorithms. The calculated eigenvalues form a degenerate system whose weight seems to follow an empirical logarithmic law.

1. Introduction

A key clue to the dynamics of exclusive reactions is the renormalization-group behavior of the hadron distribution amplitude (DA) $\Phi$, controlled by an evolution equation, derived by Brodsky and Lepage. Specifically, the momentum dependence of the nucleon DA for the valence-quark Fock state $\Phi(x_i, Q^2)$ is given by

$$\left\{Q^2 \frac{\partial}{\partial Q^2} + \frac{3C_F}{2\beta}\right\}\Phi(x_i, Q^2) = \frac{C_B}{\beta} \int_0^1 [dy] V(x_i, y_i) \Phi(y_i, Q^2).$$  \hfill (1)

Here $C_B$ and $C_F$ are the Casimir operators of the fundamental and adjoint representation of $SU(3)$, respectively, and $\beta$ is the Gell-Mann and Low function. The leading-order expression for the integral kernel $V$ has been calculated in 1.

2. Higher-order eigenfunctions of the nucleon evolution equation

It was noticed in 2 that $[\hat{P}_{13}, \int_0^1 [dy] V(x_i, y_i)] = 0$ (where $\hat{P}_{13}$ is the permutation operator applied to quarks 1 and 3). Thus it is more appropriate to introduce a symmetrized basis of Appell polynomials $F_{m,n}(x_1, x_3) = (1/2)(F_{m,n}(x_1, x_3) \pm F_{n,m}(x_1, x_3))$ for $m > n / m < n$, in terms of which the eigenfunctions $\Phi_n(x_i)$ can be explicitly represented. Within this particular basis, the evolution kernel becomes block-diagonal with respect to different polynomial orders $m + n$ and also within a particular order for different symmetry classes with respect to $\hat{P}_{13}$. A typical feature of the Appell polynomials is that their number is quadratically increasing with the polynomial order. This renders the analytic diagonalization of the evolution equation increasingly complicated. Use of the symmetrized basis allows for an analytic diagonalization up to order 7. This improves previous calculations 3 which were limited to order 3. The eigenfunctions $\Phi_n(x_i)$ are represented as a finite series of ordinary polynomials, e.g., in powers of $x_1$ and $x_3$

$$\Phi_n(x_i) = \frac{1}{\sqrt{N_n}} \sum_{ij} a_{ij}^n x_i^1 x_i^3$$  \hfill (2)

[with expansion coefficients $a_{ij}^n$ and normalization factors $N_n$, given in 2,4]. It is precisely this peculiar property of the eigenfunctions, which allows one to express them in terms of their own moments (as it is known from Taylor expansions). Another important property of the eigenfunctions of the evolution equation is that they form a commutative algebra analogous to angular momentum eigenfunctions obeying the triangle relation:

$$\Phi_k(x_i) \Phi_n(x_i) = \sum_{l=0}^{\infty} F_{kl}^n \Phi_l(x_i) \text{ with } |O(k) - O(n)| \leq O(l) \leq O(k) + O(n),$$  \hfill (3)

where $O(k)$ denotes the polynomial order of eigenfunction $\Phi_k(x_i)$. The structure
coefficients \( F_{k+}^l \) of the group are calculated via

\[
F_{k+}^l = N_l \int_0^1 [dx] x_1 x_3 (1 - x_1 - x_3) \hat{\Phi}_k(x_1) \hat{\Phi}_n(x_i) \hat{\Phi}_l(x_i),
\]

with the particularly important case \( F_{kk}^0 = \frac{N_k}{N_k^2} \). The utility of this method has been shown in \(^4\) in studying the evolution behavior of a more complicated ansatz for the nucleon DA of the Brodsky-Lepage-Huang-Mackenzie-type which incorporates higher-order Appell series contributions. In the present contribution we focus on the large-order behavior of the three-quark anomalous dimensions \( \gamma_n \). These are related to the eigenvalues of the evolution equation and are explicitly given by

\[
\gamma_n = -\left( \frac{3}{2} \frac{\partial N_k}{\partial N_k} + \eta_n \frac{\partial N_k}{\partial N_k} \right),
\]

where \( \eta_n \) are the zeros of the characteristic polynomial that diagonalizes the evolution kernel. These anomalous dimensions belong to multiplicatively renormalizable three-quark operators appearing in the operator product expansion \(^5\). Our analytical results
up to order 7 are given in \textsuperscript{2,4}. The emerging pattern of anomalous dimensions up to this order seems to follow a power-law behavior empirically fitted (dashed line in Fig. 1) to $\gamma_n = 0.37 \, O(n)^{0.565}$. However, in order to determine the asymptotic behavior of the anomalous dimension pattern, one has to take into account the eigenvalues of much higher orders. This becomes possible by combining analytic and high-precision numerical algorithms \textsuperscript{2}. Our new results are compiled in Fig. 1.

3. Discussion and conclusions

Interpreting the above results, one observes that the multiplet structure already indicated in \textsuperscript{4} is also confirmed for higher orders and exhibits an inherent symmetry of the spectrum of eigenfunctions, not appreciated previously. Comparison of symmetric and antisymmetric eigenvalues effects a strong degeneracy which increases with the polynomial order. This makes it apparent that a separation of symmetry classes is a necessary prerequisite to obtain results for higher orders. Furthermore, the resulting pattern indicates that the asymptotic behavior of both symmetry classes may be identical. It is also important to emphasize that the power-law behavior of eigenvalues suggested in \textsuperscript{4} on the basis of lower-order Appell polynomials is not confirmed if higher-order terms are included [c.f. (dotted curve) $\gamma_n = 0.52 \, O(n)^{0.417}$]. Indeed, up to order 20 the spectrum of $\gamma_n$-eigenvalues is better described by a logarithmic behavior of the form (solid line): $\gamma_n = 1.25 \, \log^{1.32} (O(n) + 1.37)$. In order to accurately separate power law from logarithmic behavior, one has to extend the calculation, at least, to polynomial order 100. Such a program is in progress. These estimates are of particular importance to check against results extracted from Wilson-loop calculations \textsuperscript{6} which also predict a logarithmic behavior of such anomalous dimensions.

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