$\eta' \rightarrow 3\pi$ decay width is calculated in QCD taking into account $U_A(1)$ anomaly.

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It has been shown recently that $[1] \eta \rightarrow 3\pi$ decay width can be satisfactorily calculated in QCD by taking into account $\Upsilon_A(1)$ anomaly. The purpose of this note is to calculate the decay width of $\eta' \rightarrow 3\pi$ in QCD. Gell-Mann-Okubo mass formula gives mass of $\eta_8$-meson close to the mass of physical $\eta$-meson. This implies negligible mixing between $\eta_8$ and $\eta_1$ mesons ($\eta_1$ is $SU(3)$ singlet). On the other hand, the decay of $\eta$ and $\eta'$ into two photons implies a mixing angle between $15^\circ$ - $20^\circ$. We discuss the $\eta$ and $\eta'$ decay to three pions taking into account the mixing and $\Upsilon_A(1)$ anomaly.

We start with the $\Sigma$-term for $\pi\eta_8$ and $\pi\eta_1$ scattering. The $\Sigma$-term is given by

$$\sum_{\pi A} (0) = \frac{2m}{\sqrt{3}} |A| \delta_{\pi A} + \sqrt{2}\delta_{\pi A}|A|$$

where $A = \eta_8$ or $\eta_1$. From Gell-Mann-Okubo mass formula

$$< \eta_8 | S_8 | \eta_8 > = -\frac{1}{\sqrt{3}} \left( \frac{m^2_8 - m^2_s}{m_s - \bar{m}} \right)$$

$$< \eta_1 | S_8 | \eta_1 > = 0$$

where $m_s$ is the mass of strange quark and $\bar{m} = \frac{1}{2}(m_u + m_d)$; $m_u$ and $m_d$ are the masses of $u$ and $d$ quarks respectively. $S_i = \frac{2i}{3} q_i$, where $q_i$'s are Gell-Mann matrices and $\lambda_0 = \sqrt{2/3}I$.

We assume nonet symmetry so that

$$< \eta_8 | \sqrt{2} S_8 | \eta_8 > = \frac{2}{\sqrt{3}} \left( \frac{m^2_8 - m^2_s}{m_s - \bar{m}} \right)$$

Thus we get

$$\sum_{\pi \eta_8} (0) = \frac{2\bar{m}}{3} \left( \frac{m^2_8 - m^2_s}{m_s - \bar{m}} \right)$$

$$\sum_{\pi \eta_1} (0) = \frac{4\bar{m}}{3} \left( \frac{m^2_8 - m^2_s}{m_s - \bar{m}} \right)$$
Hence we get for the scattering amplitudes the values

\[ T(\pi\eta_8 \rightarrow \pi\eta_8) = \frac{2m(m^2_K - m^2_\pi)}{F^2_\pi 3(m_\pi - \bar{m})} \]  
\[ T(\pi\eta_1 \rightarrow \pi\eta_1) = \frac{4m(m^2_K - m^2_\pi)}{F^2_\pi 3(m_\pi - \bar{m})} \]  

(6a)  

(6b)

where \( F_\pi \) is the pion decay constant \( (F_\pi = 93\text{MeV}) \).

The eight pseudoscalar mesons are regarded as Nambu–Goldstone bosons. Their masses are given by well known relations [2,3]

\[ F_\pi m_{\pi}^2 = 2\bar{m}v \]  
\[ F_K^2 m_{K^+}^2 = (m_\pi + m_\pi)v \]  
\[ F_K^2 m_{K^0}^2 = (m_d + m_s)v \]  
\[ F_\eta^2 m_{\eta_8}^2 = \frac{2}{3}(\bar{m} + 2m_\pi)v \]  

(7a)  

(7b)  

(7c)  

(7d)

where \( -v = < \bar{u}u > = < \bar{d}d > = < \bar{s}s > \). Using \( F_K = F_\eta = F_\pi \), we get the relations 

\[ m_{\eta_8} = \frac{(4m^2_K - m^2_\pi)}{3}(\text{the Gell-Mann–Okubo mass ratio}) \]  

and

\[ \frac{(m^2_K - m^2_\pi)}{(m_\pi - \bar{m})} = \frac{m^2_\pi}{2\bar{m}} \]  

(8)

where \( m^2_K = \frac{1}{2}(m_{K^+}^2 + m_{K^0}^2) \).

Using Eq. (8), we can express Eqs. (6) as

\[ T(\pi\eta_8 \rightarrow \pi\eta_8) = \frac{m^2_\pi}{3F^2_\pi} \]  
\[ T(\pi\eta_1 \rightarrow \pi\eta_1) = \frac{2m^2_\pi}{3F^2_\pi} \]  

(9a)  

(9b)

Eq (9a) was first obtained in reference [4].
In QCD, the mass Hamiltonian $H_M$ can be written

$$H_M = m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s$$

$$= \sqrt{2/3}(2\bar{m} + m_s)S_z + \frac{2}{\sqrt{3}}(\bar{m} - m_s)S_3 + (m_u - m_d)S_3$$  \hspace{1cm} (10)

We note that, using eqs. (2),(3) and (4)

$$< \eta_8 | H_M | \eta_8 > = \frac{(\bar{m} + 2m_s)m^2_\pi}{m}$$  \hspace{1cm} (11)

$$< \eta_1 | H_M | \eta_1 > = \frac{2\bar{m} + m_s}{m}(m^2_\pi)$$  \hspace{1cm} (12)

Comparing Eq. (11) with Eqs. (7), we see that Eq. (11) is exactly the same as Eq. (7d). This justifies the use of nonet symmetry. Moreover, we note that entire mass of $\eta_8$ meson arises from its quark content.

On the other hand using the values [3,5]

$$\frac{m_s}{m} = 25.8, \frac{m_d/m_u} = 1.80$$  \hspace{1cm} (13)

and $m^2_\pi = 0.018 GeV^2$, we get from Eq. (12)

$$< \eta_1 | H_M | \eta_1 > \approx 0.167 GeV^2$$  \hspace{1cm} (14)

Let us define the physical mesons $\eta$ and $\eta'$

$$\eta = \cos \theta \eta_8 - \sin \theta \eta_1$$

$$\eta' = \sin \theta \eta_8 + \cos \theta \eta_1$$  \hspace{1cm} (16)

Since $m^2_{\eta_1} = m^2_{\eta'} + m^2_{\eta} - m^2_{\eta_8} = 0.894 GeV^2$, it is clear that about 17% of $M_{\eta_1}^2$ arises from the quark content of the $\eta_1$ meson; the rest of the contribution is from its gluon content.

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This is expected since SU(3) singlet axial vector current $A_{\mu}$ is not conserved in the chiral limit. In fact

$$\partial_{\mu}A_{\mu} = \frac{2}{3}(2\bar{m} + m_s)P_0 + \frac{\sqrt{2}}{3}(2\bar{m} - 2m_s)P_8 + \sqrt{2/3}(m_u - m_d)P_3 + \sqrt{6} \frac{\alpha}{4\pi} GG$$  \hspace{1cm} (17)

where $P_{\pm} = i\gamma^\lambda \gamma_5 q$ and $GG = Tr(G_{\mu\lambda}G_{\nu\lambda})$. From Eq. (17), we get [5] (neglecting the term with $(m_u - m_d)$):

$$F_0 m_{\eta_1}^2 = \frac{2}{3} \left( \frac{2\bar{m} + m_s}{2\bar{m}} \right) F_0^2 m_s^2 + \sqrt{6} < 0|\frac{\alpha_s}{4\pi} GG|\eta_1 >$$  \hspace{1cm} (18)

$$0 = \frac{2\sqrt{2}}{3} \left( \frac{\bar{m} - m_s}{2\bar{m}} \right) F_0^2 m_s^2 + \sqrt{6} < 0|\frac{\alpha_s}{4\pi} GG|\eta_8 >$$  \hspace{1cm} (19)

We first note that if we use nonet symmetry via $F_0 = F_8 = F_\pi$, the first term on the right hand side of Eq. (18) is just the same as given in Eq. (12).

We write $|\eta_1 > = |\eta_1 >_G + |\eta_1 >_q$, where $G$ and $q$ signify the gluon and quark contents of $\eta_1$. We note that $|\eta_8 > = |u\bar{u} + d\bar{d} - 2s\bar{s} >$ and $|\eta_1 >_q = |u\bar{u} + d\bar{d} + s\bar{s} >$. Then comparing Eqs. (19) and (18), we get

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi} GG|\eta_1 >_q = \frac{2}{3} \left( \frac{2\bar{m} + m_s}{2\bar{m}} \right) m_s^2 = -0.167 GeV^2$$  \hspace{1cm} (20)

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi} GG|\eta_1 >_G = m_{\eta_1}^2$$  \hspace{1cm} (21)

$$\frac{\sqrt{6}}{F_\pi} < 0|\frac{\alpha_s}{4\pi} GG|\eta_8 > = \frac{2\sqrt{2}}{3} \left( \frac{m_s - \bar{m}}{2\bar{m}} \right) m_s^2 = 0.210 GeV^2$$  \hspace{1cm} (22)

We now discuss the decays $\eta \rightarrow 3\pi^0$ and $\eta' \rightarrow 3\pi^0$. The decay amplitude $T_\eta$ can be written [1]

$$T_\eta = -\frac{i}{\sqrt{6}} (m_d - m_u) \frac{1}{m} [4T(\eta\eta \rightarrow \pi\pi) \frac{1}{m_{\eta_1}} \sqrt{6} < 0|\frac{\alpha_s}{4\pi} GG|\eta >]$$  \hspace{1cm} (23)

From Eqs. (1), (2),(3),(4),(8) and (16), we get

$$T(\eta\eta \rightarrow \pi\pi) = \frac{1}{F_{\pi^2}} \sum_{\pi\pi} (0) = \frac{m_{\pi_0}^2}{3F_{\pi^2}} (\cos \theta - \sqrt{2}\sin \theta)^2$$  \hspace{1cm} (24)
\[ T(\eta'\eta' \to \pi\pi) = \frac{i}{F_\pi^2} \sum_{\pi \eta'} \langle 0 | = \frac{m_\pi^2}{3F_\pi^2}(\sin \theta + \sqrt{2}\cos \theta)^2 \] (25)

Hence we get

\[ T_\eta = -\frac{i}{\sqrt{6}} \frac{m_d - m_u}{2m} \frac{8m_\pi^2}{3F_\pi^2} (\cos -\sqrt{2}\sin \theta)^2 \frac{1}{m_\eta^2} \times \]

\[ [\frac{\sqrt{6}}{F_\pi} < 0] \frac{\alpha_s}{4\pi} G \bar{G} |\eta >] \] (26)

\[ T_{\eta'} = -\frac{i}{\sqrt{6}} \frac{m_d - m_u}{2m} \frac{8m_\pi^2}{3F_\pi^2} (\sin \theta + \sqrt{2}\cos \theta)^2 \frac{1}{m_{\eta'}^2} \times \]

\[ [\frac{\sqrt{6}}{F_\pi} < 0] \frac{\alpha_s}{4\pi} G \bar{G} |\eta' >_n] \] (27)

It may be noted that Eq.(26) gives \( T_\eta \) as in reference [1] for \( \theta = 0 \) and \( F_n = F_\pi \). Also note that due to OZI rule; in Eq.(27) only the quark part of the anomaly contributes to \( \eta' \to 3\pi \) decay. Using Eq.(16), and \( \theta = -18^\circ \) (see below), we get

\[ \frac{\sqrt{6}}{F_\pi} < 0] \frac{\alpha_s}{4\pi} G \bar{G} |\eta > = 0.148 GeV^2 \] (28)

\[ \frac{\sqrt{6}}{F_\pi} < 0] \frac{\alpha_s}{4\pi} G \bar{G} |\eta' > = -0.224 GeV^2 \] (29)

Hence we get

\[ |T_\eta| = 0.61 \] (30)

to be compared with the experimental value \( (0.69 \pm 0.84) \) [6] and

\[ |T_{\eta'}| = 0.17 \] (31)

to be compared with the experimental value \( 0.20 \pm 0.02 \) [6].

We conclude that our calculation gives rather a good agreement with the experimental results both for \( \eta \) and \( \eta' \) decays.

In order to discuss \( \eta_8 - \eta_1 \) mixing, we make the assumption that mixing between \( \eta_8 \) and \( \eta_1 \) arises from quark content of these mesons viz mixing term arises from the commutator.
of type $[F_0^s, \partial_\mu A_{8\mu}]$ and $[F_8^s, \partial_\mu \tilde{A}_{0\mu}]$, where

$$\partial_\mu \tilde{A}_{0\mu} = \partial_\mu A_{0\mu} - \sqrt{6} \frac{\alpha_s}{4\pi} G\tilde{G}$$  \hspace{1cm} (32)

In this case, the mixing term is given by

$$F_0 F_8 \frac{m_{\eta_1}^2}{m_\eta} - \frac{2\sqrt{2}}{3} \left( \frac{m_\eta}{m_s} - \frac{m_s}{m_\eta} \right) F_0^2 m_s^2 \frac{m^2_{\eta_1}}{2m_\eta}$$  \hspace{1cm} (33)

Hence from Eqs. (7), (18) and (33), we get [5,7]

$$\tan 2\theta = - \frac{2\sqrt{2}(m_s - \bar{m})/F_0 F_8}{\sqrt{2} \left[ \frac{2m_{\eta_1} m_s + \sqrt{6} \frac{\alpha_s}{4\pi} G\tilde{G} |\eta_1|}{F_0^2 m_s^2} \right] \frac{1}{1 + \frac{2m_{\eta_1}}{m_s}} \frac{1}{F_0}}$$  \hspace{1cm} (34)

We note that with $F_0 = F_8 = F_8^s$ and with no anomaly, we get $\tan 2\theta = 2\sqrt{2}$ i.e. we get nonet mixing angle as is the case for $w - \phi$ mixing. We can put Eq.(28) in the form

$$\tan 2\theta = \frac{2\sqrt{2}(m_\eta/\bar{m} - 1)}{(1 + 2m_{\eta_1}/m_s)^2 - 3m_{\eta_1}^2/m_s^2}$$  \hspace{1cm} (35)

Putting the numerical values for $m_s/\bar{m}$ and $m_{\eta_1}^2/m_s^2$, we get $\theta \approx -18^\circ$. This is the value of $\theta$ we have used in our calculation.

The mixing angle $\theta \approx -18^\circ$ is consistent with the value of $\theta$ obtained from the decays $\eta, \eta' \rightarrow 2\gamma$. We conclude that taking into account $U_A(1)$ anomaly in QCD and $\eta, \eta'$ mixing, reasonably good values for the decay widths $\eta \rightarrow 3\pi^0$ and $\eta' \rightarrow 3\pi^0$ can be obtained. I would like to thank Professor Riazuuddin for useful discussions.
References

1. Riazuddin and R.E. Marshak “Solution of the $U(1)$ problem in QCD without the $\theta$-vacuum” Virginia Tech Particle Theory preprint 91/1


