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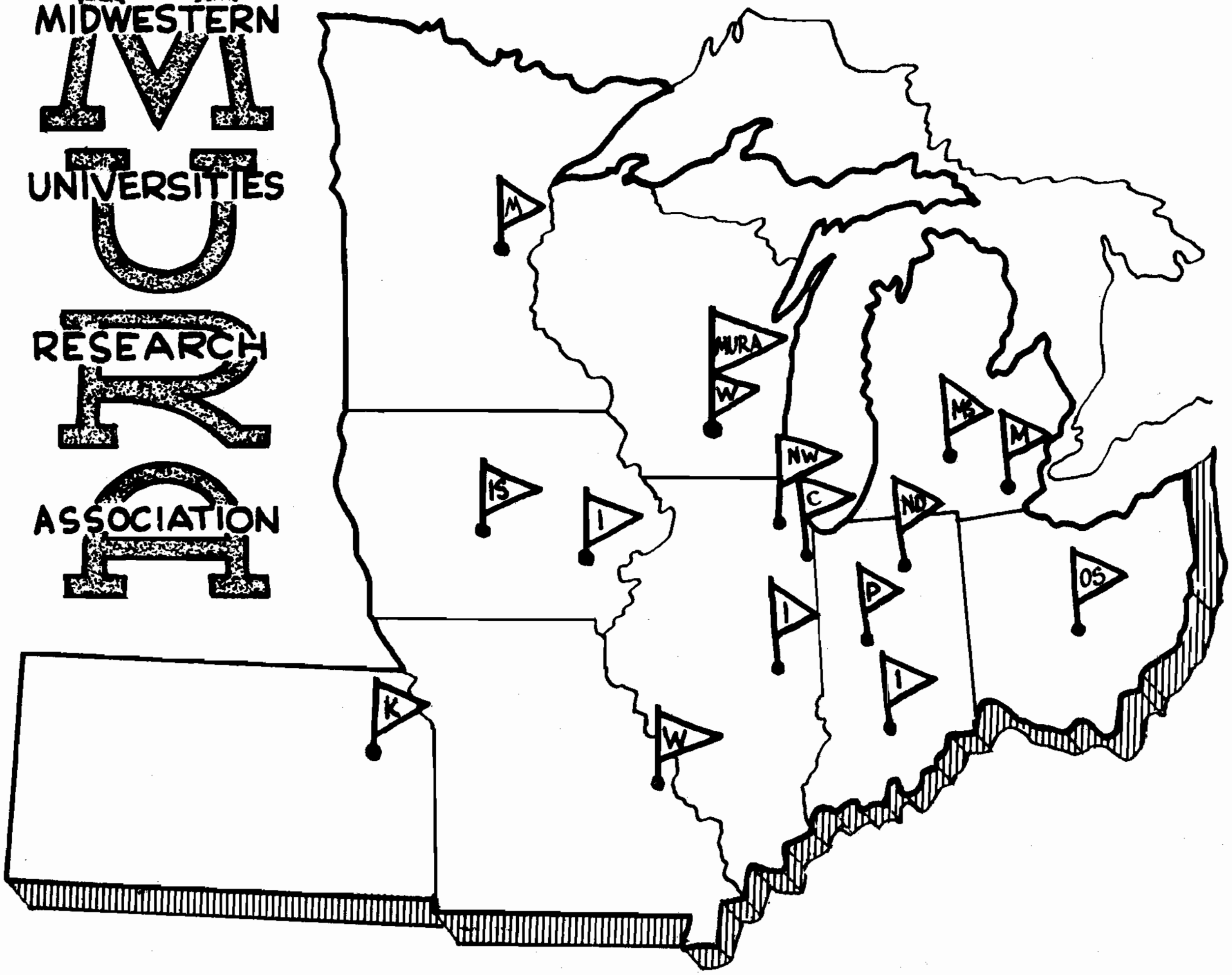

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ACHIEVING HIGHER BEAM DENSITIES BY SUPERPOSING EQUILIBRIUM ORBITS

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ABSTRACT

Fixed field accelerators can produce a beam density high enough for colliding beam experiments only if a large number of injected pulses are spatially superimposed; this number is limited in present designs by the energy difference--hence equilibrium orbit separation--of successively "stacked" pulses. Without this separation the beam density could be increased while the beam dimensions and total beam current are made as small as desired. A simple method which achieves equilibrium orbit superposition is the addition of a perturbing field of the form $H_z = H'_M x \cos M\theta$, where x is the radial distance from a selected equilibrium orbit, and M is an integer close to ν_x. Until non-linear effects set in, the forced oscillations of other close-by equilibrium orbits are proportional to their momentum difference, so they all can be made tangent or crossing at the target region. A perturbing field which produces tangent equilibrium orbits for a simple machine geometry has been designed using the MURA IBM 704 computer.

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INTRODUCTION

In order to do colliding beam experiments with a reasonable counts-to-background ratio, it is desirable to have a high beam current density. Obtaining equal numbers of beam-beam collisions and beam-residual gas collisions, for example, would require a current density of about 50 amps/cm^2 at 10^{-8} mm Hg. To get this density it is usually necessary to spatially superimpose a large number of injected pulses, requiring rf beam stacking.[1] In this stacking process, however, depositing a beam pulse at a given energy with the rf lowers the energy of the previously stacked coasting beam by the energy width of the accelerating rf bucket, in accordance with Liouville's Theorem. This decrease in energy decreases the beam radius an amount determined by the momentum compaction factor. When this decrease in radius is larger than the maximum radial betatron oscillation, the added pulses will no longer spatially superimpose on the first ones. There will be an increase in the total stacked beam, but no further increase in beam density.

It would be useful to eliminate the spatial separation of the different energy equilibrium orbits, at least in the colliding beam region. The betatron oscillation amplitude would cease to be a limit on the achievable beam density; pulses could be spatially superposed independent of this amplitude. Such equilibrium orbit superposition is reasonably easy to achieve, at least over a limited energy range.

SUPERPOSITION OF EQUILIBRIUM ORBITS

The linearized radial equation of motion of a particle of momentum $P + \Delta P$ about the equilibrium orbit of a particle of momentum P is[2]

$$\frac{d^2x}{ds^2} + \left(\frac{1 - n(s)}{\rho(s)^2}\right) x = \frac{1}{\rho(s)} \frac{\Delta P}{P} \qquad (1)$$

where s, n and ρ are measured along the P equilibrium orbit. The equilibrium orbit of the particle of momentum $P + \Delta P$ is of course the particular solution to the above equation. For conventional weak focusing or AG machines the average equilibrium orbit displacement is $\langle x \rangle \approx \frac{r}{\nu_x^2} \frac{\Delta P}{P}$ while for a scaling FFAG machine, with field $H \sim r^k$, $\langle x \rangle = \frac{r}{k+1} \frac{\Delta P}{P}$. Now, since the equation (1) is linear, if a particular solution for a given ΔP is zero or tangent to $x = 0$ at a given azimuth, all equilibrium orbits will automatically be zero or tangent at that azimuth. So it is simply necessary to find a convenient modification of the conventional fields which preserves linearity in order to produce superposition.

The different equilibrium orbits can obviously be made tangent at particular azimuths by adding a field modification such that the off-momentum equilibrium orbits undergo a forced oscillation about their average displacement, with an amplitude equal to this average displacement. One simple method of producing a forced oscillation equal to the average displacement is with the addition of a field gradient perturbation having a Fourier component $\Delta H = [H'_M \cos M\theta]x$, where M is an integer reasonably close to ν_x, and x again is the displacement from the selected equilibrium orbit. This equilibrium orbit, of course, will be unchanged. The gradient perturbation could be produced with pole face windings or quadrupole lenses.[3]

The effect of such a field perturbation is easily calculated to a first approximation, giving some insight into the process. Measuring all distances in terms of r_o = Circumference/2π, writing $\theta = \frac{s}{r_o}$ and taking

$P = \langle H \rangle_{orbit} = 1$, Eq. (1) is

$$\frac{d^2x}{d\theta^2} + \left(\frac{1-n}{\rho^2}\right) x = \frac{\Delta P}{\rho} . \tag{2}$$

With the additional perturbing field $\Delta H = \left[H'_M \cos M\theta\right] x$, this becomes

$$\frac{d^2x}{d\theta^2} + \left[\frac{1-n}{\rho^2} + H'_M \cos M\theta\right] x = \frac{\Delta P}{\rho} \tag{3}$$

n and ρ having their original values. Taking x_2 as the particular solution to (2), and x_3 as the particular solution to (3) (the desired equilibrium orbit), expand about x_2 in (3) by defining a difference equilibrium orbit $x_4 = x_3 - x_2$. Then (3) gives the equation for x_4

$$\frac{d^2x_4}{d\theta^2} + \left[\frac{1-n}{\rho^2} + H'_M \cos M\theta\right] x_4 = -\left[H'_M \cos M\theta\right] x_2 . \tag{4}$$

To obtain an approximate solution to (4) replace the left-hand side with a harmonic oscillator of the same frequency ν and x_2 by its average value $\langle x_2 \rangle$. Then

$$\frac{d^2x_4}{d\theta^2} + \nu^2 x_4 = -\left[H'_M \cos M\theta\right] \langle x_2 \rangle \tag{5}$$

which gives as the approximate particular solution to (4)

$$x_4 \cong \frac{\left[H'_M \cos M\theta\right] \langle x_2 \rangle}{M^2 - \nu^2} . \tag{6}$$

Since the correct forced equilibrium orbit is given by $x_3 = x_2 + x_4$, replacing x_2 and x_4 by their approximate values gives

$$x_3 \cong \langle x_2 \rangle + \frac{\left[H'_M \cos M\theta\right] \langle x_2 \rangle}{M^2 - \nu^2} = \langle x_2 \rangle \left[1 + \frac{H'_M \cos M\theta}{M^2 - \nu^2}\right] . \tag{7}$$

So to this first approximation, the off-momentum equilibrium orbits can all be made tangent to zero at periodic angles $\theta_\ell = (2\ell + 1)\pi / M$ by taking

$$H'_M = M^2 - \nu^2 \quad . \tag{8}$$

The addition of the perturbing field $\left[H'_M \cos M\theta\right]x$ will open up stop bands at $\nu = M/2, 2M/2, 3M/2 \ldots.$ and will cause a perturbation of the orginial ν. The tune changes and stop bands away from the $M/2$ region, however, are small, so bringing M to the closest or next to closest integer to ν should not prove troublesome.[4]

The equilibrium orbits must superpose, with a correctly designed gradient perturbation, only as long as (3) remains the radial equation of motion, which is, of course, as long as the equation stays linear in the quantities x and ΔP. In a perfect linear field AG machine the radial equation stays essentially linear in x, and the x tune is independent of amplitude. The coefficient of x, however, depends on the momentum ΔP. An increase in ΔP will change ν, generally decreasing it; in any event it will change $M^2 - \nu^2$, hence the amplitude of the forced oscillations ((6)), and the orbits will no longer exactly superimpose, as seen from (7). In a non-linear but scaling FFAG machine, the tune is independent of ΔP, but depends on the x amplitude, so there is a change in $M^2 - \nu^2$ for the larger driven oscillations, and the orbits for large ΔP will not exactly superpose.

EXAMPLE OF SUPERPOSED ORBITS IN AN AG STORAGE RING

As an illustration, a field gradient perturbation giving superposed equilibrium orbits was designed for an AG machine, using the MURA IBM 704 computer, with the approximate result of the previous section, (8), as a guide. The unperturbed magnetic field, expanded about a circle, was

$$H \simeq 1 + \left[302 \cos 32\,\theta\right] x$$

which gives the linearized on-momentum radial equation and tune

$$\frac{d^2x}{d\theta^2} + \left[1 + 302 \cos 32\,\theta\right] x = 0 \qquad \nu_x = 7.28 \,. \tag{9}$$

Nearly exact superpostion for small ΔP was achieved with the added field gradient perturbation $\Delta H = \left[H' \cos M\,\theta\right] x = 9 \cos 8\,\theta\, x$, giving the new on-momentum equation and tune

$$\frac{d^2x}{d\theta^2} + \left[1 + 302 \cos 32\,\theta + 9 \cos 8\,\theta\right] x = 0 \qquad \nu_x = 7.30 \,. \tag{10}$$

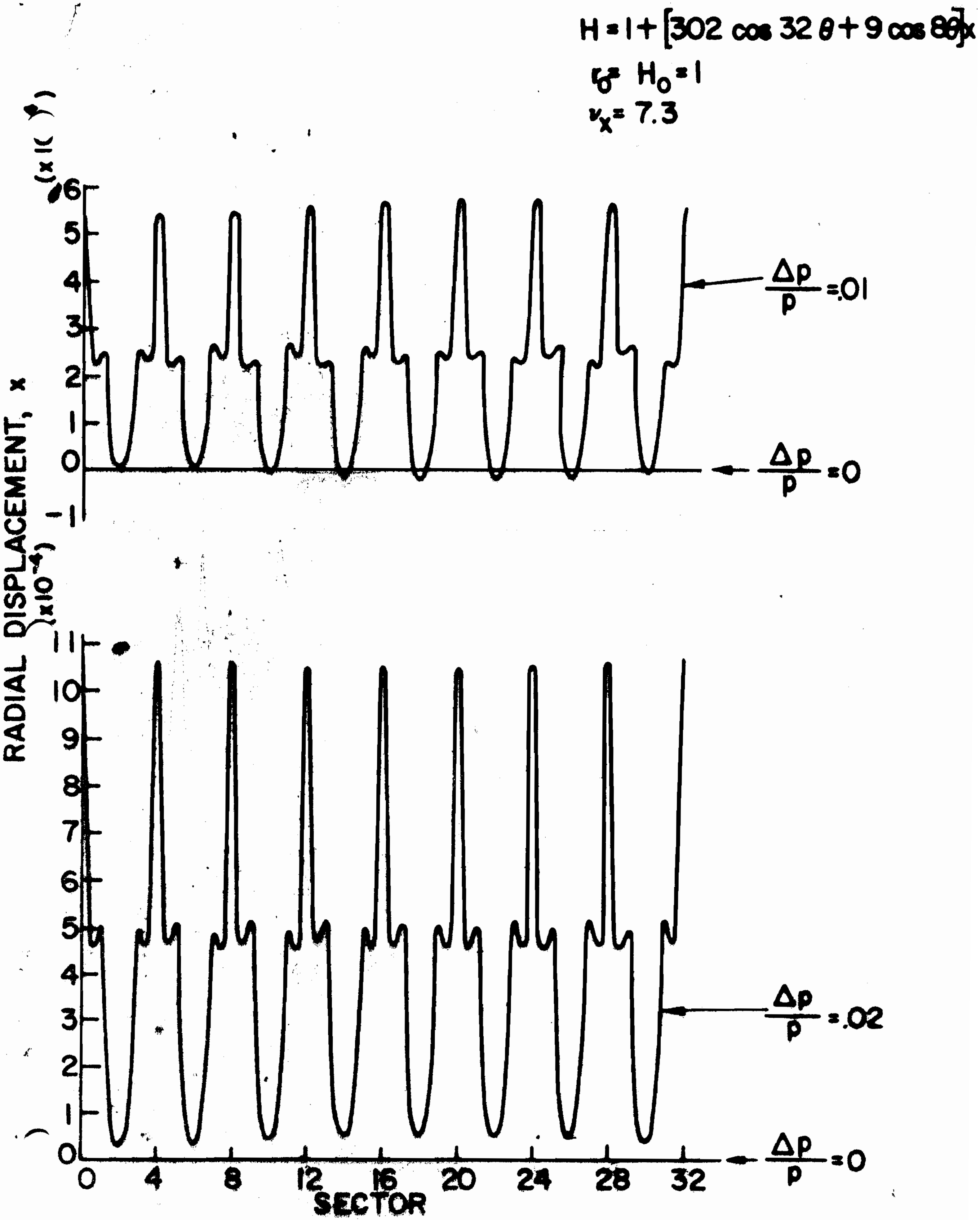
As expected, the change in tune was not large. To get superposition using the approximation (8) of the previous section, one would calculate the perturbation

$$H'_M = M^2 - \nu^2 = 8^2 - 7.3^2 = 10.7$$

reasonably close to the computer designed value $H'_M = 9$.

Off-momentum particle trajectories obtained for the given perturbed field with the computer are presented in Fig. 1. The particles have small betatron oscillations about their periodic equilibrium orbits. These equilibrium orbits are clearly driven oscillations about an average displacement, the low and high frequency periodicities being due, of course, to the $\cos 8\,\theta$ and $\cos 32\,\theta$ terms respectively. The maximum radial excursions could be made considerably less by phasing these two terms such that the peak of the perturbing gradient, hence the maximum amplitude of the driven equilibrium orbit, occurred at the middle of a negative gradient sector instead of a positive one, as in the present case.

As seen in the figure, the equilibrium orbits will superpose almost exactly up to $\Delta P = .01$. After that the decrease in the x tune increases



RADIAL MOTION OF OFF-MOMENTUM PARTICLES

$M^2 - \nu^2$, decreasing the effect of the given driving perturbation, and the orbits are no longer tangent. The density of equilibrium orbits, however, still remains higher than in the unperturbed machine, even for quite large ΔP.

For an estimate of the current density attainable, take the above AG machine to be a 10 Bev storage ring of 50-meter radius going with a 10 Bev accelerator of the same tune and radius. With an injection energy of 50 Mev, the final current density for one pulse, after betatron damping, is about .6 amps/cm^2; the density is assumed to be limited by space charge repulsion at injection, including a factor of 1/4 for rf bunching. Now from Fig. 1, it is apparent that equilibrium orbits of particles which differ in momentum by $\Delta P = \pm .01$ superpose almost exactly. These particles will have a total energy spread of 200 Mev. If the original energy resolution at 50 Mev injection was 50 Kev, the energy spread of one pulse at full energy would be $\Delta E_{inj} \frac{f}{f_{inj}}$ = 150 Kev ($\Delta E/f$ = constant, with perfect rf handling). The number of pulses (within $\Delta P = \pm .01$) which can be superposed is then 200/.150 = 1330. With the current density per pulse of .6 a/cm^2, this gives a total of 800 amps/cm^2. Two colliding beams of this density, taking an interaction cross section of 25 mb, would have a total p-p interaction rate of 4×10^7 interactions/cm^3/sec, and, at 10^{-8} mm Hg, a gas collision background from both beams of 2×10^6/cm^3/sec. There will, of course, be some decrease in this density due to rf mishandling. With a factor of 4 decrease in beam density in rf phase space, there could be 1330/4 = 330 superpositions and 200 amps/cm^2, with the same energy spread. The p-p interaction rate density would then be 2.5 x 10^6/cm^3/sec with a background from residual gas of 5 x 10^5/cm^3/sec. Although the net current density is quite high, the total current can be restricted to as small a value as desired by using small betatron oscillation amplitudes. With 200 amps/cm^2, if the betatron amplitudes were restricted to $\pm$ .1 cm, the total current would be about 8 amps. To get this 200 amps/cm^2 without equilibrium orbit superposition would require, assuming the same rf losses, that the 330 pulses be spatially superimposed by the betatron oscillations. Now assuming a momentum compaction factor of 50, the 200 Mev energy spread will correspond to a radial separation of the equilibrium orbits of 4 x 10^{-4}, and a necessary final damped betatron oscillation (for any superposition of the extreme energies) of 2 x 10^{-4}, or 1 cm at 50 meter radius. The current density per pulse of .6 amps/cm^2, the 1 cm betatron amplitude, and the 330 pulses would then require the large total current of 630 amps. To get an appreciable fraction of the current at the 200 amps/cm^2 density would require even larger total currents.

Although the example has been worked out for an AG storage ring, according to the analysis of the previous section the perturbation should have the same effect in an FFAG machine. The only difference should be that the tune will change with amplitude and not energy. The linear region, where tune does not change, is of order 3 x 10^{-4}. With a k of 200 this corresponds to a ΔP of $\pm$ .06 compared to the AG case of $\Delta P \approx \pm$ .01, thus 6 times as many pulses could be superposed as in the previous AG design (with six times as large an energy spread). However, the two-way FFAG design usually has a low ν_z ($\sim$4 compared to 7 for the AG case) and a large radius (125 meters compared to 50) so the net current density per pulse, limited by space charge at injection is down by the factor $\nu_z / r^2 \sim 1/10$, giving a net current density about 1/2 of the storage ring case.

REFERENCES

1. K. R. Symon and A. M. Sessler, CERN Symposium on High Energy Accelerators, Geneva 1956, Vol. 1, pp. 44.

   K. M. Terwilliger, L. W. Jones and C. H. Pruett, Rev. Sci. Instr. 28, 987 (1957).

2. E. D. Courant and H. S. Snyder, Annals of Physics 3, 1 (1958).

3. Another technique for achieving superposition, suggested by Courant, is to drive the off-momentum equilibrium orbits by modulating $\rho(s)$ of (1) with a periodicity close to ν_x, like the proposal of Vladimirskii and Tarasov for eliminating the transition energy.

4. A convenient chart giving stability boundaries of the Mathieu Equation, from which tune changes can be estimated, is given in McLachlan, "Theory and Application of Mathieu Functions."