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THE STRONG CP PROBLEM AND INVARIANCE UNDER LARGE GAUGE TRANSFORMATIONS

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Abstract

Recent results from string theory suggest that a truncated theory of only the light particles is not invariant under large gauge transformations. We assume that the vacuum state of such a truncated theory is of relevance to the strong CP problem and study the consequences of the relaxation of the condition of exact invariance under large gauge transformations. We find that the resulting modifications of the standard QCD vacuum state could lead to CP-invariant physics.

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1. Introduction

The strong CP problem has remained unsolved for over 16 years [1,2]. Several possible solutions have been proposed (for example, an axion [3-5], a massless quark [6,7], weak CP breaking [8,9], wormholes [10,11], and long-range interactions [12,13]) but none appears naturally attractive or phenomenologically successful in the simplest models for the theory. In fact, it may be said that very little real progress has been made except in our realization that the strong CP problem is very hard to solve and is not a problem of only passing interest; indeed, the strong CP problem is being compared to the cosmological constant problem in that the origin of the solution probably lies in physics beyond our present knowledge. For this reason, the strong CP problem may be viewed as a window through which to look for new physics and an opportunity by which to modify and extend our present theory of the strong interaction which may appear complete and consistent in all other regards (apart from computability).

In this paper, we shall consider the idea that the theory of the strong interaction (relevant to the strong CP problem) should not be exactly invariant under large (singular) gauge transformations. There is some justification for considering such an idea. In some two-dimensional string theories [14-16] (which are claimed to model four-dimensional string theories), a truncated theory of only the light states is found to be invariant under small (continuous) gauge transformations but not invariant under large gauge transformations (briefly, this is because the W-symmetries that mix the finite number of light states with the infinity of discrete massive states correspond to singular gauge transformations). However, the nice properties for an effective low-energy theory such as perturbative renomalizability, unitarity and energy conservation remain intact, and it may be that the loss of exact invariance under large gauge transformations only significantly affects the strong CP problem (this will be discussed further at the end of section 3).

The results of this paper depend only upon considering a theory of the strong interaction which is not exactly invariant under large gauge transformations. None of the results depend upon the strong interaction being a low-energy truncated effective field theory derived from string theory. However field theories, as usually constructed, are fully gauge-invariant with no distinction (as regards the form of the terms in the action) being made between large and small gauge transformations. Some readers may therefore

wonder why one should wish to consider a theory invariant under continuous gauge transformations but not under singular ones. The mixing with the higher mass string states can provide such a motivation (should such a motivation be thought necessary). No definite conclusions can be reached from considering string theory (since too much is still not known) but, in order to make the paper slightly more specific, we shall at times refer to the string theory result although, with perhaps minor modifications, any other physics that led to the low-energy theory not being invariant under large gauge transformations would serve equally as well.

For low-energy phenomena, we expect the mixing between the light and massive states to be suppressed by powers of the Planck mass. Hence we expect the breaking of large gauge invariance in the truncated light state theory to be very weak. This may or may not pose a severe problem for the solution to the strong CP problem proposed here. Further work will be necessary to answer this. However, it should be mentioned that the identical physics has been claimed [14] to lead to a modification of quantum mechanics for light particle systems and to the rapid "collapse" of the wave-function for macroscopic objects (even although full string theory is fully quantum mechanical). Thus, even although effects are suppressed by powers of the Planck mass, it need not mean that they are physically unimportant. A modification of the rules of quantum mechanics could conceivably have relevance to the strong CP problem (perhaps through a modification of the quantization process or by permitting pure states to evolve into mixed states) but we shall not pursue that idea here. The loss of full invariance under large gauge transformations has rather more definite consequences.

In section 2, we shall briefly outline the elements in the construction of the QCD vacuum that lead to the strong CP problem. The role played by invariance under large gauge transformations will be explained, together with several other assumptions that underlie this work. In section 3, we shall mention the consequences of the requirement of invariance under large gauge transformations being relaxed and various ways in which these could potentially lead to solutions of the strong CP problem. Various problems associated with these solutions are discussed. Our conclusions are briefly mentioned in section 4.

2. The standard QCD vacuum state

Since several good reviews [17,18] of the construction of the standard QCD vacuum state are available, we shall merely give a brief outline and emphasize the points most relevant to our discussion. Let us work in the temporal gauge $A_0^a = 0$ and consider states which are the gauge transforms of the classical vacuum

$$A_\mu \equiv A_\mu^a \frac{\lambda^a}{2} = \frac{i}{g} h \partial_\mu h^{-1}, \tag{2.1}$$

where

$$h(\vec{x}) = e^{i\omega^a(\vec{x})\lambda^a}, \quad \partial_0 h = 0, \tag{2.2}$$

is an element of the gauge group. Such a field defines a mapping from 3-space ($\vec{x}$) into the gauge group, and it is only necessary to consider fields for which $h(\vec{x})$ becomes constant at infinite distance; that is, points at infinity can be identified as far as the mapping from space to the gauge group is concerned. But three-space with points at infinity identified is topologically equivalent to the three-sphere S^3, and so we have a mapping from S^3 to the gauge group, or rather to an SU(2) subgroup of it. Thus the functions $h(\vec{x})$ (and so the gauge fields representing the vacuum) fall into classes labelled by an integer n, the number of times that one goes through the elements of the gauge group in going over three-space. This integer is called the winding number and it can be written as

$$n = -\frac{g^3}{96\pi^2} \int d^3\vec{x}\, \epsilon_{0\nu\lambda\rho} A_\nu^a A_\lambda^b A_\rho^c f^{abc}. \tag{2.3}$$

The winding number is invariant under all gauge transformations which can be obtained continuously from the identity. Such transformations are called 'small' gauge transformations. If R^3 is imagined to be projected onto S^3, and the group element $h(\vec{x})$ is written as $h(\vec{x}) = e^{i\omega^a(\vec{x})\lambda^a}$, then the functions $\omega^a(\vec{x})$ may always be chosen to be continuous and single-valued everywhere on S^3 if h is a small gauge transformation. Transformations for which the $\omega^a(\vec{x})$ are unavoidably singular at at least one point on S^3 are known as 'large' gauge transformations. Such transformations can change the winding number n. Gauge transformations which can do this are always singular somewhere on S^3.

We wish the vacuum states to be invariant under small gauge transformations. Such states can be formed from members of each homotopy class by adding together all the members of a class:

$$|n\rangle = \int \mathcal{D}\omega^a(\vec{x}) \left| e^{i\omega^b(\vec{x})\lambda^b} A_i^a(n) \right\rangle, \tag{2.4}$$

where $|A_i^a(n)\rangle$ is some field with winding number n and the integral is over all $\omega^a(\vec{x})$ which are regular on S^3. This state is clearly invariant under small gauge transformations. The states $|n\rangle$ are known as the n-vacua.

However, the $|n\rangle$ are not acceptable as physical vacuum states because, since the discovery of fields such as instantons, we know that starting from one value of n others can be reached by quantum tunnelling in real time. The n-vacua are not eigenstates of the Hamiltonian. However, since the n-vacua are all degenerate, the Hamiltonian has the same form as that of a one-dimensional row of points (atoms) with a particle (an electron, for example) hopping backwards and forwards along the line. The eigenstates of such a system with a displacement symmetry are

$$|\theta\rangle = \sum_n e^{in\theta} |n\rangle . \tag{2.5}$$

The states $|\theta\rangle$ are called the θ-vacua. The quantity θ is allowed to take any value because each $|\theta\rangle$ is an acceptable eigenstate and an acceptable candidate for the ground state of the theory. Also since the $|\theta\rangle$ are eigenstates of the Hamiltonian, no transitions between states of different θ can occur. The vacuum does not spontaneously relax to the state of lowest energy (which is easily seen, in the instanton dilute-gas approximation, to correspond to $\theta = 0$).

Let us now introduce the relation between the θ-vacua and CP violation. Consider the functional integral over the gauge field, starting and ending in a θ-vacuum. This can be written as

$$\langle\theta| e^{-HT} |\theta\rangle = \sum_{n_1,n_2} \int_{n_1}^{n_2} \mathcal{D}A_\mu\, e^{-S(A_\mu)} e^{-i\theta(n_2-n_1)}, \tag{2.6}$$

where the symbol $\int_{n_1}^{n_2} \mathcal{D}A_\mu$ means a functional integral over all fields whose initial winding number is n_1 and whose final winding number is n_2. The phase factor $e^{-i\theta(n_2-n_1)}$ comes from the vacuum wave-function (2.5). The change in winding number $n_2 - n_1$ may be defined in terms of the total divergence $F\tilde{F}$,

$$F_a^{\mu\nu}\tilde{F}_{a\mu\nu} \equiv \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F_a^{\mu\nu}F_a^{\alpha\beta} = \partial^\mu K_\mu, \tag{2.7}$$

where

$$K_\mu = \epsilon_{\mu\nu\lambda\rho}\left(A_\nu^a F_{\lambda\rho}^a - \frac{g}{3}f^{abc}A_\nu^a A_\lambda^b A_\rho^c\right). \tag{2.8}$$

Consider the following integral

$$\begin{aligned}\nu &= \frac{g^2}{32\pi^2}\int_V d^4x\, F_a^{\mu\nu}\tilde{F}_{a\mu\nu} \\ &= \frac{g^2}{32\pi^2}\oint d^3S\, n_\mu K_\mu,\end{aligned} \tag{2.9}$$

where n_μ is the unit outward normal to the surface S bounding the four-volume V. Let us choose the surface S to have two flat pieces at constant time, joined by a cylinder in the time direction normal to each surface of constant time. On the walls of the cylinder we may impose $F_{\mu\nu} = 0$ as a boundary condition (all the places where the fields are non-zero are well within the cylinder). Also, in the temporal gauge, the second term in the surface integral will vanish because n_μ is normal to the time direction on the walls of the cylinder. So the contribution to the surface integral from the walls of the cylinder is zero, and, assuming that the initial and final states are vacuum states ($F_{\mu\nu} = 0$), we can write

$$\begin{aligned}\nu &= \frac{g^2}{32\pi^2}\left(\int_{t_2} d^3x\, K_0 - \int_{t_1} d^3x\, K_0\right) \\ &= n_2 - n_1,\end{aligned} \tag{2.10}$$

by the definitions (2.3) and (2.8)

Thus the functional integral (2.6) may be written as

$$\langle\theta|\, e^{-HT}\,|\theta\rangle = \sum_{n_1,n_2}\int_{n_1}^{n_2}\mathcal{D}A_\mu\, e^{-S(A_\mu)-i\theta(g^2/32\pi^2)\int d^4x, F\tilde{F}}. \tag{2.11}$$

Therefore, as far as the functional integral is concerned, a phase θ in the vacuum wave-function has exactly the same effect as a term $i\theta(g^2/32\pi^2)$ $\int d^4x\, F\tilde{F}$ in the action. Under the action of parity (P) or time-reversal (T), the quantity $F\tilde{F}$ changes sign. So the breaking of the symmetry between

fields of opposite topological charge ν would imply the breaking of both P and T. Since the theory would still be invariant under CPT, the breaking of T-invariance implies the presence of CP-violation. Also, under T, $|\theta\rangle \to |-\theta\rangle$ and if θ is not zero or π the vacuum state is not an eigenstate of T.

So, we have seen that the θ-vacuum leads to the introduction of an effective $i\theta F\tilde{F}$ term in the Lagrangian and, since fields of non-trivial topological charge ν such as instantons and anti-instantons do exist, such a term potentially leads to strong CP violation. However, there are other possible sources for an $F\tilde{F}$ term. For example, if the quark masses from the electro-weak sector are complex, a chiral axial U(1) transformation is required to make the mass terms real. Through the chiral anomaly this results in the appearance of a term in the Lagrangian proportional to $iArgdetMF\tilde{F}$, where M is the original quark mass matrix. The effective coefficient of $F\tilde{F}$ therefore becomes

$$\bar{\theta} \equiv \theta + ArgdetM. \tag{2.12}$$

Experimentally $\bar{\theta} \leq 10^{-9}$. Since both θ and $ArgdetM$ are arbitrary, and most likely of order unity, this experimental absence of observed CP violation is termed the strong CP problem. This paper is primarily concerned with the θ-vacuum structure and, when we wish to refer to other sources of CP violation, we shall refer to them as sources of explicit $F\tilde{F}$ terms in the QCD Lagrangian. We must consider these other sources at times since the solution to the strong CP problem must involve the neutralization of all potential sources of CP violation and not just a modification of the consequences of the θ-vacuum structure.

At this stage, before proceeding further, it may be useful to mention two points which, although not difficult, may not be well-known. Firstly, we have grouped states which were gauge transforms of the classical vacuum into homotopy classes labelled by the winding number. From these states of definite winding number, the θ-vacuum were constructed. What about fields that are not gauge transforms of the classical vacuum? Can winding number, or analogies of the $|n\rangle$- and $|\theta\rangle$-states be defined for such fields? The answer is yes. The concepts can be extended to a more generalized non-integer winding number which is invariant under small gauge transformations and changes by an integer under large gauge transformations. Furthermore, the requirement that physical states be invariant under small gauge transformations is true in general, and not just for pure gauge states. This means that analogues of the θ-vacuum can be constructed for non-vacuum fields. However, for the sake of simplicity, we shall not go beyond the semiclassical approach to QCD which is sufficient for our needs.

Secondly, the construction of the θ-vacua depended on the distinction between large and small gauge transformations. This distinction depends upon the topology of three-space. For example, if space were discrete, or had 'holes' in it, the concept of winding number will be meaningless and there will be no distinction between large and small gauge transformations. All states in this space will be combined with the same phase, and there will be no scope for putting an angle θ into the vacuum wave-function. The solutions to the strong CP problem discussed in this paper require the inclusion of the vacuum angle θ and hence require space to be continuous and to have no boundaries. This assumption may be non-trivial (and, presumably, the two-dimensional string theory result also depends crucially on the continuity of space).

Some readers may be slightly surprised that we wish the θ-vacuum structure to exist. It may be thought that removing the θ-vacuum structure would increase the chances of a solution to the strong CP problem since the θ-vacuum leads to a source of CP violation. For some proposed solutions (such as the soft CP-breaking mechanism [8,9] which requires θ to be effectively zero) this would be true. However, we wish to construct a modified θ-vacuum which conserves CP, and also cancels off potential CP violation from an explicit $F\tilde{F}$ term in the Lagrangian. We wish to use the existence of the θ parameter to enable a solution to the strong CP problem, and not as a contributing factor to the problem.

3. The consequences of non-invariance under large gauge transformations

Let us begin this section with an explanation in clear physical terms of the essential role played by invariance under large gauge transformations in the physics of the standard QCD vacuum. In Fig.1, we give a schematic representation of the standard QCD vacuum state, The winding number n runs from $-\infty$ to $+\infty$. The n-vacua are all degenerate, and under large gauge transformations the winding number changes by an integer. A representative

of a gauge transformation that shifts the winding number by +1 is given by

$$G_1(\vec{x}) = \frac{\vec{x}^2 - \rho^2}{\vec{x}^2 + \rho^2} + \frac{2i\rho\vec{\sigma}.\vec{x}}{\vec{x}^2 + \rho^2}, \tag{3.1}$$

where ρ is arbitrary (reflecting the classical scale invariance of QCD). A representative of transformations that shift the winding number by m can easily be constructed

$$G_m(\vec{x}) = \left(G_1(\vec{x})\right)^m. \tag{3.2}$$

One will notice from the figure that the Hamiltonian H is invariant under $n \to -n$ and so H is invariant under P and T. Why then, one may ask, is the eigenstate $|\theta\rangle$ not invariant under P and T. One answer to this question is because n runs from $-\infty$ to $+\infty$. Consider the analogy of a particle moving in a one-dimensional array, similar to the n-vacua. The particle can travel in one direction forever with fixed momentum because there is nothing to reflect it back. It is the same with the θ-vacuum. A CP-invariant state would equally contain $|\theta\rangle$ and $|-\theta\rangle$. However such a state is not a physically acceptable eigenstate of the QCD Hamiltonian. The problem is that because of invariance under large gauge transformations no operator has a non-zero matrix element between states of different θ. Hence, even if the Universe were initially in a state $|\theta\rangle + |-\theta\rangle$ (with θ non-zero), after one measurement of for example the electric dipole moment of the neutron, the value of θ would be fixed at either θ or $-\theta$ for all subsequent measurements. So, despite the fact that the Hamiltonian is invariant under CP, invariance under large gauge transformations forces the physical states to be CP non-invariant because transitions between different θ-vacua are forbidden.

Let us now give the simple proof of why invariance under large gauge transformations forbids transitions between different θ-vacua. Consider a large gauge transformation G_1 that shifts the winding number by +1 i.e. $G_1|n\rangle = |n+1\rangle$. From the definition (2.5), we have

$$G_1|\theta\rangle = e^{-i\theta}|\theta\rangle. \tag{3.3}$$

The Hamiltonian being gauge-invariant implies

$$G_1 H = H G_1, \quad \text{and also} \quad G_1 e^{-HT} = e^{-HT} G_1. \tag{3.4}$$

Thus

$$\begin{aligned}\langle\phi| e^{-HT}|\theta\rangle &= \langle\phi| G_1^{-1}e^{-HT}G_1|\theta\rangle \\ &= \langle\phi| G_1^{\dagger}e^{-HT}G_1|\theta\rangle \\ &= e^{-i(\theta-\phi)}\langle\phi| e^{-HT}|\theta\rangle\end{aligned} \tag{3.5}$$

and hence the matrix element will vanish if θ is not equal to ϕ.

If we relax the condition of exact invariance under large gauge transformations, two related consequences are possible. Firstly, transitions between different θ-vacua may be possible and so a θ-vacuum may relax to one of lowest energy $|\theta = 0\rangle$ which is CP-conserving (or $\bar{\theta} = 0$ if there is an explicit $F\tilde{F}$ term in the Lagrangian). Secondly, the eigenstates of the new Hamiltonian H' may be eigenstates of CP, and the expectation value of a CP-odd operator vanishes in an eigenstate of CP. To show this, let $|\psi\rangle$ be an eigenstate of CP with eigenvalue $\eta = \pm 1$, and let $\hat{O}$ be a CP-odd operator. That is, $\hat{O}(CP) = -(CP)\hat{O}$. Then it follows that

$$\begin{aligned}\langle\psi|\hat{O}|\psi\rangle &= \langle\psi|\hat{O}(CP)|\psi\rangle\eta \\ &= -\langle\psi|(CP)\hat{O}|\psi\rangle\eta \\ &= -\langle\psi|\hat{O}|\psi\rangle\eta^2 = -\langle\psi|\hat{O}|\psi\rangle,\end{aligned} \tag{3.6}$$

since $\eta^2 = 1$. So we understand the reason why operators that are CP-odd can have non-zero expectation values in the θ-vacua is because these states are not eigenstates of CP.

Let us consider first the relaxation mechanism. A possible problem with this concerns Lorentz invariance should θ be allowed to change in time but not in space. Also, it will be very difficult to think of how θ may relax to zero should it have to change everywhere simultaneously. So, the question of whether θ must be constant over space is of crucial importance. In standard QCD, θ is constant in space as a result of invariance under small gauge transformations and we must re-examine the argument to see whether the relaxation of invariance under large gauge transformations permits any loopholes. A proof of why θ must normally be constant in space is as follows [19].

Imagine a volume $2V$ of QCD vacuum in the state $|\theta\rangle$, and consider it in two pieces, each of volume V. Now a state $|n\rangle$ can be written as

$$|n\rangle = \int dm\, |m\rangle_1 |n-m\rangle_2 , \tag{3.7}$$

where the product of the two kets represents a state where one volume contains winding number m and the other contains $n-m$. (The winding number can be written as the integral of a local density, so this division of the vacuum into two pieces with their own winding numbers does make sense. Also, while n is an integer, m need not be.) This allows us to write the state $|\theta\rangle$ as

$$\begin{aligned}|\theta\rangle &= \sum_n e^{in\theta}|n\rangle \\ &= \sum_n e^{in\theta}\int dm\, |m\rangle_1 |n-m\rangle_2 \\ &= \sum_n \int dm\, |m\rangle_1 e^{im\theta}|n-m\rangle_2 e^{i(n-m)\theta}.\end{aligned} \tag{3.8}$$

Changing the sum over n to an integral (by allowing n to be continuous because the volume 2V is itself only part of a bigger volume), we can write

$$\begin{aligned}|\theta\rangle &= \int dn\, e^{in\theta}|n\rangle \\ &= \int dm\, e^{im\theta}|m\rangle_1 \int dm'\, e^{im'\theta}|m'\rangle_2 \quad (\text{where } m' = n-m) \\ &= |\theta\rangle_1 |\theta\rangle_2 .\end{aligned} \tag{3.9}$$

If the volume $2V$ is in the state $|\theta\rangle$, then each of the subspaces 1 and 2 is also in this state. This explains how a global object like θ can have a local effect; every subspace of the total volume is itself in the state θ.

What prevents different subspaces having different values of θ? This, as we have mentioned, turns out to be invariance under small gauge transformations. Let us suppose that one of the angles in one of the subspaces can be different from θ, say ϕ. Then let us consider a large gauge transformation $G_1^{(1)}$ which takes the state $|n\rangle_1$ to the state $|n+1\rangle_1$ but leaves the other half of the vacuum unaffected. Under $G_1^{(1)}$ the state $|\theta\rangle_1$ goes to $e^{-i\theta}|\theta\rangle_1$ and $|\phi\rangle_2$ is unchanged. Similarly, let us define $G_{-1}^{(2)}$ so that $G_{-1}^{(2)}|n\rangle_1|m\rangle_2 = |n\rangle_1|m-1\rangle_2$. Then $G_{-1}^{(2)}|\theta\rangle_1|\phi\rangle_2$ is $e^{i\phi}|\theta\rangle_1|\phi\rangle_2$. Now let us consider the gauge transformation $G_1^{(1)}G_{-1}^{(2)}$ acting on one of the n-vacua of the total system

$$\begin{aligned}G_1^{(1)}G_{-1}^{(2)}|n\rangle &= G_1^{(1)}G_{-1}^{(2)}\int dm\, |m\rangle_1 |n-m\rangle_2 \\ &= \int dm\, |m+1\rangle_1 |n-m-1\rangle_2 \\ &= \int dm'\, |m'\rangle_1 |n-m'\rangle_2 \quad = \quad |n\rangle .\end{aligned} \tag{3.10}$$

Since the state $|n\rangle$ is unchanged, $G_1^{(1)}G_{-1}^{(2)}$ is a small gauge transformation for the whole system (the transformation has merely shifted winding number across the boundary between the two subspaces, without changing the total). Now $|\theta\rangle$ is constructed out of the n-vacua and so the vacuum state of the total system must be invariant under all small gauge transformations. Hence $G_1^{(1)}G_{-1}^{(2)}|\theta\rangle = \theta$. But, if $|\theta\rangle = |\theta\rangle_1|\phi\rangle_2$, we have

$$G_1^{(1)}G_{-1}^{(2)}|\theta\rangle = G_1^{(1)}G_{-1}^{(2)}|\theta\rangle_1|\phi\rangle_2 = e^{-i(\theta-\phi)}|\theta\rangle_1|\phi\rangle_2\,, \tag{3.11}$$

and so the total vacuum state is invariant only if $\theta = \phi$. Therefore, the requirement of invariance under small gauge transformations demands that all subspaces should have the same value of θ as each other. The quantity θ cannot be a function of position. (We note in passing that this does not apply to the coefficient of an explicit $F\tilde{F}$ term in the Lagrangian; the coefficient of such a term can be position-dependent as happens, for example, in the case of an axion field.)

How is this proof that θ must be constant in space affected if we relax the condition of invariance under large gauge transformations? We are considering a truncated theory of only the light states (the QCD fields). This truncated theory is not invariant under large gauge transformations because large gauge transformations mix the light and heavy states. If this mixing is not included, the equivalent of the n-vacua (described in terms of only the light states) will not be degenerate although the energy differences will be suppressed by powers of the Planck mass and therefore very small; the translational symmetry will be lost, and a θ-state (if one considers such a state despite the fact that θ is no longer a good quantum number) will not change by only a phase under a large gauge transformation. Without a particular detailed physical picture for what is happening during the mixing caused by a large gauge transformation, it is perhaps difficult to be more specific; the discussion of what happens under a large gauge transformation necessarily requires the inclusion of the massive states and this we do not know how to do. We shall simply say that the proof is no longer applicable (the step (3.10) would appear to be most suspect) and the problems with regard to Lorentz invariance may perhaps be circumvented. The vacuum may be able to relax to a CP-conserving minimum energy state via bubble formation, the rate of relaxation being related to the degree of violation of exact invariance under large gauge transformations in the truncated theory.

Let us now turn to consider what is necessary to ensure that the eigenstates of the Hamiltonian H' are eigenstates of CP. This can be done by adding a term $\epsilon f(n)$ to the normal QCD Hamiltonian H, where $f(n)$ is an even function in n (to make sure that H' commutes with CP) and non-constant (to make sure that all eigenstates of H' are eigenstates of CP). The non-constancy of $f(n)$ implies that large gauge invariance has been broken. A simple example is given in Fig.2 where it is clear that the 'particle' will be reflected from either end and hence the eigenstates will equally contain θ and $-\theta$. The eigenstates of such an H' will be either even or odd under CP (remember that H was CP-invariant but the physically acceptable eigenstates were not eigenstates of CP only because H was a constant function of n). If the Universe is in an eigenstate of H', then CP violation from the (truncated theory) vacuum wave-function will not be observed. The vacuum wave-function will be of the form [19]

$$|0\rangle = \int_{-\pi}^{\pi} d\theta\, g(\theta)\,|\theta\rangle\,, \tag{3.12}$$

where $g(-\theta) = \pm g(\theta)$. The integral over θ is from $-\pi$ to π because the state $|\theta + 2\pi\rangle$ is the same state as $|\theta\rangle$. So the expectation value of a CP-odd operator $\hat{O}$ is

$$\begin{aligned}\langle 0|\,\hat{O}\,|0\rangle &= \int d\theta' d\theta\, g^*(\theta')g(\theta)\,\langle\theta'|\,\hat{O}\,|\theta\rangle \\ &= 0,\end{aligned} \tag{3.13}$$

since $g^*(\theta')g(\theta)$ is even when θ and θ' change sign, while $\langle\theta'|\,\hat{O}\,|\theta\rangle$ is odd.

Will CP violation still arise if there is an explicit $F\tilde{F}$ term in the Lagrangian? The answer is no. The addition of a term like $f(n)$ to the Hamiltonian automatically removes all CP-violation. Suppose that a CP-violating angle θ_E appears in the action. This has precisely the same effect on the physics as a phase θ_E in the vacuum wave-function. So the matrix element of an operator $\hat{O}$ between the states $|\theta\rangle$ and $|\theta'\rangle$, evaluated with an angle θ_E in the action, is

$$\langle\theta'|\,\hat{O}\,|\theta\rangle = O(\theta' + \theta_E, \theta + \theta_E), \tag{3.14}$$

where $O(\theta', \theta)$ is the matrix element of $\hat{O}$ evaluated when the action is CP-even ($\theta_E = 0$). In particular, (3.14) will apply to the Hamiltonian. So the eigenstates of a Hamiltonian with the $\theta_E F\tilde{F}$ term in the action will just be those of the old Hamiltonian, with θ replaced by $\theta + \theta_E$:

$$|0\rangle' = \int_{-\pi}^{\pi} d\theta\, g(\theta + \theta_E)\, |\theta\rangle \qquad \Big(g(\theta + 2\pi) = g(\theta)\Big). \tag{3.15}$$

Hence, if $\hat{O}$ is a CP-odd operator,

$$\begin{aligned}\langle 0|'\, \hat{O}\, |0\rangle' &= \int_{-\pi}^{\pi} d\theta' d\theta\, g^*(\theta' + \theta_E) g(\theta + \theta_E)\, \langle \theta'|\, \hat{O}\, |\theta\rangle \\ &= \int_{-\pi}^{\pi} d\theta' d\theta\, g^*(\theta' + \theta_E) g(\theta + \theta_E) O(\theta' + \theta_E, \theta + \theta_E) \\ &= \int_{-\pi}^{\pi} d\theta' d\theta\, g^*(\theta') g(\theta) O(\theta', \theta) \quad \text{(as the theory is periodic in } \theta) \\ &= 0 \quad \text{from equation (3.13).}\end{aligned} \tag{3.16}$$

The vacuum wave-function has changed by precisely the correct amount to cancel off the external CP-violation.

The above result is very encouraging but is such a function $f(n)$ physically reasonable? One problem is that terms such as $|n|$ and n^2 are non-local. (The only function of n that turns out to be local and invariant under small gauge transformations is n itself. The inclusion of such a term has been shown to lead to CP-conserving physics [19] but we shall not consider that solution here.) However, that problem may not really be relevant. The full string theory is (most probably) invariant under large gauge transformations and the energy of states related by large gauge transformations should be equal. The reason our truncated Hamiltonian H' is different is because we are describing it only in terms of the light fields and hence some contributions are not being included. These terms that are not included are functions of the massive string states and we do not know what they are. Hence our philosophy is that H' is not the result of a function $f(n)$ of the QCD fields being added but rather the result of some function of the massive string states not being included. All that is necessary is that H' not be constant in n and be invariant under $n \to -n$. We consider this not to be physically unlikely: the more the winding, the greater is perhaps the mixing between the light and massive states and hence the greater the energy contained within the terms that are functions of the massive states; also the energy is perhaps likely to depend on the magnitude of the winding and not on its sign, and hence H' could remain symmetric under $n \to -n$. Thus we consider it possible

that the H' resulting from the neglect of some, as yet unknown, functions of the massive string states will have the same CP invariance properties as a Hamiltonian to which an even function of n has been included.

To be honest, we must conclude that the results of this section are far from satisfactory. That the physics of a truncated theory may be of relevance to the strong CP problem appears reasonable. That a truncated theory is not invariant under large gauge transformations appears interesting. However, showing that the assumption of these facts alone leads to a natural solution to the strong CP problem appears to be difficult; there is just too much that is unknown. The elements of a solution seem to exist, and it must just be hoped that these elements survive a greater understanding of the full physics involved. Anyway, let us procede to discuss some other aspects of the solution to the strong CP problem proposed here.

A natural question some may ask is how do these solutions to the strong CP problem affect the U(1) problem; for example, does the η' remain massive and is there any need for a physical light pseudoscalar state to ensure CP invariance. The chiral Ward identities of standard QCD [20,21] demand that, if the quarks all have non-zero current masses and $\bar{\theta} \neq 0$, a light pseudoscalar state must couple to $F\tilde{F}$ if the theory is to be CP-conserving (the chiral Ward identities were a major problem for the proposed long-range interaction mechanism [12,13]; the η' would remain massive but the existence of a new light pseudoscalar state would be demanded if the mechanism were realized physically).

Let us first consider the relaxation possibility. In this case the physics is clear; the theory is CP-conserving and there is no need for a new light state. The phases of the current masses (after appropriate chiral rotations), the phases of the quark condensates, and $\bar{\theta}$ will all be effectively zero, This is a CP-conserving theory. The η' mass corresponds to the energy associated with fluctuations in the phases of the quark condensates with respect to $\bar{\theta}$ and the η' will remain massive. Briefly, the solution to the U(1) problem is unaffected, and the chiral Ward identities satisfied, because the ground state of the theory is equivalent to standard QCD with $\bar{\theta} = 0$ ($\theta = -Arg det M$).

The case where the vacuum state is an eigenstate of CP, and we average over values of θ with opposite sign, is only slightly more complicated. For each θ-state, the U(1) problem is solved (the η' is massive [22,23]) but there is CP violation. However, upon averaging over θ-states, the theory is CP-conserving. Hence, briefly, the η' is massive because it is massive in each θ-state before the averaging, and no light pseudoscalar state is required because there is no modification of the chiral Ward identities within each θ-state before the averaging; the CP-violating effects cancel due to the averaging of contributions with opposite sign and not because of any light state coupling to $F\tilde{F}$.

Finally, are there any other phenomenological problems related to the ideas discussed in this paper? Without a specific detailed model for how invariance under large gauge transformations is violated, it is difficult to determine whether transitions between different θ-states would be sufficiently rapid for CP-conserving physics to be observed experimentally, and we shall leave such matters for future investigation. However, since it is only the non-perturbative vacuum structure that is being modified, there may be no significant phenomenological consequences other than for the strong CP problem. The perturbative structure, renormalizability, unitarity and energy conservation will not be affected, and chiral symmetry breaking (and possibly confinement) may be affected quantitatively but probably not qualitatively. The only direct evidence for the role of fields of non-trivial topology involves the η' mass and the U(1) problem, and this has already been discussed; the existence of instanton-like solutions is independent of the existence of the θ-vacuum (for example, on a lattice, instanton solutions exist but the θ-vacuum does not exist since there is no distinction between large and small gauge transformations). Practically speaking (though perhaps not conceptually), invariance under large gauge transformations would appear to be almost irrelevant to all aspects of QCD other than the strong CP problem. There would appear to be no experimental tests of invariance under large gauge transformations and this in itself may be sufficient reason for the ideas expressed in this paper not to be immediately rejected out of hand without good cause.

4. Conclusions

Motivated by the two-dimensional string theory result that a truncated theory of only the light states is invariant under small gauge transformations but not under large gauge transformations, we have described in two different, but related, ways how the relaxation of the condition of exact invariance

under large gauge transformations could lead to a solution of the strong CP problem. The investigation has assumed throughout that the vacuum structure of the truncated theory is relevant to the strong CP problem. This assumption was made because the results appear somewhat successful, because the strong CP problem is a problem related to the QCD fields, and because the non-perturbative vacuum structure to be derived from full string theory is not known.

This work must be considered as preliminary and incomplete since nothing has been proven nor made quantitative, and indeed the assumptions underlying the paper may themselves be suspect. However, what has been demonstrated is that an arbitrarily small relaxation of the condition of invariance under large gauge transformations can potentially have a profound effect on the vacuum wave-function and lead to CP-invariant physics. If the strong CP problem is to be taken as a fundamental problem for the standard theory of the strong interaction, then perhaps it is an indication that some minor but fundamental change is needed in the construction of the QCD vacuum and that at least one of the basic assumptions underlying that construction is flawed; the theory of the strong interaction may yet have some surprises in store and the solution to the strong CP problem may be more subtle than hitherto imagined. The truncated low-energy theory may also have modified quantum mechanics and this may lead to mechanisms beyond those discussed for the mixing of θ-states, but the general consequences of the relaxation of invariance under large gauge transformations will be the most crucial ingredient should the solution to the strong CP problem lie along the lines described in this paper.

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Figure Captions

Fig.1 A schematic representation of the standard QCD vacuum. E(n) is the energy of a state of winding number n.

Fig.2 A simple example of a modification of the standard QCD vacuum that would lead to CP-conserving physics since the eigenstates of the Hamiltonian will be eigenstates of CP.

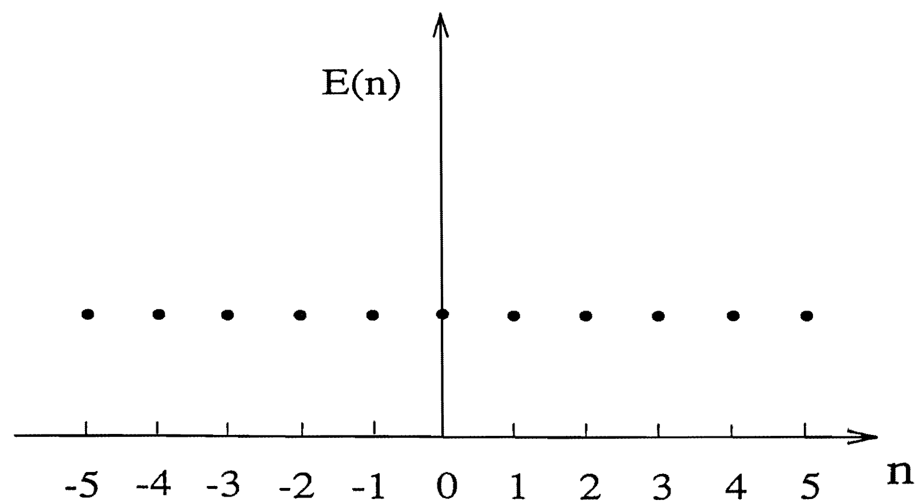

Fig. 1

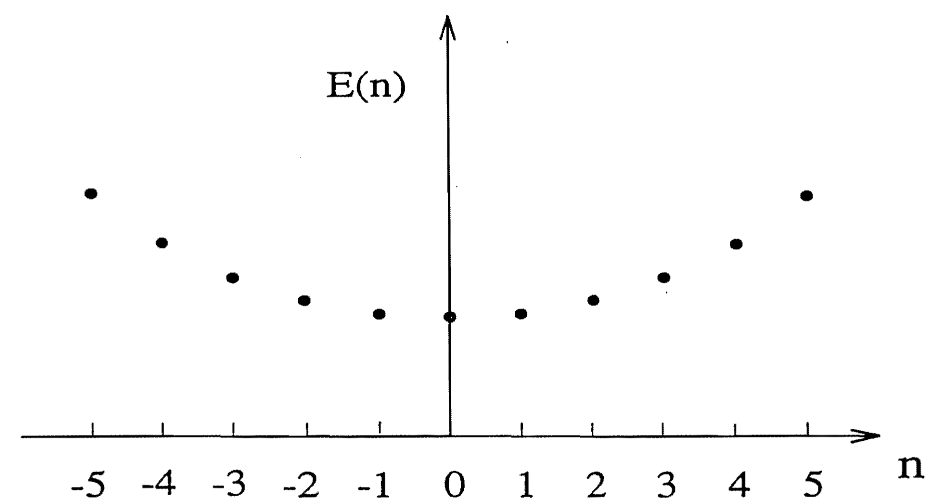

Fig. 2