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Coupled WZNW-Toda models and Covariant KdV hierarchies

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Abstract

WZNW models covariantly coupled to Toda theories are constructed using the method of conformal Hamiltonian reduction. These models have both Kac-Moody and W_N symmetry. The associated hierarchies of integrable nonlinear evolution equations are found and turn out to be covariant generalized KdV hierarchies.

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Introduction

In [1] Drinfeld and Sokolov (DS) described the Hamiltonian reduction of integrable systems of Lie algebra valued functions to integrable systems with scalar fields. This is done by constraining the fields corresponding to simple roots in the negative Borel subalgebra to non-zero constants, and the fields corresponding to negative non-simple roots, to zero. The reduced phase space is constructed from this first class constrained manifold by dividing out the local gauge invariance. The Hamiltonian structures of the original system reduce to the so called Gelfand-Dickii (GD) brackets on the reduced phase space.
If the above procedure is applied to the Lie algebra sl_2 then the resulting (scalar) integrable system is the well known Korteweg de Vries (KdV) hierarchy the second (GD) Hamiltonian structure of which is the Virasoro algebra. More generally, it has been shown [2,3,4] that the second GD bracket for sl_N is the W_N algebra, which has also been studied in the context of conformal field theory [5,6]. On the level of physical models, the above procedure can be seen as a reduction of the WZNW model to Toda theory, which indeed has W_N symmetry [4,7,8].
Recently it was realized that the DS reductions are not the only reductions leading to conformal algebras. In [9] Polyakov considered a reduction of sl_3 where in contrast to [1] he constrained the field corresponding to the non-simple root $\alpha_1 + \alpha_2$ to unity, and the fields corresponding to the simple roots to zero. He showed that this leads to a bosonic counterpart of the N=2 algebra. In [10] Bershadsky studied this algebra and its quantization in detail, and found that it was not likely to have unitary representations. More recently, Bakas and Depireux, using the self-dual Yang-Mills eqs., found the hierarchy of evolution equations associated to this specific constrained system [11].
In [12] it was shown that in fact there is a conformal reduction associated to every sl_2 embedding into g (the DS and Polyakov-Bershadsky reductions are special examples of this). Furthermore the structure of the reduced algebras is determined in quite some detail by the branching rules of the sl_2 embedding. For some representative examples the algebras were constructed explicitly.
In this paper the reductions associated to the so called 'product'-embeddings $\underline{NM} \to N\underline{M}$ are discussed in more detail. In section 1 we briefly review the results of [12] relevant to the present paper. In section 2 the physical models which underly the reduced conformal algebras associated with the aforementioned embeddings are described. In section 3 we switch our point of view and derive the hierarchies of non-linear evolution equations in Lax form of which the reduced conformal algebras are the second (GD like) Hamiltonian structure. These hierachies turn out to be covariant versions of the generalized KdV hierarchies constructed by DS [1].

1 Reduction

In this section we briefly review some results of [12]. Consider the KM current algebra

$$\{J^a(x), J^b(y)\} = f^{ab}_c J^c(y)\delta(x-y) + g^{ab}\delta'(x-y) \tag{1}$$

for currents $\mathcal{J} = J^a(x) I_a$, where I_a are the generators of some simple Lie algebra g (which we take to be sl_N for convenience), f^a_{bc} are the structure constants and

$g_{ab} = \mathrm{Tr}(I_a I_b)$. It was shown that to every embedding of sl_2 into g there is associated a reduction of this algebra leading to a conformal algebra.
If $\mathbf{T} = \{T_3, T_+, T_-\}$ is an sl_2 subalgebra of g then the adjoint representation of g decomposes under $\mathbf{T}$ into sl_2 multiplets of spin j_k $k = 1, 2, .., p$. This means that the current J can be written as

$$\mathcal{J}(x) = \sum_{k=1}^{p} \sum_{m=-j_k}^{j_k} U^{k,m}(x) T_{k,m}$$

where $T_{k,m} \in g$ is the element of the sl_2 irrep. with spin j_k and grade m (take $T_{1,1} = T_+; T_{1,0} = T_3; T_{1,-1} = T_-$).
We now constrain the algebra by putting $U^{1,-1}(x)$ to 1 and all $U^{k,m}$ for $m < 0$ to 0. This set of constraints always generates enough gauge invariance to bring the constrained currents into the form

$$\mathcal{J}_{fix}(x) = T_- + \sum_{k=1}^{p} U^{k,k}(x) T_{k,k} \tag{2}$$

This gauge is called the highest weight gauge (HWG) since the $T_{k,k}$ are the highest weight vectors of their multiplets.
On the set of currents of the form (2) there exists a Poisson bracket which is induced by (1), the so called Dirac bracket. The reduced algebra is defined to be the Dirac bracket algebra of the fields $U^{k,k}(x)$. In [12] the following facts were established.

- $T = \frac{1}{2}\mathrm{Tr}(J_{fix}^2)$ is a Virasoro algebra w.r.t. the Dirac bracket.
- The fields $U^{k,k}(x)$ are primary w.r.t. T and have conformal weights $j_k + 1$.

Let $\underline{N}_N$ denote the fundamental representation of sl_N and $\underline{2j+1}$ the $2j+1$ dimensional rep. of sl_2. The branching rule of the fundamental rep. of sl_N can then be written as

$$\underline{N}_N \rightarrow \bigoplus_{\{j\}} N_j \underline{2j+1}$$

where the N_j denote the degeneracy of the sl_2 rep. with spin j.

- The reduced algebra contains an $\bigoplus_j sl_{N_j}$ KM current subalgebra.
- For every spin j occurring in the branching of the fundamental rep. the reduced algebra contains a W_{2j+1} subalgebra commuting with the KM current subalgebra.

Note that the reduced algebra will contain as many fields as there are sl_2 multiplets in the braching of the adjoint rep.
The DS type reductions leading to W_n algebras correspond to the so called principal embeddings $\underline{N}_N \rightarrow \underline{N}$. The Polyakov- Bershadsky reduction of sl_3 corresponds to the embedding $\underline{3}_3 \rightarrow \underline{2} + \underline{1}$. The algebras considered in [13] by Romans correspond to the embeddings $\underline{N+2} \rightarrow \underline{2} + N\underline{1}$. All these cases were considered in detail in [12].
In the present paper we will be concerned with reductions associated with the sl_2 embeddings under which the fundamental representation of sl_{NM} branches as

$$\underline{NM}_{NM} \rightarrow N\underline{M},$$

i.e. the fundamental representation of sl_{NM} branches into direct sum of N, M dimensional, representations of $\mathbf{T}$. The reduced algebra will have $MN^2 - 1$ generators and contains two commuting subalgebras, being an sl_N KM current algebra and a W_M subalgebra. We will refer to this algebra as $\mathcal{CW}_M^N$.
In [12] the algebras $\mathcal{CW}_2^N$ were exlicitly constructed. The gauge fixed currents in highest weight gauge have the form

$$\mathcal{J}_{fix} = \begin{pmatrix} J & T \\ 1 & J \end{pmatrix} \tag{3}$$

where J and T are sl_N and gl_N matrices of currents respectively, i.e.

$$\begin{aligned} J(x) &= J^a(x)I_a \\ T(x) &= T^a(x)I_a + T^0(x) \end{aligned}$$

and runs from 1 to N.
The Dirac bracket algebras $\mathcal{CW}_2^N$ were found to be most efficiently summarized in terms of covariant variations. Namely, let $h = h^a(x)I_a$, $t = t^0 + t^a I_a$ and consider the quantities

$$\begin{aligned} \int g_{ad}h^d(y)\{J^a(x), J^b(y)\}T_b dy &= \delta_h J \\ \int g_{ad}h^d(y)\{J^a(x), T^b(y)\}T_b dy &= \delta_h T \end{aligned}$$

Upon defining the covariant derivative $D = \partial + ad_J$, the algebras $\mathcal{CW}_2^N$ can elegantly be summarized as

$$2\delta_h J = DJ \tag{4}$$

$$2\delta_h T = [h, T] \tag{5}$$

$$2\delta_t T = -D^3 t + 2\{T, Dt\} + \{DT, t\} \tag{6}$$

where $\{.,.\}$ here denotes the ordinary matrix anticommutator. Writing out these relations in full detail can be extremely cumbersome (see [12]) and is not very illuminating. In appendix A the algebra $\mathcal{CW}_3^N$ is given as well.

2 Covariant Toda theories

In this section we describe the physical theories that underly the symmetry algebras $\mathcal{CW}_M^N$ described in the previous section. Motivated by the case where the symmetry algebra is a pure W_M symmetry we pick a convenient parametrization of sl_{NM}

$$g := N_+ N_0 N_-$$

where

$$N_+ = \begin{pmatrix} 1 & g_{12} & \dots & g_{1M} \\ 0 & 1 & & \vdots \\ \vdots & & & g_{M-1M} \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad N_- = \begin{pmatrix} 1 & 0 & \dots & 0 \\ g_{21} & 1 & & \vdots \\ \vdots & & & 0 \\ g_{M1} & \dots & g_{MM-1} & 1 \end{pmatrix},$$

$$N_0 = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & & \vdots \\ \vdots & & & 0 \\ 0 & \dots & 0 & g_M \end{pmatrix}. \tag{7}$$

The matrix elements of the $N_\pm$ and N_0 are $N \times N$ matrices. The decomposition (7) is a valid local decomposition that can be extended globally throughout sl_{NM}.
The WZNW action for g can be decomposed using the Polyakov-Wiegmann identity

$$S_{WZNW}[g_1 g_2] = S_{WZNW}[g_1] + S_{WZNW}[g_2] + \int d^2x\, \overline{J}_1 J_2 \tag{8}$$

where we have defined the currents

$$J = (\partial_+ g)g^{-1}, \qquad \overline{J} = -g^{-1}(\partial_- g)$$

and $\kappa = -k/4\pi$. Clearly the WZNW actions of $N_\pm$ vanish, such that the total sl_{NM} action decomposes into a WZNW action for the diagonal blocks g_i that are coupled to the $N_\pm$ by the last term in eq.(8).
The constraints (2) are now easily implemented on both the left and right chiral algebra, resulting in an action that depends on N_0 only [14]

$$S_{constr.} = S_{WZNW}[N_0] + \int d^2x\, \mathrm{Tr}(N_0 T_- N_0^{-1} T_+) \tag{9}$$

Note that the number of currents on which the action explicitly depends is $2NM^2 - 1$, whereas we started with $2N^2M^2 - 1$ currents and we have constrained only $N(N-1)M^2$ of them. Therefore there are $N(N-1)M^2$ currents on which the action does not explicitly depend, i.e. the constraints have introduced a local gauge freedom. This local gauge freedom can be fixed, but it will not alter the action.
For $N = 1$, parametrization of g_i by $exp(-\phi_i + \phi_{i-1})$ ($\phi_0 = \phi_M = 0$) reduces the action (14) to the well known A_{M-1} Toda model, which has a W_M symmetry. As we will show, for arbitrary N, an sl_N symmetry will couple to this W_M symmetry, giving rise to a covariant version of the W_M symmetry, which we refered to as $\mathcal{C}\mathcal{W}_M^N$. The field equations

$$\partial_-(\partial_+ N_0 N_0^{-1}) = [T_-, N_0 T_+ N_0^{-1}] \tag{10}$$

can be written in Lax form $[\partial_+ + A_+, \partial_- + A_-] = 0$, where

$$A_+ = -(\partial_+ N_0)N_0^{-1} - T_- \tag{11}$$

$$A_- = N_0 T_+ N_0^{-1} \tag{12}$$

These equations have already been solved in [15], be it in a different context (see also [14]). The variation of the total Lagrangian in (9) can be written as

$$\delta\mathcal{L} = \mathrm{Tr}(N_0^{-1}\delta N_0[\partial_+ + A_+, \partial_- + A_-]) \tag{13}$$

For a variation to be a symmetry (13) has to vanish. To see how this works in the simplest case, we again consider the case $M = 2, N$ arbitary. For this case the action reads

$$S_{constr.} = S_{WZNW}[g_1] + S_{WZNW}[g_2] + \int d^2x\, \mathrm{Tr}(g_1^{-1} g_2) \tag{14}$$

where g_1 and g_2 are in principle gl_N valued matrices. However, due to (7) the overall determinant $\det(g_1 g_2) = 1$. Nevertheless, since this corresponds only to an overall $U(1)$ mode that is easily seen to decouple completely, we leave it in to avoid messy notations. Variation w.r.t. g_1 and g_2 gives the field equations

$$\partial_-(\partial_+ g_1 g_1^{-1}) = -g_1 g_2^{-1} \quad (15)$$

$$\partial_-(\partial_+ g_2 g_2^{-1}) = g_1 g_2^{-1} \quad (16)$$

Therefore, the original KM currents J_1 and J_2 are no longer chiral. On the other hand, the current

$$J(x^+) = \frac{1}{2}(\partial_+ g_1 g_1^{-1} + \partial_+ g_2 g_2^{-1})$$

is still chiral, and is also easily checked to be a symmetry of the action (14). A similar expression exists for $\overline{J}(x^-)$

The action possesses however a larger symmetry. Differentiating (15) and (16) w.r.t. x_+ we find

$$\partial_- \left(\partial_+^2 g_1 g_1^{-1} - \partial_+ g_1 g_1^{-1} J\right) = 0$$

$$\partial_- \left(\partial_+^2 g_2 g_2^{-1} - J \partial_+ g_2 g_2^{-1}\right) = 0$$

From this we can construct another chiral operator independent of J. With considerable hindsight we take the combination

$$T(x^+) = (J_-)^2 + DJ_-$$

where $J_- \equiv (J_1 - J_2)/2$, $D \equiv \partial + J$, and J is in the adjoint representation. Using

$$\{J^a(x), \mathrm{Tr}(g_2 g_1^{-1})(y)\} = \mathrm{Tr}(I^a g_2 g_1^{-1})(y)\delta(x-y)$$

it is easily checked explicitly that the $T^a \equiv \mathrm{Tr}(I^a T)$ indeed generate a symmetry of (14). Crucial ingredient in the proof is the fact that the symmetry operators J and T are chiral, since that automatically assures the invariance of the kinetic term $S_W[g_1] + S_W[g_2]$. In appendix B the proof for arbitrary N and M is given.

The relation between the chiral operators J and T on the one hand, and the current algebras J_1 and J_2 on the other hand suggest that the Fateev-Lukyanov quantization of the W_M algebras [6] can be generalized directly. In particular this would mean that we should postulate Kac-Moody algebra for the currents J_1 and J_2 and calculate from these the algebra of J and T. This is an interesting question which certainly deserves further study.

However, for the complete quantum theory of the covariant Toda model we would be more satisfied if we knew the partition function. The recently discovered intriguing relation between the partition function of the Liouville model and the KdV hierarchy [16] makes room for the conjecture that a similar (covariant) hierarchy might exist for the partition function of the generalized Toda models. For the matrixmodels, it clearly requires an adaption of the usual way of dealing with the integration over the infinite matrices, i.e. instead of summing over the eigenvalues, some block structure is to be maintained. A direct construction of the associated hierarchy is still under investigation, it may however be interesting to note that it is possible to construct the covariant version of the KdV hierarchy for the product embedding along the lines of Drinfeld and Sokolov.

3 Covariant KdV hierarchies

A crucial ingredient in the construction of integrable hierarchies is the so called Lax formulation

$$\frac{dL}{dt} = [M, L] .$$

where L and M are differential operators with space and time dependent coefficients. The reason is that it signals the existence of infinitely many conserved quantities in involution. In [1] it was shown that it is possible to map the constrained KM system to such a Lax system by associating to each gauge equivalence class of constrained currents a (scalar) Lax operator of a certain form. This map is called the 'Miura map'.
The Miura map in the present case can be calculated as follows. Consider the differential eqn.

$$(\partial - \mathcal{J}_{fix})\psi = 0 \tag{17}$$

where ψ is a 2-vector whose entries are $N \times N$ matrices and $\mathcal{J}_{fix}$ is given by (3). Since the KM subalgebra J acts in these reductions in the adjoint representation we must let J act on ψ_i also in the adjoint rep. (see eq.(5) and [12]), i.e. $J.\psi_i = [J, \psi_i] \equiv -ad_J\psi_i$. Eliminating the component ψ_1, eq.(17) reduces to

$$(D^2 - T)\psi_2 = 0$$

The Lax operator is then $L = D^2 - T$ which is a covariant version of the KdV Lax operator. L is invariant under gauge transformations.
The covariant derivative D is a derivation, i.e.

$$D(AB) = (DA)B + A(DB) \tag{18}$$

The operator D has a formal inverse D^{-1} within the algebra of pseudo-differential operators with matrix coefficients. The first few terms of D^{-1} are

$$D^{-1} = \partial^{-1} - ad_J\partial^{-2} + (ad_{J'} + ad_J^2)\partial^{-3} + ... \tag{19}$$

It is easily checked that $DD^{-1} = D^{-1}D = 1$. This means that $D^{-1}D(AB) = AB$. Using the derivation property of D and on defining $C = (DB)$ we find

$$D^{-1}(AC) = AD^{-1}C - D^{-1}((DA)D^{-1}C) \tag{20}$$

Iterating this equation we find the following permutation rule

$$D^{-1}A = \sum_{i=0}^{\infty}(-1)^i(D^iA)D^{-i-1} \tag{21}$$

which is identical to the one for ∂^{-1}.
From these considerations we find that we might as well work with the algebra of covariant pseudo-differential operators. This will simplify calculations drastically and also reveals more structure.
We found above that the Lax operator associated to the $\underline{2N} \to N\underline{2}$ reduction is $L = D^2 - T$. Using (21) we can determine the formal root of this operator. It reads

$$L^{1/2} = D - \frac{1}{2}TD^{-1} + \frac{1}{4}(DT)D^{-2} + ...$$

Using (19) one can easily check that $L^{1/2}$ in terms of ordinary differential operators is

$$L^{1/2} = \partial + ad_J - \frac{1}{2}T\partial^{-1} + \frac{1}{4}(T' + \{ad_J, T\})\partial^{-2} + ...$$

which indeed squares to $L = \partial^2 + 2ad_J\partial + (ad_J^2 + ad_{J'} - T) = D^2 - T$.
The hierarchy of evolution equations is given by

$$\frac{dL}{dt_{2k+1}} = [(L^{2k+1/2})_+, L] \quad (k = 0, 1, 2, ...) \tag{22}$$

where the + means that we are to consider only the positive power part of $L^{2k+1/2}$ (w.r.t. the covariant derivative). Ofcourse, for this hierarchy to be anywhere near completely integrable, the different flows must commute, i.e.

$$\frac{d^2}{dt_i dt_j}L = \frac{d^2}{dt_j dt_i}L \tag{23}$$

We will now check if this is the case.
It is not difficult to show [1] that if eq.(22) holds then also

$$\frac{dL^{i/2}}{dt_{2k+1}} = [(L^{2k+1/2})_+, L^{i/2}] \tag{24}$$

Using this equation we find

$$\frac{d^2}{dt_i dt_j}L = [[(L^{i/2})_+, L^{j/2}]_+, L] + [(L^{j/2})_+, [(L^{i/2})_+, L]] \tag{25}$$

Therefore

$$(\frac{d^2}{dt_i dt_j} - \frac{d^2}{dt_j dt_i})L = [[(L^{i/2})_+, L^{j/2}]_+ - [(L^{j/2})_+, L^{i/2}]_+ + [(L^{j/2})_+, (L^{i/2})_+], L] \tag{26}$$

where we used the Jacobi identity. Using

$$[L^{i/2}, L^{j/2}] = 0$$

it follows that

$$[(L^{i/k})_+, L^{j/k}]_+ = [(L^{j/k})_+, (L^{i/k})_-]_+$$

Inserting this into eq.(26) we indeed find eq.(23).
The first two equations of the hierarchy are

$$\frac{dT}{dt_1} = DT \tag{27}$$

$$\frac{dT}{dt_3} = \frac{1}{4}(D^3T) - \frac{3}{4}\{DT, T\} \tag{28}$$

while the connection components J do not evolve w.r.t. any time, i.e.

$$\frac{dJ}{dt_k} = 0$$

This can be seen as follows. The Lax eqn. (24) was so constructed ([1]) that its right hand side is a differential operator of order 0. However, the left hand side has in the covariant case order 1 which means that the coefficient of the order 1 term must be zero. This coefficient is exactly $2d(ad_J)/dt_{2k+1}$.
Obviously the hierarchy constructed above is a covariant generalization of the KdV hierarchy. If again we write $T = T^0 + T^a I_a$ then eq.(27) becomes

$$\frac{dT^0}{dt_1} = \partial T^0 \tag{29}$$

$$\frac{dT^a}{dt_1} = \partial T^a + f^a_{bc} T^b J^c \tag{30}$$

The eqs. (28) are far more involved and we will not display them in full detail. The trace part however reads

$$\frac{dT^0}{dt_3} = \frac{1}{4}\partial^3 T^0 - \frac{3}{2} T^0 \partial_x T^0 - \frac{3}{2N} g_{ab} T^a \partial_x T^b \tag{31}$$

which is the KdV equation modified by an extra term arizing from the covariant structure. Note that this equation remains different from the KdV even for $J = 0$.
The quantities

$$H_k = \int \mathrm{Tr} Res(L^{k/2}) dx \tag{32}$$

where $Res(\sum_i A_i D^i) = A_{-1}$ are conserved quantities of the hierarchy, i.e.

$$\frac{dH_k}{dt_i} = 0 \tag{33}$$

In order to see this consider

$$Res[AD^k, BD^l] \tag{34}$$

This quantity is 0 if $l, k > 0$ or $l, k < 0$. The only interesting case is $k > 0, l < 0$. Take $l = -p$ where $p > 0$ then $Res[AD^k, BD^{-p}] = 0$ if $p > k+1$ and

$$(\frac{k}{p+1})(A(D^{k-p+1}B) + (-1)^{k-p}(D^{k-p+1}A)B) \tag{35}$$

Also note that if

$$g = (\frac{k}{p-1}) \sum_{i=0}^{k-p} (-1)^i (D^i A) D^{k-p-i} B \tag{36}$$

then

$$Dg = (\frac{k}{p-1})(AD^{k-p+1}B + (-1)^{k-p}(D^{k-p+1}A)B) \tag{37}$$

which means that

$$\mathrm{Tr}(Dg) = \mathrm{Tr} Res[AD^k, BD^{-p}] \tag{38}$$

However $\mathrm{Tr}(Dg) = \partial \mathrm{Tr}(g)$ from which it follows that $\mathrm{Tr} Res[AD^k, BD^{-p}]$ is a total derivative. Using this and eq.(24), eq.(33) follows.
The first few Hamiltonians are

$$H_1 = \frac{1}{2} \int T^0(x) dx \tag{39}$$

$$H_2 = \frac{1}{2} \int ((T^0)^2 + \frac{1}{N} g_{ab} T^a T^b) dx \tag{40}$$

This program can be extended to the general case $\underline{NM} \rightarrow N\underline{M}$ without difficulty whatsoever. The hierarchy for $N = 3$ is a covariant version of the Boussinesq hierarchy. The reduced algebra obtained by Hamiltonian reduction of the KM current algebra has in this case $3N^2 - 1$ generators. It is given in the appendix A together with the covariant Boussinesq hierarchy.

4 Discussion and Outlook

In this work we have studied a certain class of physical theories that are conformal reductions of WZNW models. These models turned out to be coupled WZNW-Toda models. The associated Hamiltonian reduction of the multicomponent (Sacharov-Shabat) integrable system was shown to yield a covariant version of the so called Gelfand-Dickii hierarchy of which the KdV and Boussinesq hierarchies are the lowest order ones.
The reductions studied in this paper are associated to the so called 'product embeddings'. Reductions associated to different embeddings (see [12]) are now under investigation. In particular the so called 'sum embeddings' $\underline{N+M} \rightarrow \underline{N} + M.\underline{1}$ generalizing the Polyakov-Bershadsky reduction, seem to exhibit some new features in terms of the Lax formulation of the associated hierarchies. We will come back to this in a future publication.
One of the open problems is to give a classification of all reductions of WZNW models. There are some indications that the reductions related to sl_2 embeddings constitute all inequivalent conformal reductions. A proof of this statement is however lacking at the moment.
Another point of interest is the Hirota bilinear formulation of the covariant KdV hierarchies. The latter method seems to be relevant to the formulation of 2D quantum gravity in terms of matrix models.

Appendix A

We will give some more details on the $\underline{3N} \rightarrow N\underline{3}$ reduction of sl_{3N} leading to the $\mathcal{CW}_3^N$ algebras.
The gauge fixed currents in the highest weight gauge have the form

$$\mathcal{J}_{fix} = \begin{pmatrix} J & T & W \\ \mathbf{1} & J & T \\ 0 & \mathbf{1} & J \end{pmatrix} \tag{41}$$

The Dirac bracket algebra can be calculated by the algorithm explained in [12]. Using the notation introduced in section 1 it reads

$$3\delta_h J = Dh \tag{42}$$
$$3\delta_t J = [t, T] \tag{43}$$
$$3\delta_w J = [w, W] \tag{44}$$
$$2\delta_t T = -D^3 t + \frac{1}{2}\{t, DT\} + \{Dt, T\} + \frac{1}{2}[t, W] \tag{45}$$
$$\delta_w T = \frac{1}{2}[D^2 T, w] + \frac{5}{4}[DT, Dw] + \frac{5}{6}[T, D^2 w] + \frac{1}{3}[w, T^2] + \frac{1}{2}\{DW, w\} + \frac{3}{4}\{W, Dw\}$$

$$\begin{aligned}\delta_w W &= \frac{1}{6}D^5 w - \frac{5}{6}\{D^3 w, T\} - \frac{5}{4}\{D^2 w, DT\} - \frac{3}{4}\{Dw, D^2T\} - \frac{1}{6}\{D^3T, w\} + \\ &+\frac{7}{12}[D^2w, W] + \frac{7}{12}[Dw, DW] + \frac{1}{6}[w, D^2W] + \frac{10}{3}T(Dw)T + \frac{5}{3}(DT)wT + \\ &+\frac{5}{3}Tw(DT) - \frac{1}{3}\{Dw, T^2\} - \frac{1}{3}t(DT)T - \frac{1}{3}T(DT)w - \frac{4}{3}TwW - \frac{1}{3}TWw + \\ &\frac{4}{3}WwT + \frac{1}{3}wWT + \frac{1}{3}wTW - \frac{1}{3}WTw\end{aligned}$$

where $h = h^a I_a$, $t = t^0 + t^a I_a$ and $w = w^0 + w^a I_a$ are associated to J, T and W respectively.

Clearly, J is again a KM current algebra while T^0, W^0 form a W_3 subalgebra. From eqs.(43,44) it follows that T^a and W^a transform in the adjoint representation under J. This structure can be found in all algebras $\mathcal{CW}_M^N$.

The Miura map for the gauge fixed currents (41) yields in this case

$$L = D^3 - 2TD - (DT) - W \tag{46}$$

as can easily be checked. The formal kube root of this operator is

$$L^{1/3} = D - \frac{2}{3}TD^{-1} + \frac{1}{3}(T' - W)D^{-2} + \ldots \tag{47}$$

The covariant Boussinesq hierarchy is then

$$\frac{dL}{dt_k} = [(L^{k/3})_+, L] \tag{48}$$

Again one can prove that all these flows commute. The first few equations are

$$\begin{aligned}\frac{dT}{dt_1} &= DT && (49)\\ \frac{dW}{dt_1} &= DW && (50)\\ & && (51)\end{aligned}$$

and

$$\begin{aligned}\frac{dT}{dt_2} &= DW && (52)\\ \frac{dW}{dt_2} &= -\frac{1}{3}D^3T + \frac{4}{3}\{T, DT\} - \frac{4}{3}[T, W] && (53)\end{aligned}$$

which is a covariant version of the Boussinesq equation. As in the case of the covariant KdV hierarchy the connection components J do not evolve w.r.t. any time. Note that the last term in equation (53) is not present in the ordinary Boussinesq eqn. since in that case T and W are commutative variables. The Hamiltonians of the covariant Boussinesq are again the quantities $H_k = \int \mathrm{Tr} Res(L^{k/3})$.

Appendix B

In this appendix it is proven that the generalized Toda model (9) is invariant under a $\mathcal{CW}_M^N$ symmetry transformation. In fact, it will be shown that the two seperate terms of the Lagrangian will vary by a total derivative when acted upon by $\mathcal{CW}_M^N$. The action of an infinitesimal chiral variation $R_{\mathcal{CW}}(x^+)$ on N_0 is given by

$$\delta_{\mathcal{CW}} N_0 = R_{\mathcal{CW}} N_0 \tag{54}$$

In particular it means that the action of $\mathcal{CW}$ on the current $\mathcal{J}_{fix} = T_- + \partial_+ N_0 N_0^{-1}$ is given by

$$\delta(T_- + \partial_+ N_0 N_0^{-1}) = [R_{\mathcal{CW}}(x^+), T_- + \partial_+ N_0 N_0^{-1}] + \partial_+ R_{\mathcal{CW}}(x^+) \tag{55}$$

This current is locally gauge equivalent to the current (2) in the highest weight gauge. Therefore, with equation (55) we have in a sense linearized the $\mathcal{CW}$ symmetry, and we are in a position to calculate the variation of the action.
Using (54) in (13) we find

$$\begin{aligned}
\delta L_{\mathcal{CW}} &= \mathrm{Tr}\left(R_{\mathcal{CW}}(x^+)(\partial_-(\partial_+ N_0 N_0^{-1} - [T_-, N_0 T_+ N_0^{-1}])\right) \\
&= \text{tot. der.} - \mathrm{Tr}\left(([R_{\mathcal{CW}}, T_-] N_0 T_+ N_0^{-1})\right) \\
&= \text{tot. der.} + \mathrm{Tr}\left(([R_{\mathcal{CW}}, \partial_+ N_0 N_0^{-1}] N_0 T_+ N_0^{-1} - \partial_+ R_{\mathcal{CW}} N_0 T_+ N_)^{-1})\right)
\end{aligned}$$

where we have used (55). The last term is easily seen to be a total derivative as well.

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