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ON THE GALILEAN COVARIANCE OF CLASSICAL MECHANICS

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Abstract:

A Galilean covariant approach to classical mechanics of a single interacting particle is described. In this scheme constitutive relations defining forces are rejected and acting forces are determined by some fundamental differential equations. It is shown that the total energy of the interacting particle transforms under Galilean transformations differently from the kinetic energy. The statement is illustrated on the exactly solvable examples of the harmonic oscillator and the case of constant forces and also, in the suitable version of the perturbation theory, for the anharmonic oscillator.

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I. Introduction

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According to the principle of relativity the laws of physics do not depend on the choice of the reference frame in which these laws are formulated. The change of the reference frame induces only a change in the language used for the formulation of the laws. In particular, changing the reference frame we change the space-time coordinates of events and functions describing physical quantities but all these changes have to follow strictly defined ways. When that is achieved we talk about a covariant formulation of a given theory and only in such formulation we may satisfy the requirements of the principle of relativity. The non-covariant formulations of physical theories are deprived from objectivity and it is difficult to judge about their real physical meaning.

The main purpose of any theory is to describe symmetries of the physical world. These symmetries are customarily divided into spatio-temporal symmetries which describe the geometry of space-time and internal or higher ones which reflect the structure of physical observables. For low energy physics all space-time symmetries are contained in the Galilei group and all theories with or without higher symmetries should be Galilean covariant at low energies. In particular, Galilean covariance is the most important property of both classical and quantum mechanics (erroneously called non-relativistic) but unfortunately these two theories have always lacked a detailed treatment of this covariance at each step of their development. As a matter of fact, there does not exist any Galilean covariant formulation of both these theories already on the level of a single particle system. This was the main motivation for writing our paper devoted to the problems of Galilean covariance of interacting mechanical systems.

At the period of extensive development of foundations of mechanics only little attention has been paid to the symmetry properties of this theory. The main pur-

pose was always to write down the equations of motion and to find their solutions. A meaningful discussion of the invariance properties of various theories started with the birth of relativity theory. The need for such turn in theoretical physics comes from the long time lack of basic equations in many modern theories. For a long time only the group theoretical methods have been the source of stable information on the structure of these theories. In the main stream of relativistic theories the Galilean covariant ones were treated as marginal and, in some sense, not very interesting.

It is clear that we shall not give a comprehensive description of the Galilean covariant formulation of all branches of mechanics because this would require to write many volumes. We have restricted ourselves to the analytical mechanics and our attention was concentrated on the single particle systems. But even within this relatively simple case we have discovered a lot of things which are not commonly known to the physicists community and it was the second motivation writing this paper.

Among new things we have found we would like to list the following ones:

i) for each particle it is possible to introduce a new notion of mass [1,2], the Galilean mass of the particle and there is no general argument that this mass has to be equal to the inertial one,

ii) for special class of many particle systems there exist non-dynamical correlations [3] called by us Medvedev correlations [4] which already on the level of classical physics allow to introduce the notion of "confinement" [5], i.e. a specific property of many particle systems which forbids to disjoin them into individual particles,

iii) the Galilean transformation rule of the total energy of the interacting single particle is different from the well-known Galilean transformation rule for the kinetic energy.

The last statement is the most striking new result in classical mechanics. We show here, and this is one of the main results of this paper, that the transformation rule for the total energy depends on the interaction. The same is true for the action integral although up tp now we cannot give any constructive example. Since the action integral forms a five-vector together with the spatial distance and time duration of the motion, this result strongly suggests that the space-time structures in which live interacting particles have to be described for each type of interaction by different invariants. It would be interesting to investigate the relativistic consequences of this fact.

All existing ways of naive covariantizing the non-covariant formalisms of classical mechanics are not unique. To avoid this trouble we propose here new covariant formulation of mechanics. The essence of our approach consists in rejecting all non-covariant constitutive relations which are customarily used to close the equations of motion. Instead of them we propose here to treat the acting forces dynamically and to determine them from Galilean covariant differential equations which we treat as mechanical reminiscences of field equations used in more advanced theories. Such an approach allows not only to find the covariant expression for the total energy of the interacting particle but it also allows to extend the class of mechanical problems because the differential equations for forces may have more solutions than we expect. Since the explanation of the origin of forces is evidently out of the scope of mechanics, the structure of the defining equations must be postulated as an input of any mechanical model. We hope that the equations for forces may reproduce all known mechanical forces or at least approximate them.

The biggest advantage of our approach consists in its role in the construction of Galilean covariant formulation of quantum mechanics of interacting particles and this formulation, in turn, is a part of our general strategy to construct relativistic quantum mechanics. This circumstance is the third motivation for writing

the present paper. The size of the paper forced us to divide it into two parts. The present, first part, is devoted to those problems of classical mechanics which are necessary for understanding the corresponding problems of quantum mechanics. The main result of the second part consists in the introduction of the new notion of quantum mechanical force. In order to achieve that we will introduce a new kind of wave functions which, apart from the space-time variables depend also on variables which realize the new notion of force.

II. The transformation rules of basic mechanical quantities.

The aim of this section is to provide the reader with a brief review of the Galilean transformation rules of the basic quantities of classical mechanics. As it is well-known [6] the Galilean transformations map the space-time coordinates $(\vec{x}, t)$ of any event measured in one inertial reference frame onto the space-time coordinates $(\vec{x}\,', t')$ of the same event measured in a second inertial reference frame and they have the form

$$\vec{x}\,(t) \rightarrow \vec{x}\,'(t') = R\vec{x}\,(t) + \vec{u}t + \vec{a} \tag{2.1}$$

$$t \rightarrow t' = t + b \tag{2.2}$$

where R is 3×3 orthogonal matrix which describes the rotation of coordinate axes used by observers connected with the considered reference frames $[(R\vec{x})_j = \sum_{k=1}^{3} R_{jk}x_k]$, $\vec{u}$ is the velocity of the second reference frame with respect to the first one, the vector $\vec{a}$ describes the translation of the origins of the coordinate systems and b is the time translation of clocks rigidly connected with each reference frame. The trajectory of a particle in each reference frame is described by vector valued function of time and if $\vec{x}\,(t)$ and $\vec{x}\,'(t')$ are functions describing a selected

trajectory in two different reference frames then according to (2.1) and (2.2) we must have the following relation between the functions $\vec{x}\,(t)$ and $\vec{x}\,'(t')$:

$$\vec{x}\,'(t') = R\vec{x}\,(t) + \vec{u}t + \vec{a} \tag{2.3}$$

Therefore, in a manifestly Galilean covariant mechanics we always look for solutions of the equations of motion which satisfy the transformation rule (2.3). It is easy to satisfy this requirement for free motion for which the trajectories are given by the functions:

$$\vec{x}\,(t) = \frac{\vec{x}_1 t_2 - \vec{x}_2 t_1}{t_2 - t_1} + \frac{\vec{x}_2 - \vec{x}_1}{t_2 - t_1} t \tag{2.4}$$

which are linear in time and where $\vec{x}_i\,(i = 1, 2)$ are two selected positions (the initial and the final ones) of the particle at two times t_i, respectively. Transforming the selected space-time coordinates $(\vec{x}_i\,, t_i)$ according to the rules (2.1) and (2.2) we get the transformation rule (2.3). Unfortunately, a short glance at solutions of the interacting mechanical systems in classical mechanics shows that these solutions cannot satisfy the transformation rule (2.3). Consequently the free motion is the only known realization of the Galilean covariant classical mechanics. In classical mechanics the root of that lies in the notorious use of Galilean noncovariant constitutive relations which are needed to complete the Newton equations and to convert them into a closed system of differential equations for the trajectory. In the next section we shall show how to bypass this trouble.

From the transformation rule (2.3) it follows that the functions $\vec{v}\,(t)$ and $\vec{v}\,'(t')$, which describe the velocity of the particle in two inertial reference frames, are related by the transformation rule

$$\vec{v}\,'(t') = R\vec{v}\,(t) + \vec{u} \tag{2.5}$$

because otherwise the basic relation of mechanics

$$\frac{d\vec{x}(t)}{dt} = \vec{v}(t) \tag{2.6}$$

would not be Galilean covariant.

A little more complicated is the problem of the transformation rule of momentum $\vec{p}(t)$. In the standard approach to classical mechanics this fundamental quantity of mechanics is defined through the relation

$$\vec{p}(t) = m\vec{v}(t) \tag{2.7}$$

where m is the inertial mass of the particle. From that definition and from (2.5) it follows that under the Galilean transformations the momentum transforms according to the rule

$$\vec{p}(t) \to \vec{p}'(t') = R\vec{p}(t) + m\vec{u} \tag{2.8}$$

This way of operating with momentum suggests however that momentum is not a primary quantity of mechanics but a "derived" one. We consider momentum as a quantity in "its own right" for which the fundamental equation of mechanics

$$\frac{d\vec{p}(t)}{dt} = \vec{F} \tag{2.9}$$

is satisfied with $\vec{F}$ being the force acting on the particle. In such approach momentum may be or not be related to the velocity in the way given by (2.7) and we must establish for it the Galilean transformation rule independently from (2.7). To do that let us first remind that the acting force $\vec{F}$ always transforms according to the simple rule

$$\vec{F} \to \vec{F}'(t') = R\vec{F} \tag{2.10}$$

since otherwise we would not have equal magnitudes of the forces in all inertial reference frames. From (2.9) it follows then that the transformation rule for momentum is of the form

$$\vec{p}(t) \to \vec{p}'(t') = R\vec{p}(t) + \vec{C}(R, \vec{u}, \vec{a}, b) \tag{2.11}$$

where $\vec{C}$ is time independent vector which, as indicated by our notation, may depend on the parameters of the transformation. Repeating twice the transformation rule (2.11) we get a relation for the vector-valued function $\vec{C}(R, \vec{u}, \vec{a}, b)$ in the form

$$\vec{C}(R_1, \vec{u}_1, \vec{a}_1, b_1) + R_1\vec{C}(R_2, \vec{u}_2, \vec{a}_2, b_2) = \vec{C}(R_{12}, \vec{u}_{12}, \vec{a}_{12}, b_{12}) \tag{2.12}$$

where $R_i, \vec{u}_i, \vec{a}_i, b_i$ $(i = 1, 2)$ are the parameters of two subsequent Galilean transformations and

$$\begin{aligned} R_{12} &= R_1 R_2 \\ \vec{u}_{12} &= \vec{u}_1 + R_1\vec{u}_2 \\ \vec{a}_{12} &= \vec{a}_1 + R_1\vec{a}_2 + \vec{u}_1 b_2 \\ b_{12} &= b_1 + b_2 \end{aligned} \tag{4.13}$$

are the parameters of the composed Galilean transformation. Comparing (2.12) and (2.13) we see that

$$\vec{C}(R,\vec{u},\vec{a},b) = M\vec{u} \tag{2.14}$$

where M is an arbitrary parameter with the dimension of mass. In this way we come to the transformation rule of the momentum in the form

$$\vec{p}'(t') = R\vec{p}(t) + M\vec{u} \tag{2.15}$$

where M is a mass parameter which need not be equal to the inertial mass m.

Using abstract group theoretical methods [7] we also arrive at the transformation rule (2.15) where M is a mass parameter which determines the representation of the Galilei group associated with the particle. In this way we see that M is a mass parameter characteristic for the considered particle and we may call this parameter to be the Galilean mass of the particle. From (2.15) we see that the Galilean mass M and not the inertial mass m, defined as the proportionality factor between the force and acceleration, determines the Galilean transformation rule for one of the basic quantities of the particle, namely of its momentum. The transformation rule (2.15) with the Galilean mass M, in contradistinction to the rule (2.8) being a consequence of the constitutive relation (2.7), has a general meaning and there is no general argument which would imply the equality of the Galilean mass to the inertial one. On the example of gravitational and inertial masses we know that mass parameters defined independently a priori need not be equal and therefore the equality $m = M$ should be experimentally verified. In principle it is possible [8] to develop mechanics without the equality $m = M$ but here, for simplicity, we shall take this equality as an assumption. As it was shown in [1] the relation (2.7) is then the most general relation between momentum and velocity.

Apart from the trajectory, velocity, momentum and the acting force the most important and basic quantities of mechanics are the kinetic and total energies of

the bodies. Starting from the balance equation for the kinetic energy $T(t)$ in the form

$$\frac{dT(t)}{dt} = \vec{F}\cdot\vec{v}(t) \tag{2.16}$$

and using (2.5) together with (2.10) we may show that the transformation rule of $T(t)$ must have the form

$$T(t) \to T'(t) = T(t) + R\vec{p}(t)\cdot\vec{u} + C(R,\vec{u},\vec{a},b) \tag{2.17}$$

where $C(R,\vec{u},\vec{a},b)$ is now time independent scalar depending in general on the parameters of the transformation. Applying (2.17) twice and using (2.15) we get a relation

$$C(R_1,\vec{u}_1,\vec{a}_1,b_1) + C(R_2,\vec{u}_2,\vec{a}_2,b_2) + M\vec{u}_1\cdot(R\vec{u}_2) = C(R_{12},\vec{u}_{12},\vec{a}_{12},b_{12}) \tag{2.18}$$

and taking (2.13) into account we conclude that the only way to satisfy this relation is to put

$$C(R,\vec{u},\vec{a},b) = \frac{1}{2}M\vec{u}^{\,2} \tag{2.19}$$

In this way we get

$$T'(t) = T(t) + R\vec{p}(t)\cdot\vec{u} + \frac{1}{2}M\vec{u}^{\,2} \tag{2.20}$$

where again the Galilean mass is present. In the case of the assumed equality $m = M$ we may freely use the relations

$$T(t) = \frac{1}{2m}\vec{p}^{\,2}(t) = \frac{1}{2}\vec{v}^{\,2}(t) = \frac{1}{2}\vec{p}(t)\cdot\vec{v}(t) \tag{2.21}$$

while, for $m \neq M$ such relations are not valid [1].

Finally, we come to the most important point of this section, namely, to the discussion of the transformation rule for the total energy E of bodies which, contrary to the kinetic energy, is a conserved quantity and therefore in each reference frame it should satisfy the conservation law

$$\frac{dE}{dt} = 0 \tag{2.22}$$

From this conservation law it is immediately seen that the total energy E cannot transform like the kinetic energy does because it would imply that

$$\frac{dE'}{dt'} = \frac{dE}{dt} + R\vec{F}\cdot\vec{u} \tag{2.23}$$

and for $\vec{F} \neq 0$ we cannot simultaneously satisfy the conservation law (2.22) in all reference frames. It means,contrary to a common belief that in the standard mechanical relation

$$E = T(t) + V(t) \tag{2.24}$$

the potential energy $V(t)$ cannot be a scalar with respect to the Galilean transformations because it would give the total energy E the same transformation rule as the kinetic energy T has. But we have just shown that this contradicts the conservation law for the total energy. The transformation rule for the potential energy must compensate the "wrong" transformation rule for the kinetic energy.

The problem however is that we do not know the transformation rule for V as well as for E. The explicit expressions of the potential energy as functions of the position $\vec{x}\,(t)$ all have noncovariant meaning and therefore may be valid only in one inertial reference frame. For that reason we shall use the relation (2.24) only in one reference frame as verification of the correctness of our construction of the total energy.

The conservation law (2.22) does not give any hint neither on the transformation rule for the total energy nor on its form. In our construction we shall require that the total energy should satisfy the following conditions:

i) it should be conserved in all reference frames,

ii) in the absence of the interaction it should coincide with the kinetic energy of the free motion,

iii) in one chosen reference frame it should coincide with the usual expression (2.24).

Unfortunately, these requirements do not allow to give a general construction of the total energy. But in each case, as we shall show below, they are enough to find E. Having constructed the total energy we may look for its transformation properties under the Galilean transformations. The examples considered by us below allow to see that under the Galilean transformations the total energy transforms according to the rule

$$E \to E' = E + R\vec{P}_0\cdot\vec{u} + \frac{1}{2}\mu\vec{u}^{\,2} \tag{2.25}$$

where $\vec{P}_0$ is a fixed, time independent momentum given by the formula depending on the type of the interaction and on the initial conditions. The mass parameter μ may not be equal to the mass of the particle. In this way we see that the transformation rule for the total energy, contrary to the transformation rule for

kinetic energy, is different for different acting forces. For a free motion the momentum $\vec{P}_0$ is equal to the constant momentum of the particle and, as expected, (2.25) coincides with (2.20). For all interacting particles the transformation rule (2.25) is essentially different from the rule (2.20). This fundamental fact of different Galilean transformation rules for the kinetic and total energies was overlooked during the whole history of mechanics.

The total energy E is a global quantity. If it had the transformation rule (2.20) it will be, together with the local quantity $\vec{p}(t)$, a member of the same geometrical object - the so called five-vector - with components $(E, \vec{p}(t), M)$ and obviously it is impossible. The argument that the constant mass M enters the five-vector $(T, \vec{p}(t), M)$ composed from local quantities $T(t)$ and $\vec{p}(t)$ does not show any contradiction because the mass always is a local quantity which is constant only in the simplest case. On the contrary, the total energy is always global quantity which depends only on initial conditions and as such it cannot form the same geometrical object with local quantities which varies during the motion.

The statement that the transformation properties of the global quantities of mechanics are different for each type of interaction becomes stronger on the example of the action integral. For free motion the action is given by

$$S = \frac{m(\vec{x}_2 - \vec{x}_1)^2}{2(t_2 - t_1)} \tag{2.26}$$

where $\vec{x}_2$ is the final position of the particle at time t_2 while $\vec{x}_1$ is its initial position at time t_1. From (2.26) and transformation rules (2.1) and (2.2) it is easy to derive that under Galilean transformations we have

$$S \to S' = S + mR(\vec{x}_2 - \vec{x}_1) \cdot \vec{u} + \frac{m}{2} \vec{u}^{\,2}(t_2 - t_1) \tag{2.27}$$

It means that the action integral S divided by the mass m, the distance $\vec{x}_2 - \vec{x}_1$

passed by the particle and the time duration of the motion $t_2 - t_1$ form a five vector $(\frac{S}{m}, \vec{x}_2 - \vec{x}_1, t_2 - t_1)$. The invariant length of this five vector is equal to

$$\Delta = \frac{S}{m}(t_2 - t_1) - (\vec{x}_2 - \vec{x}_1)^2 \tag{2.28}$$

Unfortunately, all the action integrals for interacting particles have noncovariant meaning because they are calculated with the use of non-covariant expressions for the potential energies. For this reason we cannot explicitly write down their transformation rules. But if action integrals for interacting particles will have the same transformation properties as the action integral for the free motion has then they will also form five-vectors with the corresponding distance $\vec{x}_2 - \vec{x}_1$ passed by the particle and time interval $t_2 - t_1$. So the invariant length of these five-vectors will always be given by (2.28) with possible different values of Δ. It follows from here that the action integrals will always be of the form

$$S = m\,\frac{(\vec{x}_2 - \vec{x}_1)^2}{2(t_2 - t_1)} + S_0 \tag{2.29}$$

where

$$S_0 = \frac{m\Delta}{2(t_2 - t_1)} \tag{2.30}$$

is some invariant constant. But this means that the action integrals for any interaction will always essentially coincide with the free action integral which is obviously not true. Therefore, our assumption on the transformation properties of the action integrals for the interacting particles turns out to be false. Whatever the action integrals for the interacting particles are they should lead to Galilean invariants different from (2.28). The invariant (2.28) contains the space-time notions

of distance and time duration and, as a matter of fact it is the Galilean analogue of the relativistic space-time interval. Therefore, if for the interacting particles we have to use different space-time intervals, this means that the entire structure of space-time for the interacting particles, at least for the Galilean space-time, should be described by invariants different from (2.28) and only for the free motion we may have the simple quadratic interval. For all other cases the intervals should be more complicated functions of the space-time coordinates. We hope to come back to this fascinating problem in future.

III. The Galilean covariant specification of forces.

The relations (2.6) and (2.9) are the basic equations of the Newton's mechanics. To obtain a particular system of differential equations for the trajectory $\vec{x}(t)$ we must add to them some additional relations between the basic mechanical quantities. Additional relations, called constitutive relations of the theory, have less general meaning than the equations (2.6) and (2.9) have, because they are valid only for particular objects and their forms depend on the character of the objects. One kind of constitutive relations is provided by the relation between momentum and velocity which, under the assumed equality of the Galilean and inertial masses, is of the form (2.7). It contains the mass m of the particle as a parameter and for a given value of m the relation (2.7) is valid only for the particles which mass equals m. Another kind of constitutive relations is given by the rules which specify the acting forces either as given vector valued functions of time or express the forces in terms of vector valued functions $\vec{x}(t)$ and $\vec{v}(t)$. In the first case we have to do with the so called external forces because their time dependence is independent from time evolution of the system and is regulated by external sources. In order to have a Galilean covariant formulation of mechanics

we must specify the parameters of the functions representing the external forces in such a way that the transformation rule (2.10) is satisfied. For example, the time independent force should be represented by a frame dependent constant vector $\vec{F}_0$ which under Galilean transformations transforms according to the rule

$$\vec{F}_0 \to \vec{F}_0' = R\vec{F}_0 \tag{3.1}$$

On the contrary, for the widely used periodic driving force

$$\vec{F}(t) = \vec{F}_0 \sin\omega t \tag{3.2}$$

it is not possible to satisfy the transformation rule (2.10). This is just an example of introducing into the theory the non-covariant representation of the acting force. Solving the equations of motion with such a force we shall not get a trajectory $\vec{x}(t)$ with the required transformation rule (2.3). In order to save the situation for periodic external forces we must use the representation

$$\vec{F}(t) = \vec{F}_0 \sin(\omega t + \varphi) \tag{3.3}$$

where $\vec{F}_0$ transforms according to (3.1) and φ transforms under the Galilean transformations according to the rule

$$\varphi \to \varphi' = \varphi - \omega b \tag{3.4}$$

Clearly, we may choose $\varphi = 0$ only in one reference frame but not in all. Similarly, for other functions of time representing the external forces we must always

remember about using such representations of these functions for which the transformation rule (2.10) may be satisfied. Since this is rather obvious we do not continue to discuss it here.

In the second case of constitutive relations for forces, called the case of internal forces, it is customary to use non-covariant relations. For example, in the best known case of harmonic oscillator we use the relation

$$\vec{F}(t) = -k\vec{x}(t) \tag{3.5}$$

where k is the elasticity constant of the oscillator. The non-covariant character of such a relation is obvious if one compares the transformation rules (2.3) and (2.10). Therefore, the relation (3.5) is valid only in one particular reference frame in which the so-called center of the oscillator is at rest. The standard way of making the relation (3.5) Galilean covariant is to take into account the motion of the oscillator center. If this motion in an arbitrary reference frame is described by the trajectory $\vec{x}_c(t)$ then instead of (3.5) we should use he relation

$$\vec{F}(t) = -k[\vec{x}(t) - \vec{x}_c(t)] \tag{3.6}$$

Since $\vec{x}(t)$ and $\vec{x}_c(t)$ transform according to the rule (2.3) their difference transforms just like the force should do. Therefore, the right-hand side of (3.6) and not that of (3.5) is the correct representation for the force acting on the Galilean covariant oscillator. We should like to stress here the fact that the individual terms in (3.6) do not have the meaning of force. Only the difference on the right-hand side of (3.6) has a physical meaning of a force and it is erroneously to think that (3.6) is a difference of two forces. Such treatment will immediately lead to a contradistinction with the second law of mechanics because in the inertial reference

frame, in which the center of the oscillator is at rest, the second term in (3.6) vanishes and consequently in this reference frame it does not cause any acceleration. The magnitude of the acceleration is however the same in all reference frames and therefore if it vanishes in one reference frame it is identically zero in all reference frames. From this it follows that the second term in (3.6) does not represent any physical force.

The price which we have paid for getting the right Galilean transformation property for the oscillator force is additional time dependence of this force introduced by the time dependent vector $\vec{x}_c(t)$. Since every nonzero function of time contains its own parameters we introduce in fact into the theory some amount of new parameters which describe new degrees of freedom of the oscillator. In the standard non-covariant treatment of the oscillator all such parameters and therefore all such degrees of freedom are disregarded and the price for the gained comfort is the lack of the Galilean covariance. Since Galilean covariance is the fundamental symmetry of the whole low energy physics we consider this price as too high.

Quite similarly, as for the harmonic oscillator, we may covariantize all other forces. In any case it is sufficient to find some characteristic reference point, like the center of the oscillator, introduce its trajectory $\vec{x}_c(t)$ and replace the function $\vec{x}(t)$ by the difference $\vec{x}(t) - \vec{x}_c(t)$. For example, for the anharmonic oscillator, instead of using the relation

$$\vec{F}(t) = \lambda[\vec{x}^{\,2}(t)]^n \vec{x}(t) \tag{3.7}$$

we may use the relation

$$\vec{F}(t) = \lambda([\vec{x}(t) - \vec{x}_c(t)]^2)^n [\vec{x}(t) - \vec{x}_c(t)] \tag{3.8}$$

On that example we see that such a covariantization of the nonlinear forces is not unique, because instead of choosing a single reference point, described by the trajectory $\vec{x}_c(t)$, we may in fact choose $(n+1)$ different reference points, described by trajectories $\vec{x}_j(t)$, and replace (3.7) by the relation

$$\vec{F}(t) = \lambda \prod_{j=1}^{n} [\vec{x}(t) - \vec{x}_j(t)]^2 [\vec{x}(t) - \vec{x}_{n+1}(t)] \tag{3.9}$$

The same serious non-uniqueness exists for velocity dependent forces for which we must also replace each velocity $\vec{v}(t)$ by some difference $\vec{v}(t) - \vec{v}_0(t)$, where $\vec{v}_0(t)$ is some reference velocity which need to be physically specified.

It is doubtful whether the situation described above may satisfy anybody. For each force we must invent its own way to make it Galilean covariant, we must give physical interpretation to new introduced parameters, we cannot use the standard canonical formalism because we have to do with time dependent and in general nonpotential forces. In addition, we never have to do with a one particle problem because we need to introduce into the theory the trajectories of the reference points. Such trajectories cannot be treated dynamically because we neither know the masses associated with the reference points nor the forces acting on them. Therefore, there is a strong need to find a new approach to the Galilean covariant classical mechanics.

The main motivation for our new approach comes from the following analogy with classical electrodynamics. The great success of this theory has its roots in the fact that this theory organized the complicated interparticle electromagnetic interactions in many particle systems in terms of a single notion - the electromagnetic field. In terms of this field it is possible to single out an individual particle from any many particle system and formulate for it a consistent single particle dynamical theory in which all acting forces are given by the field. The field by

itself is specified by a set of differential equations - the Maxwell equations - and knowing the sources of the fields we may formulate a complete theory of charged particles. Guided by this analogy we shall take, as our main assumption, the idea that not only electromagnetic field but all forces in Nature are specified by systems of fundamental differential equations which determine their space and time development. In mechanics we have to deal with forces which develop only in time and according to our main idea we assume that each mechanical force $\vec{F}_j(t)$ in the Galilean covariant mechanics is given by a fundamental equation of the type

$$\begin{aligned} a\ddot{\vec{F}}_j(t) + b\dot{\vec{F}}_j(t) + c\vec{F}_j(t) = &\vec{\varrho}(\vec{F}_1(t), \vec{F}_2(t), \ldots; \dot{\vec{F}}_1(t), \dot{\vec{F}}_2(t), \ldots; \\ &\vec{x}_1(t), \vec{x}_2(t), \ldots; \vec{v}_1(t), \vec{v}_2(t), \ldots) \end{aligned} \tag{3.10}$$

where the left hand-side, with specified coefficients a, b and c expresses the linear part of the constitutive relations which are necessary for the specification of forces and the right-hand side contains all information on nonlinear part of these constitutive relations. In (3.10) $\vec{\varrho}_j$ is a local function of its arguments when no "memory" effects are involved. Otherwise we should use functionals which will describe all retardations in the system.

We have indicated on the right-hand side of (3.10) the fact that $\vec{\varrho}_j$ may depend not only on all remaining forces acting in the system but also on trajectories and velocities of all particles belonging to the system. To ensure the correct transformation properties of each force $\vec{F}_j(t)$ it is necessary that the vector valued functions or functionals $\vec{\varrho}_j$ transform according to the rule

$$\vec{\varrho}'_j(\cdots') = (R\vec{\varrho})_j(\cdots) \tag{3.11}$$

where $(\cdots')$ and $(\cdots)$ denotes all the arguments of $\vec{\varrho}_j$ in the primed and unprimed

reference frames respectively.

For one particle problems we have only one trajectory and one velocity and due to their transformation properties they cannot be present among the arguments of $\vec{\varrho}_j$. In this case (3.10) provides a closed system of differential equations for each force $\vec{F}_j(t)$ with the right-hand sides depending only on the forces themselves. Solving these equations and giving the right transformation properties to the arising integration constants we automatically shall get each force $\vec{F}_j(t)$ as a Galilean covariant function of time, i.e. with the transformation property (2.10). This fact is just the essence of our new approach to a Galilean covariant mechanics.

For many particle problems, the right-hand side of (3.10) may depend on the differences $\vec{x}_j(t) - \vec{x}_k(t)$ and $\vec{v}_j(t) - \vec{v}_k(t)$ for some j and k. In this case (3.10) may even reduce to the usual form of the constitutive relations which directly express the forces in terms of the coordinates. In particular, for two particles interacting through an oscillator force we have the relations

$$\begin{aligned} \vec{F}_1(t) &= k[\vec{x}_1(t) - \vec{x}_2(t)] \\ \vec{F}_2(t) &= k[\vec{x}_2(t) - \vec{x}_1(t)] \end{aligned} \tag{3.12}$$

and these relations perfectly lie in the class of relations of the type (3.10) for which $a = b = 0$. We shall however prefer to work with differential equations of forces because this approach makes mechanics more similar to all field theories in which quantities connected with interaction are determined from differential equations which describe the space-time propagation of the interaction. The forces in our approach acquire dynamical character and realize degrees of freedom independent from spatial ones. To specify the state of the system at a given instant of time we must specify not only the positions and velocities of the particles at that time but also the forces acting on the particles and their time derivatives at the same time.

This is a consequence of determining the forces from differential equations rather than from non-differential constitutive relations. The integration constants which appear in the process of solution of the differential equations for forces should be specified in the same way as the integration constants for the trajectories in the standard approach. This means that these constants should be specified by the initial conditions for the forces and their time derivatives. Since we give the correct Galilean transformation properties to initial conditions and the equations (3.10) are Galilean covariant, we always get as a result forces with the right transformation rules. This is the main advantage of our approach over the standard one. Moreover, our approach is universal and does not distinguish the internal and external forces. We consider such distinction as artificial one because a physical body contrary to animated bodies feels only the force and not its sources. This point of view is in agreement with the electromagnetic point of view according to which the charge feels the electromagnetic field and not the sources of the fields. The sources are necessary to create the field just like the right-hand sides of (3.10) determines all acting forces.

Finally we would like to note that our approach is not orthogonal to the standard one because we may use the latter as a hint to derive the equations (3.10). For external forces it is sufficient to know the differential equation satisfied by the functions of time which represent the forces. For internal forces we may invert the standard constitutive relation

$$\vec{F} = \vec{F}(\vec{x}(t), t) \tag{3.13}$$

to get the relation

$$\vec{x}(t) = \vec{x}(\vec{F}, t) \tag{3.14}$$

and using the usual equation of motion

$$m\ddot{\vec{x}} = \vec{F} \tag{3.15}$$

we may derive the corresponding equation for the force. The resulting equations for the forces may have complicated form. For example, for the case (3.7) we get the equation

$$\ddot{\vec{F}}(t) + \frac{4n(3n+1)}{(2n+1)^2}\left[\frac{\vec{F}\cdot\dot{\vec{F}}(t)}{\vec{F}^2(t)}\right]^2\vec{F} - \frac{2n}{2n+1}\frac{\dot{\vec{F}}(t)\cdot\dot{\vec{F}}(t)}{\vec{F}^2(t)}\vec{F} -$$

$$-\frac{2n}{2n+1}\frac{\vec{F}\cdot\ddot{\vec{F}}(t)}{\vec{F}^2(t)}\vec{F} - \frac{4n}{2n+1}\frac{\vec{F}\cdot\dot{\vec{F}}(t)}{\vec{F}^2(t)}\dot{\vec{F}}(t) - \frac{\lambda}{m}\left[\frac{\vec{F}^2(t)}{\lambda^2}\right]^{\frac{n}{2n+1}}\vec{F} = 0 \tag{3.16}$$

which has very complicated nonlinear structure and simplifies a little only for $n = 0$ which is the case of the harmonic oscillator considered in the next section and for $n = -\frac{1}{3}$ which corresponds to the Kepler problem which we shall consider in a separate paper.

IV. The Galilean covariant classical harmonic oscillator.

Before going further, we shall discuss the example of the harmonic oscillator, which is the most extensively studied physical system [10] because it serves as the first approximation to all periodic dynamics with positive energy, independently on the nature of the system. The more complicated non-linear dynamics we shall discuss in separate papers.

A particular relativistic generalization of the harmonic oscillator problem has been considered in [11]. In the limit $c \to \infty$ (c-velocity of light) it reproduces the

standard non-relativistic oscillator. But, as it is well known [12], the limit $c \to \infty$ of the relativistic physics should reproduce the Galilean covariant physics and not the standard non-relativistic one. It is therefore quite important to have Galilean covariant models of the harmonic oscillator because they may, in particular, serve as correct etalons for testing relativistic physics in the limit $c \to \infty$.

As we have already mentioned in the previous section, in the standard way of making the harmonic oscillator Galilean covariant, we explicitly introduce into the consideration the trajectory $\vec{x}_c(t)$ of the center of the oscillator. If there exists an inertial reference frame in which such center is at rest the trajectory $\vec{x}_c(t)$ is a linear function of time, i.e. we have

$$\vec{x}_c(t) = \vec{A} + \vec{B}t \tag{4.1}$$

where the constant vectors $\vec{A}$ and $\vec{B}$ which determine the trajectory are the frame dependent with the following Galilean transformation rules

$$\vec{A} \to \vec{A}' = R\vec{A} - (R\vec{B})b - \vec{u}b + \vec{a} \tag{4.2}$$

$$\vec{B} \to \vec{B}' = R\vec{B} + \vec{u} \tag{4.3}$$

From this transformation rules it is clear that the conditions $\vec{A} = \vec{B} = 0$ and therefore $\vec{x}_c(t) = 0$ may be satisfied only in one reference frame. The constants $\vec{A}$ and $\vec{B}$ are usually expressed by the initial conditions for the center of the oscillator and we may do this in two ways: either to fix the position $\vec{X}_0$ and the velocity $\vec{V}_0$ of the center at some instant of time T_0 or to fix the positions $\vec{X}_1$ and $\vec{X}_2$ of the center at two different instances T_1 and T_2. In the first case we obtain

$$\vec{A} = \vec{X}_0 - \vec{v}_0 T_0 \tag{4.4}$$

$$\vec{B} = \vec{V}_0 \tag{4.5}$$

while in the second case we have

$$\vec{A} = \frac{\vec{X}_1 T_2 - \vec{X}_2 T_1}{T_2 - T_1} \tag{4.6}$$

$$\vec{B} = \frac{\vec{X}_2 - \vec{X}_1}{T_2 - T_1} \tag{4.7}$$

The trajectory (4.1) has then the form

$$\vec{x}_c\,(t) = \vec{X}_0 - \vec{v}_0(t - T_0) \tag{4.8}$$

for the first case and

$$\vec{x}_c\,(t) = \frac{\vec{X}_1 T_2 - \vec{X}_2 T_1}{T_2 - T_1} + \frac{\vec{X}_1 T_2 - \vec{X}_2 T_1}{T_2 - T_1}\, t \tag{4.9}$$

for the second case. The transformation rules (4.1) and (4.2) are now guaranteed by the Galilean transformation rules for the positions $\vec{X}_i$ and times T_i $(i = 0, 1, 2)$ and by the transformation rule for the velocity $\vec{V}_0$. We should however note that the trajectory $\vec{x}_c\,(t)$ is determined by six frame dependent constants constituting the vectors $\vec{A}$ and $\vec{B}$ while using the expressions (4.4) - (4.7) we introduce into the theory more parameters, namely, seven parameters in (4.4) - (4.5) and eight in (4.6) - (4.7). This new parameters are redundant and physically correspond to the freedom of fixing the initial data at different instants of time for the same trajectory.

The general solution of the oscillator equation of motion

$$m\frac{d^2\vec{x}\,(t)}{dt^2} = -k[\vec{x}\,(t) - \vec{x}_c\,(t)] \tag{4.10}$$

has the form

$$\vec{x}\,(t) = \vec{A} + \vec{B}t + \vec{C}\sin\omega t + \vec{D}\cos\omega t \tag{4.11}$$

where, as usually

$$\omega^2 = \frac{k}{m} \tag{4.12}$$

and $\vec{C}$ and $\vec{D}$ are frame dependent integration constants with the following Galilean transformation rules

$$\vec{C} \to \vec{C}' = R\vec{C}\cos\omega b + R\vec{D}\sin\omega b \tag{4.13}$$

$$\vec{D} \to \vec{D}' = -R\vec{C}\sin\omega b + R\vec{D}\cos\omega b \tag{4.14}$$

$\vec{C}$ and $\vec{D}$ be fixed from the initial or final conditions for the oscillator. Fixing the oscillator position $\vec{x}_0$ and velocity $\vec{v}_0$ at the instant t_0 we get

$$\vec{x}\,(t) = \vec{A} + \vec{B}t + (\vec{x}_0 - \vec{A} + \vec{B}t_0)\cos\omega(t - t_0) + \frac{\vec{v}_0 - \vec{B}}{\omega}\sin\omega(t - t_0) \tag{4.15}$$

where for $\vec{A}$ and $\vec{B}$ we should substitute either (4.4)-(4.5) or (4.6) - (4.7). Fixing the positions of the oscillator at two times t_1 and t_2 we get

$$\vec{x}\,(t) = \vec{A} + \vec{B}t + \frac{(\vec{x}_1 - \vec{A} + \vec{B}t_1)\sin\omega(t - t_2) - (\vec{x}_2 - \vec{A} + \vec{B}t_2)\sin\omega(t - t_1)}{\sin\omega(t_1 - t_2)} \tag{4.16}$$

where again we have two possibilities of expressing the constants $\vec{A}$ and $\vec{B}$.

Discussed above standard approach to the problem hardly may be called one particle problem because the solution of the equation of motion for the oscillator always depends on the parameters of another point, the center of the oscillator. This point however cannot be treated as a material point because we do not know its mass and the forces acting on it. In addition among the parameters of the general solution (4.11) one pair $\vec{A}$ and $\vec{B}$ has been introduced by the constitutive relation (3.6) while the other pair $\vec{C}$ and $\vec{D}$ appears as integration constants. This disadvantages are absent in the new approach proposed in the previous section in which we complete the equations of motion not by the constitutive relations but by differential equations for the acting forces. For the harmonic oscillator it is easy to see that the acting force $\vec{F}(t)$ satisfies the differential equation

$$\ddot{\vec{F}}(t) + \omega^2 \vec{F}(t) = 0 \tag{4.17}$$

and we may take this equation as our starting point. Solving it we get

$$\vec{F}(t) = \vec{\alpha} \sin \omega t + \vec{\beta} \cos \omega t \tag{4.18}$$

where the integration constants are fixed from the value of the force $\vec{F}_0$ and its time derivative $\vec{G}_0$ at some instant t_0. Doing this we get

$$\vec{F}(t) = \vec{F}_0 \cos \omega (t - t_0) + \frac{\vec{G}_0}{\omega} \sin \omega (t - t_0) \tag{4.19}$$

Since under Galilean transformations $t - t_0$ is invariant and

$$\begin{aligned} \vec{F}_0 &\to \vec{F}_0' = R\vec{F}_0 \\ \vec{G}_0 &\to \vec{G}_0' = R\vec{G}_0 \end{aligned} \tag{4.20}$$

we automatically obtain the correct transformation rule for the force $\vec{F}(t)$. Substituting (4.19) into the equation of motion

$$m \frac{d^2 \vec{x}(t)}{dt^2} = \vec{F}(t) \tag{4.21}$$

we get the solution in the form

$$\vec{x}(t) = \vec{\gamma} + \vec{\delta} t + \frac{\vec{F}_0}{k} \cos \omega (t - t_0) + \frac{\vec{G}_0}{k\omega} \sin \omega (t - t_0) \tag{4.22}$$

where $\vec{\gamma}$ and $\vec{\delta}$ are some another integration constants. Fixing them from the initial position and velocity of the oscillator at the some instant t_0 we finally get

$$\vec{x}(t) = \left(\vec{v}_0 + \frac{1}{k} \vec{G}_0 \right) t_0 + \frac{\vec{F}_0}{k} + \left(\vec{v}_0 + \frac{1}{k} \vec{G}_0 \right) t - \frac{\vec{F}_0}{k} \cos \omega (t - t_0) - \frac{\vec{G}_0}{k\omega} \sin \omega (t - t_0) \tag{4.23}$$

where all parameters refer solely to the oscillator. We stay therefore strictly in the framework of one particle theory. It is also easy to check that the solution (4.23) has the correct transformation rule (2.3) under the Galilean transformations.

Our new approach to the covariant mechanics does not require the notion of the center of the oscillator. Instead of that we may introduce the notion of the average motion of the oscillator given in terms of the trajectory $\vec{x}_{av}(t)$ defined by the formula

$$\vec{x}_{av}(t) = \frac{\omega}{2\pi} \int_t^{t + \frac{2\pi}{\omega}} \vec{x}(\tau)\, d\tau = \vec{x}_0 + \frac{\vec{F}_0}{k} + \left(\vec{v}_0 + \frac{\vec{G}_0}{k} \right) \left(\frac{\pi}{\omega} - t_0 \right) + \left(\vec{v}_0 + \frac{\vec{G}_0}{k} \right) t \tag{4.24}$$

The point which moves along such a trajectory is a hypothetical not a real one and there is no need to ascribe to it any mass. Comparing the shape of the force (4.19) with the shape of the trajectory (4.23) we may write several forms of covariant constitutive relations between the force and position. For example, we have the relations

$$\vec{F}(t) = -k\vec{x}(t) + \vec{F}_0 + k\vec{x}_0 + k\left(\vec{v}_0 + \frac{\vec{G}_0}{k}\right)(t - t_0) \tag{4.25}$$

or

$$\vec{F}(t) - \vec{F}(t_0) = -k\left[\vec{x}(t) - \vec{x}(t_0) - \left(\vec{v}_0 + \frac{\vec{G}_0}{k}\right)(t - t_0)\right] \tag{4.26}$$

all the above relations take particularly simple forms in the reference frame in which

$$\vec{x}_0 = \frac{\vec{F}_0}{k} \tag{4.27}$$

$$\vec{v}_0 = \frac{\vec{G}_0}{k} \tag{4.28}$$

because in that frame we have

$$\vec{x}(t) = \vec{x}_0 \cos\omega(t - t_0) + \frac{\vec{v}_0}{\omega}\sin(t - t_0) \tag{4.29}$$

$$\vec{x}_{av}(t) = 0 \tag{4.30}$$

$$\vec{F}(t) = -k\vec{x}(t) \tag{4.31}$$

Clearly these are the formulae known from the non-covariant oscillator. Therefore, our approach allowed to characterize the particular reference frame which should

be used to simplify the covariant approach in such a way that we may use the formulae known from the non-covariant formalism. This is the frame in which the initial conditions (4.27) are (4.28) are satisfied.

As we have already mentioned there is one exception in using the non-covariant expressions. This is the case of the usual expressions for the total energy for which we do not know the transformation rule. According to our philosophy we shall search the expression for the total energy in the form

$$E = \alpha\vec{p}^{\,2} + \beta\vec{x}^{\,2} + \gamma\vec{F}^{\,2} + \delta\vec{G}^{\,2} + \varepsilon(\vec{x}\cdot\vec{p}) + \varphi(\vec{x}\cdot\vec{F}) + \lambda(\vec{x}\cdot\vec{G}) + \mu(\vec{p}\cdot\vec{F}) + \eta(\vec{p}\cdot\vec{G}) + \xi(\vec{F}\cdot\vec{G}) \tag{4.32}$$

The conservation law (2.22) requires

$$\beta = \varepsilon = \phi = \lambda = \mu = \xi = 0$$

$$2\alpha - \omega^2\lambda = 0 \tag{4.33}$$

$$2\gamma - 2\delta\omega^2 + \lambda = 0$$

and we get

$$E = \alpha\left(\vec{p}^{\,2} + \frac{2\vec{p}\cdot\vec{G}}{\omega^2} + \frac{\vec{G}^{\,2}}{\omega^4}\right) + \gamma\left(\vec{F}^{\,2} + \frac{\vec{G}^{\,2}}{\omega^2}\right) \tag{4.34}$$

Now this expression for the free case, i.e. for $\vec{F}(t) = \vec{G}(t) = 0$ should reduce to the usual expression (2.21) for the kinetic energy. This will happen if

$$\alpha = \frac{1}{2m} \tag{4.35}$$

Since in (4.34) the term proportional to γ is Galilean invariant we may easily derive the Galilean transformation rule for the energy given by (4.34). Using (2.10) and (2.15) we get

$$E \to E' = E + R\left(\vec{p} + \frac{\vec{G}}{\omega^2}\right) \cdot \vec{u} + \frac{1}{2} m \vec{u}^{\,2} \tag{4.36}$$

where we introduced the momentum

$$\vec{P} = \vec{p} + \frac{\vec{G}}{\omega^2} \tag{4.37}$$

which is constant in time because of Newton's law and definition of $\vec{G}_0$. Therefore the expression (4.34) is conserved in all reference frames. Comparing the transformation rule (4.36) with that for the kinetic energy (2.20) we explicitly see the difference in the transformation rules for the total and kinetic energies. This proves our statement that the total and kinetic energies have different transformation rules.

Substituting the solutions (4.19) and (4.23) into (4.34) we obtain

$$E = \frac{m}{2}\left(\vec{v}_0 + \frac{\vec{G}_0}{k}\right)^2 + \gamma\left(\vec{F}_0^{\,2} + \frac{\vec{G}_0^2}{\omega^2}\right) \tag{4.38}$$

and in order to get in the reference frame specified by (4.27) and (4.28) the usual expression for the total energy

$$E = \frac{m}{2}\left(\vec{v}_0^2 + \omega^2 \vec{x}_0^2\right) \tag{4.39}$$

we must choose

$$\gamma = \frac{1}{2m\omega^2} = \frac{1}{2k} \tag{4.40}$$

It is interesting to see that the value (4.39) is not the lowest value of the total energy (4.38). Minimal value of E is reached for

$$\vec{G}_0 = -\frac{k}{2}\vec{v}_0 \tag{4.41}$$

and if in addition (4.27) is satisfied we get

$$E_{min} = \frac{m}{2}\left(\frac{\vec{v}_0}{2} + \frac{\vec{G}_0}{k}\right)^2 + \gamma\left(\vec{F}_0^{\,2} + \frac{\vec{G}_0^2}{\omega^2}\right) \tag{4.42}$$

V. The case of constant force.

Before passing to more complicated interactions we shall briefly consider the case of a constant force. In our approach this particular problem is specified by the equation

$$\dot{\vec{F}}(t) = 0 \tag{5.1}$$

with an obvious solution

$$\vec{F}(t) = \vec{F} = \overrightarrow{\text{const}} \tag{5.2}$$

From the equation of motion we get then the following expression for the function $\vec{x}(t)$ which describes the trajectory

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0(t - t_0) + \frac{\vec{F}}{2m}(t - t_0)^2 \tag{5.3}$$

where m is particle mass and $\vec{x}_0$ and $\vec{v}_0$ are the initial position and velocity measured at time t_0. The solution (5.3) obviously has the transformation property

specified by (2.3) provided we shall correctly transform the initial data $\vec{x}_0$ and $\vec{v}_0$ taken at the instant t_0 and the force $\vec{F}$.

In all applications of the considered case we always assume that on each trajectory there exists a point with the space-time coordinates $(\vec{X}, T)$ at which the potential energy vanishes. Taking this into account we have the following expression for the total energy

$$E = \frac{\vec{p}^2(t)}{2m} - \vec{F} \cdot \left[\vec{x}\,(t) - \vec{X}\right] \tag{5.4}$$

which satisfies all the requirements stated in section II. The presence of the term $\vec{F} \cdot \vec{X}$ in (5.4) is however not only a matter of convenience but it is necessary for the cancellation of the unwanted term $R\vec{F} \cdot \vec{a}$ which otherwise would arise from the transformation of the function $\vec{x}\,(t)$ under space translation. This term does not introduce any additional time dependence because the coordinates $(\vec{X}, T)$, contrary to the previously considered coordinates of the center of the oscillator which in each reference frame has at least variable time coordinate, are coordinates of a fixed event and consequently the space coordinate $\vec{X}$ transforms as

$$\vec{X} \to \vec{X}' = R\vec{X} + \vec{u}T + \vec{a} \tag{5.5}$$

so the energy (5.4) obeys the transformation rule

$$E \to E' = E + R\left[\vec{p}(t) + \vec{F}(T-t)\right] \cdot \vec{u} + \frac{1}{2}m\vec{u}^{\,2} \tag{5.6}$$

Comparing this transformation rule with that for the total energy of the harmonic oscillator given by (4.35) we explicitly see the dependence of the transformation rule for the total energy on the type of interaction. This proves our statement claimed in the Introduction.

The momentum

$$\vec{P} = \vec{p}(t) + \vec{F}(T-t) \tag{5.7}$$

present in the transformation rule (5.6) is indeed constant in time and equals to

$$\vec{P} = m\vec{v}_0 + \vec{F}(T-t_0) \tag{5.8}$$

Similarly, the value of the energy (5.4) is equal to

$$E = \frac{m\vec{v}_0^{\,2}}{2} + \vec{F} \cdot \left[\vec{X} - \vec{x}_0\right] \tag{5.9}$$

Here we would like to warn the reader against making any conclusion on the dependence of the total energy (5.4) on the acting force $\vec{F}$. Indeed, from (5.9) it seems that the energy depends at most linearly on $\vec{F}$ because all other terms depend only on the initial conditions or on the condition put on the potential. For $\vec{X} = \vec{x}_0$ we may even see that the total energy does not depend on $\vec{F}$. Fixing, instead of $\vec{x}_0$ and $\vec{v}_0$, the initial position of the particle at the point $\vec{x}_1$ at the time t_1 and its final position at $\vec{x}_2$ at the instant t_2 we shall get instead of (5.9) the following formula

$$E = \frac{m\vec{v}_{av}^{\,2}}{2} + \frac{\vec{v}_{av} \cdot \vec{F}}{2}(t_2 - t_1) + \frac{\vec{F}^2}{8m}(t_2 - t_1)^2 \tag{5.10}$$

where for simplicity we have chosen $\vec{X} = \vec{x}_2$ and

$$\vec{v}_{av} = \frac{\vec{x}_2 - \vec{x}_1}{t_2 - t_1} \tag{5.11}$$

denoting the average velocity of the particle. The expression (5.10) is evidently quadratic in $\vec{F}$.

To resolve the obtained paradox let us first note that the requirement of vanishing of the potential energy at some point $(\vec{X},T)$ is a part of the definition of the total energy and for different choices of that point we get in fact different definitions of the total energy. If the potential energy vanishes at some point the total energy is equal to the kinetic energy at this point and since kinetic energy varies on the trajectory we get different values for the total energy using different choices of $(\vec{X},T)$. In the special case when $\vec{X} = \vec{x}_0$ we arbitrarily fix both the position of the particle and its velocity which determines the kinetic energy at $\vec{x}_0$ but also simultaneously we put the potential energy to be zero at the same point $\vec{x}_0$. The total energy in this case is therefore completely determined by the value of $\vec{v}_0$ and cannot feel the acting force. Therefore for $\vec{X} = \vec{x}_0$ the total energy is independent on $\vec{F}$. In the case when $\vec{X} \neq \vec{x}_0$ we fix the kinetic energy at $\vec{x}_0$ while the potential energy is equal to zero at a different point $\vec{X}_0$. The total energy is now equal to the kinetic energy at the point $\vec{X}$ which during the time $T - t_0$ has changed from the value ascribed at t_0.

It is clear now that the total energy must depend on the acting force and this dependence is linear. It is so because the difference between the kinetic energy at $\vec{X}$ (which equals to the total energy) and the initial kinetic energy (which equals to the first term in (5.9)) is equal to the work done by the force.

The quadratic dependence on $\vec{F}$ in (5.10) arises from the fact that fixing the initial and final positions of the particle we do not determine the momentary velocity at some particular point on the trajectory and therefore we never fix the kinetic energy at some point. If the total energy has to be equal to the kinetic energy at the point $(\vec{X},T)$, the velocity at this point is equal to

$$\vec{v}_T = \vec{v}_{av} + \frac{\vec{F}}{2m}(2T - t_1 - t_2) \tag{5.12}$$

and squaring this velocity we just get the quadratic dependence on $\vec{F}$. Only in the special choice

$$T = \frac{t_1 + t_2}{2} \tag{5.13}$$

we again get the total energy independent from $\vec{F}$.

VI. Perturbation theory for nonlinear interactions.

As we have seen, the different equations for the acting forces may have , in general, a very complicated non-linear structure and it is hardly to believe that these equations may be exactly solved. We must therefore look for some approximations which simplify the problem.

The easiest way of simplifying complicated equations is to use perturbation theory. In the standard perturbation theory of mechanics we expand the functions $\vec{x}(t)$ describing the trajectories of particles into power series with respect to the coupling constants of non-linear interactions. Doing this we assume that each next term in this approximation is smaller in magnitude than the sum of all previous ones. In this sense we call the higher terms in the expansion to be perturbations of the lowest terms. The trouble, however, lies in the fact that in a Galilean covariant theory such a requirement of smallness cannot be always satisfied because the first term of the perturbation series always transforms according to the rule (2.3) while all the next terms of the perturbation series undergo only rotations. This is so because the Galilean transformations do not depend on the coupling constants of any interaction. We may therefore invariantly compare the magnitudes of all

terms next to the first one and order them according to their smallness. We cannot however invariantly compare the magnitudes of the next terms with respect to the magnitude of the first term of the perturbation series because the magnitude of the latter depends on the choice of the reference frame. Therefore, there is no invariant way of saying that the contribution of all terms higher than the first one is a perturbation of the first term. In this sense the usual perturbation theory of mechanics is not Galilean covariant.

The situation is quite different in our approach to mechanics because the equation which we want to simplify do not concern the functions describing trajectories but they concern functions describing the acting forces. We shall show in a moment that for forces it is always possible not only compare them in an invariant way but also we may invariantly and uniquely establish the notion of strong and weak forces. This applies to all forces which appear in a perturbation series for the complicated acting total force and, as a result, we get a well physically founded Galilean covariant perturbation theory of mechanics.

In fact, since under Galilean transformations every force undergoes only rotation, each term in the perturbation series

$$\vec{F}(t) = \sum_{n=1}^{\infty} \vec{F}_n(t) \tag{6.1}$$

transforms in the same way, including the first unperturbed one. Therefore, the relation

$$|\vec{F}_j(t)| < |\vec{F}_k(t)| \tag{6.2}$$

for all $j < k$, including $j = 0$, has the same meaning in all reference frames. In addition, in our approach, the forces are independent physical quantities for which

we freely may fix the initial conditions and if $\vec{F}_0(t_0)$ is the initial value of the unperturbed force $\vec{F}_0(t)$ (the first term in (6.1)) then, for all terms in (6.1), we may define an invariant measure of the relative magnitude

$$\delta\vec{F}_j = \sup_t \frac{|\vec{F}_j(t) - \vec{F}_0(t_0)|}{|\vec{F}_0(t_0)|} \tag{6.3}$$

If $\delta\vec{F}_j \ll 1$ for all $j \geq 1$ we may say that all the forces $\vec{F}_j(t)$ in (6.1) with $j \geq 1$ are small perturbations of the main unperturbed term $\vec{F}_0(t)$. In this sense we have a tool to divide all forces into the weak and strong ones in the scale of forces established by the initial force $\vec{F}_0(t_0)$.

There is still one question which needs clarification in any perturbation theory, namely, we must define in what sense the presence of perturbing forces influences the notion of the state of the system. In the absence of any other guess we shall assume for simplicity that the entire state of the perturbed system coincides with the state of the unperturbed one. This means that at any initial time t_0 the state of the particle is determined by the set $\left\{\vec{x}_0(t_0), \vec{p}_0(t_0), \vec{F}_0(t_0), \dot{\vec{F}}_0(t_0) = \vec{G}_0(t_0)\right\}$ where all quantities are the quantities taken from the zeroth term in the corresponding perturbation series. The perturbing forces are not seen at the initial instant t_0 but during the motion develop themselves in the system due to some complicated processes of self-induction and regulation. Thus for all $j \geq 1$ we must take as initial conditions the equalities

$$\vec{F}_j(t_0) = \dot{\vec{F}}_j(t_0) = 0 \tag{6.4}$$

Having defined the perturbation series for the acting force and giving to each force $\vec{F}_j(t)$ of this series its own physical meaning we may, instead of the single equation (2.9), consider the set of equations

$$\frac{d\vec{p}_j(t)}{dt} = \vec{F}_j(t) \tag{6.5}$$

and represent the momentum $\vec{p}(t)$ of the particle in the form of a series

$$\vec{p}(t) = \sum_{n=1}^{\infty} \vec{p}_j(t) \tag{6.6}$$

which is the perturbation series for $\vec{p}(t)$ where all terms $\vec{p}_j(t)$ with $j \geq 1$ satisfy the initial conditions

$$\vec{p}_j(t_0) = 0 \tag{6.7}$$

The term $\vec{p}_0(t)$ is the unperturbed momentum and according to tour assumptions satisfies the initial condition

$$\vec{p}_0(t_0) = m\vec{v}_0 \tag{6.8}$$

where $\vec{v}_0$ is the initial value of the velocity of the particle.

The series (6.6) gives the perturbation series of the velocity $\vec{v}(t)$ through the constitutive relation (2.7) and instead of the single equation (2.6) we may consider the set of equations

$$\frac{d\vec{x}_n(t)}{dt} = \vec{v}_n(t) \tag{6.9}$$

and obtain the perturbation series for the trajectory $\vec{x}(t)$ in the form

$$\vec{x}(t) = \sum_{j=0}^{\infty} \vec{x}_j(t) \tag{6.10}$$

where all terms $\vec{x}_j(t)$ with $j \geq 1$ satisfy the initial conditions

$$\vec{x}_j(t_0) = 0 \tag{6.11}$$

while

$$\vec{x}_0(t_0) = \vec{x}_0 \tag{6.12}$$

where $\vec{x}_0$ is the initial position of the particle.

To illustrate our perturbation approach let us consider two examples concerning a particle moving in a field of forces being the superposition of periodic forces $\vec{F}_j(t)$ which form a perturbation series. For simplicity we shall assume that the unperturbed force $\vec{F}_0(t)$ is the simple harmonic oscillator force which satisfies the equation (4.17) with the solution (4.19) where the frequency ω is now denoted by ω_0. If during the motion of the particle some anharmonic effects may appear we shall assume that a new force was induced which in the first approximation may be described by the equation

$$\begin{aligned}\ddot{\vec{F}}_1(t) + \omega_1^2 \vec{F}_1(t) = \lambda\left(\vec{F}_0{}^2(t), \vec{G}_0{}^2(t), \vec{F}_0(t)\cdot\vec{G}_0(t)\right)\vec{F}_0(t) + \\ + \mu\left(\vec{F}_0{}^2(t), \vec{G}_0{}^2(t), \vec{F}_0(t)\cdot\vec{G}_0(t)\right)\vec{G}_0(t)\end{aligned} \tag{6.13}$$

where ω_1 is a possible new frequency generated during the motion by the perturbation mechanism and the terms on the right-hand side of (6.13) are the general Galilean covariant terms which may be constructed from the unperturbed force. The choice of functions λ and μ determines the assumed particular model of anharmonic perturbations.

Similarly, in second approximation we shall assume that next perturbing force $\vec{F}_2(t)$ appears during the motion of the particle which may be associated with a possible new frequency ω_2 and which satisfies the equation

$$\ddot{\vec{F}}_2(t) + \omega_2^2 \vec{F}_2(t) = \lambda_0(\ldots)\vec{F}_0(t) + \mu_0(\ldots)\vec{G}_0(t) + \\ + \lambda_1(\ldots)\vec{F}_1(t) + + \mu_1(\ldots)\vec{G}_1(t) \tag{6.14}$$

where the functions λ_i and μ_i determine the model in the next approximation. The arguments of these functions are now taken from the set $\left\{ \vec{F}_0{}^2(t), \vec{G}_0{}^2(t), \vec{F}_1{}^2(t), \vec{G}_1{}^2(t), \vec{F}_0(t)\cdot\vec{G}_0(t), \vec{F}_0(t)\cdot\vec{F}_1(t), \vec{F}_0(t)\cdot\vec{G}_1(t), \vec{G}_0(t)\cdot\vec{F}_1(t), \vec{G}_0(t)\cdot\vec{G}_1(t) \right\}$.

The procedure may be continued and we get a sequence of forces and a sequence of characteristic frequencies which characterize the motion of the particle. We may terminate this sequence at the place at which the required accuracy in the description of the motion is achieved. We have therefore no problem with the convergence of the perturbation series because everything is understood in the sense of asymptotic calculus what means that we deal with sums of forces which form asymptotic series and the infinite asymptotic series may always be approximated by a finite sum of forces with an arbitrary given accuracy.

Let us now take the simplest model for which

$$\lambda = \text{const} \qquad \mu = 0 \tag{6.15}$$

which physically corresponds to the case when perturbing force $\vec{F}_1(t)$ in the first approximation is driven by the unperturbed force $\vec{F}_0(t)$. The general solution of the equation (6.13) which satisfies the initial conditions (6.4) is given by

$$\vec{F}_1(t) = \frac{\lambda}{\omega_1^2 - \omega_0^2}\left\{ \vec{F}_0 \left[\cos\omega_0(t-t_0) - \cos\omega_1(t-t_0)\right] + \right. \\ \left. + \frac{\vec{G}_0}{\omega_0\omega_1}\left[\omega_1 \sin\omega_0(t-t_0) - \omega_0 \sin\omega_1(t-t_0)\right] \right\} \tag{6.16}$$

The first corrections to the momentum and the trajectory have then the form

$$\vec{p}_1(t) = \frac{\lambda}{\omega_0\omega_1(\omega_1^2 - \omega_0^2)}\left\{ \vec{F}_0\left[\omega_1 \sin\omega_0(t-t_0) - \omega_0 \sin\omega_1(t-t_0)\right] + \right. \\ \left. + \frac{\vec{G}_0}{\omega_0\omega_1}\left[\omega_1^2 \cos\omega_0(t-t_0) - \omega_0^2 \cos\omega_1(t-t_0) + \omega_0^2 - \omega_1^2\right] \right\} \tag{6.17}$$

$$\vec{x}_1(t) = -\frac{\lambda}{\omega_0\omega_1(\omega_1^2 - \omega_0^2)}\left\{ \vec{F}_0\left[\omega_1^2 \cos\omega_0(t-t_0) - \omega_0^2 \cos\omega_1(t-t_0) + \omega_0^2 - \omega_1^2\right] + \right. \\ \left. + \frac{\vec{G}_0}{\omega_0\omega_1}\left[\omega_1^3 \sin\omega_0(t-t_0) - \omega_0^3 \sin\omega_1(t-t_0)\right] \right\} \tag{6.18}$$

It is easily seen that under the Galilean transformation all the vector-valued functions which appear in the first approximation undergo only rotation. This fact ensures the correct transformation rule of all quantities because the necessary inhomogeneous terms are already present in the zeroth approximation.

We are now ready to write down the expression for the total energy in the first approximation. Before doing that we note the kinetic energy $T(t)$ is always exactly given by the expression (2.21) and it is therefore a mixture of terms of different order of magnitude including terms of the second order of smallness. This is so because the right-hand side of the defining equation (2.16) is also such a mixture. At this point we would like to remember that in our approach we approximate only the acting forces while all the remaining relations of mechanics we take as exact

and we do not apply to them any further approximations. The only exception is just the expression for the total energy because the general scheme of mechanics does not provide a clear definition of this quantity. Since it is a conserved quantity and in order to prove that fact we must use the equations for acting forces which we only known up o the first order of perturbation theory, the expression for the total energy should contain only terms up to this order of approximation. This and the requirements for the total energy listed in sec. II is satisfied by the following expression

$$E = E_0 + \alpha\Big[\vec{p}_1(t)\cdot\vec{G}_1(t) + \frac{\omega_1^2}{\omega_0^2}\,\vec{p}_1(t)\cdot\vec{G}_0(t) + \omega_1^2\,\vec{p}_0(t)\cdot\vec{p}_1(t) - \\ -\frac{\lambda}{2}\,\vec{p}_0{}^2(t) + \frac{1}{\omega_0^2}\vec{G}_0(t)\cdot\vec{G}_1(t) - \frac{\lambda}{2\omega_0^2}\vec{F}_0{}^2(t)\Big] \tag{6.19}$$

where E_0 is the energy of the harmonic oscillator given by (4.37) and α is an arbitrary constant which has to be fixed from some other condition than the requirement (iii) of sec. II because we do not know, for the present model, any non-covariant expression for the total energy. In order to do this let us first notice that, since E is conserved, we may calculate it at the time t_0 at which all terms $\vec{p}_1(t), \vec{F}_1(t)$ and $\vec{G}_1(t)$ vanish. Then we get

$$E = E_0 - \frac{\alpha\lambda}{2}\left[\vec{p}_0{}^2(t_0) + \frac{1}{\omega_0^2}\vec{F}_0{}^2(t_0)\right] \tag{6.20}$$

In the reference system in which the condition (4.27) is satisfied this expression takes the form

$$E = E_0(1 - \alpha\lambda m) \tag{6.21}$$

We see now that the constant $\alpha\lambda$ determines the change of the total energy connected with the presence of the perturbing force. Defining this relative change by

$$\eta = \frac{\delta E}{E_0} \tag{6.22}$$

we may replace the unknown parameter $\alpha\lambda$ by a physically measurable quantity η and finally rewrite the expression for the total energy in the form

$$E = E_0 + \frac{\eta}{2m}\left[\vec{p}_0{}^2(t_0) + \frac{1}{\omega_0^2}\vec{F}_0{}^2(t_0)\right] \tag{6.23}$$

valid in all reference frames. From this expression we infer that in the transformation rule for the total energy the momentum $\vec{P}$ is given by

$$\vec{P} = (1+\eta)\vec{p}_0(t_0) + \frac{\vec{G}_0{}^2(t_0)}{\omega_0^2} \tag{6.24}$$

while the mass M is given by

$$M = m(1+\eta) \tag{6.25}$$

Before passing to the next example we would clarify the point why the total energy given by (6.19) apart from the coupling constant does not depend on the quantities describing the perturbation. It is simply the consequence of our assumption that he state of the perturbed particle coincides with the state of the unperturbed one. All the global quantities of the perturbed particle, and the total energy is such, must therefore be determined by the same quantities as for the unperturbed situation. The only information on the presence of the perturbations is in the different shapes

of the corresponding expressions and in the presence of the coupling constants. The perturbing forces change the dynamics of the particle and therefore all local quantities essentially depends on the perturbing force. As an example we may once again give the kinetic energy, which is a local quantity and therefore must depend on the perturbation.

The second example which we shall consider is essentially non-linear and is specified by the choice

$$\begin{aligned} \lambda(\ldots) &= \lambda \vec{F}_0{}^2(t) \\ \mu(\ldots) &= 0 \end{aligned} \tag{6.26}$$

for the functions in (6.13). The more general choice

$$\begin{aligned} \lambda(\ldots) &= \lambda_1 \vec{F}_0{}^2(t) + \lambda_2 \left(\vec{F}_0(t) \cdot \vec{G}_0(t)\right) + \lambda_3 \vec{G}_0{}^2(t) \\ \mu(\ldots) &= \mu_1 \vec{F}_0{}^2(t) + \mu_2 \left(\vec{F}_0(t) \cdot \vec{G}_0(t)\right) + \mu_3 \vec{G}_0{}^2(t) \end{aligned} \tag{6.27}$$

with many coupling constants which measure the strength of the anharmonic effects leads essentially to the same results as that given by (6.26).

The general solution of the equation (6.13) with the choice (6.26) is given by

$$\begin{aligned} \vec{F}_1(t) = {} & \vec{A}\left[\cos^3\omega_0(t-t_0) - \cos\omega_1(t-t_0)\right] + \vec{B}\sin^3\omega_0(t-t_0) + \\ & + \vec{C}\sin^2\omega_0(t-t_0)\cos\omega_0(t-t_0) + \\ & + \vec{D}\left[\cos^2\omega_0(t-t_0)\sin\omega_0(t-t_0) - \frac{\omega_0}{\omega_1}\sin\omega_1(t-t_0)\right] \end{aligned} \tag{6.28}$$

where

$$\vec{A} = \lambda \frac{\left[\left(\omega_1^2 - 7\omega_0^2\right)\vec{F}_0{}^2 - 2\vec{G}_0{}^2\right]\vec{F}_0 - 4\left(\vec{F}_0 \cdot \vec{G}_0\right)\vec{G}_0}{\left(\omega_1^2 - \omega_0^2\right)\left(\omega_1^2 - 9\omega_0^2\right)} \tag{6.29}$$

$$\vec{B} = \lambda \frac{\left[\left(\omega_1^2 - 7\omega_0^2\right)\vec{G}_0{}^2 - 2\omega_0^4\vec{F}_0{}^2\right]\vec{G}_0 - 4\omega_0^4\left(\vec{F}_0 \cdot \vec{G}_0\right)\vec{F}_0}{\omega_0^3\left(\omega_1^2 - \omega_0^2\right)\left(\omega_1^2 - 9\omega_0^2\right)} \tag{6.30}$$

$$\vec{C} = \lambda \frac{\left[\left(\omega_1^2 - 3\omega_0^2\right)\vec{G}_0{}^2 - 6\omega_0^4\vec{F}_0{}^2\right]\vec{F}_0 + 2\left(\omega_1^2 - 3\omega_0^2\right)\left(\vec{F}_0 \cdot \vec{G}_0\right)\vec{F}_0}{\omega_0^2\left(\omega_1^2 - \omega_0^2\right)\left(\omega_1^2 - 9\omega_0^2\right)} \tag{6.31}$$

$$\vec{D} = \lambda \frac{\left[\left(\omega_1^2 - 7\omega_0^2\right)\vec{F}_0{}^2 - 6\vec{G}_0{}^2\right]\vec{G}_0 + 2\left(\omega_1^2 - 3\omega_0^2\right)\left(\vec{F}_0 \cdot \vec{G}_0\right)\vec{F}_0}{\omega_0\left(\omega_1^2 - \omega_0^2\right)\left(\omega_1^2 - 9\omega_0^2\right)} \tag{6.32}$$

and we see that the non-linear character of the present model finds its reflection in the presence of higher harmonics in (6.28) and in a new resonance frequency $\omega_1 = 3\omega_0$ in the amplitudes (6.29) - (6.32).

The first corrections to the momentum and the trajectory are now given by

$$\begin{aligned} \vec{p}_1(t) = {} & \frac{2\vec{A}+\vec{C}}{3\omega_0}\sin^3\omega_0(t-t_0) - \frac{2\vec{B}+\vec{D}}{3\omega_0}\cos^3\omega_0(t-t_0) + \\ & + \frac{\vec{A}}{\omega_0}\sin\omega_0(t-t_0)\cos^2\omega_0(t-t_0) - \frac{\vec{B}}{\omega_0}\sin^2\omega_0(t-t_0)\cos\omega_0(t-t_0) - \\ & - \frac{\vec{A}}{\omega_1}\sin\omega_1(t-t_0) + \frac{\omega_0\vec{D}}{\omega_1^2}\cos\omega_1(t-t_0) + \frac{2\vec{B}\omega_1^2 + \vec{D}\left(\omega_1^2 - 3\omega_0^2\right)}{3\omega_1\omega_1^2} \end{aligned} \tag{6.33}$$

$$\begin{aligned} \vec{x}_1(t) = {} & \frac{7\vec{A}+2\vec{C}}{9m\omega_0^2}\cos^3\omega_0(t-t_0) - \frac{7\vec{B}-2\vec{D}}{9m\omega_0^2}\sin^3\omega_0(t-t_0) + \\ & - \frac{2\vec{A}+\vec{C}}{3m\omega_0^2}\sin^2\omega_0(t-t_0)\cos\omega_0(t-t_0) - \frac{2\vec{B}+\vec{D}}{3m\omega_0^2}\sin\omega_0(t-t_0)\cos\omega_0(t-t_0) - \\ & - \frac{\omega_0}{m\omega_1^3}\vec{D}\sin\omega_1(t-t_0) + \frac{\vec{A}}{m\omega_1^2}\vec{D}\cos\omega_1(t-t_0) + \frac{7\vec{B}+2\vec{C}}{9\omega_0^2} \end{aligned} \tag{6.34}$$

It is clear that under the Galilean transformations the vector valued coefficients $\vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ undergo only rotations and this ensures the correct transformation rules for $\vec{F}_1(t), \vec{p}_1(t)$ and $\vec{x}_1(t)$

For the total energy of the particle we get now the expression

$$
\begin{aligned}
E = E_0 + \\
&+ \alpha\left[-\frac{\lambda}{2}\left(\vec{F}_0{}^2(t)\right)^2 + \frac{\lambda}{6\omega_0^4}\left(\vec{G}_0{}^2(t)\right)^2 + \left(\vec{G}_0(t)\cdot\vec{G}_1(t)\right) + \right. \\
&+ \omega_0^2\left(\vec{p}_0(t)\cdot\vec{G}_1(t)\right) + \omega_0^2\omega_1^2\left(\vec{p}_0(t)\cdot\vec{p}_1(t)\right) + \omega_1^2\left(\vec{p}_1(t)\cdot\vec{G}_0(t)\right) + \\
&+ \frac{2\lambda}{3}\left(\vec{p}_0(t)\cdot\vec{F}_0(t)\right)\left(\vec{F}_0(t)\cdot\vec{G}_0(t)\right) + \frac{\lambda}{3}\left(\vec{p}_0(t)\cdot\vec{G}_0(t)\right)\vec{F}_0(t)^2 + \\
&+ \left.\frac{2\lambda}{3\omega_0^2}\left(\vec{p}_0(t)\cdot\vec{G}_0(t)\right)\vec{G}_0(t)^2\right] + \\
&+ \beta\left[\vec{F}_0{}^2(t) + \frac{1}{\omega_0^2}\vec{G}_0{}^2(t)\right]^2 + \gamma\left[\left(\vec{F}_0(t)\cdot\vec{G}_0(t)\right)^2 - \vec{F}_0^2(t)\vec{G}_0^2(t)\right] \quad (6.35)
\end{aligned}
$$

where E_0 is again the energy of the harmonic oscillator given by (4.37) and α, β, γ are free parameters which have to be fixed from the comparison of (6.35) with some known expression for the total energy in a particular reference frame. For example, we may choose the reference frame in which $\vec{G}_0(t_0)$ and $\vec{F}_0(t_0)$ is connected with $\vec{x}_0$ by the usual harmonic oscillator relation. Under these conditions (6.35) reduces to

$$
E = E_0 + m^4\omega_0^8\left(\beta - \frac{\alpha\lambda}{2}\right)\left(\vec{x}_0{}^2\right)^2 \quad (6.36)
$$

and comparing that with the total energy of the usual anharmonic oscillator with quartic interaction we find a condition for the parameters α and β. We shall not elaborate this point here because we are not interested in any application of the model since we want only to extract the transformation rule for the total energy

and for that purpose the value of the parameters β and γ are irrelevant. In fact, under the Galilean transformations the expression (6.35) transforms like (4.35) with

$$
\vec{P} = \vec{p}_0 + \frac{\vec{G}_0}{\omega_0^2} + \frac{\alpha\lambda m}{3}\left[\vec{F}_0\left(\vec{F}_0\cdot\vec{G}_0\right) + \vec{G}_0\left(\vec{F}_0{}^2\right)\right] \quad (6.37)
$$

where $\vec{p}_0, \vec{F}_0, \vec{G}_0$ denote the initial values of the momentum, force and time derivative of the force, respectively. Again, as previously, the product $\alpha\lambda$ is connected with the change introduced by the perturbing interaction.

VII. Conclusions.

To solve the mechanical problem we need to know the forces which act on the bodies. In the standard exposition of mechanics the forces are determined by constitutive relations which express the forces in terms of positions and velocities of the bodies. This however leads to the breakdown of the Galilean covariance which, as it was mentioned in the Introduction, we consider one of the most important properties of mechanics. In order to maintain this covariance we propose use differential equations for determination of forces. Such equations, although derived from the non-covariant constitutive relations, are Galilean covariant. In this way we achieve a formulation of mechanics which at every step respects Galilean covariance.

Unfortunately, we are yet not able to propose a covariant canonical formalism. The reason for that is the trouble with the definition of the total energy of the interacting particle. One way of going out from this trouble is to define the energy by the method described in the paper. From this method we may see that the energy has the same structure as it has in modern field theories, namely, it is

composed from the kinetic energy of a particle, from the energy of the field of forces and from the interaction between the particle and the field. Adopting this fact as definition we may get a powerful method of unified construction of the expression for total energy for each particular model. This creates some hope for finding the correct canonical formalism for the Galilean covariant formulation of mechanics.

The canonical formalism provides a bridge between classical and quantum mechanics. But quantum mechanics can be built independently from its classical prototype. We shall follow this way in a second part of our paper and then use the quantum canonical structure as a hint for the corresponding classical Galilean covariant canonical formalism. Such approach is not way around because quantum mechanics is more fundamental theory than classical mechanics and its formalism may be more suitable to investigate some problems. This concerns particularly the problems which significance is seen from larger perspective of more general theories and which up to now were omitted or badly understood in the standard mechanical considerations. We think that our approach follows the usual scientific way: we start from the particulars, generalize them, construct the most general scheme which does not contradict the original particulars and having this general scheme we look for its various limiting and special cases. So we started with the classical mechanics in order to improve our non-covariant intuition in such a way that the Galilean principle of relativity always is seen as a primary feature of the theory. Using these results we are able to construct quantum version of theory obeying all the demands of Galilean covariance and to solve within this scheme the problems which we could not solve in the framework of classical theory. The agreement of partial classical results with the limiting procedure applied to quantum case may be additional information which enables us to give unique general solutions of classical problems and to formulate the classical theory in general way.

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