Microlensing: Statistical Approach

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The information from microlensing events is not full, we first discuss its mass-and-distance ranges and estimate probabilities for events to be caused by objects either of Galaxy halo or of Great Magellanic Cloud (GMC). Then we study a question what information can be obtained either from the Fourier-analysis of the observed brightness curve of a star or from its autocorrelation function. The answer is: practically the same as from a couple of single events. Finally an observational procedure is suggested of higher information output: the brightness curve of a GMC pulsar in the radio waverange will reveal a difraction pattern which could give information on both mass of and distance to the lensing object.

The recent observational confirmation [1,2] of microlensing (i.e. gravitational focusing of light from a distant star by a small invisible body) is a scientific event of rather unusual significance: this phenomena was theoretically predicted by A.Einstein nearly 60 years ago [3], all this period the absence of its experimental prove was not a subject of great concern, moreover, finally being found, it looks like not leading to some valuable consequences. Actually microlensing is an obvious outcome of the light deviation by a heavy body, this fundamental effect itself was experimentally proved very soon after its prediction. As for the instrumental capacity of microlensing and its outcome for our knowledge about lens bodies, they are rather limited: observations of a single event cannot say us definitely what are the mass m and relative transverse velocity v of the lensing body, how far it is located (what is its distance D), and what was the minimal distance x_0 from the lens axe, the line between the source star and the observer (the Earth). Thus the usefulness of this phenomena either to prove or disprove the MACHO (Massive Compact Halo Objects) concept of dark matter is rather limited. Here we will try to use a statistical approach to analyse the brightness curve and finally suggest another observational procedure.

An effective radius around the lens body where rays, which finally come to the observer, pass the lens (so called the Einstein radius) is equal to $r_e = 2\sqrt{GmD}/c$ (here G is the gravitational constant and c is the light velocity). The intensity increase $q = I/I_0 > 1$ at any moment of time depends on the ratio of the distance x of the lensing body from the optical axe to the Einstein radius. Einstein [3] wrote the answer not wasting much place for its derivation: in the case the source star distance a is much greater than the lens body distance D, the amplification is equal to:

$$q_{m,D,x} = \frac{(x/r_e)^2 + 2}{(x/r_e)\sqrt{(x/r_e)^2 + 4}}.$$

High amplifications occur in cases when $x \ll r_e$, when $q \simeq (r_e/x)^{-1}$; at great x this expression tends to unity as $q \simeq 1 + 2(r_e/x)^4$,

Since some relative motion of the source, lens body and observer the transverse distance x depends on time as $x = \sqrt{x_0^2 + v^2t^2}$, where v is the relative transverse velocity of the lensing body in respect to the lensing axe.

.1 Mass-distance Diagram

Thus, to describe a microlensing event one needs four parameters: m, D, $,v$ $,x_0$. However, only two independent values can be derived from observations. A increase of brightness $I(t)$ of a star when a lensing object occurs near its ray, comes back to its initial level symmetrically in regard to the moment of its maximum and similar in all wavebands, so an amplification $q = I_{max}/I_0$ and a characteristic time $\tau = x_0/v$ of the event are just two available values.

However, not four but three parameters are badly unknown. It is possible to obtain a reasonable estimate for the relative transverse velocity v of the lens body. Average velocities of bodies of the Galaxy halo are $v \simeq 200$ km/s, the same order is the Earth (Sun) velocity around the Galaxy center. Position of the source star can be chosen as a fixed point, the Earth motion can be eliminated too by slow rotation of the reference system. The ratio of the measured characteristic time τ to velocity $v \sim 200$ km/s gives the order of the distance $x_0 = v/\tau$.

This distance should be compared with the Einstein radius. Since it depends on the product $m \times D$ the lens body mass and its distance cannot be separately derived from observations. However, some limitations can help:

- Mass m should be much less than the solar mass, otherwise the probability for the lens body to be a visible star will be rather high.
- The gravitational radius of this body should be much greater than λ, the wavelength of the light: $r_g = 2Gm/c^2 \gg \lambda$, otherwise there will be no significant focusing. For visible light ($\lambda = 5 \cdot 10^{-7}$ m) this low mass limit is about 10^{21} kg. Take into account that all bodies of grater mass are spherical.
- The lens body cannot be located too close to the observer, in other words its Einstein radius must be grater than its actual radius. Using the fact that average internal density ρ_i of planet-like bodies (excluding neutron stars and white dwarfs) lays in a logarithmically small range $\rho_i = (1 \text{ to } 3)$ g/cm^3 this constrain can be rewritten in terms of mass and distances:

$$D > 0.1924 \frac{c^2}{G\rho_i^{2/3} m^{1/3}} \simeq 0.0065 \left(\frac{m_S}{m}\right)^{1/3} \text{ ps.}$$

The mass-distance diagram (Fig. 1) shows these limitations.

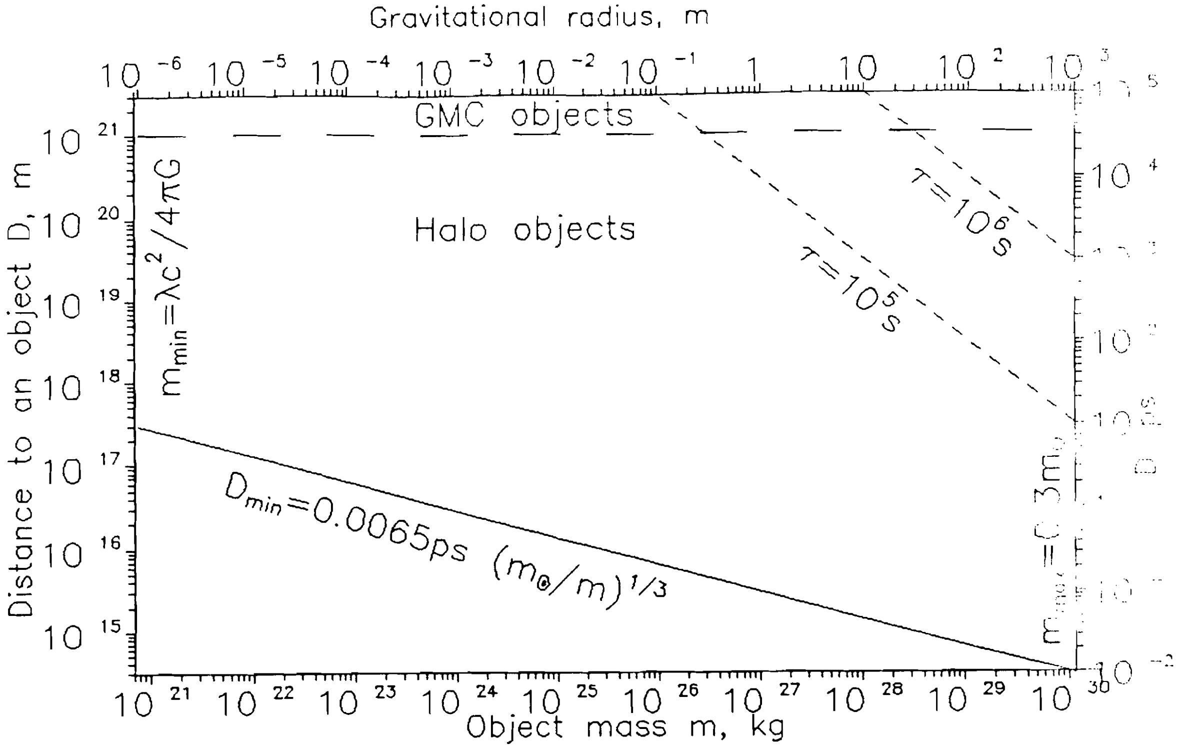

Figure 1: Mass-distance diagram for possible ranges of microlensing events. For limits of masses and distances (solid lines and the diagram boundaries) see text. Dashed lines with notations of characteristic periods correspond to undistinguishable events of the same appearance. Long dashed line separates bodies of our Galaxy from that of GMC.

.2 Mass Distribution of Lensing Bodies and Amplification Distribution

Let us introduce a mass distribution of bodies $f_m(D)$ depending on distance D from the observer. By definition, $f_m(D)dm\ dV$ is a number of bodies in the volume dV with masses about m in the range dm. Evidently, the distributed density $\rho(D)$ can be expressed as the integral

$$\rho(D) = \int_0^\infty f_m(D) m\ dm.$$

This condition actually plays role of the mass distribution normalization.

Suppose, dw is a probability for random observation for amplification q in the range dq. Evidently, the differential distribution of the observed amplifications $dw(q)/dq$ can be expressed as an integral over the mass distribution:

$$\frac{dw}{dq} = \int\int\int f_m(D)\ dm\ dD\ 2\pi x dx \delta(q - q_{m,D,x}).$$

Here the volume differential in the definition is replaced by $dV = 2\pi x dx\ dD$. (Actually, one should introduce also the velocity distribution and take an average of the integrand over velocities too, but the result will not critically depend on velocities of Galaxy bodies distribution.). The integration limits here should coincide with that of diagram of the Fig.1, but practically only the upper limit for the distance $D_{max} = a$ should be taken into account, other limits can be taken as either zero or infinity. Two from three

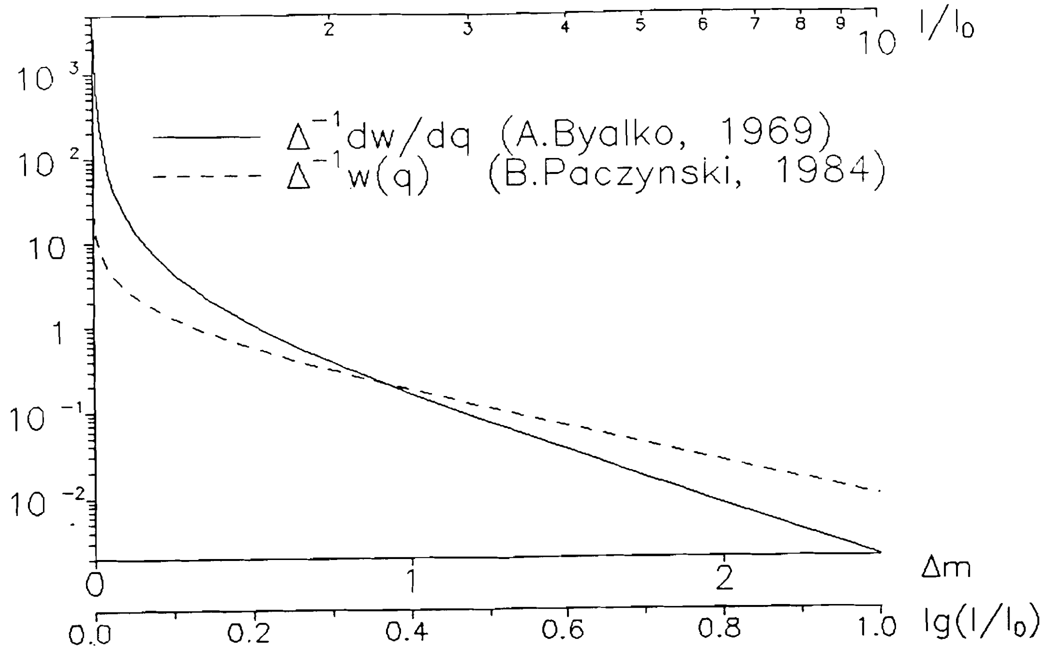

Figure 2: Differential (dw/dq, solid curve) and integral (accumulated) ($w(q)$, dashed curve) distributions of amplitudes q for microlensing events.

integrals here occur to be easily evaluated, one due to delta-function, the second — due to normalization. This result was first obtained a quarter a century ago [5]:

$$\frac{dw}{dq} = \Delta \frac{2}{(q^2-1)^{3/2}},$$

where

$$\Delta = \frac{4\pi G}{c^2} \int_0^a \rho(D) D \, dD$$

is dimensionless parameter which later was called the optical depth for microlensing (by analogy with general optical depth).

Such nomination was introduced but 18 years later by B.Paczynski, who received (with indistinct derivation) the integral amplification distribution:

$$w(q) = \int_q^\infty \frac{dw}{dq} dq = 2\Delta \left[\sqrt{1+\frac{1}{q^2-1}} - 1\right]$$

when suggesting a procedure for the experimental prove by observations of neighboring galaxy stars. Both distributions are shown at Fig. 2.

Thus, only the product mD can be evaluated in analysis of a single focusing and just one parameter results in statistical averaging of microlensing amplitudes. However, can we obtain something more from the statistics of microlensing fluctuations?

.3 Correlation Function and Fourier-Transformation of Star Brightness

Set of observations of millions GMC stars [1,2] during a few years actually represent a rich statistics of microlensing events, but most of them are either too short to be fixed with rare (less than once a day) observations or too weak to be distinguished as significant signal-to-noise events. If we forgot changeable whether conditions, the set of multiple stars observations is practically equivalent to a long observation of a single GMC star during about 10^7 years. Suppose we have such a long amplification curve of a single star and make its Fourier analysis. What information can be obtained from its Fourier-spectrum? (We use here word "Fourier" as an extra prefix not to mix it with general optical spectra.)

Different microlensing events will be treated here as independent ones. Taking into account rather high probability of double stars one should keep in mind a possibility to meet a double or multiple system of invisible lensing bodies. So this assumption is not absolutely correct: scattering on a double system cannot always be treated as two independent events. The result of double lensing occurs as a linear combination of single events, but they can be time-correlated. Here we will neglect this possibility.

We define the autocorrelation function $K(t)$ for microlensing as

$$K(t) = \int_{-\infty}^{\infty} [q(t') - 1][q(t' + t) - 1]dt',$$

where $q(t)$ means actual amplification of a star. Pay attention that dimension of our correlation function is time. Such definition slightly differs from that of Hawkins [7], who averages observed magnitude deviations, i.e. logarithms instead of linear deviations. This discrepancy prevents from comparison of his correlation function with results of given paper, but it will be possible using original data.

The "observable" correlation function can be compared with its theoretical analog, i.e. correlation of a single lensing event (with fixed parameters m, D, x)

$$K_{mD,x}(t) = \int_{-\infty}^{\infty} [q_{mD,x}(t') - 1][q_{mD,x}(t' + t) - 1]dt',$$

averaged over the mass distribution. At small $t \ll x/v$, r_e/v the function $K_{mD,x}(t)$ has a logarithmic behavior, at great t it decreases as t^{-4}. The full correlation function results in averaging:

$$K(t) = \int\int\int K_{mD,x}(t) f_m(D)\ dm\ dD\ 2\pi x\ dx.$$

Any correlation function is tightly connected with the averaged Fourier-spectrum, i.e. squared Fourier-transform of a time-dependence of some process. We define it here as $S_\omega =< |q_\omega|^2 >$ with averaging either by mass distribution (in the case of theoretical amplification) or by set of realizations (for observed star brightnesses). Then

$$K(t) = (2\pi)^{-1} \int S_\omega e^{-i\omega t} dt;$$

$$S_\omega = \int K(t) e^{i\omega t} dt.$$

The Fourier-spectrum of microlensing is easier to evaluate theoretically that the correlation function. Again, the Fourier-transformation of a single microlensing event is

$$q_\omega(mD, x) = \int_{-\infty}^{\infty} (q(t) - 1) e^{i\omega t} dt = 2 \int_{0}^{\infty} (q(t) - 1) \cos \omega t dt.$$

This integral can be evaluated analytically in the most interesting case when ($x \ll r_e$):

$$q_\omega(mD, x) = 2\frac{r_e}{v} K_0(\frac{x\omega}{v}); \quad (x \ll r_e)$$

where K_0 is the modified Bessel function.

Now let us average this single event Fourier-spectrum over the lens body mass distribution:

$$S_\omega = \int\int\int |q_\omega|^2 f_m(D)\ dm\ dD\ 2\pi x\ dx.$$

It seems strange, but this triple integral can be evaluated exactly: the very simple result is expressed through the same microlensing depth Δ as the focusing amplitude distribution:

$$S_\omega = \frac{\Delta}{\omega^2}.$$

Thus, the microlensing spectrum looks like "red noise": a probability of high level events increases when total time of observation grows.

However, this expression is not valid at very small frequences. We can estimate the low-frequency spectrum limit by replacing the averaging integration xdx by the order of its upper limit r_e^2. Then

$$S_{\omega\to 0} \simeq \frac{2\pi G^2}{c^4 v^2} \int f_m(D)\ m^2 dm\ D^2 dD\ .$$

Any reasonable distribution function $f_m(D)$ leads to the conclusion that this integral is determined by its upper physical limits: masses about of a star (solar) mass $m \simeq m_S$ and maximum distances $D \simeq a$, i.e. by the source distance. Such evaluation gives possibility to estimate the low frequence ω_{min}, where the obtained spectrum becomes $S_\omega \simeq$const. Its inverse corresponds to maximum times τ_{max} of microlensing:

$$\tau_{max} = \omega_{min}^{-1} \sim \frac{cv}{Gm_S a}.$$

For stars of GMC ($a \simeq 10^{21}$ m) this estimate gives $\tau_{max} \simeq 0.3$ yr (compare with the diagram 1 - see its upper right corner). For distant quasars ($a \simeq 10^{26}$ m and $v \simeq 10^6$ m/s) the estimate gives just slightly higher estimate $\tau_{max} \simeq 10$ yr. Compare it with the observational study [7] which gives a characteristic time of correlations about of that order.

However, neither spectrum S_ω nor its lower frequency limit give us any significant information about the mass-distributions function f_m of small invisible bodies, the information we urgently need to solve the problem of dark matter. The information capacity

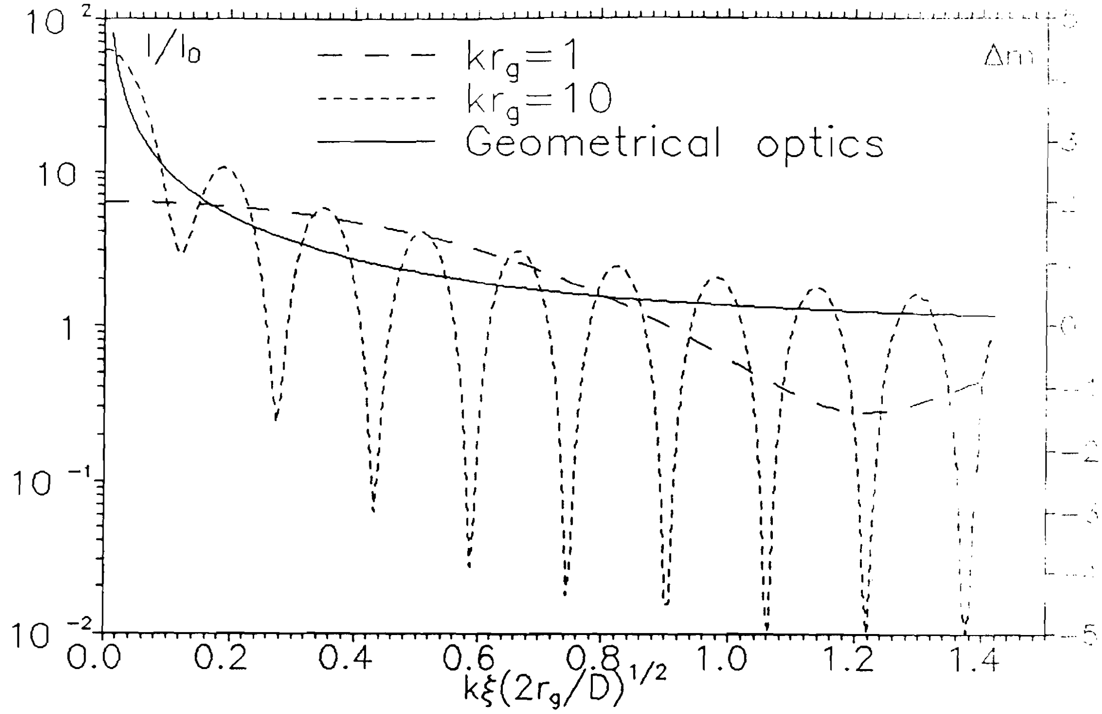

Figure 3: Difraction pattern of microlensing for two values of kr_g (dashed curves) in comparison with the geometric optics case (solid line).

of Fourier-analysis of brightness occurs to be about the same as for the time-amplitude analysis of single events: both methods reveal the depth for microlensing and some natural limitations for masses and distances, but not the distribution function itself..

That is why a new idea for its measuring I am going to suggest, probably will be of interest.

.4 Pulsar Observations for Microlensing Search

The probability distribution of microlensing was a secondary by value result of the paper [5], but its main goal was the expression for lensing intensity in the case of wave optics:

$$q(x,m,D,k) = \pi kr_g \frac{\exp(kr_g)}{\text{sh}kr_g} |F(ikr_g, 1, ikr_g \frac{x_o^2 + v^2t^2}{2r_gD}|^2.$$

Here $k = 2\pi/\lambda$ is a wave number of light (electromagnetic wave) and $r_g = 2mG/c^2$ - the gravitational radius of the lensing body. The time-dependence of these oscillations is shown at Fig.3. In the case when $kr_g \gg 1$ this expression goes to more simple one:

$$q = 2\pi kr_g J_0^2(kx\sqrt{2r_g/D}),$$

which after oscillations averaging leads to the result of the geometrical optics: $q = \sqrt{2r_gD/x}$.

In 1967 when I reported this calculations to Ya.B.Zel'dovich, his comment was: this problem is a pure theoretical one, since size of any real star kills the difraction pattern, and no difference how far is the star, it cannot be treated as a point source. He was right at the moment: pulsars were discovered later. Actually, conditions of application for the wave optics case are so tough, that pulsars only can be treated as a point source for this problem (may be quasars too).

Now suppose, we observe a distant pulsar in our Galaxy or GMC in the radio wave range. If there is a small body in between us, we will see a general microlensing changes of brightness but modulated with oscillation of a high frequency

$$\Omega = \frac{8\pi v}{c\lambda}\sqrt{\frac{Gm}{D}}.$$

For example, if the wavelength is $\lambda = 1$ cm, the body mass is about that of the Earth and $D = 1$ kps then the oscillations period would be $T = 2\pi/\Omega = 10^3$ s. Set of equal (m/D)-ratios cut the mass-distance diagram from the low left to the upper right. Thus, in principle we can independently determine from such observations both distance D and mass m of the lensing body. A good statistics of such microwave lensing events will lead to evaluation of the full mass distribution $f_m(D)$. But...

The main difficulty here is again connected with low probabilities for the phenomena observation: the amplification distribution q is approximately the same as for geometrical optics, i.e. proportional to $\Delta \simeq 10^{-6}$. However, there is a positive side of story: one needs much less time to fix oscillations of the difraction pattern, thus to be aware that an event is going. The number of found pulsars is less but of the order of 10^3, suppose about 10^2 radio-antennas are available to follow each of them at least an hour per day (whether conditions do not prevent radio-observations). Then the overall probability for discovery still will be less than 0.1 per year. Is it enough to waste money?

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