

DEPARTMENT OF PHYSICS, UNIVERSITY OF JYVÄSKYLÄ

RESEARCH REPORT No. 2/1971

PARTICLE KINEMATICS

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Jyväskylä, Finland

March 1971

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(Chapters I-VI, X)

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March 1971

Preface

The purpose of this book is to give a systematic account of relativistic kinematics of particle reactions. There are two needs which the text attempts to satisfy. It contains an introduction to the basic principles, practical applications and known methods of those aspects of elementary particle physics which are based purely on kinematics. Also, it is hoped to be useful as a reference book for anyone actively working in experimental or theoretical high energy physics.

In recent years the importance of kinematical considerations has rapidly grown. The main reason is the shift of interest, both in theory and experiment, to particle reactions of increasingly high multiplicities. When the number of particles increases, the number of kinematical variables increases rapidly. A thorough understanding of the complex kinematics is required to be able to isolate properly the essential dynamical features of the reactions. Otherwise one may be lead, for example, to tedious calculations or incorrectly interpreted experimental effects. Appreciation of the simple and attractive internal structure and consistency of relativistic kinematics allows one to attain essential conceptual and practical simplifications.

The only existing book on this subject is "Relativistic Kinematics" by R. Hagedorn. In addition, lectures on particle kinematics have been given in several summer schools. Many important new developments have appeared in recent times which are not treated consistently anywhere in the literature. These include various ways of choosing variables and plotting data in many-particle reactions, kinematical reflections, effects of peripherality, longitudinal phase space, missing mass techniques, etc. Of decisive practical significance are various numerical methods. We hope that any success in filling this gap in literature will compensate for any possible shortcomings of the book.

As prerequisites, familiarity with some basic notions of elementary particle physics and the conceptual aspects of special relativity are required. A detailed account of the technical aspects of special relativity is given in Chapter II. The book can be used as text for an advanced undergraduate or graduate course in particle kinematics. In order to facilitate this use a number of exercises are included at the end of the text. The concepts and methods are whenever possible illustrated and motivated by practical examples. An essential ingredient of the course is learning an effective use of the numerical methods explained in the last chapter.

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I. Introduction

Particle kinematics is here understood as the application of special relativity to elementary particle reactions. The theory of relativity is one of the most fundamental and best established constituents of modern physics. The reader is assumed to be acquainted with its basic principles and essential implications, such as the concept of inertial frame, constancy of light velocity, invariance of physical laws under Lorentz transformations, transformation properties of four-vectors, etc. Some relevant practical aspects of special relativity will be elucidated in detail in the next chapter.

g. I.1.1 From the point of view of pure kinematics, particles are completely characterized by their energy and momentum, i.e. their four-momentum p . Observable particle reactions are either decays or collisions, and they are symbolically drawn as in Fig. 1.1. A basic property of four-momentum is conservation: the sum of all four-momenta in the initial state is equal to the sum in the final state. This simple law has a wealth of consequences in applications of kinematics.

Internal quantum numbers are irrelevant to elementary particle kinematics. Thus the properties parity, isospin, charge, baryon or lepton number, strangeness, etc. do not enter our discussion. A special case is ordinary spin.

Invariance under Lorentz transformations implies definite statements about spin. Formally, the spin is a property of a representation of the inhomogeneous Lorentz group, which includes both Lorentz transformations and translations. Thus particle states are characterized by momentum and spin. The transformation properties of spin are well defined, but turn out to be rather complicated unless spin is zero [Werle , Fonda 1970]. Conforming to common practice, we shall in this book not consider spin. Thus all the particles are treated as they had spin zero.

Another way to characterize kinematics is to contrast it with dynamics. Dynamics is concerned with what happens within the interaction circle in Fig. 1.1., while kinematics relates to the asymptotic description in terms of four-momenta. Dynamics is thus, at the moment, largely unknown while kinematics contains no difficulties in principle. However, kinematics and dynamics are inextricably mixed. Consider, for instance, the distribution in some energy variable, Fig. 1.2.

I.1.2. Dynamics determines the exact shape of the curve, but a constraint imposed by kinematics always exists: the energy has to be larger than some minimum value (threshold) and smaller than some maximum value (due to energy conservation). This forces the distribution to zero at $E_{\min}$ and $E_{\max}$, and sometimes also the behaviour near the limit is a kinematical property. There is a formal way of separating kinematics and

dynamics but, in practice, they have to be applied simultaneously.

To roughly indicate the types of applications of kinematics, we list some examples of problems one encounters in particle physics. They range from ones in a simple form to cases of great technical and computational difficulty.

- 1) Experiments are carried out in the laboratory frame, while theoretical analysis may be simple in the center-of-momentum frame. One has to find the transformation laws from one frame to the other.
- 2) The problem of placing detectors in the laboratory, when one wants them to register certain type of phenomena.
- 3) Identify by kinematical fitting a definite reaction channel among a large number of different possibilities.
- 4) A theory should presumably be invariant under Lorentz-transformations. Find the suitable Lorentz-invariant variables and their physical region.
- 5) Calculate an experimental prediction from a given theory. This will often involve a complicated many-dimensional integration over many final-state particles.
- 6) Determine the most suitable kinematical variables to isolate a given dynamical effect. This problem is particularly complicated when many particles and many variables are involved.
- 7) Suppose a strong dynamical effect is seen in some variable. By energy and momentum conservation this effect is transmitted to other variables. Describe these kinematical reflections.

II. Lorentz transformations and invariance

1. Lorentz transformations

According to special relativity theory (see e.g. [Rindler 60]), the laws of physics are the same in all inertial systems and the speed of light in free space has the same value c in all inertial systems. A transformation of coordinates from one inertial frame to another, which keeps the velocity of light constant, is called a Lorentz transformation. Physical laws must be formulated so that they are invariant under Lorentz transformations.

A point in space-time is defined by its coordinates x, y, z, t in a given frame S . The relation between the coordinates x, y, z, t of the point in S and its coordinates x', y', z', t' in a second inertial frame S' must be linear due to the homogeneity of space-time. Assume S' moves with a constant relative velocity $\underline{v}$ in S . To obtain a simple expression for the transformation, choose the z -axis of S and the z' -axis of S' along the constant relative velocity $\underline{v}$ and the corresponding coordinate planes parallel (Fig. 1.1.).

1.1.1. The constancy of light velocity then implies that the transformation must be

$$x' = x$$

$$y' = y$$

$$z' = \frac{z - vt}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{t - zv/c^2}{\sqrt{1 - v^2/c^2}}$$

These eqs. assume that the origins of the coordinate systems coincide at $t = t' = 0$. The Lorentz transformation is then homogeneous.

The quantity

$$\gamma = \gamma(v) = (1 - v^2/c^2)^{-1/2}$$

has a central significance in special relativity. In terms of γ the Lorentz transform and its inverse are

$$\begin{aligned} x' &= x & x &= x' \\ y' &= y & y &= y' \\ z' &= \gamma z - \gamma vt & z &= \gamma z' + \gamma vt' \\ t' &= -\frac{\gamma v}{c^2} z + \gamma t & t &= \frac{\gamma v}{c^2} z' + \gamma t' \end{aligned} \tag{1.1}$$

We note that the components x and y , perpendicular to the velocity $\underline{v}$, remain unchanged. Also the inverse of a transform is obtained simply by changing the sign of v . Eqs. (1.1) are constantly used and it is convenient to memorize them.

For small velocities v , the light velocity c disappears from the Eqs. (1.1) and they approach the nonrelativistic Galilean transformation. On the macroscopic

scale only some astronomical phenomena involve velocities so large that (1.1) need be applied. The velocities in particle physics are mostly so close to c that deviations from the Galilean transformation are large. In fact, particle physics offers numerous opportunities to verify the validity of (1.1).

Example 1. As an application of (1.1) we shall consider time dilatation and its verification in particle physics. Assume that S' is again moving with a velocity v in S and that in the origin of S' is a stationary clock showing the time t' in S' . Consider two events at times t'_1 and t'_2 in S' . Use now (1.1) with $z' = 0$. A clock which is stationary in S will measure these events at the times

$$\begin{aligned} t_1 &= \gamma t'_1 \\ t_2 &= \gamma t'_2 \end{aligned} \tag{1.2}$$

The corresponding time intervals $T_0 = t'_1 - t'_2$ and $T = t_1 - t_2$ then satisfy

$$T = \gamma T_0 = \frac{T_0}{\sqrt{1-v^2/c^2}} \geq T_0 \tag{1.3}$$

T_0 is the time measured by a clock in its rest system, T is the same time interval as seen in a system in which the clock is moving. Since $T \geq T_0$ time goes faster in the rest system of the clock. Seen from S , the moving clock goes slow. This effect is called time dilatation.

A quantitative verification of time dilatation is obtained as a byproduct of the CERN g-2 experiment [Bailey 1970]. Leaving aside the purpose of this

experiment, it involves the storage of muons in a circular ring with radius 2.5 m. The velocity of the muons is such that $\gamma(v) = 12.7$. One can follow the gradual decrease of the number of muons and determine their lifetime in the laboratory. The lifetime of the muons in their rest system is $T_0 \approx 2.2 \mu s$ and the lifetime measured for the circulating muons agrees with $\gamma(v) \cdot T_0 \approx 27 \mu s$.

Note that this result also gives an experimental proof of the fact that the surprising "twin paradox" involves a true effect. Of two twins, the one who has left the earth in a space vehicle will at his return appear younger to an observer on the earth. This result has been verified by muons, the decay of which forms an ideal clock. The circulating muon is subject to accelerations similar to those of a space vehicle making a round trip.

Example 2. An even more pragmatic example of the effects of time dilatation is given by the design of low energy $K^\pm$ beams. The lifetime T_0 of the charged K is such that $cT_0 \approx 3.7$ m. This distance is too short to allow one to separate a pure K beam. Thus very low energy (and velocity) K beams can not be produced. At higher energies the factor $\gamma(v)$ lengthens the time and distance available in laboratory to the separation of the K beam. Qualitatively similar remarks apply for construction of beams of any other unstable particles, e.g. pions or hyperons.

Let us generalize the previous situation by allowing the velocity $\underline{v}$ of S' in S to have an arbitrary direction (Fig. 1.2.) The appropriate transformation equations between $\underline{x} = (x, y, z)$ and $\underline{x}' = (x', y', z')$ can now be derived by demanding that the component of $\underline{x}$ perpendicular to $\underline{v}$,

II.1.3.

$$\underline{x}_{\perp} = \underline{x} - \underline{v} \frac{\underline{v} \cdot \underline{x}}{v^2},$$

should remain invariant and the component of $\underline{x}$ parallel to $\underline{v}$,

$$\underline{x}_{\parallel} = \underline{v} \frac{\underline{v} \cdot \underline{x}}{v^2},$$

should transform as z in (1.1). We thus have

$$\begin{aligned}\underline{x}'_{\perp} &= \underline{x}_{\perp} \\ \underline{x}'_{\parallel} &= \gamma(v)(\underline{x}_{\parallel} - \underline{v}t).\end{aligned}$$

Together these give

$$\begin{aligned}\underline{x}' &= \underline{x}'_{\perp} + \underline{x}'_{\parallel} \\ &= \underline{x} - \underline{v} \left[\frac{\underline{v} \cdot \underline{x}}{v^2} (1 - \gamma) + \gamma t \right].\end{aligned}$$

In these transformations the dimensionless quantity $\underline{\beta} = \underline{v}/c$ appears frequently. The quantity $\gamma = (1 - \beta^2)^{-1/2}$ satisfies the identities

$$\begin{aligned}\beta^2 + \gamma^{-2} &= 1, \\ \gamma^2 \beta^2 &= (\gamma - 1)(\gamma + 1).\end{aligned}\tag{1.4}$$

The transformation formula of $\underline{x}$ can then be written in the final form

$$\underline{x}' = \underline{x} + \gamma \underline{\beta} \left(\frac{\gamma}{\gamma + 1} \underline{\beta} \cdot \underline{x} - ct \right).\tag{1.5}$$

The time coordinate transforms according to (1.1) with z replaced by $|\underline{x}_{||}|$,

$$t' = \gamma(t - \frac{v}{c^2} |\underline{x}_{||}|) .$$

This is simply

$$t' = \gamma(t - \frac{\underline{v} \cdot \underline{x}}{c^2}) \tag{1.6}$$

The formulas (1.5) and (1.6) are very important, although in many cases one may choose the configuration so that the simpler formulas (1.1) apply. Eq. (1.5) is rather untransparent and one should keep in mind the derivation which indicates its real meaning.

2. Transformation of three-velocities

A three-velocity in a frame S with t and $\underline{x}$ as time and space coordinates is evidently defined by

$$\underline{w} = \frac{d\underline{x}}{dt} \quad (2.1)$$

When going from S to another frame S' (Fig. 1.2), both $\underline{x}$ and t transform according to (1.5) and (1.6) so that $\underline{w}$ is transformed to

$$\underline{w}' = \frac{d\underline{x}'}{dt'} \quad (2.2)$$

Since the differential operator is linear, the transformation equations of $d\underline{x}$ and dt are obtained from (1.5) and (1.6) by differentiation:

$$d\underline{x}' = d\underline{x} + \frac{1}{c} \gamma \underline{v} \left(\frac{\gamma}{\gamma+1} \underline{v} \cdot d\underline{x} - c dt \right)$$

$$dt' = \gamma \left(dt - \frac{1}{c^2} \underline{v} \cdot d\underline{x} \right)$$

By dividing one obtains immediately

$$\underline{w}' = \frac{\underline{w} + \gamma \underline{v} \left(\frac{\gamma}{\gamma+1} \frac{\underline{v} \cdot \underline{w}}{c^2} - 1 \right)}{\gamma \left(1 - \frac{\underline{v} \cdot \underline{w}}{c^2} \right)} \quad (2.3)$$

Here $\underline{v}$ is the velocity of the frame S' as seen from the frame S , while $\underline{w}$ and $\underline{w}'$ are velocities in S and S' , respectively. The inverse relation giving $\underline{w}$ in terms of $\underline{w}'$ is obtained from (2.3) by interchanging $\underline{w}$ and $\underline{w}'$ and changing the sign of $\underline{v}$. In this inverted form, the resulting equations are often called

relativistic velocity addition formulas, since they can be regarded as giving the result $\underline{w}$ of adding the two velocities $\underline{v}$ and $\underline{w}'$ (Fig. 1.2).

The general form (2.3) is fairly untransparent. It is clarified by looking at some of its properties. We know that the component of $\underline{x}$ perpendicular to $\underline{v}$ is invariant under Lorentz transformations. To see how the parallel and perpendicular components of $\underline{w}$ transform, assume that $\underline{v}$ is parallel to the z-axis (Fig. 2.1).

II.2.1.

Then (2.3) is equivalent to the three equations

$$\begin{aligned} w'_x &= \frac{w_x}{\gamma(1-vw_z/c^2)} \\ w'_y &= \frac{w_y}{\gamma(1-vw_z/c^2)} \\ w'_z &= \frac{w_z - v}{1-vw_z/c^2} \end{aligned} \quad (2.4)$$

The perpendicular components w_x and w_y of $\underline{w}$ thus also get transformed and $\underline{w}$ does not behave like $\underline{x}$. We shall soon construct a relativistic generalization of $\underline{w}$ which really behaves like $\underline{x}$.

The angle θ between $\underline{w}$ and $\underline{v}$ is transformed to an angle θ' between $\underline{w}'$ and $\underline{v}$. The transformation equation is obtained from (2.4) by assuming that $w_y = 0$ and writing $w_x = w \sin \theta$, $w_z = w \cos \theta$, $w'_x = w' \sin \theta'$, $w'_z = w' \cos \theta'$. The result is

$$\operatorname{tg} \theta' = \frac{w \sin \theta}{\gamma(w \cos \theta - v)} \quad (2.5)$$

This equation will later be analyzed in detail.

Assume finally that even w_x vanishes, so that all velocities are parallel. The inverted form of the last of the equations (2.4) then gives

$$w = \frac{w' + v}{1 + w'v/c^2} \quad (2.6)$$

With this equation it is simple to prove the following properties of addition of velocities: 1) the sum of velocities does not exceed c ; 2) for $v \ll c$, velocities are simply additive.

3. Four-vectors

The Lorentz-transformation equations can be written in a both technically and conceptually simpler form by combining t and $\underline{x}$ to a single entity, the (contra-variant) four-vector $x = (x^\mu)$, $\mu = 0,1,2,3$. Write

$$\begin{aligned} x^0 &= ct \\ x^1 &= x \\ x^2 &= y \\ x^3 &= z \\ x &= (x^0, \underline{x}) = (ct, \underline{x}) \end{aligned} \quad (3.1)$$

and introduce the metric tensor

$$g = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (3.2)$$

A general Lorentz transformation transforms the four-vector x to another four-vector x' so that the quadratic form

$$x^2 = x \cdot x = \sum_{\mu} x_{\mu} x^{\mu} = \sum_{\mu\nu} g_{\mu\nu} x^{\mu} x^{\nu} = (x^0)^2 - \underline{x}^2 = (ct)^2 - \underline{x}^2 \quad (3.3)$$

remains invariant. The Lorentz transformation can be represented by a 4x4 matrix Λ ,

$$x^{\mu} \rightarrow x'^{\mu} = \sum_{\nu=0}^3 \Lambda^{\mu}_{\nu} x^{\nu} \quad (3.4)$$

Then x^2 remains invariant, if Λ satisfies the matrix equation

$$\Lambda^T g \Lambda = g, \quad (3.5)$$

where Λ^T is the transpose of the matrix Λ . This can be seen by inserting (3.4) to (3.3). Notice, in particular, that a three-dimensional rotation effecting only the space components is also a Lorentz transformation.

In general, any object which transforms like x is called a four-vector. If a is another four-vector, the scalar product

$$a \cdot x = \sum_{\mu} a_{\mu} x^{\mu} = \sum_{\mu\nu} g_{\mu\nu} a^{\nu} x^{\mu} = a^0 x^0 - \underline{a} \cdot \underline{x} \quad (3.6)$$

is also invariant. Also, the square of the sum $x+a = (x^{\mu}+a^{\mu})$ is invariant:

$$(x+a)^2 = x^2 + a^2 + 2a \cdot x. \quad (3.7)$$

The equations transforming an arbitrary four-vector a from the frame S to a frame S' , moving in S with the velocity $\underline{v}$, are obtained from (1.4) and (1.5) by replacing ct by a^0 and $\underline{x}$ by $\underline{a}$:

$$\begin{aligned} \underline{a}' &= \underline{a} + \gamma \underline{\beta} \left(\frac{\gamma \underline{\beta} \cdot \underline{a}}{\gamma + 1} - a^0 \right) \\ a^{0'} &= \gamma (a^0 - \underline{\beta} \cdot \underline{a}) \end{aligned} \quad (3.8)$$

The four-vectors a can be divided in three classes:

- timelike ($a^2 > 0$)
- lightlike ($a^2 = 0$)
- spacelike ($a^2 < 0$).

The coordinate four-vector, x can be of any of these types. The type is related to the question whether or not the point x can be reached from the origin by a light

signal. Other four-vectors we shall use will be of definite type. By a suitable Lorentz transformation followed by a rotation a can be transformed in one of the forms:

- a timelike: $(a_\mu) = (a_0, \underline{0})$
- a lightlike: $(a_\mu) = (1, 0, 0, 1)$
- a spacelike: $(a_\mu) = (0, 0, 0, b)$.

The Lorentz transformations Λ_μ^ν that appear in particle kinematics satisfy in addition to (3.5) the following conditions:

- $\det \Lambda = +1$. This means that no discontinuous reflections are included.
- $\Lambda_0^0 \geq 1$. The sign of the 0-component of a time-like vector is invariant.

In order to illustrate the matrix formulation we shall write the basic equations (1.1) in matrix form. Let us replace v by a parameter r defined by

$$\frac{v}{c} = \tanh \frac{r}{c}. \quad (3.9)$$

The parameter r is called rapidity. The relation (3.9) maps the limited range of variation of v ($-c \leq v \leq c$) to an infinite range of r ($-\infty \leq r \leq \infty$). Rapidity has more than a notational significance, since e.g. in collinear Lorentz transformations rapidities are additive while velocities transform according to the more complicated rule (2.6). Writing (2.6) in the form

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2} \quad (3.10)$$

and introducing $v_i = c \tanh(r_i/c)$, the right hand side of (3.9) becomes $\tanh[(r_1+r_2)/c]$, by the addition theorem of $\tanh$. Thus, in fact,

$$r_3 = r_1 + r_2 . \quad (3.11)$$

The equation (3.11) also emphasizes the symmetry of the situation, which is not so apparent from (3.10).

Eq. (3.9) implies

$$\gamma(v) = \frac{1}{\sqrt{1-\tanh^2 \frac{r}{c}}} = \cosh \frac{r}{c} \quad (3.12)$$

$$\frac{v}{c} \gamma(v) = \sinh \frac{r}{c}$$

With this notation Eq. (1.1) can immediately be written in the matrix form (3.4) with

$$\Lambda = \Lambda(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \frac{r}{c} & -\sinh \frac{r}{c} \\ 0 & 0 & -\sinh \frac{r}{c} & \cosh \frac{r}{c} \end{pmatrix} \quad (3.13)$$

If the more general Lorentz transformation (3.7) is expressed in matrix form, all matrix elements are non-vanishing. An arbitrary Lorentz transformation Λ can be decomposed into two rotations $\Lambda(R_1)$ and $\Lambda(R_2)$ and a pure Lorentz transformation $\Lambda(r)$:

$$\Lambda = \Lambda(R_2) \Lambda(r) \Lambda(R_1) .$$

4. Four-velocity

The three-velocity $\underline{w}$ was defined previously by $\underline{w} = d\underline{x}/dt$. We saw that $\underline{w}$ does not transform like the space component of a four-vector. The reason was that t is not an invariant variable. To construct a velocity four-vector, one must take the derivative of $x = (x^\mu)$ with respect to some invariant variable related to time. A natural choice is the proper time τ defined by

$$d\tau^2 = \frac{1}{c^2} dx^2 = \frac{1}{c^2} (c^2 dt^2 - dx^2 - dy^2 - dz^2) \quad (4.1)$$

Here $dx^2 = dx_\mu dx^\mu$ is evidently an invariant so that $d\tau$ is invariant. We can further write

$$d\tau^2 = dt^2 \left[1 - \frac{1}{c^2} \frac{dx^2 + dy^2 + dz^2}{dt^2} \right]$$

so that

$$d\tau = dt \left[1 - \frac{1}{c^2} \left(\frac{d\underline{x}}{dt} \right)^2 \right]^{\frac{1}{2}} = dt / \gamma(w) \quad (4.2)$$

In the rest frame defined by $w = 0$, the times are equal which explains the term proper time. The factor γ in (4.2) is, of course, the same γ as that appearing in the time dilatation equation (1.2).

The four-velocity $u = (u^\mu)$ will thus be defined by

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \cdot \frac{dt}{d\tau} = \gamma(w) \frac{dx^\mu}{dt} \quad (4.3)$$

It is then $u = \gamma(w)(c, \underline{w})$. The space component of u^μ differs from $\underline{w}$ by the γ -factor:

$$\underline{u} = \gamma(w) \underline{w} \quad (4.4)$$

In the rest frame $w = 0$ u^μ has only the 0-component:
 $u^\mu = (c, \underline{0})$.

Since u is a four-vector, u^2 must be an invariant.
 That this is the case can be seen by an explicit calculation:

$$u^2 = \gamma^2(w) \cdot (c^2 - w^2) = c^2 .$$

The length of the four-velocity is thus c and it is a timelike four-vector.

The transformation equations (3.8) of an arbitrary four-vector $a = (a^0, \underline{a})$ can be written in a more compact form, if $\underline{v}$ is replaced by the associated four-velocity

$$u = \gamma(v) (c, \underline{v}) . \quad (4.5)$$

Replacing in (3.8) γ by u^0/c and $\gamma \underline{\beta}$ by $\underline{u}/c$, (3.8) becomes

$$\underline{a}' = \underline{a} + \frac{\underline{u}}{c} \left(\frac{\underline{u} \cdot \underline{a}}{u^0 + c} - a^0 \right) \quad (4.6)$$

$$a^{0'} = \frac{1}{c} (a^0 u^0 - \underline{a} \cdot \underline{u}) = \frac{1}{c} a \cdot u . \quad (4.7)$$

Notice, in particular, that $a^{0'}$ has been written in a formally invariant form, although it is not an invariant. We have been able to do this since the velocity parameter has been introduced in the covariant from (4.5). Later on it is seen on many occasions that writing quantities in an invariant form often gives an economical way to carry out Lorentz transformations. Even here one may go further and write $|\underline{a}|$ in terms of invariants. Since $a^2 = a^{02} - \underline{a}^2$ is an invariant, we have immediately

$$|\underline{a}'|^2 = (a^{0'})^2 - a^2 = \frac{1}{c^2} (a \cdot u)^2 - a^2 .$$

Using $u^2 = c^2$ this is symmetrically

$$\begin{aligned} |\underline{a}'|^2 &= \frac{1}{c^2} \left[(a \cdot u)^2 - a^2 u^2 \right] \\ &= - \left(g_{\mu\nu} - \frac{u_\mu u_\nu}{c^2} \right) a^\mu a^\nu \end{aligned} \quad (4.8)$$

When $\underline{v} = 0$ or $u^\mu = (c, 0)$ the transformation should be an identity and (4.7) and (4.8) reduce to a^0 and $|\underline{a}|^2$, respectively. This is a convenient way of memorizing the signs and constants.

As an illustration of the preceding formulae we reconsider the transformation of three-velocities. The velocities involved were the velocity $\underline{v}$ of S' in S , a velocity $\underline{w}$ in S and its transform $\underline{w}'$ in S' . Each of these is now associated with a four-velocity,

$$\begin{aligned} u^\mu &= \gamma(v)(c, \underline{v}) \\ u_1^\mu &= \gamma(w)(c, \underline{w}) \\ u_2^\mu &= \gamma(w')(c, \underline{w}') \end{aligned} \quad (4.9)$$

The velocity transformation is now immediately obtained from (4.6) by changing $\underline{a}$ to $\underline{u}_1$ and $\underline{a}'$ to $\underline{u}_2$:

$$\underline{u}_2 = \gamma(w')\underline{w}' = \gamma(w)\underline{w} + \gamma(w)\gamma(v)\underline{v} \left[\frac{\gamma(v)}{\gamma(v)+1} \frac{\underline{v} \cdot \underline{w}}{c^2} - 1 \right]. \quad (4.10)$$

To apply this one first has to calculate $|\underline{w}'|$ or $\gamma(w')$. These can be obtained by squaring (4.10), but much more simply directly from (4.8). This gives

$$\begin{aligned} |\underline{u}_2|^2 &= \frac{\underline{w}'^2}{1 - \underline{w}'^2/c^2} = \frac{1}{c^2} (u_1 \cdot u)^2 - c^2 \\ &= c^2 \gamma^2(v) \gamma^2(w) (1 - \underline{v} \cdot \underline{w}/c^2)^2 - c^2. \end{aligned}$$

After some simple algebraic manipulations one obtains

$$|\underline{w}|^2 = \frac{|\underline{v}-\underline{w}|^2 - |\underline{v} \times \underline{w}|^2 / c^2}{(1 - \underline{v} \cdot \underline{w} / c^2)^2} \quad (4.11)$$

or equivalently

$$\gamma(w') = \gamma(v)\gamma(w)(1 - \underline{v} \cdot \underline{w} / c^2) . \quad (4.12)$$

These formulae were derived already by Einstein. When (4.12) is inserted to (4.10) one arrives at the previously derived transformation formula (2.3) of three-velocities.

Using Eq. (4.12) one can show that the geometry of the space of relativistic velocities is that of a sphere with imaginary constant curvature. Eqs. (3.12) and (4.12) imply

$$\cosh \frac{r'}{c} = \cosh \frac{s}{c} \cosh \frac{r}{c} - \sinh \frac{s}{c} \sinh \frac{r}{c} \cos \theta . \quad (4.13)$$

The rapidities related to $\underline{v}$, $\underline{w}$ and $\underline{w}'$ are s , r and r' , and θ is the angle between $\underline{v}$ and $\underline{w}$. Comparing with the cosine theorem of spherical trigonometry in Appendix B, one sees that (4.13) is identical with it for imaginary r, r' and s .

5. Four-momentum

The states of a single free particle in relativistic quantum mechanics are characterized by their four-momentum (in addition to spin, which we neglect). The basic four-vector in particle kinematics is thus four-momentum $p = (p^\mu)$. In order to define, in a consistent manner, the four-momentum of a particle or of a system of particles, the concepts of mass and energy have to be subjected to a careful analysis. This can be found in text-books of special relativity [Rindler 60]. The four-momentum can be defined as rest mass m of the particle times its four-velocity:

$$p = mu = m\gamma(v)(c, \underline{v}) \quad (5.1)$$

Alternatively, p is expressed in terms of energy and three-momentum as follows:

$$p = \left(\frac{E}{c}, \underline{p}\right) . \quad (5.2)$$

The equivalence of the definitions (5.1) and (5.2) is not a trivial matter since it involves, for instance, the equivalence of energy and matter. With proper interpretation of m , $\underline{v}$, E and $\underline{p}$ these equations apply also to a system of particles.

The simultaneous validity of (5.1) and (5.2) implies the equations

$$E = m\gamma(v)c^2 \quad (5.3)$$

$$\bar{p} = m\gamma(v)\underline{v} . \quad (5.4)$$

Thus, in the equation relating three-velocity and three-momentum, the velocity-dependent quantity

$$m\gamma(v) = \frac{m}{\sqrt{1-v^2/c^2}}$$

has replaced the rest mass. In special relativity, the rest mass m and the inertial mass $m\gamma(v)$ are different for $v \neq 0$. According to (5.3) the inertial mass is essentially equivalent to the total energy of the particle. For $v = 0$ the rest mass corresponds to a rest energy $E = mc^2$, which is the classic relation derived by Einstein.

In particle physics, the energy and momentum are the basic quantities. The velocity $\underline{v}$ and $\gamma(v)$ are obtained in terms of E and $\underline{p}$ by inverting (5.3) and (5.4):

$$\begin{aligned} \underline{\beta} &= \underline{v}/c = c\underline{p}/E \\ \gamma &= \gamma(v) = E/mc^2 \end{aligned} \quad (5.5)$$

$$\underline{\beta}\gamma = \underline{p}/mc.$$

The definition of $\underline{p}$ also implies that its invariant length is given by

$$\underline{p}^2 = m^2 \underline{u}^2 = m^2 c^2 = \left(\frac{E}{c}\right)^2 - \underline{p}^2 \quad (5.6)$$

so that $\underline{p}$ is a timelike four-vector. Equivalently,

$$E^2 = c^2 \underline{p}^2 + (mc^2)^2. \quad (5.7)$$

This equation relates energy, momentum and rest mass and it is perhaps the most frequently used equation in particle kinematics. When m is given and E and $\underline{p}$

satisfy (5.7), $\underline{p}$ is said to be on the mass shell. For small $|\underline{p}|$, energy contains mainly rest energy,

$$E = mc^2 + \frac{\underline{p}^2}{2m} + \dots \quad (5.8)$$

For large $|\underline{p}|$ the contribution from the mass is small,

$$E = c|\underline{p}| + \frac{m^2 c^3}{2|\underline{p}|} + \dots \quad (5.9)$$

Since $\underline{p}$ is a four-vector, the Lorentz transformation equations (4.6) and (4.7) apply if $\underline{a}$ is replaced by $\underline{p}$. In many cases the moving frame S' is the rest frame of a particle (or of a system of particles). Then, instead of u , it may be convenient to use the momentum $\underline{p} = mu$ of the moving particle as a parameter describing the motion of S' in S . Replacing in (4.6) and (4.7) u^0 by E/mc and $\underline{u}$ by $\underline{p}/m$ one obtains

$$\underline{a}' = \underline{a} + \underline{p} \left[\frac{\underline{a} \cdot \underline{p}}{m(E+mc^2)} - \frac{a^0}{mc} \right] \quad (5.10)$$

$$a^{0'} = \frac{1}{mc} \underline{a} \cdot \underline{p} \quad (5.11)$$

Here $\underline{a} = (a^0, \underline{a})$ is expressed in the stationary frame S and $\underline{a}' = (a^{0'}, \underline{a}')$ in the rest frame of the particle moving in S with momentum $\underline{p}$. In practice, $\underline{a}$ may be the four-momentum of some other particle.

Let us denote the Lorentz transformation contained in (5.10) and (5.11) by $L(\underline{p})$:

$$\underline{a}' = L(\underline{p})\underline{a} \quad (5.12)$$

Then it is easy to see by explicit calculation that

$a = p = (E/c, \underline{p})$ implies

$$L(p)(E/c, \underline{p}) = (mc, \underline{0}) . \quad (5.13)$$

Conversely, the Lorentz transformation $L^{-1}(p)$ is obtained from (5.10-11) by interchanging a' and a and reversing the sign of $\underline{p}$. It transforms the particle state $(mc, \underline{0})$ to motion:

$$L^{-1}(p)(mc, \underline{0}) = (E/c, \underline{p}) .$$

For this reason $L^{-1}(p)$ is often called a boost.

6. Units and conventions

To facilitate the transition from special relativity to particle kinematics the light velocity c has been explicitly written in all formulae. From here on we shall everywhere set $c = 1$, as is customary in particle physics. This is just a formal simplification and simply means choosing c as the unit of velocity.

According to the relation $E = mc^2 = m$, the rest mass is equivalent to energy and thus mass is expressed in energy units. The basic energy unit is $J = Nm = \text{kgm}^2/\text{s}^2$, but in high energy physics the electron volt is more common:

$$1 \text{ GeV} = 10^3 \text{ MeV} = 10^9 \text{ eV} = 1.602 \cdot 10^{-10} \text{ J} \quad (6.1)$$

The inverse relation is

$$1 \text{ erg} = 10^{-7} \text{ J} = 624 \text{ GeV} = 624 \cdot 10^9 \text{ eV}.$$

Accelerators with energies in the 100 GeV range give macroscopic energies ($\approx$ erg) to microscopic particles. The conversion factor between kg and GeV is obtained by solving $1 \text{ GeV} = mc^2 = m$ for m . The result is

$$1 \text{ GeV} = 1.7827 \cdot 10^{-27} \text{ kg}. \quad (6.2)$$

For instance, the proton mass $m_p = 1.673 \cdot 10^{-27} \text{ kg}$ is 0.938 GeV in eV units; similarly for other particles.

The energies of accelerated particle beams are always expressed in eV units. The choice of the energy variable is not unique and the following alternatives are in use:

1. Kinetic energy of the particle, $T = E - m$, is mostly used in the domain where the rest energy is larger than the kinetic energy. T is the normal variable nuclear physics.
2. Total energy E of the particles is used in the high energy domain ($E \gtrsim 1$ GeV).
3. Momentum p of the particles (in units MeV/c or GeV/c) is normally used to express the energy of an experiment. The separators producing mono-energetic particle beams separate most directly in momentum and not in energy. Note the convention of writing the unit in the form GeV/c to indicate a momentum.

All these variables are, of course, equivalent. They become asymptotically equal when $E \gg m$, i.e. for $E \gtrsim 1$ GeV for hadrons, $E \gtrsim 1$ MeV for electrons and always for photons.

A convention analogous with $c = 1$ is to $\hbar = 1$. This is not relevant to pure kinematics where only momentum vectors appear. It is convenient when one calculates lengths (cross sections) or lifetimes. As previously $E = mc^2 = m$ permitted one to express mass in kg or GeV, the relations

$$\lambda = \frac{\hbar}{mc} = \frac{1}{m} \quad (6.3)$$

and

$$\frac{\lambda}{c} = \frac{\hbar}{mc^2} = \frac{1}{m} \quad (6.4)$$

can now be used to express length (meter) and time (sec) in $1/\text{GeV}$. To calculate the conversion factors, the value of m in kg corresponding to 1 GeV from (6.2) is inserted to (6.3) and (6.4), together with known values of $\hbar$ and c . The results are

$$1/\text{GeV} = 0.19733 \text{ fm} = 0.19733 \cdot 10^{-15} \text{ m} \quad (6.5)$$

$$1/\text{GeV} = 6.5822 \cdot 10^{-25} \text{ sec} \quad (6.6)$$

In harmony with these relations one often tabulates $\hbar c$ and $\hbar$ as follows

$$\hbar c = 197.33 \text{ MeV} \cdot \text{fm}$$

$$\hbar = 6.5822 \cdot 10^{-22} \text{ MeV sec} .$$

Length in particle physics is mostly needed in connection with areas or cross sections. The normal unit of cross sections is millibarn = $\text{mb} = 10^{-31} (\text{meter})^2 = 0.1 \text{ fm}^2$. From (6.5) one then obtains the very practical relation

$$\left(\frac{1}{\text{GeV}}\right)^2 = 0.38939 \text{ mb} = \frac{1}{2.568} \text{ mb} . \quad (6.7)$$

With the known values of proton and pion masses m_p and m_π one has, equivalently

$$\frac{1}{m_p^2} = 0.44232 \text{ mb}$$

$$\frac{1}{m_\pi^2} = 19.987 \text{ mb} .$$

7. Reference frames for collision processes

We have so far considered general properties of Lorentz transformations. In this section, a set of reference frames defined by the properties of the initial state of a collision process is introduced. Later on, several frames depending on the reaction products will also be considered.

In a two-particle collision process, two particles a and b with four-momenta $p_a = (E_a, \underline{p}_a)$ and $p_b = (E_b, \underline{p}_b)$ collide. The values of p_a and p_b are normally, up to some small experimental errors, fixed by the experimental conditions. Different frames can be defined by requiring $\underline{p}_a$ or $\underline{p}_b$ to have some special values. The following are most frequently used:

Fig.II.7.1.

1. Laboratory system (LS) is defined as the system in which the experiment is carried out and all energies and momenta measured. It is fixed by the experimental outlay, which may involve either a beam of particles hitting a stationary target or colliding beams (see below). LS is, in a sense, the primary system. From this the momenta and energies are then, for different reasons, transformed to other systems. We shall denote LS quantities by an index L : $E^L, \underline{p}^L$.

2. Center-of-momentum system (CMS) is defined as a system in which

$$\underline{p}_a^* + \underline{p}_b^* = 0 . \quad (7.1)$$

The CMS quantities will be denoted by an asterisk.

3. Target system (TS) is defined as a system in which

$$\underline{p}_b^T = 0 \quad (7.2)$$

Most experiments are carried out on stationary targets and for all these TS coincides with LS. In fact, in standard terminology our TS is called the laboratory system. However, for colliding beam experiments TS and LS do not coincide and we have preferred to keep the terminology unambiguous.

4. Beam system (BS) is defined by

$$\underline{p}_a^B = 0 \quad (7.3)$$

From a theoretical point of view BS and TS are equivalent.

5. Colliding beam system (CBS) is defined as a frame in which two particles of equal mass and momentum

Fig.II.7.2. $(\underline{p}_a^{CB} = \underline{p}_b^{CB})$ collide so that their momenta form an angle $\pi - \vartheta$, as shown in Fig. 7.2. For colliding beam experiments CBS coincides with LS, for $\vartheta = 0$ it even coincides with CMS.

In practice one constantly needs to transform from one system to another. For this purpose one has to find the relative velocities between the frames. We shall see that this transformation is most simply carried out by using invariants.

8. Relations between quantities in center-of-momentum and target systems

The convention $c = 1$ puts (5.5) in the form

$$\begin{aligned}\underline{v} &= \underline{p}/E \\ \gamma &= E/m \\ \gamma \underline{v} &= \underline{p}/m .\end{aligned}\tag{8.1}$$

In writing a transformation between two frames, (8.1) give the relevant factors most simply.

The invariant mass (or energy) of a two particle system a, b is the total four-momentum squared

$$\begin{aligned}s &= (\underline{p}_a + \underline{p}_b)^2 = (E_a + E_b)^2 - (\underline{p}_a + \underline{p}_b)^2 \\ &= m_a^2 + m_b^2 + 2(E_a E_b - \underline{p}_a \cdot \underline{p}_b) .\end{aligned}\tag{8.2}$$

In particular, in CMS (8.2) shows that $\sqrt{s}$ is equal to total CMS energy

$$E_a^* + E_b^* = \sqrt{s} .\tag{8.3}$$

In CMS the system a, b , having rest mass $\sqrt{s}$, is at rest. Its velocity in TS is the relative velocity of CMS and TS. By definition $\underline{p}_b^T$ vanishes, and (8.1) give

$$\underline{v} = \frac{\underline{p}_a^T}{E_a^T + m_b} \tag{8.4}$$

CMS $\rightarrow$ TS

$$\gamma = \frac{E_a^T + m_b}{\sqrt{s}} \tag{8.5}$$

The simplest of these parameters is, in this case,

$$\gamma \underline{v} = \underline{p}_a^T / \sqrt{s} .$$

An experiment is usually specified quoting the momentum p_a^T of the incident beam in laboratory. According to (8.2) the invariant mass is

$$s = m_a^2 + m_b^2 + 2m_b E_a^T. \quad (8.6)$$

In going from TS to CMS, the target particle gets the momentum $-m_b \gamma \underline{v}$ and energy $m_b \gamma$, or

$$\begin{aligned} p_b^* &= -\frac{m_b}{\sqrt{s}} p_a^T. \\ E_b^* &= \frac{m_b^2 + m_b E_a^T}{\sqrt{s}} \end{aligned} \quad (8.7)$$

In CMS, of course, p_a^* equals $-p_b^*$. Then (8.3), (8.6) and (8.7) give

$$E_a^* = \frac{m_a^2 + m_b E_a^T}{\sqrt{s}}. \quad (8.8)$$

The transformation between the beam system BS and CMS results from (8.4), (8.5) simply by exchanging a and b . For $TS \rightarrow BS$ the parameter γ is E_a^T/m_a .

Next we consider then the transformation of the energy E^T and momentum $\underline{p}^T$ of an arbitrary final state particle from the target system to the center-of-mass system. Choosing again p_a^T parallel to the z -axis, $\underline{p}^T$ may be expressed either in Cartesian or polar coordinates (Fig. 7.3):

$$\underline{p}^T = (p_x^T, p_y^T, p_z^T) \quad (8.9)$$

$$\underline{p}^T = p^T (\sin\theta^T \cos\phi^T, \sin\theta^T \sin\phi^T, \cos\theta^T) \quad (8.10)$$

The transformation of the Cartesian components to the CMS is

$$\begin{aligned}
 p_x^* &= p_x^T \\
 p_y^* &= p_y^T \\
 p_z^* &= \gamma (p_z^T - v E^T) \\
 E^* &= \gamma (-v p_z^T + E^T)
 \end{aligned} \tag{8.11}$$

where v and γ are given in (8.4-5). Among the polar coordinates φ^T is defined by $\text{tg}\varphi^T = p_y^T/p_x^T$ so that

$$\varphi^* = \varphi^T \tag{8.12}$$

In general, the azimuthal angle about an axis is invariant under pure Lorentz transformations along this axis. The transform θ^* of θ^T is obtained by requiring the transverse component $\sqrt{(p_x^T)^2 + (p_y^T)^2}$ of $\underline{p}^T$ to be invariant and the longitudinal component p_z^T to transform according to (8.11):

$$\underline{p}^* \sin\theta^* = \underline{p}^T \sin\theta^T \tag{8.13}$$

$$\underline{p}^* \cos\theta^* = \gamma (\underline{p}^T \cos\theta^T - v E^T) . \tag{8.14}$$

By dividing one obtains after some modifications

$$\text{tg}\theta^* = \frac{\sin\theta^T \sqrt{1-v^2}}{\cos\theta^T - v/v_p} \tag{8.15}$$

where $v_p = \underline{p}^T/E^T$ is the velocity of the particle in question in CMS. Eq. (8.15) shows how a polar angle with respect to an axis transforms under Lorentz transformations along this axis. We shall analyze this formula in more detail later. The transform $\underline{p}^*$ of $\underline{p}^T$ is, finally, obtained most simply from (8.11).

9. Energies and momenta in terms of invariants

According to the relativity principle, physical laws are invariant under Lorentz transformations. To explicitly guarantee that this is obeyed, one writes physical quantities and relations between them in terms of invariants. A second significant use of invariants is to write combinations of non-invariant quantities in terms of invariants. Once a sufficient number of the non-invariants are fixed by choosing a frame, the rest can be found.

The scalar product pp' of two four-vectors is invariant. Out of n four-vectors one can write $\frac{1}{2}n(n+1)$ such quantities $p_1^2, p_1p_2, p_1p_3, \dots$. An equivalent set is formed by the one- and two-particle invariant masses $m_i^2 = p_i^2, s_{ij} = (p_i + p_j)^2$. These are clearly unique, but are in general not independent. The inverse problem of finding when there exist four-vectors $p_1, \dots, p_n$ corresponding to given invariants, and of explicitly constructing them, is of considerable difficulty. It will be discussed later on in length. We here describe some simple uses of the invariants m_a^2, m_b^2, s related to the four-momenta p_a, p_b of the initial state of the collision.

The four-momenta p_a, p_b define three independent invariants, $p_a^2 = m_a^2, p_b^2 = m_b^2$ and $(p_a + p_b)^2 = s$. One should thus be able to characterize all the kinematics

of the initial state in terms of s and the masses. We shall work out the connection between these invariant variables and noninvariant variables E_a, p_a and E_b, p_b separately for TS and CMS. We adhere to the convention that the length of a three-vector is denoted by the corresponding capital letter.

a) Target system.

For TS we have, to begin with, $p_b^T = 0$ and $E_b^T = m_b$. From (8.6) we have also

$$E_a^T = \frac{s - m_a^2 - m_b^2}{2m_b} \quad (9.1)$$

so that

$$\begin{aligned} (p_a^T)^2 &= (E_a^T)^2 - m_a^2 \\ &= \frac{(s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2}{4m_b^2} . \end{aligned}$$

We write this result in the form

$$p_a^T = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2m_b} \quad (9.2)$$

Here appears the kinematical function

$$\lambda(x, y, z) = (x - y - z)^2 - 4yz \quad (9.3)$$

$$= x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \quad (9.4)$$

$$= [x - (\sqrt{y} + \sqrt{z})^2] [x - (\sqrt{y} - \sqrt{z})^2] \quad (9.5)$$

$$= (\sqrt{x} - \sqrt{y} - \sqrt{z})(\sqrt{x} + \sqrt{y} + \sqrt{z})(\sqrt{x} - \sqrt{y} + \sqrt{z})(\sqrt{x} + \sqrt{y} - \sqrt{z})$$

$$(9.6)$$

$$= x^2 - 2(y+z)x + (y-z)^2 . \quad (9.7)$$

Useful special cases of these are

$$\lambda(x,y,y) = x(x-4y) \quad (9.8)$$

$$\lambda(x,y,0) = (x-y)^2 \quad (9.9)$$

From (9.4) it is seen that λ is invariant under all permutations of its arguments. The basic motivation for the introduction of λ will be clarified in section 11. Sometimes λ is called the triangle function since $\frac{1}{4}\sqrt{-\lambda(x,y,z)}$ is the area of a triangle with sides $\sqrt{x}$, $\sqrt{y}$ and $\sqrt{z}$.

According to (9.5) we have

$$\lambda(s, m_a^2, m_b^2) = [s - (m_a + m_b)^2][s - (m_a - m_b)^2]. \quad (9.10)$$

Thus p_a^T in (9.2) is real if

$$\sqrt{s} \geq m_a + m_b. \quad (9.11)$$

The threshold value $m_a + m_b$ is the smallest value $\sqrt{s}$ can attain. The pseudothreshold value $m_a - m_b$ is important for some more advanced kinematical considerations. The threshold $m_a + m_b$ also appears if one writes the kinetic energy T_a of particle a in TS in terms of s :

$$T_a = E_a^T - m_a = \frac{s - (m_a + m_b)^2}{2m_b}$$

b) Center-of-momentum system

In CMS $\underline{p}_a^* + \underline{p}_b^* = 0$ so that

$$p_a^* = p_b^* = p^* \quad (9.12)$$

and, according to (8.3),

$$\sqrt{s} = E_a^* + E_b^*. \quad (9.13)$$

The invariant energy is thus equal to the total energy in CMS. Inserting (9.12) to (9.13) one obtains

$$\sqrt{s} - \sqrt{(p^*)^2 + m_a^2} = \sqrt{(p^*)^2 + m_b^2}.$$

Squaring once this gives

$$E_a^* = \frac{s + m_a^2 - m_b^2}{2\sqrt{s}}. \quad (9.14)$$

Squaring a second time results in

$$p^* = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2\sqrt{s}}. \quad (9.15)$$

The remaining energy E_b^* is then obtained from

$$E_b^* = \sqrt{s} - E_a^* :$$

$$E_b^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}}. \quad (9.16)$$

Note how in (9.14) and (9.16) E_i^* has in the numerator a plus-sign in front of the mass m_i^2 .

It is easy to verify that the expressions so obtained for TS and CMS quantities satisfy the relations (8.8) and (8.10) obtained by explicit Lorentz transformations.

Example 1. It is useful to have an idea of the numerical magnitudes of various kinematical quantities for normal experiments, which are characterized by the value p_a^T of the incident momentum. When the momenta involved are sufficiently large, for instance, $p_a^T \gtrsim 5 \text{ GeV}/c$, one may neglect the rest masses and assume that $E = P$.

Then (8.9) gives $s \approx 2m_b p_a^T$. If one takes into account

(8.6)

that the target is - practically without exception - a nucleon ($m_b \approx 1 \text{ GeV}$) one may estimate

$$s \approx 2p_a^T \quad (\text{in GeV units}) \quad (9.17)$$

For $p_a^T = 19 \text{ GeV}/c$ this gives $s \approx 38 \text{ GeV}^2$ while the correct value is $s = 37.45 \text{ GeV}^2$. For the CMS quantities one has similarly

$$E_a^* \approx E_b^* \approx p_a^* \approx p_b^* \approx \frac{1}{2}\sqrt{s} \approx \sqrt{p_a^T/2} \quad (9.18)$$

For a pp initial state with $p_a^T = 19 \text{ GeV}/c$ the exact values are $E_a = E_b = 3.06 \text{ GeV}$, $p_a = p_b = 2.91 \text{ GeV}/c$ while the approximation (9.18) would give $\sqrt{p_a^T/2} = 3.08 \text{ GeV}$. For qualitative purposes the accuracy is adequate.

Example 2. According to (9.17) the useful energy $\sqrt{s}$ of an accelerator behaves like $\sqrt{s} \approx \sqrt{2p_a^T}$. Increasing the incident momentum of a particle hitting a stationary target by a factor four, for instance, will increase the useful energy $\sqrt{s}$ only by a factor two. The rest will go to the useless energy of the motion of CMS in TS. This simple consideration lies behind the motivation for constructing colliding beams. If two beams with particles of momentum p^{CB} collide, head-on in laboratory, the total energy $\sqrt{s}$ will be within the approximation of Example 1 equal to $2p^{CB}$. According to (9.17) this corresponds to an effective momentum p_{eff}^T of a beam hitting a stationary target which equals

$$p_{eff}^T \approx 2(p^{CB})^2 \quad (9.19)$$

For $p^{CB} \approx 28 \text{ GeV}/c$, the effective momentum would be $p_{\text{eff}}^T \approx 1570 \text{ GeV}/c$. The gain is impressive.

Example 3. Starting from $v = p/E$, $\gamma = E/m$ we can write the velocity of CMS in TS in an invariant form

$$v = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{s - m_a^2 + m_b^2} \quad (9.20)$$

CMS $\rightarrow$ TS

$$\gamma = \frac{E_b^*}{m_b} = \frac{s - m_a^2 + m_b^2}{2m_b \sqrt{s}} \quad (9.21)$$

Similarly, the velocity v_a^T of particle a in TS is given by

$$v_a^T = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{s - m_a^2 - m_b^2} \quad (9.22)$$

$$\gamma_a^T = \frac{s - m_a^2 - m_b^2}{2m_a m_b} = \frac{p_a \cdot p_b}{m_a m_b} \quad (9.23)$$

These are also the parameters for the transform BS $\rightarrow$ TS. Note how v and v_a^T only differ in the sign of m_b^2 ; both are for large s very close to one. On the other hand, for large s γ is proportional to $\sqrt{s}$ and γ_a^T to s .

10. Colliding beam system

The introduction of the colliding beam system (CBS) has become useful only now that one is able to carry out experiments with colliding beams. Our definition (Fig.II.7.1) of the CBS describes directly, for instance, the experimental situation in the CERN Intersecting Storage Rings (ISR), if the two colliding proton beams have equal momenta. For unequal momenta slight and obvious modifications are necessary. In the ISR the maximum momenta are 28 GeV/c and the angle of intersection is $\vartheta = 14.77^\circ \approx 0.2578$ rad.

One should again emphasize that for colliding beam experiments of the ISR type the CBS coincides with the laboratory system. From the CBS one may then Lorentz transform the measured momenta and energies to any frame, for instance to CMS or to the target system (which in pre-ISR times was mostly called the laboratory system). We shall in the following only consider the transformation to CMS.

For the transformation to CMS one again needs the velocity of the CMS in the CBS (= LS). As the velocity of the CMS in any frame is $(\underline{p}_a + \underline{p}_b)/(E_a + E_b)$, the velocity v_{CB} of the CMS in CBS is according to Fig.(8.1) given by

$$v^{CB} = v_a \sin \frac{\vartheta}{2} \quad (10.1)$$

where $v_a (= v_b)$ is the velocity of particle a in the laboratory frame. Correspondingly, with $\gamma_a^{-2} = 1 - v_a^2$.

$$\gamma^{CB} = \frac{\gamma_a}{\sqrt{\sin^2 \frac{\vartheta}{2} + \gamma_a^2 \cos^2 \frac{\vartheta}{2}}} \quad (10.2)$$

The velocity is oriented along $\underline{p}_a + \underline{p}_b$ as shown in Fig. 8.1. and the Lorentz transformation is easy to carry out.

For a colliding beam experiment the invariant s is given by

$$s = (E_a + E_b)^2 - |\underline{p}_a + \underline{p}_b|^2 = 4m^2 + 4(P^{CB})^2 \cos^2 \frac{\vartheta}{2} \quad (10.3)$$

where $P^{CB} = P_a^{CB} = P_b^{CB}$ is the laboratory momentum of the particles a and b . To second order in

$$\sqrt{s} = 2E^{CB} \left(1 - \frac{\vartheta^2}{8}\right) \quad (10.4)$$

where E^{CB} is the laboratory energy of the particles a and b and the approximation $P^{CB} \approx E^{CB}$ has been used. Since $\vartheta^2/8 \approx 0.0083$ for the ISR, the correction to the total CMS energy $\sqrt{s}$ arising from the fact that the two beam intersect at an angle ϑ is numerically very small. Inverting (10.3) we have, finally

$$\begin{aligned} P^{CB} &= \frac{\sqrt{s - 4m^2}}{2 \cos \frac{\vartheta}{2}} \\ E^{CB} &= \frac{\sqrt{s - 4m^2} \sin^2 \frac{\vartheta}{2}}{2 \cos \frac{\vartheta}{2}} \end{aligned} \quad (10.5)$$

Exercises to Chapter II

1. The Stanford linear electron accelerator is 3 km long and accelerates the electrons to an energy of $E = 20 \text{ GeV}$. The energy of the electron is linearly proportional to the distance it has covered, its mass is $m_e = 0.511 \text{ MeV}$. What is the total length of the accelerator seen by the electron?
2. Derive the transformation formula (2.3) of the three-velocity by using the Lorentz transformation properties of the four-velocity.
3. a) What are the maximum values of v and γ if one wants to write $E = m + p^2/2m$ so that the error is less than $\epsilon \cdot p^2/2m$? b) What are the minimum values of v and γ , if one wants to write $E = p$ so that the error is less than $\epsilon \cdot p$? Formulate in words the conditions for the validity of the non-relativistic and relativistic approximations if $\epsilon = 1 \%$.
4. An electron, a pion and a proton have each a momentum $1 \text{ GeV}/c$. What are the times these particles need to cover a distance of 3 m ?
5. The four-acceleration a^μ is defined by the equation

$$a^\mu = \frac{du^\mu}{d\tau}$$

- a) Determine a^0 and $\underline{a}$, b) Prove that $a \cdot u = 0$, c) Evaluate the invariant $a \cdot a$ if the particle to which a^μ refers follows a straight path.

6. In a proton-proton experiment at 19 GeV/c ($\sqrt{s} = 6.12$ GeV, $m_p = 0.938$ GeV) one observes in the final state a proton with a momentum 4 GeV/c at an angle 30° relative to the beam axis. What are the energy and momentum of the produced proton in the center-of-momentum system?
7. Consider in a proton-proton experiment at a fixed incident momentum P_a^T a momentum vector perpendicular to the beam direction in the center-of-momentum system. If its length varies between zero and its maximum value (determine this maximum), how does the corresponding momentum vector vary in the target system? Draw a figure with the numerical values of Problem 6.
8. Check the correctness of the calculations in Problems 6 and 7 by computing the magnitude of the transformed momentum vector both from $P = (p_x^2 + p_y^2 + p_z^2)^{1/2}$ and from $P = (E^2 - m^2)^{1/2}$.
9. Suppose two particles have equal velocities (in magnitude and direction) in some Lorentz system. How are their velocities in any other Lorentz frame related? What if velocities are replaced by three-momenta?

III. Phase Space

1. Definition of phase space

So far we have essentially only considered properties of the initial state and Lorentz transformations of unconstrained final state momentum vectors. Turning now our attention also to the final state of a particle reaction $p_a + p_b \rightarrow p_1 + \dots + p_n$ (Fig. I.1.1) we have to impose on the final state momentum vectors four-momentum conservation:

$$\begin{aligned} E_a + E_b &= \sum_{i=1}^n E_i \\ p_a + p_b &= \sum_{i=1}^n p_i \end{aligned} \quad (1.1)$$

$$\text{with } E_\ell^2 = p_\ell^2 + m_\ell^2, \quad \ell = a, b, 1, \dots, n$$

The m_i are fixed particle masses. Due to four-momentum conservation the n momentum vectors p_i cannot vary arbitrarily for a fixed initial state, but have to satisfy the four conditions (1.1). We shall call the $3n$ -dimensional space of the unconstrained final state momentum vectors p_i the momentum space. The conditions (1.1) define in this space a $(3n-4)$ -dimensional surface which will be called the phase space. Sometimes the terms momentum space and phase space are used synonymously for the $3n$ -dimensional space and the $(3n-4)$ -dimensional space is called a surface of constant energy and momentum. We

have adopted the definitions above in order to fix a concise and unique terminology. The structure of the momentum space is simple, while the structure of phase space is extremely complicated. Much of our subsequent effort will go to clarifying this structure.

There is a formal analogy between the phase space in statistical physics and that in particle physics. This is due to the fact that both in relativistic quantum statistics and in the study of final states of particle collisions a state is determined by a set of four-momenta $p_1, \dots, p_n$. In order to discuss this connection we shall review here some concepts of statistical mechanics. These remarks are not indispensable for later developments in this book and they are included here only to give some wider perspective.

a) Classical statistical mechanics. Consider n particles in a box of volume V . Each of the particles has a momentum coordinate p_i and a position coordinate x_i , $i = 1, \dots, n$. The phase space is then defined as the $6n$ -dimensional space of the x_i and p_i . At a fixed time the state of the particles in the box corresponds to a point in the phase space. In statistical mechanics, the state of the system is not followed in detail but instead one considers the probability density of the points in phase space. For an isolated system this density $P = P(E)$ only depends on the total energy $E = E(p_i, x_i)$ of the system. Different densities define different ensembles:

microcanonical ensemble (constant energy): $P(E) = \delta(E-E_0)$,
canonical ensemble (constant temperature): $P(E) = e^{-E/T}$.

In the microcanonical ensemble the basic function from which all thermodynamical quantities can be derived is the volume of the accesible phase space:

$$N(E_0) = \int \prod_i d^3x_i d^3p_i \delta(E-E_0) \quad (1.2)$$

For an ideal gas, $E = \sum_i p_i^2/2m_i$ does not depend on x_i and

$$N(E_0) = V^n \int d^3p_i \delta(E-E_0) . \quad (1.3)$$

In the canonical ensemble the basic function is the partition function

$$Z(T) = \int \prod_i d^3x_i d^3p_i e^{-E/T}$$

For an ideal gas, $Z(T)$ is the Laplace transform of $N(E)$:

$$Z(T) = \int_0^\infty dE e^{-E/T} N(E) . \quad (1.4)$$

b) Quantum statistics. In quantum mechanics the uncertainty principle ($\Delta^3x \Delta^3p = h^3$) divides the previously considered phase space into cells of volume h^3 . Each cell contains one state, which can be defined e.g. by giving the momentum p . In problems of particle physics the states are so closely spaced that the discrete sum can be replaced by an integral over a continuous variable. This is seen as follows. Consider again free particles in a box of volume $V = L^3$. Their wave functions

are $V^{1/2} \exp(i\mathbf{p} \cdot \mathbf{x})$. Boundary conditions require that the components of $\mathbf{p}$ satisfy $p_x = 2\pi n_x/L$, $n_x = 0, \pm 1, \dots$, etc. By simple counting one sees that the number of states in the volume element d^3p is

$$d^3n = \frac{V}{(2\pi)^3} d^3p . \quad (1.5)$$

This is a very large number if the box is macroscopic. For instance, if V is the volume of a bubble chamber, $V \approx 1 \text{ m}^3$, and dp_x , etc, is taken to be a typical measuring accuracy of 10 MeV, then using (II.6.5) one obtains $d^3n = 5 \cdot 10^{38}$ states. Thus the quantization of states is within this experimental accuracy totally unobservable. Sums over discrete states may be replaced by integrals over $\mathbf{p}$:

$$\sum_{\substack{\text{values} \\ \text{of } \mathbf{p}}} = \frac{V}{(2\pi)^3} \int d^3p . \quad (1.6)$$

2. Integrations over phase space, cross section formulas

According to our present knowledge on dynamics of particle reactions, the transition probability from an initial state $p_a + p_b$ to a final state with definite momenta p_i is obtained from the matrix element

$$\langle p_1, \dots, p_n | T | p_a, p_b \rangle \equiv A(p_i) \quad (2.1)$$

The purpose of experiments is to clarify the structure of $A(p_i)$ and the conclusions are theoretically described in the form of different dynamical models specifying $A(p_i)$. Later on we shall present some general properties $A(p_i)$ is known to satisfy, but for the moment it is sufficient to consider it as some unknown function of the p_i .

In order to obtain measurable quantities (for $n > 2$), $|A(p_i)|$ has to be integrated over a set of allowed values of the p_i . The total reaction cross section is obtained, if the integration is carried over all possible values of the p_i , i.e. over the entire $(3n-4)$ -dimensional phase space. The corresponding quantity for a decay is the lifetime. If the integration is restricted to a subset of the phase space, a differential cross section or, if the normalization is inessential, a distribution is obtained. How exactly $|A(p_i)|$ and the cross sections are related, depends on some normalization conventions. The derivations can be found in textbooks on high energy physics (Bjorken 64, Källén 64). For our purposes it is sufficient just to state the results. Denoting the total reaction cross section by $\sigma_n \equiv \sigma_n(\sqrt{s}) \equiv \sigma_n(\sqrt{s}; m_i)$ we have

$$\sigma_n = \frac{1}{F} I_n(\sqrt{s}) \quad , \quad (2.2)$$

where

$$F = 2\sqrt{\lambda(s, m_a^2, m_b^2)} \cdot (2\pi)^{3n-4} \quad (2.3)$$

is a flux factor and

$$I_n(\sqrt{s}) = \int \prod_{i=1}^n \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_i p_i) |A(\underline{p}_i)|^2 \quad (2.4)$$

contains the integration over the phase space. The conservation of four-momentum has been accounted for by introducing in the integrand a four-dimensional δ -function, which is a product of four δ -functions corresponding to the four components p^μ . The dependence on m_i is suppressed in the notation. We shall presently clarify the reasons for writing the definition of $I_n(\sqrt{s})$ in the form above. It is important to emphasize that (2.2) as a matter of fact defines the normalization of $A(\underline{p}_i)$, i.e. what constants and $\sqrt{s}$ -dependent factors are attached to $A(\underline{p}_i)$ by convention. In this sense the derivation of (2.2) is not essential; it can be understood without any derivation. We shall return to this question of normalization in more detail in connection with two-body to two-body scattering. The formula for the lifetime τ_a of an unstable particle with mass m_a is very much similar to (2.2):

$$\frac{1}{\tau_a} = \frac{1}{2m_a} \frac{1}{(2\pi)^{3n-4}} I_n(m_a) \quad , \quad (2.5)$$

where $I_n(m_a)$ is given by

$$I_n(m_a) = \int \prod_{i=1}^n \frac{d^3 p_i}{2E_i} \delta^4(p_a - \sum_i p_i) |\langle \underline{p}_1, \dots, \underline{p}_n | T | \underline{p}_a \rangle|^2. \quad (2.6)$$

If $x = x(p_i)$ is any variable depending on the p_i , the differential cross section $d\sigma_n/dx$ is obtained by transforming the integral in (2.4) so that x appears as a variable and then omitting the integration over x . In practice, this can be most simply carried out by inserting the constraint $x = x(p_i)$ in the integrand as a δ -function so that

$$\frac{d\sigma_n}{dx} = \frac{1}{F} \int \prod_i \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_i p_i) \delta(x - x(p_i)) |A(p_i)|^2. \quad (2.7)$$

This satisfies trivially $\int dx \cdot (d\sigma_n/dx) = \sigma_n$. Eq. (2.7) is very convenient and will be used frequently. Higher-order differential cross sections $d^2\sigma/dxdy$, etc. are obtained similarly.

Consider now in more detail the integral $I_n(\sqrt{s})$. In its definition (2.4) we have separately included the factor $\prod_i (2E_i)$ in the integration over the p_i . This factor could, in principle, be included in A , but it is kept separate since the quantity $d^3p/2E$ is invariant under Lorentz transformations. We shall prove this invariance in two different ways:

1. By explicit computation one finds from the transformation formulas of four-momentum that

$$\begin{aligned} dp_x &= dp'_x \\ dp_y &= dp'_y \\ dp_z &= \gamma(dp'_z + v dE') \\ &= \gamma dp'_z (1 + vp'_z/E') = dp'_z E/E', \end{aligned} \quad (2.8)$$

since $dE'/dp'_z = p'_z/E'$ and $E = \gamma(E' + vp'_z)$. The volume element $d^3p = dp_x dp_y dp_z$ thus satisfies

$$\frac{d^3p'}{E'} = \frac{d^3p}{E} \quad (2.9)$$

so that the combination d^3p/E is invariant.

2. The invariance of d^3p/E can be made still more explicit by writing it in the manifestly invariant form

$$\int \frac{d^3p}{2E} = \int d^4p \delta(p^2 - m^2) \theta(p_0) \quad (2.10)$$

where the second integral is extended over all values of the components p^μ , $\mu = 1, \dots, 4$. The function $\theta(p_0)$ is zero for $p_0 \leq 0$ and 1 for $p_0 \geq 0$. It is also invariant under the orthochronous Lorentz transformations considered here. Writing $p^2 = p_0^2 - \underline{p}^2$ and $E^2 = \underline{p}^2 + m^2$ and using the following property of the δ -function integrations:

$$\delta(f(x)) = \frac{1}{|f'(x_0)|} \delta(x - x_0) \quad (2.11)$$

$$f(x_0) = 0$$

one can easily prove (2.10). Eq. (2.10) also explains the factor 2 added conventionally to the invariant d^3p/E . Formula (2.10) is often used to write the integrals (2.4), (2.6) and (2.7) in other equivalent forms, e.g.

$$I_n(\sqrt{s}) = \int \prod_{i=1}^n d^4p_i \delta(p_i^2 - m_i^2) \delta^4(p_a + p_b - \sum_{i=1}^n p_i) |A|^2 .$$

In (2.4), the δ -function is a singular function. For many purposes, for instance if one wants to calculate (2.4) numerically, it is important to eliminate the δ -function. This can be done in many different ways. After the elimination one has $3n-4$ variables which are only constrained by limits of integration and not by any singular constraints. Calling this set of variables ϕ we shall write (2.4) in the form

$$I_n(\sqrt{s}) = \int d\phi f_n(\phi) , \quad (2.12)$$

where $d\phi$ is a volume element in the $(3n-4)$ -dimensional phase space and $f_n(\phi)$ contains all the factors arising from transforming from the p_i in (2.4) to the variables ϕ . These include the integrand of (2.4), factors arising from integrations over the δ -functions according to (2.11) and a Jacobian. The exact relation between (2.4) and (2.12) will be more transparent after we have given concrete examples.

3. The phase space integral

If the matrix element (2.1) is identically 1, the integral $I_n(\sqrt{s})$ defined in (2.4) is often called the phase space integral. Denoting I_n in this case by R_n , we have

$$R_n(\sqrt{s}) = \int \prod_i \frac{d^3 p_i}{2E_i} \delta^4(p - \sum p_i) \quad (3.1)$$

where $s = p^2$. There is no deeper theoretical reason for giving $A \equiv 1$ a special treatment, it is just the simplest possible choice. In fact, from experimental evidence one knows that at high energies A may vary considerably in the phase space. On the other hand, many technical developments like transformations of variables are independent of A . They can be conveniently presented by using (3.1). For this the relativistic invariance of (3.1) is also crucial.

Similarly, all distributions $d\sigma/dx$, $d^2\sigma/dxdy$, etc. derived from (2.7) with the assumption $A = 1$ are called phase space distributions. Again, the higher the energy, the more any experimentally measured distribution will deviate from the phase space distribution. In spite of this, the case $A = 1$ may be conveniently used to derive the boundaries of the physical region in the variable x , on the xy -plane, etc. The specification of the physical region only involves four-momentum conservation and the form of A does not enter.

Historically, the phase space distributions have played a significant role as resonance backgrounds in connection with a search for resonances. The assumption was made that $A = 1$ somehow gives the consequences of pure kinematics and any deviation from this denotes a dynamical effect, for instance, a resonance. This works at lower energies where the matrix element really is rather constant, but at higher energies the separation of kinematical and dynamical effects becomes involved. This question will be discussed later on in connection with kinematical reflections (Chapter VII).

Other special choices of A may also be given a special treatment. One example is the non-covariant phase space integral $R_n(p^\mu)$, defined by

$$R_n(p^\mu) = \int \prod_i d^3 p_i \delta^4(p - \sum p_i) \quad (3.2)$$

or by choosing $A = \prod_i (2E_i)$. For many purposes (3.2) would be as convenient as (3.1), but due to its non-covariance it is far more difficult to handle.

4. The unitarity relation

The unitarity relation is a very deep and involved nonlinear constraint on reaction amplitudes following from the unitarity of the S-matrix. It is purely dynamical in character, but contains phase space integrations which we are going to analyse in some detail. To this end we formulate here the unitarity relation using the same normalization conventions (Källén 64) as those leading to the cross section formulas given previously. When applied to the scattering amplitude $\langle p_c p_d | T | p_a p_b \rangle$ of a 2+2 reaction $p_a + p_b \rightarrow p_c + p_d$ it reads:

$$\begin{aligned} \text{Im} \langle p_c p_d | T | p_a p_b \rangle & \quad (4.1) \\ &= \sum_{n=2}^{\infty} \frac{1}{2(2\pi)^{3n-4}} \left(\prod_i \frac{d^3 p_i}{2E_i} \right) \delta^4(p_a + p_b - \sum_i p_i) \langle p_c p_d | T | p_1 \dots p_n \rangle \\ & \quad \langle p_1 \dots p_n | T^\dagger | p_a p_b \rangle, \end{aligned}$$

where $T^\dagger$ is the adjoint of T . This equation is best understood by looking at Fig. III.4.1; the imaginary part of a 2+2 amplitude is given in terms of all possible 2+n amplitudes. Since kinematics only depends on the multiplicity, sums over intermediate states with different internal quantum numbers but same n have been omitted from (4.1). Further relations are obtained by putting some other amplitudes on the left-hand side of (4.1). In this way unitarity connects everything with everything, which is the reason for its great complexity.

The integrand in (4.1) is much more complicated than that in the cross section formula (2.4), since it 1) is a complex number, 2) depends on both the initial and final states of the reaction $p_a + p_b \rightarrow p_c + p_d$. When the initial and final states coincide, the integrand in (4.1) simplifies to that of (2.4) and one obtains the optical theorem

$$\text{Im} \langle p_a p_b | T | p_a p_b \rangle = \sqrt{\lambda(s, m_a^2, m_b^2)} \sigma_{\text{tot}}(s) . \quad (4.2)$$

This relates the total cross section

$$\sigma_{\text{tot}}(s) = \sum_{n=2}^{\infty} \sigma_n(s) \quad (4.3)$$

to the imaginary part of the forward 2→2 amplitude.

Exercises to Chapter III

1. a) What are the dimensions of the n-particle production amplitude $A(p_i)$ according to Eq. (III.2.2) if σ is to have the dimensions of mb?
b) What are the dimensions of the phase space integral R_n ?
2. Evaluate the non-relativistic phase space integral $I(E)$ defined by

$$I(E) = \int \prod_{i=1}^n d^3 p_i \delta(E - \sum_i \frac{p_i^2}{2m})$$

Hint: use the formulas

$$\begin{aligned} \delta(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{ixt} \\ \int_0^{\infty} dt t^b e^{-at^2} &= \Gamma(b+1)/2 / 2a^{(b+1)/2} \\ \int_{-\infty}^{\infty} dt e^{ixt} \frac{1}{(it)^a} &= \frac{2\pi}{\Gamma(a)} x^{a-1} \cdot \theta(x) \end{aligned}$$

where Γ is the gamma-function.

3. The entropy S of an ideal gas in the microcanonical ensemble is defined by $S = \log [N(E, V, n)/n!]$, where

$$N(E, V, n) = \frac{V^n}{(2\pi)^{3n}} \int \prod_{i=1}^n d^3 p_i \theta(E - \sum_i \frac{p_i^2}{2m})$$

is the number of states having the total energy

less than E . Calculate S by using $\theta'(x) = \delta(x)$, the results of Exercise 2 and Stirling's formula $\log n! = n(\log n - 1)$.

4. Calculate the partition function

$$Z(T, V, n) = \frac{V^n}{(2\pi)^{3n}} \int_0^\infty dE e^{-E/T} I(E)$$

where $I(E)$ is defined in Exercise 2.

5. The entropy in the canonical ensemble is defined by $S = \frac{\partial}{\partial T} [T \log Z(T, V, n)/n!]$. If E and T are connected by $E = \frac{3}{2}nT$, show that the entropies in the microcanonical and canonical ensembles coincide up to terms vanishing like $\log n/n$ for $n \rightarrow \infty$.

IV. Two-particle final states

1. Decay of one particle to two particles

In this section, we consider the two-particle final state without any reference to the initial state besides four-momentum conservation. The properties of the initial state may thus be compressed in its total four-momentum $p = (E, \underline{p})$, which may either be the four-momentum of a decaying particle p_a , the total four-momentum $p_a + p_b$ of an initial collision state, etc. However, for convenience we shall use the terminology associated with decay.

The two-particle phase space integral is needed in connection with cross section and life-time formulas and with developing a set of convenient variables for n-particle final states. According to the definition of R_n :

$$R_2(E, \underline{p}) = \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \delta(E - E_1 - E_2) \delta^3(\underline{p} - \underline{p}_1 - \underline{p}_2) \quad (1.1)$$

As it stands, R_2 is known by Lorentz invariance to be a function of

$$s = E^2 - \underline{p}^2 \quad (1.2)$$

only and one could for its evaluation go to the frame $p = (\sqrt{s}, 0)$. However, we wish to be more general for later use and evaluate (1.1) in an arbitrary frame.

Integrating first over $d^3 p_2$ by the second δ -function, one has

$$R_2(E, \underline{p}) = \int d\Omega_1 \int_0^\infty \frac{dP_1 P_1^2}{4E_1(E-E_1)} \delta(f(P_1)) \quad (1.3)$$

where one has to set everywhere $\underline{p}_2 = \underline{p} - \underline{p}_1$. The angles $\Omega_1 = (\cos\theta_1, \varphi_1)$ define the orientation of $\underline{p}_1$ with respect to $\underline{p}$ (Fig. IV.1.1) and we have denoted

Fig.IV.1.1.

$$f(P_1) = E - E_1 - (\vec{P}^2 + \vec{P}_1^2 - 2\vec{P}\vec{P}_1 \cos\theta_1 + m_2^2)^{1/2}. \quad (1.4)$$

The constraint $f(P_1) = 0$ defines the length of $\underline{p}_1$ as a function of $\cos\theta_1$, or the angle between $\underline{p}$ and $\underline{p}_1$. The solution $P_1 = P_1^\pm$ is two-valued and we give it later in Eqs. (1.11-12). Since the integral over P_1 goes from 0 to ∞ , only the positive solutions contribute. Calculating $f'(\underline{p})$ from (1.4):

$$f'(P_1) = \frac{1}{E_1(E-E_1)} (P_1 E - P E_1 \cos\theta_1) \quad (1.5)$$

and using the standard δ -function integration formulas one has, finally

$$R_2(E, \underline{p}) = \frac{1}{4} \int d\Omega_1 \frac{P_1^2}{|E P_1 - P E_1 \cos\theta_1|}, \quad (1.6)$$

where the solutions are given in (1.11-13). Eq. (1.6) is complicated and it is not simple to proceed further with the evaluation of (1.6) in the above form (apart from the trivial integration over φ_1). Since we have integrated only over the δ -functions, (1.6) is valid

unchanged also if one introduces a matrix element $A(p_1, p_2)$, which due to the δ -functions only depends on $\cos\theta_1$. Eq. (1.6) gives essentially the differential cross sections $d\sigma_2/d\Omega_1$ or $d\sigma_2/d\cos\theta_1$ in an arbitrary system, in practice mostly the target system.

In the rest frame $p = (\sqrt{s}, 0)$, Eq. (1.6) simplifies greatly and one has

$$R_2(\sqrt{s}) = \frac{P_1^*}{4\sqrt{s}} \int d\Omega_1^* \quad (1.7)$$

$$= \frac{\pi P_1^*}{\sqrt{s}} \quad (1.8)$$

The values of the "decay momentum" P_1^* and the corresponding energies E_1^* and E_2^* in terms of the "decaying mass" $\sqrt{s}$ are obtained from the results of Section II.9 or directly from $f(P_1) = 0$ for $P = 0$. One has

$$P_1^* = P_2^* = \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{2\sqrt{s}} \quad (1.9)$$

The values of these decay momenta are tabulated in the Elementary Particle Tables for different observed resonances and their two-particle decay modes. Introducing (1.9) to (1.8) R_2 can be written in the invariant form

$$R_2(\sqrt{s}) = \frac{\pi \sqrt{\lambda(s, m_1^2, m_2^2)}}{2s} \quad (1.10)$$

Eq. (1.7) can be used even if one introduces an integrand depending on the orientation of p_1^* with respect to some separately defined axis - since $p=0$ in this system $\underline{p}$ cannot be used as an axis. All the equations for R_2

should actually contain the θ -function $\theta[\sqrt{s}-(m_a+m_b)]$ specifying that R_2 vanishes below threshold.

The solution of $f(P_1) = 0$ can after squaring twice be found to be

$$\frac{P_1}{m_1} = \frac{v\gamma_1^* \cos\theta_1 \pm \sqrt{\gamma_1^{*2} v_1^{*2} - \gamma^2 v^2 \sin^2\theta_1}}{\gamma(1-v^2 \cos^2\theta_1)} \quad (1.11)$$

or

$$\frac{E_1}{m_1} = \frac{\gamma_1^* \pm v \cos\theta_1 \sqrt{\gamma_1^{*2} v_1^{*2} - \gamma^2 v^2 \sin^2\theta_1}}{\gamma(1-v^2 \cos^2\theta_1)} \quad (1.12)$$

where $v = P/E$, $v_1^* = P_1^*/E_1^*$, $\gamma = E/\sqrt{s}$ and $\gamma_1^* = E_1^*/m_1$ are θ_1 -independent. Adding, one obtains

$$EP_1^+ - E_1^+ P \cos\theta_1 = \pm m_1 \sqrt{s} \sqrt{\gamma_1^{*2} v_1^{*2} - \gamma^2 v^2 \sin^2\theta_1} \quad (1.13)$$

Equating Eqs. (1.6) and (1.7) in differential form one has

$$\frac{d\Omega_1}{d\Omega_1^*} = \frac{P_1^*}{\sqrt{s}} \frac{(EP_1 - PE_1 \cos\theta_1)}{P_1^2}$$

But also

$$\frac{d\Omega_1}{d\Omega_1^*} = \frac{d\cos\theta_1}{d\cos\theta_1^*} = \frac{\sin\theta_1}{\sin\theta_1^*} \frac{d\theta_1}{d\theta_1^*} = \frac{P_1^*}{P_1} \frac{d\theta_1}{d\theta_1^*} \quad (1.14)$$

where the equation $P_1 \sin\theta_1 = P_1^* \sin\theta_1^*$ has been used.

Thus

$$\frac{d\theta_1}{d\theta_1^*} = \frac{1}{\sqrt{s}} \frac{(EP_1 - PE_1 \cos\theta_1)}{P_1} \quad (1.15)$$

Inserting the θ_1 -dependence of P_1 and E_1 from (1.11-13) one finds after some calculations

$$\frac{d\theta_1}{d\theta_1^*} = \gamma(1-v^2 \cos^2 \theta_1) \frac{\sqrt{v_1^{*2} \gamma_1^{*2} - v^2 \gamma^2 \sin^2 \theta_1}}{v \gamma_1^* \cos \theta_1 \pm \sqrt{v_1^{*2} \gamma_1^{*2} - v^2 \gamma^2 \sin^2 \theta_1}} \quad (1.16)$$

The $\pm$ -sign is due to the two-valuedness of the mapping (E.q. II.8.15) from θ_1 to θ_1^* . If $v < v_1^*$ the mapping is single-valued and only the $+$ -sign is used. We shall rederive (1.16) and simultaneously clarify the meaning of this equation later.

Eq. (1.16) can also be interpreted as giving the distribution $w(\theta_1)$ normalized to unity in the angle θ_1 between $\mathbf{p}$ and $\mathbf{p}_1$ (Fig. IV.1.1). We have namely

$$w(\theta_1) = \frac{1}{R_2} \frac{dR_2}{d\theta_1} = \frac{1}{R_2} \frac{dR_2}{d\theta_1^*} \frac{d\theta_1^*}{d\theta_1} \equiv w(\theta_1^*) \frac{d\theta_1^*}{d\theta_1} \quad (1.17)$$

so that if the distribution is constant in the decay frame (all directions equivalent) or $w(\theta_1^*) = \text{const}$ one has

$$w(\theta_1) = \frac{d\theta_1^*}{d\theta_1} = \frac{1}{\gamma(1-v^2 \cos^2 \theta_1)} \left[\frac{v \gamma_1^* \cos \theta_1}{(v_1^{*2} \gamma_1^{*2} - v^2 \gamma^2 \sin^2 \theta_1)^{1/2}} + 1 \right] \quad (1.18)$$

If there is a θ_1 -dependent matrix element $A(\theta_1)$, the corresponding distribution is (apart from normalization) obtained from (1.18) by multiplying by $A(\theta_1)$. As it stands, (1.18) is derived for $A=1$ so that it corresponds to an isotropic decay in the CMS.

In order to obtain the distribution $w(\theta_{12})$ in the opening angle θ_{12} defined in Fig. IV.1.2 it is convenient to apply the general formula (III.2.) for the differential cross section with $x = \cos\theta_{12} = p_1 \cdot p_2 / p_1 p_2$. We have again

$$w(\theta_{12}) = \frac{1}{R_2} \sin\theta_{12} \frac{\partial R_2}{\partial \cos\theta_{12}} \quad (1.19)$$

where

$$\frac{\partial R_2}{\partial \cos\theta_{12}} = \int \prod_{i=1}^2 \frac{d^3 p_i}{2E_i} \delta^4(p - p_1 - p_2) \delta(\cos\theta_{12} - \frac{p_1 \cdot p_2}{p_1 p_2})$$

To evaluate this, integrate first over $d^4 p_2$:

$$\begin{aligned} \frac{\partial R_2}{\partial \cos\theta_{12}} &= \int \frac{d^3 p_1}{2E_1} \delta[(p - p_1)^2 - m_2^2] \delta(\cos\theta_{12} - \frac{p_1 \cdot (p - p_1)}{p_1 p_2}) \\ &= \int_0^\infty dP_1 \frac{P_1^2}{2E_1} \int d\varphi_1 d\cos\theta_1 \delta(s + m_1^2 - m_2^2 - 2EE_1 + 2PP_1 \cos\theta_1) \times \\ &\quad \times \delta(\cos\theta_{12} - \frac{PP_1 \cos\theta_1 - P_1^2}{P_1 p_2}) \end{aligned}$$

where θ_1 is the angle between p and p_1 . The integration over φ_1 is trivial and the integration over $\cos\theta_1$ can be carried out by using the first δ -function.

Writing $P_1 dP_1 = E_1 dE_1$ we have

$$\frac{\partial R_2}{\partial \cos\theta_{12}} = \frac{\pi}{2P} \int_{m_1}^\infty dE_1 \delta(\cos\theta_{12} - \frac{2EE_1 - 2E_1^2 - s + m_1^2 + m_2^2}{2\sqrt{(E_1^2 - m_1^2)}[(E - E_1)^2 - m_2^2]}) \quad (1.20)$$

where the dependence on the integration variable E_1 has been stated explicitly. In a concise form the argument of the δ -function is

$$f(E_1) = \cos\theta_{12} - \frac{E_1 E_2 - \frac{1}{2}(s - m_1^2 - m_2^2)}{P_1 P_2}.$$

The equation (1.20) is rather complicated to analyze for arbitrary masses and we shall specialize to $m_1 = m_2 = 0$. Examples of processes of this type are then $\pi^0 \rightarrow \gamma\gamma$ or $e^+e^- \rightarrow \gamma\gamma$, although for the latter the matrix element has a definite $\cos\theta_1$ -dependence determined by quantum electrodynamics. Now $P_1 = E_1$ and $P_2 = E_2$ so that $f(E_1) = \cos\theta_{12} - 1 + \frac{s}{2E_1(E-E_1)}$. The solution of $f(E_1) = 0$ is then simply

$$\begin{aligned} E_1^{\pm} &= \frac{1}{2}E \pm \frac{1}{2}\sqrt{E^2 - \frac{2s}{1-\cos\theta_{12}}} \\ &= \frac{1}{2}E \left[1 \pm \frac{1}{\sin\theta_{12}} \sqrt{v^2 - \cos^2 \frac{\theta_{12}}{2}} \right] \end{aligned} \quad (1.21)$$

where $v = P/E$ is the velocity of the decaying particle. Computing $f'(E_1)$ and integrating over E_1 in (1.20) one finds

$$\frac{\partial R_2}{\partial \cos\theta_{12}} = \frac{\pi}{Ps} \frac{P_1(E-P_1)}{|E-2P_1|}.$$

It is irrelevant which of the roots (1.21) is used for P_1 . Combining with (1.19) the final result is

$$w(\theta_{12}) = \frac{\pi}{2} \frac{1-v^2}{4v} \frac{\cos^{\frac{\theta_{12}}{2}}}{\sin^2 \frac{\theta_{12}}{2}} \frac{1}{\sqrt{v^2 - \cos^2 \frac{\theta_{12}}{2}}} .$$

This distribution is plotted in Fig. IV.1.3 for some values of v .

2. Scattering of two particles into two particles

The most extensively studied reactions are the elastic and quasi-elastic reactions which involve the scattering of two particles, with possible changes of quantum numbers and mass of one or both particles. The reaction is thus in general form

$$a + b \rightarrow c + d . \quad (2.1)$$

The total energy squared in CM is

$$s = (p_a + p_b)^2 = (E_a^* + E_b^*)^2 = m_a^2 + m_b^2 + 2E_a^L m_b . \quad (2.2)$$

The expressions for energies and three momenta in CM and lab in terms of s are given by (II.9.1-2) and (II.9.14-16). For the final state, E_c^* , E_d^* , p_c^* , p_d^* are given by the same equations with a, b replaced by c, d .

To describe the scattering completely, a second invariant variable t , the momentum transfer squared, is introduced

$$t = (p_a - p_c)^2 = (p_b - p_d)^2 . \quad (2.3)$$

In CM t is related to the scattering angle (Fig. 2.1) by

$$\cos \theta^* = \frac{1}{2p_a^* p_c^*} (t - m_a^2 - m_c^2 + 2E_a^* E_c^*)$$

(2.4)

$$= \frac{s^2 + s(2t - m_a^2 - m_b^2 - m_c^2 - m_d^2) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)}{\sqrt{\lambda(s, m_a^2, m_b^2)} \sqrt{\lambda(s, m_c^2, m_d^2)}}$$

The lab quantities of a and b were discussed above. To find E_d^L we write

$$t = (p_b - p_d)^2 = m_b^2 + m_d^2 - 2m_b E_d^L$$

and thus

$$E_d^L = \frac{1}{2m_b} (m_b^2 + m_d^2 - t) .$$

(2.5)

Energy conservation gives E_c^L :

$$E_c^L = m_b + E_a^L - E_d^L = \frac{1}{2m_b} (s + t - m_a^2 - m_d^2) .$$

(2.6)

For symmetry reasons, it is convenient to introduce also the invariant momentum transfer

$$u = (p_a - p_d)^2 = (p_b - p_c)^2 .$$

(2.7)

In lab the last expression gives

$$E_c^L = \frac{1}{2m_b} (m_b^2 + m_c^2 - u) . \quad (2.8)$$

Variables s, t, u are not independent, comparison of (2.6) and (2.8) yields the important identity

$$s+t+u = m_a^2 + m_b^2 + m_c^2 + m_d^2 . \quad (2.9)$$

Eq. (2.9) follows also from

$$p_a^2 = (p_b - p_c - p_d)^2 = (p_b - p_c)^2 + (p_b - p_d)^2 + (p_c + p_d)^2 - p_b^2 - p_c^2 - p_d^2 .$$

Exchange of c and d will permute t and u . The laboratory momenta in the final state are obtained from (2.5) and (2.8) in terms of the λ -function,

$$p_c^L = \frac{1}{2m_b} \sqrt{\lambda(u, m_b^2, m_c^2)} \quad (2.10)$$

$$p_d^L = \frac{1}{2m_b} \sqrt{\lambda(t, m_b^2, m_d^2)} .$$

The first expression can be regarded as the lab momentum for a beam of $\bar{c}$ colliding with b ; this process has the invariant mass u . Similar interpretation applies to p_d^L .

The scattering angle θ^L , the angle between $\underline{p}_a^L$ and $\underline{p}_c^L$, is found from $t = (p_a - p_c)^2$ as

$$\cos \theta^L = \frac{1}{2p_a^L p_c^L} (t - m_a^2 - m_c^2 + 2E_a^L E_c^L) .$$

In terms of invariants this is

$$\cos \theta^L = \frac{(s - m_a^2 - m_b^2)(m_b^2 + m_c^2 - u) + 2m_b^2(t - m_a^2 - m_c^2)}{\sqrt{\lambda(s, m_a^2, m_b^2)} \sqrt{\lambda(u, m_b^2, m_c^2)}} . \quad (2.11)$$

For elastic scattering, $a=c$, $b=d$, the preceding equations simplify slightly. For equal masses, $m_a = m_b = m_c = m_d = m$, the invariants are

$$\begin{aligned} s &= 4(p^{*2} + m^2) \\ t &= -2p^{*2}(1 - \cos \theta^*) \\ u &= -2p^{*2}(1 + \cos \theta^*) \end{aligned} \quad (2.12)$$

The Mandelstam variables (Fig. 2.2)

$$\begin{aligned} s &= (p_a + p_b)^2 = (p_c + p_d)^2 \\ t &= (p_a - p_c)^2 = (p_b - p_d)^2 \\ u &= (p_a - p_d)^2 = (p_b - p_c)^2 \end{aligned} \quad (2.13)$$

describe symmetrically the reactions

$a+b \leftrightarrow c+d$	s-channel
$a+\bar{c} \leftrightarrow \bar{b}+d$	t-channel
$a+\bar{d} \leftrightarrow \bar{b}+c$	u-channel

In s-channel, t and u are squares of momentum transfers. In going to t-channel, the signs of p_b and p_c change, and thus t becomes energy squared and s, u the momentum transfers squared. In u-channel, u is energy and s, t momentum transfers. The symmetrical description of the reactions related by crossing is most useful due to the observed crossing symmetry of the scattering amplitude.

One of the vectors p_a, p_b, p_c, p_d is given by the other three by energy-momentum conservation. In section II.6. it was seen that kinematics with three independent vectors involves the Gram determinants Δ_1, Δ_2 and Δ_3 . The poles of Δ_1 and $\Delta_2 = -\frac{1}{4}\lambda$ are exhibited in the previous formulae. The scattering angles and the boundary of the physical region are given most naturally by Δ_3 .

The CM scattering angle is $\theta^* = \theta(p_a, p_c)$ in the frame $p_a + p_b = 0$. To obtain the usual invariants most easily, one calculates the angle $\pi - \theta^*$ between $\underline{p}_d^*$ and $\underline{p}_a^*$. From (Appendix A) it follows that

$$\sin^2 \theta^* = \frac{\Delta_1(p_a + p_b) \Delta_3(p_a + p_b, -p_a, -p_d)}{\Delta_2(p_a + p_b, -p_a) \Delta_2(p_a + p_b, -p_d)} \quad (2.14)$$

Use of definitions of λ and G results in

$$\sin^2 \theta^* = -4 \cdot \frac{s \cdot G(m_b^2, m_c^2, t, s, m_a^2, m_d^2)}{\lambda(m_b^2, s, m_a^2) \lambda(m_c^2, s, m_d^2)} \quad (2.15)$$

In lab $\theta^L = \pi - \theta(p_a^L, -p_c^L)$ is directly given by

$$\sin^2 \theta^L = \frac{\Delta_1(p_b) \Delta_3(p_b, p_a, -p_c)}{\Delta_2(p_b, p_a) \Delta_2(p_b, -p_c)} \quad (2.16)$$

$$= -4 \frac{m_b^2 G(s, u, m_d^2, m_b^2, m_a^2, m_c^2)}{\lambda(s, m_b^2, m_a^2) \lambda(u, m_b^2, m_c^2)}$$

The functions λ and G have the symmetry properties mentioned in Appendix A. These can be used to put the arguments into a standard order

$$\sin^2 \theta^* = -4 \frac{s G(s, t, m_d^2, m_a^2, m_b^2, m_c^2)}{\lambda(s, m_a^2, m_b^2) \lambda(s, m_c^2, m_d^2)} \quad (2.17)$$

$$\sin^2 \theta^L = -4 \frac{m_b^2 G(s, u, m_c^2, m_a^2, m_b^2, m_d^2)}{\lambda(s, m_a^2, m_b^2) \lambda(u, m_b^2, m_c^2)} \quad (2.18)$$

3. The physical region in terms of invariants

Real values of the momenta p_a, p_b, p_c, p_d are obtained if and only if s and t fulfill the condition

$$G(s, t, m_4^2, m_1^2, m_2^2, m_3^2) \leq 0. \quad (3.1)$$

Due to symmetry in the variables s, t, u , the same condition applies to all the three channels. It also applies to the decay of one particle into three particles.

The boundary curve is $G = 0$. Expansion of (II.6.29) results in the equation

$$stu - (\alpha s + \beta t + \gamma u) = 0 \quad (3.2)$$

where

$$K\alpha = (m_1^2 m_2^2 - m_3^2 m_4^2)(m_1^2 + m_2^2 - m_3^2 - m_4^2)$$

$$K\beta = (m_1^2 m_3^2 - m_2^2 m_4^2)(m_1^2 + m_3^2 - m_2^2 - m_4^2)$$

$$K\gamma = (m_1^2 m_4^2 - m_2^2 m_3^2)(m_1^2 + m_4^2 - m_2^2 - m_3^2)$$

with

$$K = \sum m_i^2 = s + t + u.$$

The asymptotes of the curve are $s = 0$, $t = 0$ and $u = 0$ and it intersects them on the line

$$\alpha s + \beta t + \gamma u = 0.$$

Kotanski has classified these cubic curves into fourteen types distinctive to different ratios of the masses. If two pairs of the masses are equal or two masses = 0, the curve degenerates to a line and a hyperbola. If all the masses are equal, the curve degenerates to three lines.

For exact figure we can solve t from

$$G(s, t, m_4^2, m_1^2, m_2^2, m_3^2) = 0$$

$$t^{\pm} = m_4^2 + m_2^2 + (2s)^{-1} [(-s + m_1^2 - m_2^2)(s + m_4^2 - m_3^2)$$

$$\pm \sqrt{\lambda(s, m_1^2, m_2^2)(s, m_4^2, m_3^2)}]$$

It can be drawn in Cartesian st -plane but usually we have the symmetrical representation, see Fig. 2.

The points, where the boundary curve of the physical region intersects the asymptotes, are:

$$\begin{array}{lll} s = 0 : & t = -\frac{\gamma K}{\beta - \gamma} & u = \frac{\beta K}{\beta - \gamma} \\ t = 0 : & s = -\frac{\gamma K}{\alpha - \gamma} & u = \frac{\alpha K}{\alpha - \gamma} \\ u = 0 : & s = -\frac{\beta K}{\alpha - \beta} & t = \frac{\alpha K}{\alpha - \mu} \end{array} .$$

For a rough draft we need, in addition, the tangents parallel to $s = 0$, $t = 0$ and $u = 0$.

These are

$$\begin{aligned} s &= (m_1 \pm m_2)^2, & s &= (m_3 \pm m_4)^2 \\ t &= (m_1 \pm m_3)^2, & t &= (m_2 \pm m_4)^2 \\ u &= (m_1 \pm m_4)^2, & u &= (m_2 \pm m_3)^2. \end{aligned}$$

Now in channel S

$$s \geq \max [(m_1 + m_2)^2, (m_3 + m_4)^2]$$

in channel T

$$t \geq \max [(m_1 + m_3)^2, (m_2 + m_4)^2]$$

in channel U

$$u \geq \max [(m_1 + m_4)^2, (m_2 + m_3)^2].$$

Between the tangents there can be a closed area with

$$G < 0$$

where the decay of one particle into three particles is possible, if the masses are suitable.

In Figs. (3.2)-(3.7) we show different cases of mass combinations.

The kinematic function G gives also the boundary of so-called Chew-Low plot, that is the mass of an outgoing particle (or a group of particles) as a function of the momentum transfer t . The curve $G = 0$ gives actually only the upper limit for the invariant mass M^2 . The maximum of the curve is at $M = \sqrt{s} - m_1$. The lower limit must be ≥ 0 . If we have a group of particles, the lower limit is

$$M^2 = \left(\sum_{i=1}^n m_i \right)^2 .$$

As an example, in Fig. (3.8) the Chew-Low plot of the reaction $K^-p \rightarrow (\pi^+\pi^0\pi^-)\Lambda$ is given.

The shape of the plot is the same for different s , main variation is in the size.

V. Three Particle Final State

1. Decay of one particle into three particles

We consider the decay

$$A \rightarrow 1 + 2 + 3 .$$

In the absence of spin, there is no preferred orientation in space. Of the five independent variables describing the final state, there are three angles on which the process cannot depend. A usual choice for the two essential variables is to take two of the two particle invariant masses

$$s_{ij} = (p_i + p_j)^2 , \quad i < j \quad (1.1)$$

Due to four-momentum conservation these are related, expansion of $(p_1 + p_2 + p_3)^2$ gives

$$s_{12} + s_{13} + s_{23} = m_A^2 + m_1^2 + m_2^2 + m_3^2 \quad (1.2.)$$

The momentum of particle i in the rest frame of A is obtained from the two particle decay $A \rightarrow i + (j+k)$ as

$$p_i^* = \sqrt{\lambda(m_A^2, m_i^2, s_{jk})} / 2m_A , \quad (1.3)$$

and similarly for 2 and 3. The relations between energies and s_{jk} are linear,

$$E_i^* = (m_A^2 + m_i^2 - s_{jk})/2m_A . \quad (1.4)$$

The physical region for the pair s_{12}, s_{23} is given by the requirement Δ_3 positive or

$$G(s_{12}, s_{13}, m_A^2, m_2^2, m_1^2, m_3^2) < 0 . \quad (1.5)$$

2. The Dalitz plot

A plot of experimental events with three particles in the final state in the s_{ij}, s_{jk} plane is useful for two reasons. If the matrix element is constant, the distribution of events in the Dalitz plot is constant. Any structure in the reaction is thus clearly apparent. Also because the variables are two particle, masses, resonance structure appears simply as strips with higher density.

The constant density can be derived starting from the three particle phase space integral

$$R_3 = \int \frac{d^3p_1 d^3p_2 d^3p_3}{8E_1E_2E_3} \delta^4(p - p_1 - p_2 - p_3) . \quad (2.1)$$

We first integrate (2.1) over p_3 with the result

$$R_3 = \int \frac{\delta(\sqrt{s} - E_1 - E_2 - E_3)}{2E_3} \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} , \quad (2.2)$$

$$E_3 = \sqrt{(p_1 + p_2)^2 + m_3^2}.$$

Now p_1 and p_2 are arbitrary, the energy δ -function takes care of energy conservation. The next step is to integrate over the angles: first over the directions p_2 relative to a fixed direction of p_1 , then over the directions of p_1 itself. Call θ the angle between p_1 and p_2 and put $\cos\theta = \xi$. Then the first angular integration consists in replacing d^3p_2 by

$$2\pi p_2^2 dp_2 d\xi$$

and the second angular integration (because after first integration no direction is distinguished any longer) in replacing d^3p_1 by

$$4\pi p_1^2 dp_1.$$

Hence we obtain

$$R_3 = \int \frac{\delta(\sqrt{s} - E_1 - E_2 - E_3)}{2 E_3} 8\pi^2 \frac{p_1^2 p_2^2}{2E_1 2E_2} dp_1 dp_2 d\xi \quad (2.3)$$

where

$$E_3 = \sqrt{p_1^2 + p_2^2 + 2p_1 p_2 \xi + m_3^2}.$$

We now transform the integral to the variables

$$E_1 = \sqrt{P_1^2 + m_1^2}$$

$$E_2 = \sqrt{P_2^2 + m_2^2}$$

$$E_3 = \sqrt{P_1^2 + P_2^2 + 2P_1P_2\xi + m_3^2}$$

by means of

$$dP_1 dP_2 d\xi = dE_1 dE_2 dE_3 \frac{\partial(P_1P_2\xi)}{\partial(E_1E_2E_3)}.$$

The Jacobian is easily found to be

$$\frac{\partial(P_1P_2\xi)}{\partial(E_1E_2E_3)} = \left[\frac{\partial(E_1E_2E_3)}{\partial(P_1P_2P_3)} \right]^{-1} = \frac{E_1E_2E_3}{P_1^2P_2^2}.$$

Inserting this into the integral (2.3) yields

$$\begin{aligned} d^2R &= \pi^2 \int \frac{\delta(\sqrt{s}-E_1-E_2-E_3)}{E_1E_2E_3} \frac{E_1E_2E_3}{P_1^2P_2^2} P_1^2P_2^2 dE_1 dE_2 dE_3 \\ &= \frac{(2\pi)^{+2}}{4} \int \delta(\sqrt{s}-E_1-E_2-E_3) dE_1 dE_2 dE_3 \\ &= \pi^2 \int dE_1 dE_2 \end{aligned} \quad (2.4)$$

Thus the probability density in E_1, E_2 plane is constant. Because E_i is linearly related to s_{jk} , cf. (1.4), also the s_{ij}, s_{jk} plot will have constant density.

Before starting the calculation of the boundary of the Dalitz plot we shall find the values of the effective mass of one pair corresponding to the smallest and largest possible values of the other pair of particles, that is, the coordinates of points A and B in Fig. 5.2.

The smallest value of the effective mass of a pair is equal to:

$$\sqrt{s_{jk}}^{\min} = m_j + m_k . \quad (2.5)$$

This configuration arises when the velocities of both particles are equal and equal to the velocity of the whole system "jk" (because momenta p_j and p_k are also collinear):

$$\beta_j = \beta_k = \beta_{jk} . \quad (2.6)$$

Therefore also Lorentz-factors of particles j and of the system jk are equal:

$$\frac{E_j}{m_j} = \frac{E_{jk}}{m_j + m_k} = \frac{\sqrt{s} - E_i}{m_j + m_k} \quad (2.7)$$

$$E_j = (\sqrt{s} - E_i) \frac{m_j}{m_j + m_k} .$$

The energy of the particle i is equal to:

$$E_i = \frac{s + m_i^2 - (m_j + m_k)^2}{2 \sqrt{s}} . \quad (2.9)$$

Therefore:

$$E_j = \left\{ \frac{s - m_i^2 + (m_j + m_k)^2}{2 \sqrt{s}} \right\} \frac{m_j}{m_j + m_k} . \quad (2.10)$$

Substituting (2.10) in the equation

$$s_{ik}(s_{jk}^{\min}) = s + m_j^2 - 2 \sqrt{s} E_j \quad (2.11)$$

one finds the value of s_{ik} corresponding to the smallest value of s_{jk} :

$$s_{ik}(s_{jk}^{\min}) = s + m_j^2 - \frac{m_j}{m_j + m_k} \left\{ s - m_i^2 + (m_j + m_k)^2 \right\} . \quad (2.12)$$

$\left[s_{23}(s_{13}^{\min}), s_{13}^{\min} \right]$ are the coordinates of point A_{13} ,

$\left[s_{13}(s_{23}^{\min}), s_{23}^{\min} \right]$ of point A_{23} , respectively.

One can use (1.2) which relates the mass of the pair "12" to the masses of two other pairs, in order to find the coordinates of the point A_{12} .

In a similar way we can calculate the coordinates of the points B on Fig. 2.1. The maximum possible value of $s_{jk}^{1/2}$ is:

$$(\sqrt{s_{jk}})^{\max} = s^{1/2} - m_i . \quad (2.13)$$

Particle i is now at rest in the CM system:

$$\underline{p}_i = 0 .$$

Therefore the momenta of particles j and k are equal:

$$P_j = P_k .$$

Also the system jk is now at rest in the CM system, so that the energy of particle j in the rest system of jk is the same as in the CM system. The energy of particle j in the system jk is given by:

$$E_j = \frac{(\sqrt{s} - m_i)^2 + m_j^2 - m_k^2}{2(\sqrt{s} - m_i)} \quad (2.14)$$

since the mass of jk system is equal to $s^{1/2} - m_i$.

The value of s_{ik} corresponding to the maximum value of s_{jk} is equal to:

$$s_{ik}(s_{jk}^{\max}) = s + m_j^2 - 2\sqrt{s} E_j . \quad (2.15)$$

And finally, inserting (2.14) in (2.15) one finds:

$$s_{ik}(s_{jk}^{\max}) = m_j^2 + \sqrt{s} m_i^2 - \sqrt{s} \frac{m_j^2 - m_k^2}{\sqrt{s} - m_i} . \quad (2.16)$$

Eqs. (2.13), (2.16) and (1.2) give the coordinates of all points B.

The equation for the boundary of the Dalitz plot is

$$G(s_{jk}, s_{ik}, s, m_k^2, m_j^2, m_i^2) = 0. \quad (2.17)$$

Using (II.6.) the limits of s_{ik} for constant s_{jk} are

$$s_{ik}^{\min} = s + m_j^2 - \sqrt{s} \frac{E_{jk} A \pm P_i B}{s_{jk}} \quad (2.18)$$

with

$$\begin{aligned} E_{jk} &= (s + s_{jk} - m_i^2) / 2\sqrt{s} \\ P_i &= \sqrt{\lambda(s, s_{jk}, m_i^2)} / 2\sqrt{s} \\ A &= s_{jk} + m_j^2 - m_k^2 \\ B &= \sqrt{\lambda(s_{jk}, m_j^2, m_k^2)} . \end{aligned} \quad (2.19)$$

These complicated expressions for limits of effective mass become very simple when the CM energy is much greater than the masses of all particles, i.e., when

$$m_i \rightarrow 0, \quad i = 1, 2, 3 .$$

In this case (2.18) is reduced to the form:

$$s_{ik}^{\min}(s_{jk}) = 0 \quad (2.20)$$

$$s_{ik}^{\max}(s_{jk}) = s - s_{jk}$$

and contour of Dalitz plot is reduced to the triangle (Fig. 2.3).

It is also of interest to look at the behaviour of the Dalitz plot contour for fixed masses, but changing energy. The starting point is:

$$\sqrt{s_0} = m_1 + m_2 + m_3 .$$

Then the contour is reduced to the point with coordinates:

$$s_{13} = (m_1 + m_3)^2$$

$$s_{23} = (m_2 + m_3)^2 .$$

With increasing energy, point B_{12} (Fig. 2.1) moves up and to the right from the starting point " $\sqrt{s_0}$ " (Fig. 2.4). This can be seen from (2.16), where

$$\text{for } \sqrt{s} \rightarrow \infty, \quad s_{13}(s_{12}^{\max}) \rightarrow \sqrt{s} m_3 .$$

A similar equation applies for s_{23} . Here is an apparent contradiction: both coordinates of the point B_{12} are increasing proportionally to $\sqrt{s}$, when $\sqrt{s}$ increases; on the other hand the situation at very high energy resembles the case, when the masses of all three particles are zero, and the contour reduces to the triangle with the point B_{12} located at the beginning of the coordinate system. The answer is, that the dimension of the Dalitz plot increases with increasing energy much faster, than the coordinates of the point B_{12} . In the high energy limit, when masses are small with respect to energy, the distance $A_{12} - B_{12}$ is proportional to s .

3. Scattering of two particles into three particles

The simplest collisions involving particle production are in the category

$$a+b \rightarrow 1+2+3.$$

Dynamically these are of large variety. There are e.g. quasi two body final states with subsequent decay of one constituent, diffraction dissociation of the type $a+b \rightarrow a+(2+3)$, and "pure" three body processes. However, the same invariant kinematical variables are usually most convenient for all the reactions.

Accounting for four-momentum conservation, there are $3n-4=5$ independent variables. In the absence of polarization, one of these, rotation around the beam axis, is trivial; the process cannot depend on it. The four essential variables could be chosen so that the set would be symmetrical with respect to p_1, p_2, p_3 . The quantum numbers of the final state particles, however, usually are such that one particle is related to $_a$ and one particle to $_b$. The two particle invariant masses connected with the ordering in Fig. 3.2. are then

$$\begin{aligned} s &= (p_a + p_b)^2 && \text{and the variables} \\ s_1 &= (p_1 + p_2)^2 \\ s_2 &= (p_2 + p_3)^2 \\ t_1 &= (p_a - p_1)^2 \\ t_2 &= (p_b - p_2)^2 = (p_a - p_1 - p_2)^2 \end{aligned} \tag{3.1}$$

There is a third pair of similar variables, s_{13} and t_{a2} , which are obtained from (3.1) by permuting indices:

$$\begin{aligned} s_{13} &= (p_1 + p_3)^2 \\ t_{a2} &= (p_a - p_2)^2 \end{aligned} \quad (3.2)$$

Expansion of $(p_1 + p_2 + p_3)^2$ in two particle masses gives

$$s_1 + s_2 + s_{13} = s + m_1^2 + m_2^2 + m_3^2. \quad (3.3)$$

Similar expansion of the last expression in (3.1) results in

$$t_{a2} = -s_1 - t_1 + t_2 + m_a^2 + m_1^2 + m_2^2. \quad (3.4)$$

A pair s_1, t_2 or s_2, t_1 corresponds to a quasi two particle process depicted in Fig. 3.3. The pair s_1, t_2 thus determines the momentum P_1 and angle θ_1 in a given frame. In CM for fixed s_2 , P_1 must lie on the sphere

$$P_1^* = \sqrt{\lambda(s, m_1^2, s_2)} / 2 \sqrt{s}. \quad (3.5)$$

The values of P_1 in lab for fixed s_2 are on the ellipsoid, obtained from (3.5) by a transformation in the z-direction. The result is given by (IV.1.7), or

$$\frac{p_{1x}^L{}^2}{a^2} + \frac{p_{1y}^L{}^2}{a^2} + \frac{(p_{1z}^L - p_0)^2}{b^2} = 1 \quad (3.6)$$

with

$$\begin{aligned} a &= p_1^* \\ b &= \gamma p_1^* = \frac{s - m_a^2 + m_b^2}{2m_b \sqrt{s}} \cdot p_1^* \end{aligned} \quad (3.7)$$

$$\beta_0 = \gamma \beta E_1^*$$

For a fixed t_1 , the surface formed by the corresponding p_1 is a sphere in the antilaboratory frame,

$$p_1^{AL} = \sqrt{\lambda(m_a^2, m_1^2, t_1)} / 2m_a \quad (3.8)$$

In CM this is an ellipsoid and in going further to lab it becomes usually a very slim ellipsoid. Examples of surfaces of constant s_2 and of constant t_1 in lab are drawn in Fig. 3.4. For fixed s_1 , p_3 must lie on the sphere

$$p_3^* = \sqrt{\lambda(s, m_3^2, s_1)} / 2\sqrt{s} \quad (3.9)$$

in CM. For fixed t_1 , p_3 is on a sphere in lab,

$$p_3^L = \sqrt{\lambda(m_b, m_3, t_2)} / 2m_b. \quad (3.10)$$

In designing experiments, it is useful to have some feeling as to what kind of momentum configuration p_1^L, p_2^L, p_3^L

will correspond to a given set s_1, s_2, t_1, t_2 . Suppose s_2, t_1 to be given so that $\underline{p}_1^L$ is fixed. We take the coordinate axis to be rotated around beam so that $\underline{p}_1^L$ lies in the x, z plane. There are now two remaining degrees of freedom. Thus for example, $\underline{p}_3^L$ must lie on a definite surface. The equation of this surface is given by the missing mass constraint, the particle 2 must be on its mass shell:

$$m_2^2 = (p_a + p_b - p_1 - p_3)^2 . \quad (3.11)$$

This is a surface of second degree in $p_{3x}^L, p_{3y}^L, p_{3z}^L$.

Instead of solving (3.11) through a tedious algebra, one can proceed via the frame in which (3.11) is a sphere. The decay of the system 2+3 is spherically symmetric in the frame $\underline{p}_2 + \underline{p}_3 = 0$, i.e. $\underline{p}_a - \underline{p}_1 = -\underline{p}_b$. This is related to lab by a Lorentz transformation along $\underline{p}_a - \underline{p}_1$, which is thus parallel to $\underline{p}_a^L - \underline{p}_1^L$. The transform parameters are obtained from

$$\gamma = E_b/m_b = (s_2 - t_1 + m_b^2) / 2m_b \sqrt{s_2} \quad (3.12)$$

Denote now the direction cosines of $\underline{p}_a^L - \underline{p}_1^L$ by

$$\alpha = \frac{(\underline{p}_a^L - \underline{p}_1^L)_x}{|\underline{p}_a^L - \underline{p}_1^L|} , \quad \gamma = \frac{(\underline{p}_a^L - \underline{p}_1^L)_z}{|\underline{p}_a^L - \underline{p}_1^L|} . \quad (3.13)$$

The allowed values of $\underline{p}_3^L$ then lie on the ellipsoid

$$\frac{(\gamma p_{3x}^L - \alpha p_{3z}^L)^2}{a^2} + \frac{p_{3y}^L{}^2}{a^2} + \frac{(\alpha p_{3x}^L + \gamma p_{3z}^L - \xi_0)^2}{b^2} = 1 \quad (3.14)$$

with

$$a = \sqrt{\lambda(s_2, m_2^2, m_3^2)} / 2 \sqrt{s_2}$$

$$b = (s_2 - t_1 + m_b^2) \sqrt{\lambda(s_2, m_2^2, m_3^2)} / 4 m_b s_2$$

$$\xi_0 = (s_2 - m_2^2 + m_3^2) \sqrt{\lambda(s_2, t_1, m_b^2)} / 4 m_b s_2$$

An example of the mass constraint ellipsoid (3.14) is drawn in Fig. 3.5. Interchange $2 \leftrightarrow 3$ in a, b, ξ_0 will give the parameters for the surface of p_2^L .

The points on the ellipsoid (3.14) are in two to one correspondence with points on the s_1, t_2 plane. The surface is symmetric with respect to the x, z plane, and reflection will not effect the invariants. The boundary of the region in s_1, t_2 is the image of the intersection of the ellipsoid and the x, z plane. In this case namely the vectors p_1, p_2, p_3 are linearly related.

The intersection is an ellipse. Amusingly, also the boundary of the s_1, t_2 region is an ellipse.

To construct p_1, p_2, p_3 in a given frame, the shortest way is to find p_1, θ_1 from s_2, t_1 and p_3, θ_3 from s_1, t_2 .

The relative polar angle is determined by (II.6.20).

Explicitly, substitution of the invariants gives

$$\begin{aligned} \cos(\varphi_1 - \varphi_3) &= \frac{(\underline{p}_a \times \underline{p}_1) \cdot (\underline{p}_a \times \underline{p}_3)}{|\underline{p}_a \times \underline{p}_1| |\underline{p}_a \times \underline{p}_3|} \\ &= \frac{G \begin{pmatrix} b & a & 1 \\ b & a & 3 \end{pmatrix}}{\sqrt{\Delta_3(b \ a \ 1) \Delta_3(b \ a \ 3)}} \end{aligned} \quad (3.15)$$

in which

$$\begin{aligned} G \begin{pmatrix} b & a & 1 \\ b & a & 3 \end{pmatrix} &= \frac{1}{8} \begin{vmatrix} 2m_b^2 & s - m_a^2 - m_b^2 & -t_2 + m_b^2 + m_3^2 \\ s - m_a^2 - m_b^2 & 2m_a^2 & -s - t_2 - s_1 + m_b^2 \\ -s - t_1 + s_2 + m_a^2 & -t_1 + m_a^2 + m_1^2 & s - s_1 - s_2 + m_2^2 \end{vmatrix} \\ \Delta_3(b \ a \ 1) &= \frac{1}{8} \begin{vmatrix} 2m_b^2 & s - m_a^2 - m_b^2 & -s - t_1 + s_2 + m_a^2 \\ s - m_a^2 - m_b^2 & 2m_a^2 & -t_1 + m_a^2 + m_1^2 \\ -s - t_1 + s_2 + m_a^2 & -t_1 + m_a^2 + m_1^2 & 2m_1^2 \end{vmatrix} \\ \Delta_3(b \ a \ 3) &= \frac{1}{8} \begin{vmatrix} 2m_b^2 & s - m_a^2 - m_b^2 & -t_2' + m_b^2 + m_3^2 \\ s - m_a^2 - m_b^2 & 2m_a^2 & -s - t_2 + s_1 + m_b^2 \\ -t_2 + m_b^2 + m_3^2 & -s - t_2 + s_1 + m_b^2 & 2m_3^2 \end{vmatrix} \end{aligned} \quad (3.16)$$

In using (3.16) it is preferable to first cancel the several large terms with opposite signs to avoid numerical errors.

When the reaction is dominated by a two particle resonance in the final state, the decay of this is best studied by replacing two invariants by the aximuthal and polar scattering angles in the rest frame of the resonance. The angle θ , which for the choice 2+3 for resonating particles is given by

$$\cos \theta = \frac{s_2 + s_2(2t_2 - t_1 - m_b^2 - m_a^2 - m_3^2) + (t_1 - m_b^2)(m_2^2 - m_3^2)}{\sqrt{\lambda(s_2, t_1, m_b^2)} \sqrt{\lambda(s_2, m_2^2, m_3^2)}}, \quad (3.17)$$

is called the Jackson angle. The angle φ defined as

$$\cos \varphi = \frac{[(p_2 + p_3) \times p_1] \cdot [(p_2 + p_3) \times p_2]}{|(p_2 + p_3) \times p_1| |(p_2 + p_3) \times p_2|} \quad (3.18)$$

is the Treiman-Yang angle. Application of (II.6.20) yields φ as a function of invariants.

$$\cos \varphi = \frac{G \begin{pmatrix} b & 2+3 & 1 \\ b & 2+3 & 2 \end{pmatrix}}{\sqrt{\Delta_3(b \ 2+3 \ 1) \Delta_3(b \ 2+3 \ 2)}} \quad (3.19)$$

where $G \begin{pmatrix} b & 2+3 & 1 \\ b & 2+3 & 2 \end{pmatrix}$ depends linearly on s_1 ,

$$G \begin{pmatrix} b & 2+3 & 1 \\ b & 2+3 & 2 \end{pmatrix} = \frac{1}{8} \begin{vmatrix} 2m_b^2 & s_2 - t_1 + m_b^2 & s_2 - t_1 + t_2 + m_3^2 \\ s_2 - t_1 + m_b^2 & 2s_2 & s_2 + m_2^2 - m_3^2 \\ s - s_2 + t_1 - m_a^2 & s - s_2 - m_1^2 & s_1 - m_1^2 - m_2^2 \end{vmatrix} \quad (3.20)$$

$$= V + \frac{1}{2} \Delta_2 (b \ 2+3) s_1 \quad .$$

Differentiation and use of

$$\sin^2 \varphi = \Delta_4 (b \ 2+3 \ 1 \ 2) \Delta_2 (b \ 2+3) / \Delta_3 (b \ 2+3 \ 1) \Delta_3 (b \ 2+3 \ 2) \quad (3.21)$$

yields

$$d\varphi = \frac{1}{2} ds_1 \sqrt{\Delta_2 (b \ 2+3) / \Delta_4 (b \ 2+3 \ 1 \ 2)} \quad . \quad (3.22)$$

One value of s_1 corresponds to two values of φ (φ and $2\pi - \varphi$).
For integrals this means

$$\int_0^{2\pi} d\varphi = 2 \int_0^{\pi} d\varphi = \int_{s_1^{\min}}^{s_1^{\max}} ds_1 \sqrt{\Delta_2 / \Delta_4} \quad , \quad (3.23)$$

4. The physical region in terms of invariants

Symmetry properties of the process $2 \rightarrow 3$ are exhibited more clearly by adopting the notation $p_4 = -p_b$, $p_5 = -p_a$. The variables

$$s_i = (p_i + p_{i+1})^2, \quad i = 1, \dots, 5 \quad (4.1)$$

where $p_6 = p_1$, are symmetrical in cyclic and anticyclic permutations. These are a basic set of invariants. Any other invariant can be written in terms of m_i^2 and s_i , $i = 1, \dots, 5$. To see this, it suffices to observe that for a non-neighbouring pair, e.g. 1,3, one can write

$$\begin{aligned} (p_1 + p_3)^2 &= (p_4 + p_5)^2 - (p_1 + p_2)^2 - (p_2 + p_3)^2 \\ &+ m_1^2 + m_2^2 + m_3^2. \end{aligned} \quad (4.2)$$

Kinematic variables obtained through a permutation which is not cyclic or anticyclic are linear functions of the old ones. In the reaction $2 \rightarrow 2$ the basic set is s, t and the complete set s, t, u . For $2 \rightarrow 3$, the basic set is (4.1) and the complete set contains 10 variables $(p_i + p_j)^2$.

The physical region is $\Delta_4 < 0$. The Cayley determinant representation of Δ_4 exhibits the cyclic symmetry,

$$\Delta_4 = -\frac{1}{16} \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & m_2^2 & s_2 & t_1 & m_1^2 \\ 1 & m_2^2 & 0 & m_3^2 & t_2 & s_1 \\ 1 & s_2 & m_3^2 & 0 & m_b^2 & s \\ 1 & t_1 & t_2 & m_b^2 & 0 & m_a^2 \\ 1 & m_1^2 & s_1 & s & m_a^2 & 0 \end{vmatrix} \quad (4.3)$$

$$= -\frac{1}{16} \begin{vmatrix} 2m_1^2 & s_1 + m_1^2 - m_2^2 & s - s_2 + m_1^2 & -t_1 + m_a^2 + m_1^2 \\ s_1 + m_1^2 - m_2^2 & 2s_1 & s + s_1 - m_3^2 & s_1 - t_2 + m_a^2 \\ s - s_2 + m_1^2 & s + s_1 - m_3^2 & 2s & s + m_a^2 - m_b^2 \\ -t_1 + m_a^2 + m_1^2 & s_1 - t_2 + m_a^2 & s + m_a^2 - m_b^2 & 2m_a^2 \end{vmatrix} \quad (4.4)$$

Δ_4 is a homogenous polynomial of fourth degree, and it is of second degree with respect to each of its variables.

The phase space integral R_3 is taken over the physical region. There are various useful versions of R_3 in terms of momenta and invariants. We write them for unity matrix element; general phase space integrals are then obtained simply by multiplying the expressions by the matrix element squared.

Regarding the process as two successive scatterings, Fig. 3.3.a, the integral is

$$R_3 = \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} ds_2 R_2(s; m_1^2, s_2) R_2(s_2; m_2^2, m_3^2), \quad (4.5)$$

where any representation of R_2 can be substituted. In terms of t_1, t_2 and angles (4.5) becomes (c.f. (3.17))

$$R_3 = \int ds_2 \int d\varphi_1 \int \frac{dt_1 \theta(-G(1))}{4\sqrt{\lambda(s, m_a^2, m_b^2)}} \int d\varphi_2 \int \frac{dt_2 \theta(-G(2))}{4\sqrt{\lambda(s_2, t_1, m_b^2)}} . \quad (4.6)$$

The step functions $\theta(-G(i))$ limit the contribution to the integrals to the physical region $G(i) \leq 0$. The arguments of $G(i)$ are those of the scattering processes,

$$\begin{aligned} G(1) &= G(s, t_1, s_2, m_a^2, m_b^2, m_1^2) , \\ G(2) &= G(s_2, t_2, m_3^2, t_1, m_b^2, m_2^2) . \end{aligned} \quad (4.7)$$

A change in the polar angle φ_2 of the second process is related to that of the invariant s_1 by the equation (3.22). Substitution in (4.6) gives

$$R_3 = \int \frac{d\varphi_1 ds_1 ds_2 dt_1 dt_2}{4\sqrt{\lambda(s, m_a^2, m_b^2)} \cdot 8\sqrt{-\Delta_4}} \theta(-\Delta_4) . \quad (4.8)$$

The argument of Δ_4 appear explicitly in (4.3).

5. The Dalitz, Chew-Low and t_1 - t_2 plots

Structure in the matrix element is conveniently seen in bubble chamber data by plotting the events on plane, i.e. as a plot in two variables. If s_1 and s_2 are chosen, the absence of momentum transfer variables implies that the situation is here identical with the process $A \rightarrow 1+2+3$. The plot is thus the Dalitz plot with constant probability density for constant matrix element and boundaries as discussed in Section V.2. Fig. 5.1 illustrates a typical experimental distribution, $pp \rightarrow p\pi^+n$ at 5.5 GeV/c.

A plot in a momentum transfer $t' = (p_a - p_i)^2$ and its conjugate mass $s' = (p_a + p_b - p_i)^2$ is the Chew-Low plot. The physical region in the s', t' plane is given by

$$G(s, t', s', m_a^2, m_b^2, m_1^2) \leq 0. \quad (5.1)$$

The probability density of the Chew-Low plot is a function the variables. When the system with mass $\sqrt{s'}$ consists of two particles and the matrix element of the total process is unity, the phase space integral (4.5) is from (III.4.6)

$$R_3 = 2\pi \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \int \frac{dt_1}{4\sqrt{\lambda(s, m_a^2, m_b^2)}} \frac{\sqrt{\lambda(s_2, m_2^2, m_3^2)}}{4s_2} \quad (5.2)$$

Some of the structure in the s_2 direction is due to kinematics.

Because the dependence on t_1 and s_2 in (5.2) is factorizable it is possible to even out kinematic dependence on s_2 . Let $r(x)$ be defined as

$$r(x) = \int_{(m_2+m_3)^2}^x dy \frac{\sqrt{\lambda(y, m_2^2, m_3^2)}}{y} \quad (5.3)$$

$$\begin{aligned} &= \sqrt{\lambda(y, m_2^2, m_3^2)} \\ &- \frac{1}{2}(m_2^2 - m_3^2) \log \frac{(m_2^2 - m_3^2)^2 - (m_2^2 + m_3^2)y + (m_2^2 - m_3^2)\sqrt{\lambda(y, m_2^2, m_3^2)}}{(m_2^2 - m_3^2)^2 - (m_2^2 + m_3^2)y - (m_2^2 - m_3^2)\sqrt{\lambda(y, m_2^2, m_3^2)}} \\ &- (m_2^2 + m_3^2) \log \frac{\sqrt{\lambda(y, m_2^2, m_3^2)} + y - (m_2^2 + m_3^2)}{2m_2 m_3} \end{aligned} \quad (5.4)$$

Eq. (5.2) is then

$$R_3 = \frac{\pi/2}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \int dr dt_1, \quad (5.5)$$

and the r, t_1 plot has constant event density.

Turning now the t_1, t_2 plot we note that the boundary of the physical region, $\Delta_4 = 0$, is a fourth order homogeneous polynomial which is of second order in t_1 and t_2 . In Fig. 5.2 representative examples are given and in Fig. 5.3 the effect of the incoming energy is seen. The probability density for constant matrix element is obtained from (4.6). The integrals over φ_1 and φ_2 give $(2\pi)^2$, and the integral over s_2 can be computed in closed form. The result is

$$R_3 = \frac{\pi^2/4}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \int dt_1 dt_2 \log \frac{s_2^+ - t_2 - m_a^2 + \sqrt{\lambda(s_1^+, t_2, m_a^2)}}{s_2^- - t_2 - m_a^2 + \sqrt{\lambda(s_1^-, t_2, m_a^2)}}$$

(5.6)

Here $s_2^\pm$ are the maximum and minimum physical values of s_2 , and are determined from $G(1) = 0$, $G(2) = 0$ with $G(i)$ in (4.7). An example of the phase space distribution is shown in Fig. 5.4, where the events have been generated by computer using a Monte Carlo method.

VI. Multiparticle Production

1. Choice of variables

In the process

$$a+b \rightarrow 1+2+ \dots +n \quad (1.1)$$

the particles are on the mass shell, $p_i^2 = m_i^2$, and energy and momentum are conserved. Thus there remain $3n-4$ independent variables ϕ which are needed to describe the process (see Chapter III). There are a large number of different descriptions that have been used. In fact, the choice of appropriate variables for the description of a multiparticle amplitude has posed a long-lasting problem to both theoreticians and experimentalists in particle physics. No clear choice has emerged, and it seems that there is no privileged set of variables in general.

When a process is analysed in terms of a mathematical model, a basic criterion of usefulness is simplicity. The variables should be chosen so that the function describing the process is simple and transparent. The essential functional dependences in the expression can then be interpreted as distinct physical effects. A particular example of simplicity is a factorizable function. The factors correspond to uncorrelated, independent physical phenomena.

In particle processes a common phenomenon is resonance formation. If the quantum members are suitable, the distribution in a two-particle invariant mass shows peaks at definite masses with known widths, and the production mechanism affects only the strength of the peak.

A universal feature is also peripherality. When the incident energy increases, the average transverse momentum of the emitted particles remains limited, even at cosmic ray energies, to about 0.4 GeV/c. At energies above a few GeV this means that the particles are strongly correlated along the beam axis. A second, but not equivalent, way to state peripherality is the multiperipheral model. According to this there always exists an ordering of the final state particles such that the momentum transfers t_i in Fig. VI.1.1. are small.

Relativity principle would seem to imply that invariant variables are best suited for the analysis of particle collisions. In practice this need not be true. One example is the convenient way of expressing peripherality as a limiting distribution of transverse momenta. Secondly, if n is four or larger, use of purely invariant variables leads to cumbersome mathematics. E.g. the boundary of the physical region is given by a set of nonlinear equations.

In practice, multiparticle processes are studied by measuring total cross sections and distributions in one or two variables. If data of a collision process

with n particles in the final state are plotted, this means that one has to neglect $3n-5$, $3n-6$ or $3n-7$ essential kinematic variables. In testing a dynamical model, it similarly means integration over the same number of variables. Calculation of phase space integrals is thus an important tool for the multiparticle physicist. However, one should also have more intuitive insight into the different types of phase space effects. In the case of several particles the purely kinematic effects are far from trivial. These must be properly separated from dynamic features which reflect the structure of the collision amplitude. A good presentation of data emphasizes the dynamical aspects of experimental findings. The problem of data presentation is central in the study of multiparticle processes.

We list several common sets of variables for the process (1.1):

a) Data are conveniently listed giving the momentum components of the particles in each event. Within experimental accuracy, these should be related by the four energy-momentum conservation equations.

b) Let $s_{n-1}^{1/2}, \dots, s_2^{1/2}$ to be the invariant masses of the intermediate states in the cascade decay in Fig. 1.2. These are supplemented by the decay angles θ_i, φ_i in the rest frames of $s_n, s_{n-1}, \dots, s_2$. For this set the equations defining the physical region are the simplest possible.

c) A matrix element containing resonances, multi-Regge behaviour or both (Veneziano), must include the relevant two particle invariant masses $s_{ij} = (p_i + p_j)^2$ among variables.

d) Group-theoretical analysis suggests simplicity in multiperipheral models in terms of the Toller variables (Section VII.3). These are two-particle invariant masses, momentum transfers squared, and the so-called Toller angles ω_i . The main difficulty here is that the expression for the total energy is unusually unwieldy, except in the limit of infinite energy.

e) Transverse momentum distributions are at high energy fairly independent of other properties of the reaction. This implies that $\vec{p}_{i\perp}$ are natural variables, which in some approximations can be put to zero. Van Hove has suggested (Van Hove 69) that the remaining interrelated $p_{i\parallel}$ be replaced by certain $n-2$ angles φ_i . These longitudinal phase space plots will be discussed in Chapter VIII.

f) Combinations of invariant and angular variables and consequent projections of phase space into one or two dimensional plots to analyse data have been discussed in some detail by Nyborg (Nyborg 69, 70).

Once the choice of variables is made, there are three main questions to settle to be able to use them in practice. 1) The range of variation (the physical region)

must be found for the set of variables. 2) The phase space factor in the phase space integral must be determined. If the matrix element were constant, this factor would be the density of events in these variables. 3) The relation of the variables to p_i and possibly to other usual variables must be known.

2. Recursion relations

A multiparticle reaction is often on the basis of experimental evidence known to take place via resonance formation. In the intermediate state there are less than n unstable particles which then successively decay to each other through the emission of the final state particles. Now, regardless of the actual existence of such intermediate states, an n -particle final state can kinematically always be considered as the result of a sequence of simpler processes. This is formally seen by deriving recursion formulae for R_n . There are numerous ways to this. Some of these contain kinematic variables which provide simplest possible description of the physical region of the n -particle process.

The total cross section was in Eq. (III.2.2) defined as

$$\sigma = F^{-1} \prod_{i=1}^n \int \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^n p_i) |A|^2 \quad (2.1)$$

with the flux factor

$$F = 2\lambda(s, m_a^2, m_b^2)^{1/2} (2\pi)^{3n-4} \quad (2.2)$$

To derive new representations of (2.1), it suffices to neglect F and $|A|^2$ and consider transformations of

$$R_n = \prod_{i=1}^n \int \frac{d^3 p_i}{2E_i} \delta^4(q - \sum_{i=1}^n p_i) = \prod_{i=1}^n \int d^4 p_i \delta(p_i^2 - m_i^2) \delta^4(q - p_1 - \dots - p_n) \quad (2.3)$$

To get σ one just "multiplies" (2.3) by the integrand $F \cdot |A|^2$. A distribution in a variable x is then $\partial\sigma/\partial x$ (Eq. (III.2.7)).

A covariant recursion relation based on the physical picture of sequential decay, Fig. 1.2, was first derived by Srivastava and Sudarshan (Srivastava 58). Grouping together the factors in (2.3) which refer to the system $1, \dots, n-1$ results in

$$\begin{aligned} R_n(q) &= \iint \frac{d^3 p_n}{2E_n} \prod_{i=1}^{n-1} \frac{d^3 p_i}{2E_i} \delta^4((q-p_n) - \sum_{i=1}^{n-1} p_i) \\ &= \int \frac{d^3 p_n}{2E_n} R_{n-1}(q-p_n) \end{aligned} \quad (2.4)$$

A still more perspicuous version can be written. If p is any time-like four-vector, the identities

$$1 = \int d^4 q \delta^4(p-q) \quad (2.5)$$

$$1 = \int dM^2 \delta(p^2 - M^2) \quad (2.6)$$

are valid. We introduce then the $n-1$ particle momentum $q_{n-1} = p_1 + \dots + p_{n-1}$ and its length $M_{n-1}^2 = q_{n-1}^2$ to (2.3) by these identities,

$$\begin{aligned} R_n(q) &= \int dM_{n-1}^2 \int d^4 p_n \delta(p_n^2 - m_n^2) \int d^4 q_{n-1} \delta(q_{n-1}^2 - M_{n-1}^2) \times \\ &\times \delta^4(q - p_n - q_{n-1}) \prod_{i=1}^{n-1} \int d^4 p_i \delta(p_i^2 - m_i^2) \delta^4(q_{n-1} - p_1 - \dots - p_{n-1}). \end{aligned} \quad (2.7)$$

The factor from the product sign onwards is R_{n-1} . One then has a two-body decay followed by a $n-1$ particle process,

$$R_n(q) = \int dM_{n-1}^2 \int \frac{d^3 p_n d^3 q_{n-1}}{2E_n 2E(q_{n-1})} \delta^4(q - p_n - q_{n-1}) R_{n-1}(q_{n-1}). \quad (2.8)$$

Writing consistently $q^2 = q_n^2 = M_n^2$ this is also

$$R_n(M_n) = \int_0^\infty dM_{n-1}^2 R_2(M_n; M_{n-1}, m_n) R_{n-1}(M_{n-1}). \quad (2.9)$$

Fig. VI.2.1.

Instead of taking a group of $n-1$ particles, one can choose, say ℓ particles to form the intermediate state. A few steps analogous to the preceding ones lead to a "splitting" relation as follows:

$$\begin{aligned} R_n(q_n) &= \int dM_\ell^2 \int d^4 q_\ell d^4 p_{\ell+1} \dots d^4 p_n \delta(q_\ell^2 - M_\ell^2) \delta(p_{\ell+1}^2 - m_{\ell+1}^2) \dots \delta(p_n^2 - m_n^2) \times \\ &\times \delta^4(q_n - q_\ell - \sum_{\ell+1}^n p_i) \int d^4 p_1 \dots d^4 p_\ell \delta(p_1^2 - m_1^2) \dots \delta(p_\ell^2 - m_\ell^2) \delta^4(q_\ell - \sum_1^\ell p_i) \\ &= \int dM_\ell^2 R_{n-\ell+1}(M_n; M_\ell, m_{\ell+1}, \dots, m_n) R_\ell(M_\ell; m_1, \dots, m_\ell) \end{aligned} \quad (2.10)$$

Fig. VI.2.2.

It is now obvious how to generalize these recurrence relations to an arbitrary tree graph depicting a set of successive decays. Each "unstable" particle will give one mass integration.

Let us now study these kinematical variables in detail. For definiteness we choose a fixed graph, that in Fig. 1.2. In the corresponding equation, (2.7), we can integrate over p_n and $Q_{n-1} = |q_{n-1}|$ using the four delta-functions. According to (IV.1.) the result is

$$R_n(q_n) = \int dM_{n-1}^2 d\cos\theta_{n-1} d\varphi_{n-1} \frac{\lambda(M_n^2, M_{n-1}^2, m_n^2)^{1/2}}{8 M_n^2} R_{n-1}(q_{n-1}) \quad (2.11)$$

The angles $\theta_{n-1}, \varphi_{n-1}$ are the angles of emission of the system $1, \dots, n-1$ with momentum $\underline{q}_{n-1}$ in the rest system of $1, \dots, n$ ($\underline{q}_n = 0$). The decay can here be regarded as spherically symmetric and the orientation of the coordinate system can be chosen at will.

The ranges of variation of θ, φ are $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$. Also M_{n-1} has simple limits. It must be at least larger than $m_1 + \dots + m_{n-1}$. The upper limit is obtained when the energy M_n goes totally into the rest masses M_{n-1} and m_n . We thus get

$$\begin{aligned} \mu_{n-1} &\leq M_{n-1} \leq M_n - m_n, \\ \mu_{n-1} &= m_1 + \dots + m_{n-1}. \end{aligned} \quad (2.12)$$

Iteration of (2.11) leads to lower R_ℓ and ends with $R_1(M_1^2) = \delta(M_1^2 - m_1^2)$. Carrying out the M_1^2 -integration leaves R_n as a $3n-4$ fold integral

$$\begin{aligned} R_n(M_n^2) &= \int_{\mu_{n-1}}^{(M_n - m_n)^2} dM_{n-1}^2 \int_{-1}^1 d\cos\theta_{n-1} \int_0^{2\pi} d\varphi_{n-1} \frac{\lambda(M_n^2, M_{n-1}^2, m_n^2)^{1/2}}{8 M_n^2} \times \\ &\dots \times \int_{\mu_2}^{(M_3 - m_3)^2} dM_2^2 \int_{-1}^1 d\cos\theta_3 \int_0^{2\pi} d\varphi_3 \frac{\lambda(M_3^2, M_2^2, m_3^2)^{1/2}}{8 M_3^2} \int_{-1}^1 d\cos\theta_2 \int_0^{2\pi} d\varphi_2 \times \\ &\times \frac{\lambda(M_2^2, m_1^2, m_2^2)^{1/2}}{8 M_2^2}. \end{aligned} \quad (2.13)$$

Because of the simplicity of conditions defining the physical region, eq. (2.13) has important applications. It is the basis of the Monte Carlo method of Chapter X.

The transformation from (2.3) to (2.13) is general and exact. To use (2.13) for an arbitrary matrix element A , one should express A in terms of M_k , θ_k , φ_k and put it inside the integral. Unfortunately, physical processes rarely resemble that in Fig. 1.2, and thus M_k , θ_k , φ_k are not very natural for interpretation of data or for theoretical models.

A recursion relation is an algebraic expression for the physical picture of a sequence of basic processes. In the preceeding the basic process was the decay $1 \rightarrow 2$ (one particle goes into two). Any decay $1 \rightarrow k$ or collision $2 \rightarrow k$ would serve as a basic process. The cases $2 \rightarrow 2$ and $2 \rightarrow 3$ have been discussed by us (Byckling 69a,b). In $2 \rightarrow 2$ the invariant momentum transfers t_i can be included among variables. It is the basis of the peripheral Monte Carlo and we show here the idea behind it.

Fig.VI.2.3. The multipheral graph in Fig. 1.1 is cut at the k 'th exchange and particles $1, \dots, k-1$ are grouped together. The basic process is then the scattering of m_a^2 and t_k to form M_{k-1}^2 and m_k^2 . The scattering is described by scattering angles θ, φ or equivalently by φ and momentum transfer squared t . One uses (Chapter IV) to express the scattering process and includes $R_{k-1}(M_{k-1})$ as weight function ("matrix element") of the subsequent

decay of M_{k-1} . The recursion relation is then

$$R_k(M_k, t_k) = \int_{\mu_{k-1}} dM_{k-1}^2 \int dt_{k-1} \int_0^{2\pi} d\varphi_{k-1} \frac{\theta(-G(k-1))}{4\lambda(M_k^2, t_k, m_a^2)^{1/2}} R_{k-1}(M_{k-1}, t_{k-1}) \cdot \quad (2.14)$$

The step function $\theta(-G(k-1))$ is zero outside the physical region of the scattering process. The process is physical inside the Chew-Low-plot: $M_{k-1}^2 \geq \mu_{k-1}$ and

$$G(k-1) \equiv G(M_k^2, t_{k-1}, m_k^2, m_a^2, t_k, M_{k-1}^2) \leq 0 \quad (2.15)$$

The aximuthal angle varies freely, $0 \leq \varphi \leq 2\pi$. The axis around which φ is defined is the "beam axis" of the collision, i.e. the incident direction of t_k and m_a^2 .

Iteration of (2.14) leads to

$$R_n(M_n^2, m_b^2) = \frac{1}{4\lambda(M_n^2, m_b^2, m_a^2)^{1/2}} \int_{\mu_{n-1}} dM_{n-1}^2 \int dt_{n-1} \int_0^{2\pi} d\varphi_{n-1} \frac{\theta(-G(n-1))}{4\lambda(M_{n-1}^2, t_{n-1}, m_a^2)^{1/2}} \times \dots \times \int_{\mu_2} dM_2^2 \int dt_2 \int_0^{2\pi} d\varphi_2 \frac{\theta(-G(2))}{4\lambda(M_2^2, t_2, m_a^2)^{1/2}} \int dt_1 \int_0^{2\pi} d\varphi_1 \theta(-G(1)) \quad (2.16)$$

When polarization is zero, rotation φ_{n-1} around the beam axis is irrelevant, and could be integrated to give 2π .

That would leave $3n-5$ variables. Because the t_i 's appear in (2.16), peripherality can be imposed in the sense of

the multiperipheral model. Thus (2.16) is suited to models

with e.g. $|A|^2 = \prod_{i=1}^{n-1} e^{a_i t_i}$.

The multi-Regge description is of the type

$$|A|^2 \sim \prod_{i=1}^{n-1} s_i^{\alpha_i + \alpha_i' t_i} \quad (2.17)$$

The two particle invariant masses $s_i = (p_i + p_{i+1})^2$ must be included among variables. A formula analogous to (2.13) and (2.16) can be written in terms of M_i^2, s_i, t_i (Byckling 69). The basic process is $2 \rightarrow 3$ and the integration limits will be determined by the somewhat unwieldy conditions $\Delta_4 < 0$.

3. Physical region in terms of invariants

When invariant variables are used to discuss multi-particle reactions, there are two main kinematic problems. The range of variation (physical region) must be determined, and the number of distinct momentum configurations related to one point in this region computed. Secondly, the kinematic factor appearing inside the phase space integral must be calculated by computing the Jacobian of the transformation to the invariant variables (Section 4).

The physical region is best discussed by starting from the reaction

$$0 \rightarrow 1+2+ \dots +n . \quad (3.1)$$

The physical region of any process obtained from (3.1) by crossing, e.g. $0+\bar{n} \rightarrow 1+ \dots +(n-1)$ will be shown to be limited by the same set of universal boundaries.

A natural set of invariants is formed by the scalar products

$$(ij) = p_i p_j , \quad 1 \leq i < j \leq n , \quad (3.2)$$

in addition to the fixed masses $p_i^2 = m_i^2$. The set (3.2) contains $\frac{1}{2}n(n-1)$ quantities (ij) . They are linearly related to two-particle invariant masses

$$s_{ij} = (p_i + p_j)^2 = m_i^2 + m_j^2 + 2(ij) . \quad (3.3)$$

The invariant masses of three or more particles, $s_{ijk\dots} = (p_i + p_j + p_k + \dots)^2$, are linear combinations of (ij) .

The process $0 \rightarrow 1+2+3$ involves 2 invariant variables (Chapter V), and each additional particle brings three more. The total is then $3n-7$. A second way to derive this is first to note that the reaction (3.1) involves $4(n+1)$ four-momenta. Four-momentum conservation gives four equations. The over-all Lorentz-transformation (4 parameters) and space rotations of the whole system (3 parameters) leave the process invariant. Finally, the particles 1, ... ,n must be on the mass shell (n parameters). The result is again $3n-7$.

We see that there are $\frac{1}{2}(n^2-7n+14)$ relations between the quantities (ij). One of these is the invariant equation due to four-momentum conservation

$$m_0^2 = \left(\sum_i^n p_i \right)^2 = \sum_i^n m_i^2 + 2 \sum_{i < j} (ij) . \quad (3.4)$$

Further, because in a four-dimensional space five or more vectors are always linearly dependent, the Gram determinants with $n > 4$ vanish identically, as shown in Appendix A. We then set up the equations (with notation $i \equiv p_i$)

$$\begin{aligned} \Delta_5(12345) &= 0 \\ \Delta_5(12346) &= G\left(\begin{smallmatrix} 12345 \\ 12346 \end{smallmatrix}\right) = 0 , \\ \Delta_5(12347) &= G\left(\begin{smallmatrix} 12345 \\ 12347 \end{smallmatrix}\right) = G\left(\begin{smallmatrix} 12346 \\ 12347 \end{smallmatrix}\right) = 0 , \end{aligned} \quad (3.5)$$

and so on. Now each equation contains a new relation between (ij). Their number is $\frac{1}{2}(n-3)(n-4)$, and we have thus found all the necessary equations.

Let us now compare the present situation with $n = 3, 4$. For $n=3$ the set (12),(13),(23) corresponds to Mandelstam variables s, t, u . The relation (3.4) is $s+t+u = \sum m_i^2$. For $n=4$ there are the variables (12),(23), (34),(13),(24),(14), which are connected by (3.4). One can use $(p_0 - p_4)^2 = (p_1 + p_2 + p_3)^2$ to get (13) in terms of (12),(23) and (40), and similarly for (24). Finally (3.4) gives (14) linearly in the $(i, i+1)$. Thus one finds that any set of invariant variables (of the type discussed here) is linearly related to (01),(12), (23),(34),(40) or s, t_1, s_1, s_2, t_2 in Fig. (V.3.2), and all choices are essentially equivalent.

Beginning with $n=5$ the choice of independent invariant variables ceases to be arbitrary. The reason is the four-dimensionality of space-time reflected in the non-linear equations (3.5). The complicated problem of finding the equations for the physical region in terms of invariants was first discussed in detail by Asribekov (62,a,b,c). A complete solution was given by Byers and Yang (Byers 64). In a different form the solution is given by Poon (70). A third formulation, explicit generalization of Asribekov's equations to arbitrary n , is also possible (Morrow 70). Conditions in definite sets of variables have been considered by Morrow (66) and Kumar (69,70), and the case $n=5$ by Nyborg (65), and McNeil and Morrow (McNeil 69).

We state here one set of conditions defining the physical region R and indicate its justification.

Let R_{1s} be a space with 1 time and s space dimensions. The physical space is R_{13} . If $p_1, \dots, p_n$ all lie in a plane, on a line, or vanish, $p_1, \dots, p_n$ belong to R_{12}, R_{11}, R_{10} , respectively. Thus if $p_1, \dots, p_n$ are in R_{13} , they are inside the physical region R . The boundary is given by R_{12}, R_{11} and R_{10} .

The physical region is specified by the following statement. The time-like four-vectors $p_1, \dots, p_n$ with positive energy components are in R_{13} if and only if there is such a permutation of $p_{i_1}, \dots, p_{i_n}$ that

$$\Delta_1(1) > 0, \quad \Delta_2(12) < 0, \quad \Delta_3(123) > 0, \quad \Delta_4(1234) < 0 \quad (3.6)$$

$$\Delta_5(1234k) = 0 \quad k = 5, \dots, n \quad (3.7)$$

$$G\left(\begin{smallmatrix} 1234k \\ 1234\ell \end{smallmatrix}\right) = 0 \quad 5 \leq k < \ell \leq n. \quad (3.8)$$

Let us try to understand the content of (3.6)-(3.8). Because p_1 is time-like, we can take the frame $\underline{p}_1=0$. According to (A.3), (A.8), (A.11), (A.14), we have

$$\begin{aligned} \Delta_1(1) &= m_1^2 \\ \Delta_2(12) &= -m_1^2 p_2^2, \\ \Delta_3(123) &= m_1^2 p_2^2 p_3^2 \sin^2 \theta_{23}, \\ \Delta_4(1234) &= -m_1^2 p_2^2 p_3^2 p_4^2 \sin^2 \theta_{23} \sin^2 \theta_{24} \sin^2 \varphi. \end{aligned} \quad (3.9)$$

Thus (3.6) is exactly equivalent to the existence of (at least one) linearly independent momentum configuration p_1, p_2, p_3, p_4 . Further, according linear algebra (Gantmacher 53), (3.7) and (3.8) are equivalent to the linear dependence of each of $p_5, \dots, p_n$ on $p_1, \dots, p_4$. Thus

(3.6)-(3.8) imply that $p_1, \dots, p_n$ are in R_{13} and conversely.

Next we look at what happens when (3.6)-(3.8) fail. First, if (3.7), (3.8) are not valid, $p_1, \dots, p_n$ cannot be a configuration in real space-time but a higher dimensional space is needed. Secondly, if (3.7), (3.8) are valid but there is no set of four p_i 's such that (3.6) applies, there are two alternatives. Among $\Delta_2, \Delta_3, \Delta_4$ there is always at least either one incorrect sign or, if not, then one $\Delta_k=0$. In the latter case, it is seen from (3.9) that $\Delta_k(1\dots k)=0$ implies $\Delta_{k+1}(1\dots k\ell)=0$. We thus obtain the boundaries of R:

$$R_{12}: \quad \Delta_4(ijkl)=0 \quad \text{for all } i, \dots, \ell$$

$$R_{11}: \quad \Delta_3(ijk) = 0 \quad \text{for all } i, j, k$$

$$R_{10}: \quad \Delta_2(ij)=0 \quad \text{for all } i, j$$

with lower Δ_k 's in (3.6) intact. As to the wrong signs, at least one of the physical quantities in (3.9) would then be imaginary.

A completely symmetric characterization of R is possible (Byers 64). Define the sums

$$\bar{\Delta}_m = (-1)^{m-1} \sum \Delta_m(i_1, i_2, \dots, i_m), \quad 1 \leq i_1 < \dots < i_m \leq n \quad (3.11)$$

Thus e.g.

$$\bar{\Delta}_1 = \sum_{1}^n m_1^2,$$

$$\bar{\Delta}_n = (-1)^{n-1} \Delta_n.$$

Then the necessary and sufficient condition for $p_1, \dots, p_n$ to be in R_{1s} is

$$\bar{\Delta}_1 > 0, \quad \bar{\Delta}_2 > 0, \quad \dots, \quad \bar{\Delta}_{s+1} > 0, \quad (3.12)$$

$$\bar{\Delta}_{s+2} = \dots = \bar{\Delta}_n = 0. \quad (3.13)$$

It is easy to see that (3.12), (3.13) are necessary. If $p_1, \dots, p_n$ are in a $s+1$ dimensional space, according to Appendix A $\Delta_{s+2}, \Delta_{s+3}, \dots$ vanish. Then (3.13) follows. Further, (3.9) imply directly (3.12). The sufficiency of (3.12), (3.13) are shown in (Byers 64).

The invariant mass $(p_1 + \dots + p_n)^2$ is not fixed by the conditions (3.6)-(3.8). Usually the reaction (3.1) is studied at fixed energy, and one must supplement (3.6)-(3.8) by (3.4). As shown earlier, the number of independent invariants is now the same as the number of essential geometric variables. Thus each combination of invariants corresponds to a discrete set of different momentum configurations. Two momentum configurations related by a proper orthochronous Lorentz-transformation are, of course, considered identical. The number of distinct kinematic configurations is computed next.

The relation (3.4) is linear, and each (3.8) is linear in the new quantity (k_l) . However, each (3.7) which relates $(l_k), \dots, (4_k)$ is quadratic in these variables. In solving these equations there are in all 2^{n-4} possible choices of signs. A space reflection does not change invariants, but otherwise the

correspondence between $p_1, \dots, p_n$ and the whole set of invariants (ij) is one-to-one. Thus a set of independent (ij) corresponds to 2^{n-3} distinct $p_1, \dots, p_n$.

The scattering process

$$0 + \bar{n} \rightarrow 1 + \dots + (n-1) \quad (3.10)$$

has a physical region limited by the same equations (3.4), (3.6)-(3.8). The main difference is that $E_{\bar{n}}$ is negative. The sign of an energy component E_i is reflected in (ij) . In the frame $p_i = 0$ one has $(ij) = m_i E_j > 0$. If now either i or j is incoming, the sign of p_i or p_j is changed, i.e. $(ij) < 0$. To each channel corresponds a region in the (ij) space disconnected from all others. These are all defined by (3.4), (3.6)-(3.8). There are 2^{n-1} such regions, because there is one operation which does not affect the (ij) 's: changing signs of all p_i 's.

In the process (3.1) at fixed m_0^2 there are $3n-7$ essential variables, when the energy m_0 of the decaying system is fixed. In the collision process m_0 and $m_{\bar{n}}$ as well as other masses are fixed. In addition, in real experiments the incoming energy $(p_0 + p_{\bar{n}})^2 = m_0^2 + m_{\bar{n}}^2 + 2(0\bar{n})$ is fixed. There remain $3n-8 = 3(n-1)-5$ independent invariant variables.

We have now, in principle, determined the physical region of any decay or collision process in terms of invariant variables. The equations are linear or quadratic in (ij) and thus also in $s_{i\dots k} = (p_i + \dots + p_k)^2$.

4. The phase space factor

The phase space integral (2.3) is now transformed into the variables (ij) discussed in the previous section. Then (3.3) gives trivially the transform to invariant masses and momentum transfers. There is a large range of possible combinations of integration variables. E.g. one can choose only independent quantities, or take all $\frac{1}{2}n(n-1)$ quantities (ij) and insert delta-functions to account for the relations (3.7), (3.8). Also the variables themselves allow considerable latitude.

In transforming R_n the problem lies in computation of the phase space factor, i.e. the Jacobian of the transformation. We do it step by step. To write p_i explicitly a Lorentz frame must be defined. As the first step we take the rest frame of particle 0. From (A.4) follows $\partial(01)/\partial P_1 = m_0 P_1 / E_1$ and (A.8) gives

$$\int \frac{d^3 p_1}{2E_1} = 4\pi \int \frac{dP_1 P_1^2}{2E_1} = 2\pi \int d(01) \frac{\sqrt{-\Delta_2(01)}}{m_0^2} . \quad (4.1)$$

Next we take the direction of p_1 as the reference direction. The Jacobian is $|\partial((02), (12))/\partial(P_2, \cos\theta_{12})| = m_0 P_1 P_2^2 / E_2$, and (A.8) gives

$$\int \frac{d^3 p_2}{2E_2} = 2\pi \int \frac{dP_2 P_2^2 d\cos\theta_{12}}{2E_2} = \pi \int d(02) d(12) \frac{1}{\sqrt{-\Delta_2(01)}} . \quad (4.2)$$

The quantity $d^3 p_i / 2E_i = dP_i P_i^2 d\cos\theta_i d\varphi_i / 2E_i$ can be written in terms of (0i), (1i), (2i). The angle between $\underline{p}_1, \underline{p}_2$ and $\underline{p}_1, \underline{p}_i$ planes is called φ_i , then

$$\begin{aligned} (0i) &= m_0 E_i \\ (1i) &= E_1 E_i - P_1 P_i \cos\theta_{1i} \\ (2i) &= E_2 E_i - P_2 P_i (\cos\theta_{12} \cos\theta_{1i} - \sin\theta_{12} \sin\theta_{1i} \cos\varphi_i) \end{aligned} \quad (4.3)$$

The Jacobian is easily obtained as

$$\left| \frac{\partial((0i), (1i), (2i))}{\partial(P_i, \cos\theta_i, \varphi_i)} \right| = m_0 P_1 P_2 \frac{P_i^3}{E_i} \sin\theta_{12} \sin\theta_{1i} \sin\varphi_i \quad (4.4)$$

Thus one has

$$\int \frac{d^3 p_i}{2E_i} = \frac{1}{2} \int d(0i) d(1i) d(2i) / m_0 P_1 P_2 P_i \sin\theta_{12} \sin\theta_{1i} \sin\varphi_i$$

According to (A.14) this is simply

$$\int \frac{d^3 p_i}{2E_i} = \int \frac{d(0i) d(1i) d(2i)}{2 \sqrt{-\Delta_4(123i)}} \quad (4.5)$$

Let us first study the decay

$$0 \rightarrow 1+2+\dots+n \quad (4.6)$$

Fixed incoming energy means the constraint (3.4). We denote

$$K = \frac{1}{2} m_0^2 - \frac{1}{2} \sum_{i=1}^n m_i^2 \quad (4.7)$$

R_n is then written in the form

$$R_n = m_0^{-2} \int d(12)d(13)d(23) \prod_{i=4}^n d(1i)d(2i)d(3i) F_n \delta\left(\sum_{i<j} (ij) - K\right) \quad (4.8)$$

The decays to two and three particles have been discussed in detail earlier. Let us in any case first derive F_3 in (4.8).

We integrate over p_3 in R_3 ,

$$\int \frac{d^3 p_3}{2E_3} \delta^4(p_0 - p_1 - p_2 - p_3) = \delta(p_3^2 - (p_0 - p_1 - p_2)^2) \quad (4.9)$$

Next we insert invariants from (4.1) and (4.2),

$$R_3 = 2\pi \int d(01) \frac{\sqrt{-\Delta_2(01)}}{m_0^2} \cdot \pi \int d(02)d(12) \frac{1}{\sqrt{-\Delta_2(01)}} \delta(2(01) + 2(02) - 2(12) - m_0^2 - m_1^2 - m_2^2 + m_3^2)$$

To get final state (ij) one uses the four-momentum conservation to write

$$(01) = -(23) + \frac{1}{2}(m_0^2 + m_1^2 - m_2^2 - m_3^2) \quad (4.10)$$

$$(02) = -(13) + \frac{1}{2}(m_0^2 - m_1^2 + m_2^2 - m_3^2) \quad .$$

Then R_3 becomes

$$R_3 = \frac{\pi^2}{m_0^2} \int d(12)d(13)d(23) \delta((12) + (13) + (23) - K) \quad (4.11)$$

Thus $F_3 = \pi^2$ reproduces the constant density of the Dalitz plot.

Once three vectors are given, they define a coordinate frame completely. Thus the transformation for vectors p_i , $i = 3, 4, \dots$, can be carried out using (4.3). Let us look at the decay into four particles. Analogously to the preceding case, we integrate over p_4 and use (4.1), (4.2) and (4.5) to get

$$R_4 = \frac{\pi^2}{m_0^2} \int d(01)d(02)d(12) \frac{d(03)d(13)d(23)}{2\sqrt{-\Delta_4(0123)}} \delta(X) \quad (4.12)$$

where the delta function is

$$\delta(X) = \delta\left(2\sum_{i=1}^3 (0i) - 2\sum_{1 \leq i < j \leq 3} (ij) - \sum_{i=1}^3 m_i^2 + m_4^2\right) \quad (4.13)$$

Momentum conservation can now be used to find expressions purely in final state invariants. One writes

$$(01) = -(23) - (24) - (34) + \frac{1}{2}(m_0^2 + m_1^2 - m_2^2 - m_3^2 - m_4^2)$$

and similarly for other $(0i)$. It is straightforward to check that (4.13) is one fourth of the delta-function in (4.8) and that the Jacobian from (4.12) to the final state (ij) 's is two. Thus we get

$$F_4 = \frac{\pi^2}{4\sqrt{-\Delta_4}} \quad (4.14)$$

where $\Delta_4 = \Delta_4(0123) = \Delta_4(1234)$.

In this case of decay to four particles a fairly symmetric choice of five independent variables is $s_{12}, s_{23}, s_{34}, s_{123}, s_{234}$. Integrating over (14) and making the simple linear transformation of variables gives

$$R_4 = \frac{\pi^2}{2^7 m_0^2} \int \frac{ds_{12} ds_{23} ds_{34} ds_{123} ds_{234}}{\sqrt{-\Delta_4}} \quad (4.15)$$

At the boundary, $\Delta_4 = 0$, the weight function diverges.

As discussed in Section 3, when the total number of (incoming and outgoing) particles exceeds five, the choice of variables is no longer arbitrary. The process $1 \rightarrow n$ with $n > 4$ can be described in terms of the (ij) in several ways. We indicate two possibilities.

The particles 4, 5, ... can be referred to the system p_0, p_1, p_2 just as particle 3 in the case $n=4$. Using (4.5) repeatedly gives

$$R_n = \frac{\pi^2}{m_0^2} \int d(01)d(12)d(12) \prod_{i=3}^{n-1} \frac{d(0i)d(1i)d(2i)}{2 \sqrt{-\Delta_4(012i)}} \delta(X)$$

with (4.16)

$$X = 2 \sum_{i=1}^{n-1} (0i) - 2 \sum_{1 \leq i < j \leq n-1} (ij) - \sum_{i=0}^{n-1} m_i^2 + m_n^2$$

If final state invariants are required, one uses momentum conservation to get

$$X = 2(n-2) \left[\sum_{i < j} (ij) - K \right]. \quad (4.17)$$

A straightforward computation of the Jacobian from (4.16) to (4.8) gives two. Thus we find that (4.8) applies with

$$F_n = \frac{\pi^2}{n-2} \prod_{i=3}^{n-1} \frac{1}{2} \left(-\Delta_4(12i(1+\dots+n)) \right)^{-1/2}. \quad (4.18)$$

It is also possible to write R_n symmetrically in all (ij) (Byers 64). Recalling the definition (3.11) one gets after considerable work

$$R_n = \alpha \int \prod_{i<j} d(ij) \bar{\Delta}_4^{\frac{1}{2}(n-5)} \delta(\bar{\Delta}_5) \delta(\bar{\Delta}_6) \dots \delta(\bar{\Delta}_n) \delta\left(\sum_{i<j} (ij) - K\right) \quad (4.19)$$

$$\alpha = \frac{2^{2n-5} \pi^2}{m_0^2} \prod_{v=2}^{n-4} \frac{\Gamma(v/2)}{2\pi^{v/2}}$$

Each $\bar{\Delta}_k$ contains all the (ij) 's. No application has so far appeared for (4.19).

The collision $a+b \rightarrow 1+\dots+n$ involves $3n-5$ invariant variables plus rotation φ around beam axis, if the incoming energy $s = p_0^2 = (p_a + p_b)^2$ is fixed. We take the beam system $p_a=0$, put p_b in z-direction, and use p_1 to define the x,z plane. We replace then 0,1,2 by a,b,1 in (4.2) and (4.5). The expressions of R_2 and R_3 derived earlier come out easily:

$$R_2 = \pi \int d(al) d(bl) \frac{1}{\sqrt{-\Delta_2(ab)}} \delta(m_b^2 - s - m_1^2 + 2(al) + 2(bl))$$

$$= \frac{\pi}{2} \int \frac{d(al)}{\sqrt{-\Delta_2(ab)}}.$$

The momentum transfer t is linearly related to $(a1)$ and thus $dt = -2d(a1)$. Calculation of R_3 reproduces easiest the earlier expression, if we put the vectors $p_0 = p_a + p_b, p_a, p_1$ in (4.2) and p_0, p_b, p_1 and p_3 in (4.5). Integration over p_2 then gives

$$R_3 = \frac{\pi}{2} \int \frac{d(01)d(a1)d(03)d(b3)d(13)}{\sqrt{-\Delta_2(0a)} \sqrt{-\Delta_4(0b13)}} \delta(2(13)+\dots)$$

$$= \frac{\pi}{4} \int \frac{d(01)d(a1)d(03)d(b3)}{\sqrt{-\Delta_2(ab)} \sqrt{-\Delta_4}} .$$

The identities $s_1 = (p_1 + p_2)^2 = (p_0 - p_3)^2 = s + m_3^2 - 2(03)$, etc., allow one to recover the expression (V.4.8).

Collisions $2 \rightarrow n$, $n > 3$, allow many sets of invariant variables, which are not linearly related. The choice must then be based on the dynamical model being studied. The general formula is

$$R_n = \frac{\pi}{2} \int \frac{d(01)d(a1)}{\sqrt{-\Delta_2(ab)}} \prod_{i=3}^n \frac{d(j_i, i)d(k_i, k)d(l_i, i)}{\sqrt{-\Delta_4(j_i, k_i, l_i, i)}} . \quad (4.20)$$

The vectors j_i, k_i, l_i, i and $j_i + k_i + l_i + i$ define a basic process $2 \rightarrow 3$ or $1 \rightarrow 4$. They can be any suitable linear combinations of $p_a, p_b, p_1, \dots, p_n$. One choice, which gives the multi-Regge variables $t_i, s_{i, i+1}, s_{1 \dots i}$, was referred to in the end of Section 2.

5. Distributions in masses and angles

Resonances of a group of two or more particles are best seen plotting the number of events against the invariant mass of the group. The background is roughly described by the phase space distribution. A more realistic estimate of background requires a dynamic model.

The prediction from pure phase can be simply understood, if we take the k resonating particles to form one cluster. Formally this is effected by introducing the four-momentum q_k and invariant mass M_k of the k particles,

$$\delta^4(q - \sum_{i=1}^n p_i) = \int dM_k^2 \delta(q_k^2 - M_k^2) \int d^4 q_k \delta^4(q_k - \sum_{i=1}^k p_i) \delta^4(q - q_k - \sum_{i=k+1}^n p_i) \quad (5.1)$$

Then the integrand in R_n factorizes,

$$R_n = \int dM_k^2 \int \prod_{i=1}^k \frac{d^3 p_i}{2E_i} \delta^4(q_k - \sum_{i=1}^k p_i) \int \frac{d^3 q_k}{2E(q_k)} \prod_{i=k+1}^n \frac{d^3 p_i}{2E_i} \delta^4(q - q_k - \sum_{i=k+1}^n p_i) \quad (5.2)$$

The distribution in M_k^2 is then

$$\frac{dR_n}{dM_k^2} = R_k(M_k; m_1 \dots m_k) R_{n-k+1}(\sqrt{s}; M_k, m_{k+1} \dots m_n) \quad (5.3)$$

To obtain $d\sigma/dM_k^2$ one multiplies (5.3) with the flux factor $F^{-1} \sim \lambda(s, m_a^2, m_b^2)^{-1/2}$.

The phase space distribution is seen to be product of two factors. The first, R_k , gives the probability that k particles have the total energy M_k in their CMS.

The second is probability that $M_k, m_{k+1}, \dots, m_n$ have the total energy $\sqrt{s}$ in overall CMS. When M_k approaches $\sum_k = m_1 + \dots + m_k$, $R_k(M)$ goes to zero as $(M_k - \sum_k)^{\frac{1}{2}(3k-5)}$

ig. VI.5.2. When M_k approaches $\sqrt{s} - \sum_{k+1}^n m_i$, the factor R_{n-k+1} goes to zero with the power $\frac{3}{2}(n-k+1) - \frac{5}{2} = \frac{3}{2}(n-k) - 1$.

ig. VI.5.3. The general shape of dR_n/dM_k will largely depend on the behaviour at $M_{k,min}$ and $M_{k,max}$. The distribution has a vertical tangent at $M_{k,min}$ for $k=2$ and at $M_{k,max}$ for $k=n-1$; otherwise the slope vanishes at $M_{k,min}$ and $M_{k,max}$.

Dynamical effects change the shape of dR_n/dM_k . If there is a resonance of v particles, which are all included among $1, \dots, k$, it effectively reduces R_k in (5.3) to R_{k-v+1} . This is strictly true for an infinitely narrow resonance. Similarly v resonating particles within $k+1, \dots, n$ reduce multiplicity by $v-1$. If some of the particles which participate in a resonance are included in the system $1, \dots, k$ and some are not, then dR_n/dM_k is modified in a more complicated manner (Skjeggstad 65). The analytically soluble case dR_3/dM_{23} with resonance in M_{12} is left as an exercise to the reader.

If the matrix element is constant, within a fixed n -particle channel the distribution of one particle is spherically symmetric. Distribution in the angle between two particles is not uniform due to correlations caused by four-momentum conservation. Let θ be the angle between $\underline{p}_1$ and $\underline{p}_2$ in CMS. Grouping particles 3,...,n together with momentum $\underline{q}_3$ and mass M_3 , $dR_n/d\cos\theta$ involves effectively three "particles",

$$\frac{dR_n}{d\cos\theta} = \int dM_3^2 \int \frac{d^2p_1 d^3p_2 d^3q_3}{8E_1 E_2 E_3} \delta^4(q - p_1 - p_2 - q_3) R_{n-2}(M_3) \delta(\cos\theta - \cos\theta_{12}) \quad (5.4)$$

Angular integrations over θ_1 , φ_1 and φ_2 give $8\pi^2$. Integrations over $\underline{q}_3$ and the integration variable θ_{12} are simple due to the delta-functions, and we are left with

$$\frac{dR_n}{d\cos\theta} = \pi^2 \int dM_3^2 \int \frac{dP_1 P_1^2 dP_2 P_2^2}{E_1 E_2 E_3} \delta(E - E_1 - E_2 - E_3) R_{n-2}(M_3) \quad (5.5)$$

where the energy E_3 in CMS is

$$\begin{aligned} E_3 &= [(\underline{p}_1 + \underline{p}_2)^2 + M_3^2]^{1/2} \\ &= [P_1^2 + P_2^2 + 2P_1 P_2 \cos\theta + M_3^2]^{1/2} . \end{aligned} \quad (5.6)$$

To integrate with respect to P_2 one needs the derivative of the argument of the delta-function,

$$\frac{\partial(E_2 + E_3)}{\partial P_2} = \frac{P_2}{E_2} + \frac{P_2 + P_1 \cos\theta}{E_3} . \quad (5.7)$$

The value of P_2 is the solution of

$$E = (P_1^2 + m_1^2)^{1/2} + (P_2^2 + m_2^2)^{1/2} + (P_1^2 + P_2^2 + 2P_1P_2\cos\theta + M_3^2)^{1/2}. \quad (5.8)$$

Putting these results into (5.5) gives

$$\frac{dR_n}{d\cos\theta} = \pi^2 \int \frac{dM_3^2 dP_1 P_1^2 P_2^2}{E_1 [P_2^2 E_3 + P_2 E_2 + P_1 E_2 \cos\theta]} R_{n-2}(M_3), \quad (5.9)$$

where $E_2^2 = P_2^2 + m_2^2$ and P_2 now are functions,

$$P_2 = \frac{-aP_1 \cos\theta + (E - E_1)(a^2 - 4m_2^2 b)^{1/2}}{2b}$$

$$a = (E - E_1)^2 - M_3^2 + m_2^2 - P_1^2 \quad (5.10)$$

$$b = (E - E_1) - P_1^2 \cos^2\theta.$$

The double integral (5.9) must be done numerically. If n is three, the delta-function $R_1(M_3) = \delta(M_3^2 - m_3^2)$ gives

$$\frac{dR_3}{d\cos\theta} = \pi^2 \int \frac{dP_1 P_1^2 P_2^2}{E_1 [P_2^2 E_3 + P_2 E_2 + P_1 E_2 \cos\theta]} \quad (5.11)$$

In (5.10) M_3 must then be replaced by m_3 .

Fig.VI.5.4. In Fig. 5.4. the opening angle distribution in CMS of $K \rightarrow 3\pi$ according to phase space is shown.

6. Plots for multiparticle production

A survey of publications on experimental data of multiparticle processes reveals that plots in one variable tend to fall almost exclusively into four categories.

One particle distributions. Here one chooses a specific final state particle, say π^+ , and plots the number of events as a function of some momentum variable for this particle. Common plots are $d\sigma/dp_{||}$ and $d\sigma/p_{\perp}$. If statistics is sufficient, as is especially the case in counter experiments, one may show dependence on $p_{||}$ for a limited interval $p_{\perp}$, $p_{\perp} + \Delta p_{\perp}$ in transverse momentum. In this manner the data gives $d^2\sigma/dp_{||}dp_{\perp}$. Similarly one obtains $d^2\sigma/dpd\Omega$, usually as a function of p at fixed θ . Measurements are performed in laboratory frame (TS, CBS) and can be transformed to CMS for interpretation. One particle spectra have several significant practical and theoretical aspects. The whole of Chapter IX is devoted to this topic.

Distributions in invariants. A plot in the invariant mass of a combination of final state particles reveals the existence of resonances in this subset. A second type of useful invariant is the invariant momentum transfer $t_i = (p_a - p_1 - \dots - p_i)^2$. Also there are angles which are linearly related to invariants $s_{ij\dots k}$ and are generalizations of Treiman-Yang and Jackson angles to groups of emitted particles. In all these, as in other

plots, one can make cuts in other variables to choose events in order to enhance the studied effect. If one plots in x and makes the cuts in y , $y_0 < y < y_0 + \Delta y_0$, more and more narrow, one approaches continuously a description of the double differential cross section $d^2\sigma/dx dy$. This allows detection of correlations between x and y .

Longitudinal phase space plots. Description in terms of Van Hove's ω -variables is the subject of Chapter VIII.

Correlations. Interdependence of x and y is seen in $d^2\sigma/dx dy$. Instead one can plot some average of y as a function of x . A typical case is the average transverse momentum $\langle p_T \rangle$ as a function $p_{||}$. At this writing the study of correlations and their theoretical understanding is at a rudimentary stage. One correlation of purely kinematic origin, that in the directions of two emitted particles ($d\sigma/d\cos\theta_{12}$), was touched upon in the previous section. Correlation of $p_{\perp}$, $p_{||}$ is analysed in Chapter IX. Strong correlations, kinematic reflections, which are due to peripherality and transmitted by kinematics, are studied in Chapter VII.

The only aspect of distributions in one variable that is treated here is the t -distribution. Take the process $a+b \rightarrow 1+\dots+n$ and put $t = (p_a - p_1)^2$. Grouping $2, \dots, n$ together into a system of mass $\sqrt{s'}$ gives

$$\begin{aligned}
 R_n(s) &= \int ds' R_2(s; s', m_1^2) R_{n-1}(s'; m_2^2 \dots m_n^2) \\
 &= \frac{\pi}{2} \int \frac{ds' dt}{\lambda(m_a^2, m_1^2, t)^{1/2}} R_{n-1}(s') . \quad (6.1)
 \end{aligned}$$

The physical region in the s', t plane is the inside of the Chew-Low plot. Thus phase space gives the t -distribution

$$\frac{dR_n(s)}{dt} = \frac{\pi/2}{\lambda(m_a^2, m_1^2, t)^{1/2}} \int_{\nu_{n-1}} ds' R_{n-1}(s') \theta(-G(s, t, s', m_a^2, m_b^2, m_1^2)) \quad (6.2)$$

The "F(t)-method" consists of plotting $F(t) = (\text{number of events})/(\frac{d\sigma_n}{dt})$ against t . Because $\frac{d\sigma_n}{dt}$ goes to zero at $|t|_{\min}$, this enhances the small $|t|$ region. $F(t)$ will be roughly exponential with slope comparable to the slope in two particle scattering.

Scatter plots in two variables are convenient if the bubble chamber data of a channel includes some 100-1000 events. The Peyroy plot is one-particle distribution in the $p_{||}, p_{\perp}$ plane. Its points are directly related to the points in the s', t plane (see Chapter IX). The phase space distribution in the Chew-Low s', t plot is given by (5.1).

A four-particle final state can be analysed with a M_{12}, M_{34} plot. Let us look at the boundaries and the event density for $A = 1$. Writing $a+b \rightarrow (1+2)+(3+4)$ gives

$$R_4 = \int dm_{12}^2 dm_{34}^2 R_2(s; M_{12}^2, M_{34}^2) R_2(M_{12}^2; m_1^2, m_2^2) R_2(M_{34}^2; m_3^2, m_4^2). \quad (6.3)$$

Substitution of R_2 results in

$$\frac{\partial^2 R_4}{\partial M_{12} \partial M_{34}} = \frac{\pi^3}{2sM_{12}M_{34}} \left[\lambda(s, M_{12}^2, M_{34}^2) \lambda(M_{12}^2, m_1^2, m_2^2) \lambda(M_{34}^2, m_3^2, m_4^2) \right]^{1/2}$$

(6.4)

When any one of the three decays reaches its threshold, the boundary of the plot is reached. The plot is thus a triangle,

$$\begin{aligned} M_{12} &\geq m_1 + m_2 \\ M_{34} &\geq m_3 + m_4 \\ M_{12} + M_{34} &\leq \sqrt{s} \end{aligned}$$

(6.5)

X. Numerical methods of phase space integration

1. Introduction

A central numerical problem in particle physics is the computation of the integral

$$I_n = I_n(\sqrt{s}) = \int_V \prod_{i=1}^n \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - p_1 - \dots - p_n) M_n. \quad (1.1)$$

The matrix element squared M_n is a function of $3n-4$ independent kinematic variables. The region of integration V is either the whole physical region or part of this. The total cross section is I_n multiplied by the flux factor, if V is the entire phase space. Differential cross sections are obtained integrating over parts of the entire phase space.

As earlier, it is often convenient to eliminate the δ -function in the integral (1.1) and write it in the form

$$I_n = \int_V d\vec{\Phi} f_n(\vec{\Phi}) \quad (1.2)$$

Here $\vec{\Phi}$ means a point in the $3n-4$ dimensional phase space, expressed in terms of any set of kinematical variables. Several examples have appeared in previous chapters. The domain of integration V is the domain of integration in (1.1) expressed in the phase space variables $\vec{\Phi}$. The integrand $f_n(\vec{\Phi})$ is a product of the matrix

element squared M_n , certain factors arising from the transformation of variables (the Jacobian), and factors due to integration over the delta function.

The main reason for the complexity of (1.1) is, of course, the large number of dimensions one has to integrate over. Due to the four-dimensional δ -function, which as such is very simple, the limits integration are also normally enormously complicated and interdependent. For $n = 2$ the problem is trivial, but already for $n = 3$ there are four non-trivial variables. Also the domain V may be such that there exists no set of the variables $\vec{\phi}$ in terms of which V is simply expressible.

The techniques commonly used to evaluate (1.2) fall into the following classes:

1. Direct numerical integration. This is the most straightforward idea and consists essentially of applying to each dimension successively the ordinary techniques of one-dimensional numerical integration (Simpson's rule, Gaussian integration formulas, etc.). Since all these one-dimensional methods are based on evaluating the integrand in a fixed set of points, the multi-dimensional integral (1.2) is also carried out by evaluating $f_n(\vec{\phi})$ in a predetermined set of points. If the integration interval in each variable is divided into $k-1$ subintervals, the lattice thus formed in the $3n-4$ dimensional space contains k^{3n-4} points. Since k is a fairly large number (≥ 10), k^{3n-4} grows very fast with n . The time required to compute the values of the integrand in the k^{3n-4} points limits

the use of direct integration to $n = 2, 3$, apart from some special cases. A factorizable matrix element is sometimes effectively integrated directly using recursion relations.

It is also conceivable to sample a function in a multidimensional space in a set of points not forming a regular lattice. The optimum choice of these points, which must depend on the type of the function, has not been solved.

The method of direct numerical integration was the first one to be applied to the evaluation of (1.2) (Block 56, Almgren 68, Proriol 69), in the special case $M_n = 1$.

2. The Monte Carlo methods. Since it is unknown how the points at which $f_n(\vec{\Phi})$ is evaluated should be chosen to optimize the efficiency, it appears to be convenient to go to the other extreme and choose the points at random. This is the Monte Carlo method of integration, which at present is the most efficient, versatile and practical method of evaluating (1.2). For this reason the treatment of the Monte Carlo method will be fairly detailed.

The Monte Carlo method is a widely used tool in different branches of applied science, but it involves also many deep problems of purely mathematical nature. For a general description of the method, see the book by Hammersley and Handscomb (Hammersley 67) and the review by Halton (Halton 70). In particle physics the method was first used in connection with the non-covariant phase space integral by Cerulus and Hagedorn (Cerulus 58).

Kopylov (Kopylov 58, 60, 61) was the first one to exploit the covariance properties of (1.1). The method in use to-day is essentially the same as Kopylov's.

The Monte Carlo method requires for reasonable accuracy very many random points (at least of the order of 10^3). It rose into full prominence only with the development of fast computers and standard programs simple for the general user. The most extensively used standard Monte Carlo program is FOWL (James 68, 70) written by James on the basis of work done by Lynch (Lynch 60) and Raubold. The Monte Carlo methods in particle physics have been constantly improved to optimize their efficiency for the problems occurring most frequently in practice (Byckling 69, Friedman 69, 70, Pene 69, Van Hove 70).

3. Statistical methods. When in (1.1) M factorizes,

$$M = g_1(p_1) \dots g_n(p_n), \quad (1.3)$$

some special and very efficient techniques can be used to evaluate (1.1). Due to their analogy with certain methods applicable in statistical physics we shall rather loosely call these techniques statistical methods. The name saddle point methods would also be appropriate. The value of (1.1) is, by these methods, given as an expansion in $1/n$ and the accuracy thus improves with increasing n , in contrast to all other methods.

The factorizability condition includes also the case $M = 1$ to which these statistical methods have first been applied (Fialho 57, Kolkunov 62, Lurcat 64, Campbell 67).

Lurcat and Mazur (Lurcat 64) give a particularly elegant interpretation of the method in terms of the central limit theorem of probability theory (Khinchin 49). They also calculate the first correction term and show that the resulting values are numerically remarkably accurate. The case in which each g_i in (1.3) only depends on the length r_i of the transverse momentum of particle i has been treated by Krzywicki (Krzywicki 64, 65).

2. Direct numerical integration

There are certain cases of phase space integration in which the usual direct methods of numerical integration are best. We shall consider a) three-particle final states, b) factorizable matrix elements in connection with recursion relations. Three particle final states involve four essential kinematical variables (in the absence of spin). Calculation of distributions in one or two variables, which need only three- or two-dimensional integrations, and occasionally also calculation of the total cross section, are most accurately and economically carried out using multiple Newton-Cotes or Gaussian formulae. There are also special methods for definite integrals in two and three variables (Hammer 59). A drawback in this approach is that a separate computer program must be written for each problem. The Monte Carlo method is more versatile; when the sub-program for calculation of the matrix element is written and histogramming of events is specified, the whole set of required distributions are automatically and simultaneously produced by the computer.

When the matrix element squared is factorizable in some variables in terms of which the phase space integral can be expressed recursively (Chapter VI), the numerical integration can be carried out so that the number of points goes essentially as $n \cdot k$ (n = number of dimensions, k = number of subdivisions in one dimension) and not like k^n . This implies an essential improvement in efficiency.

As examples we shall briefly treat the cases

$|A|^2 = \prod_i f(M_i, M_{i-1})$ and $|A|^2 = \prod_i f(t_i)$, where the variables have been defined in Chapter VI. Note that the factorization conditions here may be different from those of Eq. (1.3), which allowed one to apply methods based on analogy with statistical physics (Section X.6).

a) Three particle integrals

Several choices of kinematical variables for the process $a+b \rightarrow 1+2+3$ are exhibited in the integral formulae in Appendix C. Choice between such expressions is dictated by the properties of $|A|^2$ and by the quantities to be calculated. In connection with Appendix C the following facts should be taken into account. 1) The variables appearing in $|A|^2$ should be chosen as integration variables whenever possible. If $|A|^2$ is factorizable in these, computer time to find $|A|^2$ at each point is essentially decreased. In general, a proper attention to minimizing operations inside the inner loops of the computer program will pay off. 2) In the expression of Appendix C involving s_1, s_2, t_1, t_2 , the factor $(-\Delta_4)^{-1/2}$ diverges at boundaries and the last integration limits are given by the cumbersome condition $\Delta_4 = 0$. Even if $|A|^2$ vanishes sufficiently fast at boundaries, the latter property makes preferable formulae involving at least one angle and no Δ_4 . 3) When distributions in one or two variables are computed, this or these should appear as last integrations. E.g. the integrand in case E gives directly the distribution in s_1 .

Distributions in variables not appearing as integration variables can be obtained as in the Monte Carlo method, but their accuracy is not predictable due to correlation of sampling points.

b) Recursion relations

Assuming the matrix element to be unity, Eq. (VI.2.8) gives $R_n(M_n)$ recursively as follows:

$$R_n(M_n) = \int_{\Sigma_{n-1}}^{M_n - m_n} dM_{n-1} K_n(M_n, M_{n-1}) R_{n-1}(M_{n-1}) \quad (10.1)$$

where all variables have been previously defined. To evaluate $R_n(M_n)$ numerically, one selects a discrete set of k points for each M_n variable. Then R_n is reached from R_2 by $n-2$ multiplications by $k \times k$ matrix, which will involve about $(n-2) \cdot k^2$ multiplications and additions. The matrix elements are products of K_n and coefficients of a suitable integration formula (e.g. Simpson's). This method is applicable in the more general case that the matrix is of the form $\prod_i f_i(M_i, M_{i-1})$, but this rarely occurs in practice.

When n is large, the statistical methods of Section 6 yield $R_n(M_n)$ faster. On the other hand, recursion relations have the property that the distribution in s is obtained with almost the same effort as one value $R_n(M_n)$. In fact, if one starts from $R_2(M_2)$ defined in the interval $\Sigma_2 \leq M_2 \leq M_n - \Sigma_{n-1}$, then $n-2$ matrix multiplications give $R_n(M_n)$ in the whole interval Σ_n, M_n .

As a second case, if $|A|^2 = \prod_i f_i(t_i)$ the integral I_n obtained by integrating $|A|^2$ over all phase space can according to Eq. (VI.2.11) be written recursively as follows:

$$I_n(s_n, t_n) = \iint ds_{n-1} dt_{n-1} K_n(s_n, t_n; t_{n-1}) I_{n-1}(s_{n-1}, t_{n-1})$$

$$K_n = \frac{\pi}{2} \lambda(s_n, t_n, m_a^2)^{-1/2} f_{n-1}(t_{n-1}) \quad (10.2)$$

$$I_2(s_2, t_2) = \frac{\pi}{2} \lambda(s_2, t_2, m_a^2)^{-1/2} \int f_1(t_1) dt_1 ,$$

where all variables have been previously defined. The evaluation of (10.2) proceeds as that of (10.1).

A computer program of some complexity carrying out this has been written (Byckling 1971). To improve efficiency the transformations $M = \sqrt{s}$ and $v = e^{Bt}$ are substituted, with B depending on the asymptotic behaviour of $f_i(t_i)$. An advantage in this method is that a distribution in s or also in the whole s, t Chew-Low plot is efficiently generated. Computer time is comparable with Monte Carlo, so that at around 1 % accuracy one point obtained with peripheral FOWL requires similar processing time as one distribution $R_n(s)$ by the recursion method.

3. Principles of Monte Carlo integration

By far the most common way to carry out phase space integrations is the Monte Carlo method. This is due to the following main properties of Monte Carlo methods:

- (a) Within the domain of application the rate of convergence is faster than that of other methods. The error decreases proportionally $1/\sqrt{N}$ or even better, if N is the number of points at which the integrand is evaluated.
- (b) The method is very general in the sense that it can be made reasonably efficient for all matrix elements occurring in practice.
- (c) The method gives many distributions essentially at the same expense as a single distribution; the same events need only be histogrammed in different ways. Also complicated distributions like those in Treiman-Yang angles are easily obtained.
- (d) Monte Carlo computer programs can be made very simple for the general user.
- (e) The Monte Carlo method treats events exactly as they are treated, for example, in bubble chamber physics. The method is thus intimately connected with experimental particle physics.

When applying Monte Carlo methods to particle physics one does not evaluate the integrand $f_n(\phi)$ in a predetermined set of points but rather chooses these points at random. This happens by generating at random events with a known density $g(\phi)$ in phase space and evaluating $f_n(\phi)$ at these events. An event here is a set of n

momentum vectors $\underline{p}_1, \dots, \underline{p}_n$ in any given frame satisfying four-momentum conservation and the mass shell conditions $p_i^2 = m_i^2$, $i = 1, \dots, n$. The density $g(\phi)$ is defined as a usual probability density so that the probability that a random event appears in an infinitesimal neighbourhood of a point ϕ of the phase space is $g(\phi)d^n\phi$ where $d^n\phi$ is the volume element in the phase space. Different sets of $3n-4$ variables can be used to express the coordinates of the point ϕ . We shall use invariants and angles, but one can also use momentum components.

How many events have to be generated to give a reliable result depends sensitively on the statistical error or the efficiency of the Monte Carlo method being used. Qualitatively the efficiency is the better the better $g(\phi)$ approximates the matrix element squared to be integrated over. In the best cases with present Monte Carlo programs a few thousand events are often sufficient.

Notice that there are from the point of view of generation of events three different kinds of densities:

- (1) The ideal Monte Carlo generator of events is, of course, a particle reaction. The purpose of most experiments is just to chart with what density in the phase space the particles are produced.
- (2) Any model for a reaction involves the specification of a matrix element or a density of events in the phase space. These matrix elements are often very complicated functions (like those appearing in dual models) and it is not possible to use computers to generate events with these densities.
- (3) Monte Carlo programs generate events with certain densities

$g(\phi)$ which mostly are some simple functions of ϕ . There is much freedom in the choice of $g(\phi)$ but one must remember that a $g(\phi)$ may be good for one problem but inefficient for another. Thus for Monte Carlo programs in general use one has to optimize $g(\phi)$ so that it is good for as many problems as possible.

Our main goal in the following is to explain how the random generation of events can be carried out and what types of densities $g(\phi)$ occur in practice. To do this, consider first the simple one-dimensional integral (Fig.3.1)

$$m = \int_0^1 dx f(x) \quad (3.1)$$

In practice one would never apply Monte Carlo integration to one-dimensional integrals, since normal numerical methods are much faster and converge more rapidly. We denote the value of the integral by m , since it is clearly the mean value of the function $f(x)$ over the interval $(0,1)$. Similarly one defines the variance σ^2 of the function $f(x)$:

$$\sigma^2 = \int_0^1 dx [f(x) - m]^2 \quad (3.2)$$

The variance measures the fluctuations of $f(x)$ in the interval $(0,1)$. The larger σ^2 is, the more $f(x)$ fluctuates around its mean value.

Suppose now that we are able to generate ^{random numbers} r_k ; $0 < r_k < 1$, $k = 1, 2, \dots$, so that each value between 0 and 1 is equally probable, i.e. the r_k are uniformly distributed within the interval $(0,1)$. In the following r_k will always denote a random variable of this type. The r_k are basic for any Monte Carlo technique and the problem how exactly they should be generated in a computer which certainly is not a random device, is a very important one. For

this question we refer to the specialized literature mentioned earlier. Let us now generate N numbers $r_1, \dots, r_N$ and sample the integrand in (3.1) at these points. Then m is approximated by the arithmetic mean $\bar{m}$ of the N numbers $f(r_k)$,

$$\bar{m} = \frac{1}{N} \sum_{k=1}^N f(r_k). \quad (3.3)$$

This is a Monte Carlo estimate of the integral (3.1). Applying the terminology of particle physics to the Monte Carlo technique, the generated value r_k is called an "event" and $f(r_k)$ the "weight assigned to the event".

Since the r_k are random variables, $\bar{m}$ is also a random variable, i.e. if another set of r_k is generated a new value of $\bar{m}$ is obtained. For fixed N , the values of $\bar{m}$ are so distributed that the expectation value of $\bar{m}$ is just the required value m of the integral (1). The deviation of $\bar{m}$ from its most probable value is measured by the quantity

$$\bar{\sigma}^2 = \frac{1}{N-1} \sum_{k=1}^N [f(r_k) - \bar{m}]^2 \quad (3.4)$$

which approximates (3.2) in the same sense as $\bar{m}$ approximates m . The factor $N-1$ is included instead of N in order to make the expectation value of $\bar{\sigma}^2$ equal to σ . In (3.4) one measures the deviation of $f(r_k)$ from the estimated value $\bar{m}$ and not from the true value m . This has the effect of increasing the variance by $N/(N-1)$.

The formal justification of the above statements follows from the central limit theorem of probability theory. Assume that the random variables x_i , $i=1, \dots, N$, are distributed so that their mean values are m_i and variances σ_i^2 . Under fairly general conditions the distribution of the sum $x = (x_1 + \dots + x_N)/N$ for large N then approaches the normal distribution

$$N(x; m, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2} \quad (3.5)$$

with mean $m = \frac{1}{N} \sum_{i=1}^N m_i$ and variance $\sigma^2 = \frac{1}{N^2} \sum_{i=1}^N \sigma_i^2$.

In the present case the random numbers are the numbers $f_k = f(r_k)$ which are distributed so that their average value is m and variance σ^2 . Thus, for large N , $\bar{m}$ converges to m and the variance of the the distribution in $\bar{m}$ is $\frac{1}{N} \sigma^2$. The value of σ^2 is not known, of course, but $\bar{\sigma}^2$ in Eq. (3.4) is an estimate for it. The result of the Monte Carlo integration can thus be expressed as

$$m = \bar{m} \pm \frac{\bar{\sigma}}{\sqrt{N}} \quad (3.6)$$

It follows from (3.5) that for repeated Monte Carlo integrations the probability that the result deviates from the correct value of m , for example, by one or two standard deviations $\sigma/\sqrt{N}$, is 32 % or 4,5 %, respectively. Note that (3.6) tells nothing about the error in a single Monte Carlo integration.

Example: We shall in this example exhibit a simple case for which the probability density $F(\bar{m})$ of $\bar{m}$ can be evaluated explicitly. Assume that $f(x) = x$ so that from Eqs. (3.1) and (3.2) $m = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$. The Monte Carlo estimate for m is now according to (3.3)

$$m = \frac{1}{N} \sum_{k=1}^N r_k,$$

i.e. N^{-1} times the sum of N random numbers evenly distributed between 0 and 1. The probability density $F(\bar{m})$ of $\bar{m}$ can in this case be calculated by standard methods of probability theory. In fact, $F(m)$ is the integral

$$F(\bar{m}) = N \int_0^1 dr_1 \dots dr_N g(r_1) \dots g(r_N) \delta(r_1 + \dots + r_N - N\bar{m}) \quad (3.7)$$

where $g(r)$ is one for $0 \leq r \leq 1$ and zero elsewhere. Evaluation of the integral by Laplace transforms gives

$$F(\bar{m}) = \frac{N}{(N-1)!} \sum_{k=0}^N (-1)^k \binom{N}{k} \left[(Nx-k)_+^{N-1} \right] \quad (3.8)$$

where $(x)_+ = (x+|x|)/2 \equiv x\theta(x)$. The curve representing $F(\bar{m})$ is drawn for different values of N in Fig. 32. It consists of segments of N different polynomials of order $N-1$ in the intervals $(0, \frac{1}{N})$, $(\frac{1}{N}, \frac{2}{N})$, etc. One can clearly see how $F(\bar{m})$ approaches a gaussian (3.5) peaked at $\frac{1}{2}$ (= the exact value m of the integral) and how the peak becomes narrower when N increases. This gives an idea of how the accuracy of Monte Carlo integration increases when N increases.

The generalization of Eq. (3.3) to more dimensions is intuitively obvious:

$$\int_0^1 \dots \int_0^1 dx_1 \dots dx_K f(x_1, \dots, x_K) = \frac{1}{N} \sum_{k=1}^N f(r_1^{(k)}, \dots, r_K^{(k)}) \quad (3.9)$$

where the random points $(r_1^{(k)}, \dots, r_K^{(k)})$ are uniformly distributed in a K -dimensional hypercube. Previously we sampled the integrand within an interval, now we sample it within a hypercube. If the limits of integration are separately dependent on some of the x_i , they can separately be transformed to 0 or 1 and (3.9) applied. The error is given by a formula analogous to (3.6). In particular, it is proportional to $1/\sqrt{N}$ independent of the dimension K .

4. Reduction of the statistical error

The statistical error of Monte Carlo integration depends on two factors, the variance σ^2 of the function to be integrated and the number of random points at which the function is sampled. The simplest way to decrease the error is to increase N . The error is proportional to $1/\sqrt{N}$ so that to decrease the error by 10 one has to increase N by 100. The accuracy that can be obtained through increase of N is limited by the computer time available.

However, it is also possible to improve the rate of convergence so that the statistical error may even be proportional to $1/N$. The proof of the statement that the error behaves, in general, like $1/\sqrt{N}$ rests on the fact that the random numbers used are, in principle, truly random or, in computers, numbers which "look random" (pseudo-random numbers). With pseudo-random numbers one has no idea of at what points for finite N the integrand $f_n(\phi)$ is sampled. However, one may also use random numbers which on the whole are evenly distributed but which are strongly correlated so that r_{i+1} depends on r_i (quasi-random numbers). For instance, the sequence $\frac{1}{2}; \frac{1}{4}, \frac{3}{4}; \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}; \frac{1}{16}, \dots$, formed according to an evident rule, would be a sequence of quasi-random numbers. With quasi-random numbers one has some guarantee of that the integrand is sampled evenly even for finite N . In this case one can prove that the error behaves like $1/N$. Thus one is improving the result by taking a step backwards from genuine randomness towards the methods of direct numerical integration, in which the distribution of points is entirely deterministic. The distinction between pseudo- and quasi-random numbers is very important in practice and the latter are to be preferred strongly (for instance, FOWL uses them). For more details on

this important and not entirely understood question we refer to the specialized literature (Hammersley 67, Halton 70, James 68).

Another way to improve the Monte Carlo method is to reduce the variance σ^2 . According to (2.2) this is evidently possible by transforming the integral so that the function to be integrated by the Monte Carlo method is as constant as possible. To apply this technique of reducing the variance some information on the behaviour of the integrand is required. Therefore, the methods used in particle physics have been developed specifically to apply to the problems encountered in this field. Their necessity is illustrated by the following example.

Example Consider the integral

$$m = \int_0^1 dx a e^{ax} = e^a - 1 \quad (4.1)$$

where a is a constant. Integrals of this type arise in high-energy physics when integrating over an invariant momentum transfer t , since the matrix element is experimentally known to be of the form e^{at} . A simple calculation gives

$$\sigma^2 = \frac{a}{2} \left[\left(1 - \frac{2}{a}\right) e^{2a} + \frac{4}{a} e^a - \frac{2}{a} + 1 \right], \quad (4.2)$$

which increases with a . Thus the relative error $\sigma/m\sqrt{N}$ of the Monte Carlo integration increases as $\sqrt{a/2N}$ for large a . The reason is the increasing asymmetry of the integrand ae^{ax} . This function is sampled evenly in the interval $(0,1)$. At large a , the points near 0 have a small weight compared to those near 1. In evaluating the integral the former points are nearly useless. To improve the situation one should somehow generate more points in the region where the integrand is large or sample the integrand only in regions which are of most importance. In the example this

is easy to do by taking $y = e^{+ax}$ as a new variable. This results in

$$m = \int_1^{e^a} dy ,$$

which is computed by generating points uniformly in y within the interval $(1, e^a)$. Since the integrand is now a constant, the variance has, in fact, been reduced to zero. Note that a similar transformation is needed also if the range of integration in (4.1) extends from 0 to ∞ .

It is obvious from the example that if $f(x)$ varies considerably, the Monte Carlo method becomes more efficient when the random points are generated so that their density is closer to $|f(x)|$. This method is called importance sampling. To apply it, we need a way to generate random numbers distributed according to a given density $g(x)$. The density $g(x)$ is defined so that the probability that a value between x and $x+dx$ is obtained is given by $\frac{1}{G} g(x)dx$ where $G = G(+\infty)$ and

$$G(x) = \int_{-\infty}^x dt g(t) \quad (4.3)$$

Consider then the integral

$$I = \int_{x^-}^{x^+} dx f(x) , \quad (4.4)$$

where arbitrary limits of integration have been introduced for later use. Take as a new variable

$$r = \frac{G(x) - G(x^-)}{G(x^+) - G(x^-)} \quad (4.5)$$

It varies between 0 and 1 and its differential is

$$dr = g(x)dx / [G(x^+) - G(x^-)] . \quad (4.6)$$

Eq. (4.4) can now be written in the form

$$I = \int_{x^-}^{x^+} dx f(x) = \int_{x^-}^{x^+} dx g(x) \frac{f(x)}{g(x)} = [G(x^+) - G(x^-)] \int_0^1 dr \frac{f(x)}{g(x)} \quad (4.7)$$

where

$$x = G^{-1} \{ G(x^-) + r [G(x^+) - G(x^-)] \} \quad (4.8)$$

when Eq. (3.3) applied to (4.7) we obtain for I

$$I = \int_{x^-}^{x^+} dx f(x) \approx \bar{I} = \frac{G(x^+) - G(x^-)}{N} \sum_{k=1}^N \frac{f(x_k)}{g(x_k)} \equiv \frac{1}{N} \sum_{k=1}^N w_k \quad (4.9)$$

where x_k is defined in terms of r_k by Eq. (4.8) and w_k is the weight assigned to the k th event. According to (4.6) the random variables x_k are now distributed according to the density $g(x) / [G(x^+) - G(x^-)]$ which is normalized to unity over the range of integration.

In the integral over r in (4.7) only the ratio f/g appears and thus the variance of f/g is reduced if g approximates f better than a constant. Using Eqs. (3.3), (3.4) and (3.6), the result of the Monte Carlo integration (4.9) can be written in the form

$$I = \bar{I} \pm \delta \bar{I} \quad (4.10)$$

with

$$\begin{aligned}\bar{I} &= \frac{1}{N} \sum_{k=1}^N w_k \\ \delta \bar{I}^2 &= \frac{1}{N(N-1)} \left[\sum_{k=1}^N w_k^2 - \frac{1}{N} \left(\sum_{k=1}^N w_k \right)^2 \right] \\ w_k &= \left[G(x^+) - G(x^-) \right] \frac{f(x_k)}{g(x_k)}\end{aligned}\tag{4.11}$$

Notice that the transformation of variables carried out above gives formally a method of generating random numbers with a density $g(x)$, but need not always be feasible in practice. We list below the conditions under which the method of transforming variables can be used in practice and also describe one different method:

(1) The method of transforming variables is practicable if Eq. (4.8) is such that there is a simple and fast way of getting from the primary random numbers r_k to the random numbers x_k . Else the increase in labour compensates the advantage obtained with the decrease in variance. In practice this requires that all the functions $g(x)$, $y = G(x)$ and $x = G^{-1}(y)$ must be elementary functions. In particle physics the two common cases of variations of the integrand resembling the exponential or Breit-Wigner functions can be taken care of by this method. The required transformations are listed in Table 1, and can be applied by simply substituting to Eq. (4.9).

Type of variation	$g(x)$	$G(x)$	$G^{-1}(y)$
Exponential	ae^{ax}	e^{ax}	$\frac{1}{a} \log y$
Breit-Wigner	$\frac{b^2}{(x-a)^2 + b^2}$	$\frac{\pi}{2}b + b \arctan \frac{x-a}{b}$	$a + b \tan \frac{y - \frac{\pi}{2}b}{b}$

Table 1 Densities, integrated densities and inverses to be used in connection with Eq. (4.9) for importance sampling.

(2) In addition to the method of variable transformations there are other ways of generating random numbers distributed with a prescribed density $g(x)$. In these cases $g(x)$ must be one of certain special functions but it need not be integrable in closed form. Consider for instance, the case in which $g(x)$ is a Gaussian $g(x) = e^{-x^2/2\sigma}$ and the integration goes from $-\infty$ to $+\infty$. This will occur in practice if the peripherality of the matrix element is expressed in terms of transverse momenta. The weight function $g(x)$ is not integrable in closed form, but it is easy to generate random numbers which are normally distributed. For instance, one may apply the central limit theorem to the sum of n rectangularly distributed random numbers. As shown previously, this approaches a gaussian rapidly and already the density function (3.8) of

$$s = \sigma \left(\sum_{i=1}^{12} r_i - 6 \right) \quad (4.12)$$

is sufficiently close to the limit (3.5) with mean zero and variance σ . Thus, Eq. (4.9) gives directly

$$I = \int_{-\infty}^{\infty} dx f(x) \approx \frac{\sigma \sqrt{2\pi}}{N} \sum_{i=1}^N f(s_i) e^{s_i^2/2\sigma} \quad (4.13)$$

where s_i are normally distributed random numbers between $-\infty$ and $+\infty$.

5. Application of the Monte Carlo method to particle physics

When considering the application of the Monte Carlo method to particle physics one should distinguish between two sides of the problem: a) the random generation of the particle events, b) the treatment of these events. From the practical point of view, the latter data-handling aspect is as important as the first one and we shall comment on it later. Here we shall first give a reasonably detailed description of event generation.

The task is thus to apply the general principles presented above to the generation of particle events and to the evaluation of the integral over phase space (cf. (1.1) and (1.2))

$$I_n = \int_V \sum_{i=1}^n \frac{d^3 p_i}{2E_i} \delta^4(p - p_1 - \dots - p_n) M_n(\phi) = \int_V d\phi f_n(\phi) \quad (5.1)$$

where $M_n(\phi) = M_n(p_1, \dots, p_n)$ with $p_1, \dots, p_n$ satisfying four-momentum conservation. Earlier, in connection with the one-dimensional examples, the generation of "events" presented no difficulties, while now our main problem will be just to see how particle events can be generated and what densities $g(\phi)$ can be used.

In order to construct $g(\phi)$ and the corresponding events we shall carry out a transformation of variables in (5.1) so that the domain of integration, which in the $3n-4$ variables ϕ may be very complicated, becomes a $(3n-4)$ -dimensional unit hypercube. We choose V = entire physical region in order to get events everywhere in the physical region. The Monte-Carlo estimate of

this integral over the unit hypercube is obtained by generating random points within the hypercube (which we know how to do!) and by using (3.9). Let us write the result in the form

$$I_n = \bar{I}_n \pm \delta \bar{I}_n^2 \quad (5.2)$$

$$\bar{I}_n = \frac{1}{N} \sum_{k=1}^N \frac{M_n(\phi_k)}{g_n(\phi_k)} = \frac{1}{N} \sum_{k=1}^N w_k$$

where $\delta \bar{I}_n$ is given in (4.11) and where $1/g_n$ contains all factors arising from various transformations. Then, in analogy with (4.11), the function $g_n(\phi)$ is just the density of events associated with this choice of variables. Different densities $g(\phi)$ are obtained depending on what variables ϕ one starts from and on how the transformation to the hypercube is performed. The correspondence between a point in the $(3n-4)$ -dimensional unit hypercube and the event ϕ_k is one-to-one, and the random event ϕ_k can always be constructed from a random point in the hypercube.

If V is a subspace of the entire phase space, the corresponding I_n is obtained by simply restricting the sum in (5.2) to those ϕ_k which lie in V . Thus, if the distribution in some variable v or the derivative $\partial I_n / \partial v$ is requested, the range of v is divided in bins of width Δv (not necessarily of equal length). The derivative is then estimated by

$$\frac{\partial I_n}{\partial v} = \frac{\Delta I_n}{\Delta v} \quad (5.3)$$

Here ΔI_n is given by (5.2) under the restriction that only events with v inside the given bin are included.

In order to present the method in more detail we shall choose a particularly simple set of $3n-4$ variables ϕ . As shown previously, the integral (5.1) can be written in the form

$$I_n(\sqrt{s}) = \frac{1}{2\sqrt{s}} \int_{\sum_{n-1}}^{\sqrt{s}-m_n} dM_{n-1} d\Omega_n \frac{|p_n|}{2} \dots$$

$$\int_{\sum_2}^{M_3-m_3} dM_2 d\Omega_3 \frac{|p_3|}{2} \int d\Omega_2 \frac{|p_2|}{2} M_n(\phi)$$
(5.4)

where

$$\sum_k = m_1 + \dots + m_k$$

and

$$|p_k| = \frac{\sqrt{\lambda(M_k^2, M_{k-1}^2, m_k^2)}}{2M_k}$$
(5.5)

Here the $3n-4$ variables ϕ consist of

- $n-2$ invariant masses M_k , $M_k^2 = P_k^2 = (p_1 + \dots + p_k)^2$, $k = 2, \dots, n-1$ defined as the masses of the intermediate decaying particles in Fig. 5.1.
- $2(n-1)$ angles $\Omega_k = (\cos\theta_k, \varphi_k)$, $k=2, \dots, n$, defining (Fig. 5.2) the direction of the vector p_k in the frame $P_k = p_1 + \dots + p_k = (M_k, 0)$, i.e., the rest frame of the intermediate decaying particle.

With these variables the relation between $M_n(\phi)$ and $f_n(\phi)$ is

$$f_n(\phi) = \frac{1}{2\sqrt{s}} \prod_{i=2}^n \frac{|p_i|}{2} \cdot M_n(\phi)$$

The region of integration in (5.4) can be simply transformed to a unit hypercube. We shall consider the angles and masses

and the construction of the event separately.

(a) Generation of angles. The angles ϕ_k and θ_k can be generated in the corresponding rest frame $P_k = (M_k, \underline{0})$ by writing

$$\begin{aligned}\phi_k &= 2\pi r_k^{(1)} \\ \cos \theta_k &= 2(r_k^{(2)} - 1)\end{aligned}\tag{5.6}$$

where $r_k^{(1)}$ and $r_k^{(2)}$ are random numbers between 0 and 1.

The transformation (5.6) transforms the integral over Ω_k to an integral over a unit square with the Jacobian 4π . The events produced by (5.6) are uniformly distributed in ϕ_k and $\cos \theta_k$ and the choice of the polar axis is arbitrary. One may, for instance, use the direction of P_{k+1} or of p_a . However, when the integrand $M_n(\phi)$ is the production amplitude for n particles the events will be collimated along the direction of p_a and it will be necessary to importance sample in some variable describing this collimation. We shall return to this question soon.

(b) Generation of invariant masses.

The invariant masses M_k vary between the limits

$$L_k < M_k < M_{k+1} - m_{k+1} \quad k=2, \dots, n-1 \tag{5.7}$$

When (5.7) is satisfied, each two-body decay in Fig. 5.1 is physical. In the $(n-2)$ -dimensional space spanned by $M_2, \dots, M_{n-1}$ the inequalities (5.7) define a simplex, which is an interval for $n=3$, a triangle for $n=4$ (Fig. 5.3), a pyramid for $n=5$, etc. In the following two examples of how to generate the M_k within this simplex are given.

(1). The most straightforward method is to transform the simplex to an $(n-2)$ -dimensional unit hypercube by the linear relation

$$M_k = \sum_{k+r}^{(3)} (M_{k+1} - \sum_{k+1}) \quad k=2, \dots, n-1 \quad (5.8)$$

Each M_k -integration is transformed to an integral over $r_k^{(3)}$ from 0 to 1. Including the Jacobians arising from (5.6) and (5.8) the Monte Carlo estimate of the integral (5.4) can then be written in the form (5.2) with

$$1/g_n(\phi) = \frac{1}{2\sqrt{s}} \prod_{k=2}^n 2\pi |p_k| \cdot \prod_{k=3}^n (M_k - \sum_k) \quad (5.9)$$

The density $g_n(\phi)$ of events generated in the phase space with this choice of ϕ is not constant and the statistical error of the estimate (5.2) is the larger the more $f(\phi)$ deviates from $g(\phi)$. In particular, also a constant matrix element involves a definite statistical error. Notice the factors $M_k - \sum_k$ in (5.9); the density of points is infinite when this factor vanishes. Why this happens can be clearly seen from Fig. 5.3. There M_3 is generated uniformly between $\sum_3$ and $\sqrt{s} - m_4$ while M_2 is the generated uniformly on the line between $\sum_2$ and $M_3 - m_3$. The length of this line decreases linearly when M_3 decreases to $\sum_2$ and thus the density of points increases proportionally to $1/(M_2 - \sum_2)$. The situation is analogous for larger n .

The same method of transforming the limits to constants can also be used, if one wants to generate events which have Breit-Wigner distributions in some of the M_k . The equations

(4.5) and (4.6) and Table 1 are immediately applicable and it is easy to write down/^{the} resulting density $g(\phi)$. Note that this density will not contain pure Breit-Wigner terms but products of these with factors similar to those in (5.9). Near the resonance masses, however, the Breit-Wigner density will dominate. Also not all invariant mass combinations can be given a Breit-Wigner distribution but only those appearing as intermediate states in the cascade-type decay of Fig. 5.1.

The method can be generalized to include more general decay than that of Fig. 5.1, e.g. both reaction products of a decay can decay further. The intermediate states again allow Breit-Wigner sampling. A program of this type has been described by Friedman (Friedman 69) .

(2). Another way of generating the M_k uses the fact that the simplex is that part of the $(n-2)$ -dimensional hypercube

$$\sum_k \leq M_k \leq \sqrt{s} - \sum_n + \sum_k \quad k=2, \dots, n-1 \quad (5.10)$$

in which the coordinates are ordered:

$$M_k \leq M_{k+1} - m_{k+1} \quad k=2, \dots, n-1 \quad (5.11)$$

One may then first generate uniformly in the hypercube (5.10):

$$M_k = \sum_k + r_k (s - \sum_n) \quad (5.12)$$

and then order the M_k so that (5.11) is satisfied. This ordering is simply carried out by ordering the $n-2$ random numbers:

$$r_2^{(3)} \leq r_3^{(3)} \leq \dots \leq r_{n-1}^{(3)},$$

since then

$$M_{k+1} - M_k = (r_{k+1}^{(3)} - r_k^{(3)}) (\sqrt{s} - \sum_n) + m_{k+1} \geq m_{k+1}$$

Referring to Fig. 5.3 in the case $n=4$ one first generates uniformly within the square but if the point generated lies above the diagonal it is by ordering or reflection around the diagonal transformed so that it lies within the allowed triangle. Since the density is uniform within the hypercube it is also uniform in the simplex, in contrast to the previous case of transforming the simplex to a hypercube. The density of events in the phase space is now given by

$$\frac{1}{g(\phi)} = \frac{1}{2\sqrt{s}} \prod_{k=2}^n [(2\pi)^{|\underline{p}_k|}] \cdot \frac{1}{(n-2)!} (\sqrt{s} - \sum_n)^{n-2} \quad (5.13)$$

where the factor $1/(n-2)!$ gives the ratio between the volumes of the simplex and the hypercube.

Notice also that in this latter method the $(n-2)$ -dimensional unit hypercube of the random numbers $r_k^{(3)}$ is not uniformly populated. The ordering carried out above has as a consequence that $r_2^{(3)}$ has for the most small values while $r_{n-1}^{(3)}$ has large values, etc. In fact, the density of the k th random number is given by $\rho_k = (1-r)^{n-k-1}$ where r is uniformly distributed between 0 and 1. Thus the second method of generating the M_k is equivalent to the first one with the additional assumption that the random numbers $r_k^{(3)}$ in (5.8) are distributed with the density $(1-r)^{n-k-1}$.

(c) Construction of the event.

When the $n-2$ masses and $2(n-1)$ angles have been generated, the construction of the random event, i.e. the momentum configuration $\underline{p}_1, \dots, \underline{p}_n$ in any given frame, is possible. First, in the frame $\underline{p}_1 + \underline{p}_2 = (M_2, 0)$ the length of $\underline{p}_2$ is given by Eq. (5.5) and its orientation by the generated values of θ_2, φ_2 .

Thus $\underline{p}_2$ and $\underline{p}_1 = -\underline{p}_2$ are completely known. Similarly, in the frame $\underline{p}_1 + \underline{p}_2 + \underline{p}_3 = (M_3, 0)$ $\underline{p}_3$ is obtained from Eq. (5.5) and the generated values of θ_3, φ_3 . The vectors $\underline{p}_1$ and $\underline{p}_2$ in this frame are then obtained by Lorentz transforming them from the frame $\underline{p}_1 + \underline{p}_2 = (M_2, 0)$ so that $\underline{p}_1 + \underline{p}_2 = -\underline{p}_3$. Continuing in this way one obtains finally the required momentum configuration in the CM frame.

The densities of events given by Eqs. (5.9) and (5.13) above are relatively constant. The method is thus effective only if the integrand $M_n(\phi)$ also is relatively constant. This is not the case if the total energy is large and $M_n(\phi)$ is the production amplitude for not too many particles. Then $M_n(\phi)$ is large only if the particle momenta are nearly correlated along the beam axis. At high energy the Monte Carlo method must be modified so that event density is small for large transverse momenta. There are two essentially different methods in use. One can either apply importance sampling in the generation of the angles $\cos\theta_k$ in (5.4) (Byckling 69, Friedman 69) or one can start from a completely different set of $3n-4$ variables ϕ and apply it to transverse momenta (Pene 69, Van Hove 70). Since the first alternative is technically slightly simpler we shall describe it in some detail.

(a') Generation of the angles for peripheral events.

In order to carry out importance sampling in some variable describing peripherality, assume that the masses M_k have been generated and choose in the frame $\underline{P}_{k+1} = \underline{p}_1 + \dots + \underline{p}_{k+1} = (M_{k+1}, 0)$ the direction of $\underline{p}_a$ as the z-axis. Then $\cos\theta_{k+1}$ is the angle between $\underline{p}_{k+1}$ and $\underline{p}_a$ in this frame (Fig. 5.2). Now it would be very simple to importance sample in $\cos\theta_k$ and generate mainly

events for small values of θ_k . However, it is more useful to replace $\cos\theta_{k+1}$ by the invariant momentum transfer $t_k = (p_a - p_k)^2$. The result of replacing $\cos\theta_{k+1}$ by t_k in (5.4) is according to our previous results

$$I_n(\sqrt{s}) = \frac{1}{2\sqrt{s}} \cdot \frac{1}{4|p_a^{(n)}|} \int_{\Sigma_{n-1}}^{\sqrt{s}-m_n} dM_{n-1} \int_0^{2\pi} d\varphi_n \int_{t_{n-1}^{(-)}}^{t_{n-1}^{(+)}} dt_{n-1} \frac{1}{4|p_a^{(n-1)}|} \dots$$

$$\dots \int_{\Sigma_2}^{M_3-m_3} dM_2 \int_0^{2\pi} d\varphi_3 \int_{t_2^{(-)}}^{t_2^{(+)}} dt_2 \frac{1}{4|p_a^{(2)}|} \int_0^{2\pi} d\varphi_2 \int_{t_1^{(-)}}^{t_1^{(+)}} dt_1 M_n(\phi)$$
(5.14)

where

$$|p_a^{(k)}| = \frac{\sqrt{\lambda(M_{k+1}^2, t_{k+1}, m_a^2)}}{2M_{k+1}}$$
(5.15)

and all other notations have been declared previously.

As shown previously (Eqs. (4.5) and (4.6) and Table 1), one can now importance sample in each of the t_k by generating events with the normalized density $a_k e^{akt_k} / (e^{akt_k^{(+)}} - e^{akt_k^{(-)}})$. Leaving the generation of the φ_k and the M_k unchanged and collecting all factors the value of the integral (5.14) can be written in the form (5.2) with

$$1/g(\phi) = \frac{1}{2\sqrt{s}} \prod_{k=1}^{n-1} \left[\frac{\pi}{2|p_a^{(k+1)}|} \frac{e^{akt_k^{(+)}} - e^{akt_k^{(-)}}}{a_k e^{akt_k}} \right] \cdot \frac{(\sqrt{s} - \Sigma_n)^{n-2}}{(n-2)!}$$
(5.16)

Here, for definiteness, the masses M_k have been taken to be generated by the ordering method presented above.

One can see that, apart from some more slowly varying

mass-dependent factors, the density $g(\phi)$ of the generated random events in this case behaves as $g(\phi) = \exp \left| \sum_{k=1}^{n-1} a_k t_k \right|$. It is thus rather efficient in multi-Regge calculations, in which the integrand is roughly of this form.

The construction of the event is carried out as previously; now one only has to solve $\cos \theta_{k+1}$ from the generated value of t_k . The dominance of small t_k implies evidently the dominance of small θ_{k+1} .

We have now at our disposal a method to generate random events with a number of different phase space densities $g(\phi)$. For any practical applications the computer routine carrying out the generation must be completed by a set of routines designed to initiate the generation and to treat the generated events. In more detail, these additional routines have to

- read in the specification of the reaction and the matrix element $M_n(\phi)$
- read in a specification of the density $g(\phi)$ to be used
- read in how many events are to be generated and how the generated events should be histogrammed
- histogram as requested the generated events
- print out the calculated information

Standard programs like FOWL have been developed to carry out these tasks. They are very flexible and convenient for the general user, who after having learned some conventions can use them for many different problems. For a more detailed description

of all practical matters we refer the reader to the long write-up of FOWL (James 70) .

In order to exhibit the connection between the theory we have given above and a practical example, we show in Fig. 5.4 a histogram from FOWL. This histogram contains the following information:

- a) The histogram gives the distribution in the invariant mass $M_{123} = \sqrt{(p_1+p_2+p_3)^2}$ of the reaction $\pi p \rightarrow \pi\pi\pi p$ (or $p_a+p_b \rightarrow p_1+p_2+p_3+p_4$) when the laboratory momentum of the incoming π is 4 GeV/c and the matrix element is $M_n(\phi) = \exp(5t_{pp})$ with t_{pp} = invariant momentum transfer between the nucleons. Apart from the normalization, the quantity calculated is thus $\partial I_4/\partial M_{123}$, where I_4 is defined by (5.1)
- b) The columns denoted by the word INTERVAL give the upper and lower limits of the bins to which the whole range of M_{123} has been divided by the user
- c) The column denoted by the word EVENTS gives the derivative $\partial I_4/\partial M_{123}$ as estimated by (5.3). This distribution is normalized to one. In other words, the number assigned to each bin is

$$(\text{normalization}) \times \sum w_k \quad (5.17)$$

where $w_k = M_4(\phi)/g(\phi)$ and the sum goes over those events for which the value of M_{123} lies in the bin in question. In the present case, the density $g(\phi)$ of Eq. (5.17) was used

- d) The normalization is given by SUM OF WEIGHTS = $0.2045 \cdot 10^3$.
Thus

$$\sum_{\text{all events}} w_k = 204.5$$

and since $N=2000$ the value of the integral I_4 is according to (5.2) $204.5/2000 = 0.10$. The correct numerical value of $\partial I_4 / \partial M_{123}$ is similarly obtained from (5.3), although percentage distributions are normally sufficient for comparison with experiment.

- e) The column denoted by the word ERROR give the statistical error as estimated by (4.11). Actually the approximation

$$\frac{1}{N(N-1)} \left[\sum w_k^2 - \frac{1}{N} (\sum w_k)^2 \right] = \frac{1}{N^2} \sum w_k^2$$

has been used, so that the contents of each bin in the ERROR column are

$$(\text{normalization}) \times \sqrt{\sum w_k^2},$$

where the normalization is the same as in (5.18)

- f) The number N' given by EQUIVALENT UNWEIGHTED EVENTS = 1157 is evaluated from the equation

$$\frac{1}{\sqrt{N'}} = \frac{\sqrt{\sum w_i^2}}{\sum w_i},$$

where the sums now go over all the events. It gives thus the statistical error in SUM OF WEIGHTS or the value of I_4 . In the present case the estimate of this statistical error is $1/\sqrt{1157} = 3\%$. The number N' is thus very convenient when one tries to assess the statistical significance of a histogram. If N' is less than 100 the usefulness of a

histogram is rather questionable.

- g) the contents of the ERROR column are also printed out in a form of a visual display.

6. Statistical methods

In statistical mechanics, the analogue of Eq. (1.1) is the volume of accessible phase space in the microcanonical ensemble. The four-dimensional δ -function in (1.1) is there only replaced by a single δ -function requiring the total energy to be within δE at E . Due to this δ -function constraint, calculations are very complicated. The normal way out of the difficulties is to replace the microcanonical ensemble by the canonical ensemble, in which the energy may fluctuate but the temperature is constant. Within a certain approximation, valid in statistical mechanics, the two ensembles give identical results. Basically the same method will now be used to evaluate (1.1).

Another distinctive feature of the methods applied in this section is that they treat all particles on an equal footing. All other methods must fix an ordering of the particles and start numerical integration or event generation in some end of the chain. Here the method decouples the correlation between the particles due to four-momentum conservation and only those in M_n remain. However, the method is useful only if even M_n does not cause correlations between the particles, i.e. M_n factorizes.

Assume now that M_n is factorizable in the coordinates of the particles as shown in (1.3). We shall presently see why this condition is necessary. Then the only constraint in

(1.1) connecting the different particle is the δ -function. But this constraint may be decoupled by taking the Laplace transform of $I_n(P^\mu)$. The Laplace transform $\phi_n(\alpha_\mu)$ is defined by

$$\phi_n(\alpha_\mu) = \int d^4P e^{-\alpha_\mu P^\mu} I_n(P^\mu) \quad (6.1)$$

and, for arbitrary M_n , it is a function of the four-vector α_μ . Inserting here $I_n(P^\mu)$ with a factorizable $M_n = \prod_i g_i(p_i)$ we find

$$\begin{aligned} \phi_n(\alpha_\mu) &= \int d^4P \prod_i \frac{d^3 p_i}{2E_i} g_i(p_i) \delta^4(P - p_1 - \dots - p_n) e^{-\alpha_\mu P^\mu} \\ &= \int \prod_i \frac{d^3 p_i}{2E_i} g_i(p_i) e^{-\alpha_\mu P_i^\mu} \\ &= \prod_i \varphi_i(\alpha_\mu) \end{aligned} \quad (6.2)$$

where $\varphi_i(\alpha_\mu)$ is the Laplace transform of $g_i(p_i)$ defined in analogy with (6.1). Thus, if M_n factorizes, the Laplace transform of $I_n(p^\mu)$ is the product of the Laplace transforms of the $g_i(p_i)$ and we have made δ -function disappear in a symmetrical manner. If we then invert (6.1), an integral representation for $I_n(p^\mu)$ is obtained:

$$I_n(P^\mu) = \frac{1}{(2\pi i)^4} \int_{\beta_\mu - i\infty}^{\beta_\mu + i\infty} d^4\alpha e^{\alpha_\mu P^\mu} \left[\prod_i \varphi_i(\alpha_\mu) \right] \quad (6.3)$$

Here β_μ is some real four-vector with all components positive.

This equation is exact and what is called the statistical method for evaluating $I_n(p^\mu)$ amounts to developing an approximation for computing the integral (6.3).

Before proceeding further we shall specialize to the case $M = 1$, i.e. to the evaluation of the total phase space volume (Fialho 57, Kolkunov 62, Lurcat 64, Satz 65, Campbell 67, Kajantie 71). Later on we shall briefly also comment on the case in which $g_i(p_i)$ only depends on the transverse momentum r_i (Krzywicki 64, 65). This is a very important case in practice, since the g_i can then be used to describe the experimentally observed limitation of the transverse momenta.

When $M = 1$ or $g_i = 1$ the integral $I_n(p^\mu) = R_n(P)$ only depends on the length $P = p_\mu p^\mu = p_0^2 - \underline{p}^2$ of the four-vector P^μ . Then in (6.1) one can integrate over the three dimensions in the four-dimensional space of the components P^μ . One could use Lorentz-invariance and argue that $\phi_n(\alpha_\mu)$ now only depends on the length $\alpha = \alpha_0^2 - \underline{\alpha}^2$ of α_μ and carry out the integration in the frame $\alpha_\mu = (\alpha_0, \underline{0})$. However, in order to present a more convincing calculation we shall in this integration keep all the components nonzero. One has, writing d^4P in full,

$$\phi_n(\alpha_\mu) = \int d|\underline{P}| |\underline{P}|^2 dP_0 d\Omega_P e^{-\alpha_0 P_0 + \underline{\alpha} \cdot \underline{P}} R_n(P)$$

where $P^2 = p_0^2 - \underline{p}^2$. Integration over the directions of P is trivial and gives essentially the factor $2\pi/|\underline{\alpha}||P|$.

If we then introduce as new variables P and λ defined by

$$\begin{aligned} P_0 &= P \cosh \lambda \\ |\underline{P}| &= P \sinh \lambda \end{aligned}$$

we can write

$$\phi_n(\alpha_\mu) = \frac{2}{|\alpha|} \int_0^\infty dP P^2 R_n(P) \int_{-\infty}^\infty d\lambda \operatorname{sh} \lambda e^{-\alpha_0 P \operatorname{ch} \lambda + |\alpha| P \operatorname{sh} \lambda}$$

One knows that (Magnus 66)

$$\int_{-\infty}^\infty d\lambda e^{-z \operatorname{ch} \lambda - \xi \operatorname{sh} \lambda} = 2K_0(\sqrt{z^2 - \xi^2}) \quad (6.4)$$

and that

$$K'_0(z) = -K_1(z) \quad (6.5)$$

where K_0 and K_1 are modified Bessel functions (Abramowitz 65, p. 374). Thus the integral over λ can be expressed in terms of $K_1(P\sqrt{\alpha_0^2 - \alpha^2}) = K_1(\alpha P)$ and we have finally

$$\phi_n(\alpha_\mu) = \phi_n(\alpha) = \frac{4\pi}{\alpha} \int_0^\infty dP P^2 K_1(\alpha P) R_n(P) \quad (6.6)$$

This integral converges, since for large z $K_\nu(z)$ decreases exponentially:

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \dots\right), \quad (6.7)$$

while $R_n(z)$ only increases as a power of z .

On the left hand side ^{of (6.6)} ϕ_n is given by (6.2), where $\varphi_i(\alpha_\mu)$ now also only depends on α . In order to evaluate $\varphi_i(\alpha)$ we go to the frame $\alpha_\mu = (\alpha_0, \underline{0})$.

Then

$$\begin{aligned} \varphi_i(\alpha) &= \varphi_i(\alpha_0) = \int \frac{d^3 P_i}{2E_i} e^{-\alpha_0 E_i} \\ &= 2\pi \int_{m_i}^\infty dE_i \sqrt{E_i^2 - m_i^2} e^{-\alpha_0 E_i} \end{aligned} \quad (6.8)$$

But according to formula 9.6.23 of (Abramowitz 65),

$$K_1(z) = z \int_1^\infty dt e^{-zt} \sqrt{t^2 - 1}, \quad (6.9)$$

so that one finds

$$\varphi_i(\alpha) = \frac{2\pi m_i}{\alpha} K_1(m_i \alpha) \quad (6.10)$$

Eq. (6.6) in connection with (6.9) gives $\phi_n(\alpha)$ as a very simple Bessel transform (Erdelyi 52) of $R_n(P)$. This transformation can also be inverted to give $R_n(P)$ explicitly (Lurcat 64) as a one-dimensional integral. The Laplace transform $\varphi_i(\alpha)$ is clearly always positive: $\varphi_i(\alpha) > 0$. Its derivative will according to (6.8) be negative. Putting these together we find

$$-\frac{\varphi_i'(\alpha)}{\varphi_i(\alpha)} = -\frac{\partial \log \varphi_i(\alpha)}{\partial \alpha} > 0 \quad (6.11)$$

The second logarithmic is similarly always positive:

$$\frac{\partial^2 \log \varphi_i(\alpha)}{\partial \alpha^2} = \frac{1}{\varphi_i(\alpha)} \int \frac{d^3 P_i}{2E_i} \left[E_i + \frac{\varphi_i'(\alpha)}{\varphi_i(\alpha)} \right]^2 e^{-\alpha E_i} > 0 \quad (6.12)$$

The equations (6.11) and (6.12) hold evidently also for $\phi_n(\alpha)$.

Eq. (6.6) forms also the starting point of probably the simplest method of finding the nonrelativistic form $R_n^{NR}(P)$ of $R_n(P)$, i.e. the limit of $R_n(P)$ when $P \rightarrow \sum_i m_i$. Eq. (6.6) is valid for all $\alpha > 0$. Consider, in particular, the limit $\alpha \rightarrow \infty$. Since $K_1(\alpha P)$ decreases exponentially for large P , only values of P near threshold $\sum_i m_i$ contribute significantly to the integral. Replacing $\varphi_i(\alpha)$ and $K_1(\alpha P)$ by their limits for $\alpha \rightarrow \infty$ or $\alpha P \rightarrow \infty$

we have

$$(2\pi^3)^{n/2} \prod_i \sqrt{m_i} \alpha^{-3n/2} e^{-\sum m_i \alpha} = \left(\frac{2\pi}{\alpha}\right)^{3/2} \int_0^\infty dP P^{3/2} e^{-\alpha P} R_n^{NR}(P)$$

But from the formula

$$\Gamma(\mu) \alpha^{-\mu} e^{-P_0 \alpha} = \int_0^\infty dP e^{-\alpha P} (P-P_0)^{\mu-1} \theta(P-P_0)$$

one finds that

$$R_n^{NR}(P) = \frac{(2\pi^3)^{(n-1)/2}}{2\Gamma[3(n-1)/2]} \frac{(\sum_i m_i)^{1/2}}{(\sum_i m_i)^{3/2}} (P - \sum_i m_i)^{(3n-5)/2} \quad (6.13)$$

This result is complicated to obtain if the particles are treated in an unsymmetric way.

By considering similarly the limit $\alpha \rightarrow 0$ one can derive the extremely relativistic limit $R_n^{EP}(P)$. We have already obtained this result by still simpler means.

After these exact formulas we shall now proceed to approximate (6.3). The idea is to replace the integrand by its form where it is largest. To do this, write the integrand of (6.3) in the form $\exp F(\alpha_\mu)$ where

$$F(\alpha_\mu) = \alpha_\mu P^\mu + \log \phi_n(\alpha) \quad (6.14)$$

Choose then some constant real four vector β_μ with positive components and expand $F(\alpha)$ around this point:

$$F(\alpha_\mu) = F(\beta_\mu) + (\alpha_\mu - \beta_\mu) F^\mu(\beta_\mu) + \frac{1}{2!} (\alpha_\mu - \beta_\mu) (\alpha_\nu - \beta_\nu) F^{\mu\nu}(\beta_\mu) + \dots \quad (6.15)$$

where

$$F^\mu(\alpha_\mu) = \frac{\partial F(\alpha_\mu)}{\partial \alpha_\mu} = P^\mu + \frac{\partial \log \phi_n(\alpha)}{\partial \alpha_\mu} \quad (6.16)$$

$$F^{\mu\nu}(\alpha_\mu) = \frac{\partial^2 F(\alpha_\mu)}{\partial \alpha_\mu \partial \alpha_\nu} = \frac{\partial^2 \log \phi_n(\alpha)}{\partial \alpha_\mu \partial \alpha_\nu} \quad (6.17)$$

Since ϕ_n only depends on α , $\alpha^2 = g_{\mu\nu} \alpha^\mu \alpha^\nu$, it is convenient to change the derivatives with respect to α_μ in (6.16) and (6.17) to derivatives with respect to α . Using the formulas

$$\frac{\partial \alpha}{\partial \alpha_\mu} = \frac{\alpha^\mu}{\alpha}$$

$$\frac{\partial^2 \alpha}{\partial \alpha_\mu \partial \alpha_\nu} = \frac{1}{\alpha} (g^{\mu\nu} - \frac{\alpha^\mu \alpha^\nu}{\alpha^2})$$

one obtains

$$F^\mu(\alpha_\mu) = P^\mu + \frac{\partial \log \phi_n(\alpha)}{\partial \alpha} \cdot \frac{\alpha^\mu}{\alpha} \quad (6.18)$$

$$F^{\mu\nu}(\alpha_\mu) = \frac{\partial^2 \log \phi_n(\alpha)}{\partial \alpha^2} \frac{\alpha^\mu \alpha^\nu}{\alpha^2} + \frac{\log \phi_n}{\partial \alpha} \frac{1}{\alpha} (g^{\mu\nu} - \frac{\alpha^\mu \alpha^\nu}{\alpha^2}) \quad (6.19)$$

Inserting the terms shown explicitly in (6.15) to (6.3) and carrying out the integration over α_μ in closed form will constitute the leading term of the approximation. When more terms with higher-order derivatives of $F(\alpha_\mu)$ are included, the integrand and also $R_n(P)$ will be approximated better. If we take instead of α_μ , $t_\mu = -i(\alpha_\mu - \beta_\mu)$ as a variable of

integration so that each t_μ is now real and varies between $-\infty$ and $+\infty$ we can write the leading term in the form

$$R_n(P) = R_n^{(0)}(P) = \frac{e^{F(\beta_\mu)}}{(2\pi)^4} \int_{-\infty}^{\infty} d^4 t e^{i t_\mu F^\mu(\beta_\mu) - \frac{1}{2} t_\mu t_\nu F^{\mu\nu}(\beta_\mu)} \quad (6.20)$$

Here we see explicitly, how the absolute value of the integrand indeed has a maximum for real α_μ ($t_\mu=0$).

We could now perfectly well integrate (6.20) in closed form and obtain a result which is valid for any β . In order to minimize the error it is convenient to choose β_μ so that the integrand in (6.20) contains only the quadratic terms in the exponent. This point is a saddle point (Morse 56) of the integrand $\exp[F(\alpha)]$: when one moves along the real α_μ axes the integrand has a minimum at $\alpha_\mu = \beta_\mu$ while along lines (real t_μ axes) going through $\alpha_\mu = \beta_\mu$ perpendicularly to the real α_μ axes the integrand has a maximum.

The linear terms in (6.17) vanish evidently to β_μ is determined from $F^\mu(\beta_\mu) = 0$. The solution of this is

$$\beta_\mu = \beta \cdot P_\mu / P \quad (6.21)$$

if β satisfies

$$P + \frac{\partial \log \phi_n(\alpha)}{\partial \alpha} = 0 \quad (6.22)$$

Due to the positivity properties (6.11) and (6.12) this equation has one and only one solution.

Using the equation

$$K_1'(z) = -K_0(z) - \frac{1}{2} K_1(z) , \quad (6.23)$$

and the explicit form (6.10) of φ_i we find that

$$\frac{\partial \log \varphi_i(\alpha)}{\partial \alpha} = -\frac{2}{\alpha} - m_i \frac{K_0(m_i \alpha)}{K_1(m_i \alpha)}$$

The equation determining $\beta = \beta(P)$ is then

$$P = \frac{2n}{\alpha} + \sum_i m_i \frac{K_0(m_i \alpha)}{K_1(m_i \alpha)} \quad (6.24)$$

With $F^\mu(\beta_\mu) = 0$ the integration over d^4t in (6.20) gives

$$R_n^{(0)}(P) = \frac{e^{F(\beta_\mu)}}{(2\pi)^2} \frac{1}{\sqrt{\det(F_{\mu\nu})}} \quad (6.25)$$

The derivation of the relevant integration formula over a multidimensional Gaussian is a simple generalization of the well-known one-dimensional formula and can be found, for instance, in Cramer 46, p.118.

Before obtaining the final result we still have to evaluate the determinant of $F_{\mu\nu}$. This can be done most simply in the frame $\alpha_\mu = (\alpha, \underline{0})$. Then the only nonvanishing element of $\alpha^\mu \alpha^\nu / \alpha^2$ in (6.19) is a 1 in the 00-position, while $g^{\mu\nu} - \alpha^\mu \alpha^\nu / \alpha^2$ has -1 in the 11-, 22- and 33-positions and zero elsewhere. Thus

$$\det F_{\mu\nu} = - \frac{\partial^2 \log \phi_n(\alpha)}{\partial \alpha^2} \left(\frac{1}{\alpha} \frac{\partial \log \phi_n(\alpha)}{\partial \alpha} \right)^3$$

But this is needed at $\alpha = \beta$ and we can use (6.22) to replace $\partial \log \phi_n(\alpha) / \partial \alpha$ by $-P$. If we still introduce the notation

$$a(\beta) = \alpha^2 \left. \frac{\partial^2 \log \phi_n(\alpha)}{\partial \alpha^2} \right|_{\alpha=\beta} \quad (6.26)$$

$$= 4n - \beta P + \beta^2 \sum_i m_i^2 - \sum_i \left[m_i \beta \frac{K_0(m_i, \beta)}{K_1(m_i, \beta)} \right]^2 > 0$$

we can write the final result in the form

$$R_n^{(0)}(P) = \frac{1}{(2\pi)^2} e^{\beta P} \phi_n(\beta) \frac{\beta^{5/2}}{\sqrt{Pa(\beta)}} \quad (6.27)$$

Remember that β here is as a solution of (6.23) a function of P (and also of n and the masses m_i).

As it stands, Eq. (6.27) is not very transparent and the actual dependence on P is hard to see. In order to obtain some more insight we shall consider the behaviour of $\beta = \beta(P)$ and the NR and ER limits.

Due to the positivity properties (6.11) and (6.12) we know that the right hand side (RHS) of (6.24) is always positive and decreases monotonically when α increases. When $\alpha \rightarrow \infty$ the behaviour of the RHS can be calculated by replacing K_0 and K_1 by their asymptotic expansions according to (6.7). One finds

$$P = \sum_i m_i + \frac{3n}{2\alpha} \quad (\alpha \rightarrow \infty) \quad (6.28)$$

Conversely, when $P \rightarrow \sum_i m_i$ or one approaches the NR limit, the solution of (6.24) is

$$\beta_{NR}(R) = \frac{\frac{3}{2}n}{P - \sum_i m_i} \quad (6.29)$$

When $\alpha \rightarrow \infty$, the RHS of (6.24) is simply $2n/\alpha$. This then corresponds to the ER limit:

$$\beta_{ER}(P) = \frac{2n}{P} \quad (6.30)$$

The two equations (6.29) and (6.30) express $1/\beta$ essentially as energy per particle. This is in agreement with the identification of $1/\beta$ as temperature T . According to (6.29) in the NR limit

$$P = \sum_i m_i = \frac{3}{2}nT ,$$

which is just the well-known result that (kinetic) energy per quadratic (in the Hamiltonian) degree of freedom is $\frac{1}{2} T$.

For arbitrary P Eq. (6.24) must be solved numerically. The result of a calculation is shown in Fig. 6.1. The parameters used are listed in the figure caption.

A very interesting estimate for the accuracy of this method is now obtained by inserting $\beta_{NR}(P)$ and $\beta_{ER}(P)$ to (6.27) and comparing the resulting approximate forms $R_n^{NR(0)}(P)$ and $R_n^{ER(0)}(P)$ with the known exact forms of $R_n^{NR}(P)$ (Eq. (6.13)) and $R_n^{ER}(P)$ (Lurcat 64). This calculation gives plenty of new insight to the method, but we leave the details to an exercise. The result is shown in graphical form in Fig. 6.2. where the error $(= (\text{approximate} - \text{exact}) / \text{exact})$ is plotted versus n . As a function of n for fixed energy P the error follows very closely a $1/n$ -dependence. As a function of energy P for fixed n the error varies only slightly being largest at threshold and smallest for $P \rightarrow \infty$ and varying monotonically between these limits.

It is obvious from Fig. 6.2 that the leading term $R_n^{(0)}(P)$ is numerically not sufficiently accurate. However, it is a straightforward task to calculate higher order corrections to $R_n^{(0)}(P)$. One just includes more terms in the expansion (6.15), and proceeds else as before. The higher-order derivatives of $F(\alpha p)$ which enter make the calculations lengthy and we do not go into details here. Including the first-order correction (Lurcat 64) gives already a numerical accuracy which is sufficient for most purposes. The error associated with $R_n^{(1)}(P)$ is also plotted in Fig. 6.2.

Appendix A

Gram determinants

A kinematic configuration is defined, in general, by a mixture of invariants and geometrical variables. The latter are angles, boosts, lengths of three momenta, etc. in a specified frame. The relation between the two types of variables is most elegantly expressed in terms of Gram determinants. It is shown in Section VI.3. that also the equations for the boundaries of the physical region in terms of invariants are obtained requiring a certain set of Gram determinants to vanish.

The Gram determinant of four-vectors $p_1, \dots, p_n$; $q_1, \dots, q_n$ is

$$G \begin{pmatrix} p_1 \dots p_n \\ q_1 \dots q_n \end{pmatrix} = \text{Det } (p_i q_j) \quad (\text{A.1})$$

$$= \begin{vmatrix} p_1 q_1 & p_1 q_2 & \dots & p_1 q_n \\ \vdots & \vdots & & \vdots \\ p_n q_1 & p_n q_2 & \dots & p_n q_n \end{vmatrix}$$

A symmetric Gram determinant, $p_i = q_i$ ($i=1, \dots, n$), is called Δ_n :

$$\Delta_n(p_1 \dots p_n) = G \begin{pmatrix} p_1 \dots p_n \\ p_1 \dots p_n \end{pmatrix} \quad (A.2)$$

$$= \begin{vmatrix} p_1^2 & p_1 p_2 & \dots & p_1 p_n \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ p_n p_1 & p_n p_2 & & p_n^2 \end{vmatrix}$$

In applications the p_i are particle four-momenta or linear combinations of these. G and Δ_n are evidently invariants.

When n is small, G and Δ_n have a direct connection to geometric quantities. The first, Δ_1 , is simply the invariant

$$\Delta_1(p_1) = p_1^2 = m_1^2. \quad (A.3)$$

The scalar product $G \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p_1 p_2$ is simple in the rest frame of one of the particles:

$$G \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p_1 p_2 = m_1 E_2, \quad \text{for } p_1 = 0. \quad (A.4)$$

If $\underline{v}$ is the relative velocity of the two rest frames and r the corresponding rapidity, then

$$G \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = m_1 m_2 \gamma(\underline{v}) = m_1 m_2 \cosh r. \quad (A.5)$$

Eq. (A.5) gives now

$$\begin{aligned} \Delta_2(p_1 p_2) &= m_1^2 m_2^2 - (p_1 p_2)^2 = m_1^2 m_2^2 (1 - \gamma^2(\underline{v})) \\ &= - m_1 m_2 \gamma^2(\underline{v}) \underline{v}^2 \end{aligned} \quad (A.6)$$

or

$$\sqrt{-\Delta_2(p_1 p_2)} = m_1 m_2 \sinh r. \quad (\text{A.7})$$

In the rest frame of 1 this is simply

$$\Delta_2(p_1 p_2) = -m_1^2 p_2^2. \quad (\text{A.8})$$

When there are three particles, we can take the rest system of one of them, say 1. In addition to m_1, m_2, m_3 there are three parameters: p_2, p_3 and the angle θ between p_2 and p_3 . The m_i and p_i are obtained from (A.3), (A.8). The angle θ involves p_1, p_2, p_3 simultaneously and thus e.g. $G\left(\begin{smallmatrix} p_1 p_2 \\ p_1 p_3 \end{smallmatrix}\right)$. In the frame $p_1 = 0$, it is

$$G\left(\begin{smallmatrix} p_1 p_1 \\ p_1 p_3 \end{smallmatrix}\right) = \begin{vmatrix} m_1^2 & m_1 E_2 \\ m_1 E_3 & E_2 E_3 - p_2 \cdot p_3 \end{vmatrix} = -m_1^3 p_2 p_3 \cos \theta. \quad (\text{A.9})$$

There is a way to write $G\left(\begin{smallmatrix} p_1 p_2 \\ p_1 p_3 \end{smallmatrix}\right)$ in terms of rapidities, which deepens its interpretation. From (A.5)

$$G\left(\begin{smallmatrix} p_1 p_2 \\ p_1 p_3 \end{smallmatrix}\right) = \begin{vmatrix} m_1^2 & m_1 m_2 \cosh r_{12} \\ m_1 m_3 \cosh r_{13} & m_2 m_3 \cosh r_{23} \end{vmatrix}$$

The law of addition of velocities (II.4.13) implies

$$\cosh r_{23} = \cosh r_{12} \cosh r_{13} - \sinh r_{12} \sinh r_{13} \cos \theta,$$

and thus

$$G\left(\begin{smallmatrix} p_1 p_2 \\ p_1 p_3 \end{smallmatrix}\right) = -m_1^2 m_2 m_3 \sinh r_{12} \sinh r_{13} \cos \theta. \quad (\text{A.10})$$

The symmetric Δ_3 contains essentially the same information as (A.9). In the frame $p_1 = 0$ it is easily reduced to

$$\Delta_3(p_1 p_2 p_3) = \begin{vmatrix} 1 & E_2 & E_3 \\ 0 & -\underline{p}_2^2 & -\underline{p}_2 \cdot \underline{p}_3 \\ 0 & -\underline{p}_2 \cdot \underline{p}_3 & -\underline{p}_3^2 \end{vmatrix} \quad (\text{A.11})$$

$$= m_1^2 |\underline{p}_2 \times \underline{p}_3|^2 = m_1^2 p_2^2 p_3^2 \sin^2 \theta.$$

(A.9) and (A.11) are different in that $\cos \theta$ gives θ unambiguously, but $\Delta_3 \sim \sin^2 \theta$ does not distinguish between θ and $\pi - \theta$. The expression of Δ_3 in rapidities is

$$\sqrt{\Delta_3(p_1 p_2 p_3)} = m_1 m_2 m_3 \sinh r_{12} \sinh r_{13} \sin \theta. \quad (\text{A.12})$$

When 4 four-vectors are given, the various $\Delta_1, \Delta_2, \Delta_3$ fix m_i, p_i , and the angles θ_{ij} between p_i and p_j . There is one new geometric quantity which involves all four p_i simultaneously. It can be defined in the frame $p_1 = 0$ as the angle φ between the planes p_2, p_3 and p_2, p_4 . A reduction similar to (A.11) gives

$$\Delta_4(p_1 p_2 p_3 p_4) = -m_1^2 |\underline{p}_2 \cdot (\underline{p}_3 \times \underline{p}_4)|^2 \quad \text{for } p_1 = 0. \quad (\text{A.13})$$

Its geometric significance is seen writing the matrix

$$A = \begin{vmatrix} E_1 & p_1^x & p_1^y & p_1^z \\ E_2 & p_2^x & p_2^y & p_2^z \\ E_3 & p_3^x & p_3^y & p_3^z \\ E_4 & p_4^x & p_4^y & p_4^z \end{vmatrix}$$

and denoting the metric tensor as $G = (g_{\mu}^{\nu})$. Then

$$\Delta_4 = \text{Det}(AGA^T) = -(\text{Det}A)^2.$$

Thus $-\Delta_4$ is the volume squared of the four-dimensional parallelepiped with edges $p_1 \dots p_4$. Setting $p_1 = 0$ in A gives (A.13). According to elementary geometry the triple vector product in (A.13) gives

$$\Delta_4(p_1 p_2 p_3 p_4) = -(m_1 p_2 p_3 p_4 \sin \theta_{23} \sin \theta_{24} \sin \varphi)^2. \quad (\text{A.14})$$

Expressing P_i 's in terms of rapidities gives

$$\sqrt{-\Delta_4(p_1 p_2 p_3 p_4)} = m_1 m_2 m_3 m_4 \sinh r_2 \sinh r_3 \sinh r_4 \sin \theta_{23} \sin \theta_{24} \sin \varphi \quad (\text{A.15})$$

The angle φ also comes out of unsymmetric 3x3 Gram determinants. A straightforward evaluation yields

$$G \begin{pmatrix} p_1 p_2 p_3 \\ p_1 p_2 p_4 \end{pmatrix} = m_1^2 p_2^2 p_3 p_4 \sin \theta_{23} \sin \theta_{24} \cos \varphi. \quad (\text{A.16})$$

The preceeding discussion shows what are the values of momenta and angles in a fixed frame $p_1 = 0$. We list them here using the notation $p, q, r, s, Q = |\underline{q}|$,

$\theta = \angle(\underline{q}, \underline{r})$, $\varphi = \angle(\underline{qr}, \underline{qs})$:

$$\begin{aligned} \Delta_1(p)Q &= -\Delta_2(pq) \\ \Delta_2(pq)\Delta_2(pr)\sin^2 \theta &= \Delta_1(p)\Delta_3(pqr) \\ \Delta_3(pqr)\Delta_3(pqs)\sin^2 \varphi &= \Delta_2(pq)\Delta_4(pqrs) \\ [\Delta_2(pq)\Delta_2(pr)]^{1/2} \cos \theta &= G \begin{pmatrix} pq \\ pr \end{pmatrix} \\ [\Delta_3(pqr)\Delta_3(pqs)]^{1/2} \cos \varphi &= G \begin{pmatrix} pqr \\ pqs \end{pmatrix} \end{aligned} \quad (\text{A.17})$$

With $n > 4$ in Δ_n or $G(\begin{smallmatrix} p_1 \dots p_n \\ q_1 \dots q_n \end{smallmatrix})$, the situation changes completely. More than four vectors in four-dimensional space are always linearly dependent.

There are thus coefficients α_i that

$$\sum_{i=1}^n \alpha_i p_i = 0, \quad \text{not all } \alpha_i = 0. \quad (\text{A.19})$$

Taking the scalar product of (A.19) with q_1 , then with q_2 , etc. to q_n results in n homogeneous linear equations for the invariants $p_i q_j$. According to (A.19) this system has a non-trivial solution $\alpha_1, \dots, \alpha_n$ and thus the coefficient matrix $G(\begin{smallmatrix} p_1 \dots p_n \\ q_1 \dots q_n \end{smallmatrix})$ must vanish.

Adding a vector p_n to the set $p_1 \dots p_{n-1}$ gives four new parameters but n new scalar products $p_i p_j$. Thus $n-4$ equations (A.18) are to be found going from $n-1$ to n four-vectors. We just saw that 5×5 determinants vanish, and thus, denoting p_i by i ,

$$\begin{aligned} \Delta_5(12345) &= 0, \\ \Delta_5(12346) &= G(\begin{smallmatrix} 12345 \\ 12346 \end{smallmatrix}) = 0, \\ \Delta_5(12347) &= G(\begin{smallmatrix} 12345 \\ 12347 \end{smallmatrix}) = G(\begin{smallmatrix} 12346 \\ 12347 \end{smallmatrix}) = 0, \text{ etc.} \end{aligned} \quad (\text{A.20})$$

The number of equations in (A.20) is $\frac{1}{2}(n-3)(n-4)$. It is evident that if $p_1, \dots, p_n$ are four-vectors, also the following is true:

$$\begin{aligned} \Delta_5(12345) &= 0, \\ \Delta_5(12346) &= \Delta_6(123456) = 0, \\ \Delta_5(12347) &= \Delta_6(123457) = \Delta_7(1234567) = 0, \text{ etc.} \end{aligned} \quad (\text{A.21})$$

It is clear that Gram determinants of order five or more do not give values of geometric quantities, but express necessary relations between invariant quantities. Eqs. (A.20) raise the converse problem: If the invariants $p_i p_j$ are given, do the conditions (A.20) suffice to guarantee that there exist one or more sets $p_1, \dots, p_n$ with the given invariants. This complicated problem is discussed in Chapter VI.

The elements of the Gram determinants Δ_n are the invariants $p_i p_j$. In practice, more useful are m -particle invariant masses

$$s_{ijk\dots} = (p_i + p_j + p_k + \dots)^2. \quad (\text{A.22})$$

Outward directed momenta are taken positive, ingoing particles have a reversed sign in (A.22). A Gram determinant Δ_n in $p_i p_j$ can be shown to be identical to a Cayley determinant of dimension $n+2$ in $s_{ijk\dots}$.

The expression of Δ_n as Cayley determinant is

$$\Delta_n(p_1 \dots p_n) = \frac{(-1)^{n+1}}{2^n} \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & p_1^2 & p_2^2 & \dots & p_n^2 \\ 1 & p_1^2 & 0 & (p_1 - p_2)^2 & \dots & (p_1 - p_n)^2 \\ 1 & p_2^2 & (p_1 - p_2)^2 & 0 & \dots & (p_2 - p_n)^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_n^2 & (p_1 - p_n)^2 & (p_2 - p_n)^2 & \dots & 0 \end{vmatrix} \quad (\text{A.23})$$

To prove (A.23), one simply subtracts the second row from all the following rows and the second column from all the following columns.

Equation (A.23) is not yet in terms of $s_{ijk}...$

To get this one writes the identity

$$\Delta_n(p_1, p_2, \dots, p_n) = \Delta_n(p_1, p_1+p_2, \dots, p_1+p_2+\dots+p_n) \quad (\text{A.24})$$

which results from adding rows and columns in Δ_n . On the right hand side of (A.24) the difference squared of two arguments is of the form $s_{i+1, \dots, j}$.

Explicitly Δ_n is then

$$\Delta_n(p_1 \dots p_n) = \frac{(-1)^{n+1}}{2^n} \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & s_1 & s_{12} & \dots & s_{12 \dots n} \\ 1 & s_1 & 0 & s_2 & \dots & s_{23 \dots n} \\ 1 & s_{12} & s_2 & 0 & \dots & s_{34 \dots n} \\ 1 & s_{123} & s_{23} & s_3 & \dots & s_{45 \dots n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & s_{12 \dots n} & s_{23 \dots n} & s_{34 \dots n} & \dots & 0 \end{vmatrix}$$

(A.25)

Appendix B

Spherical Trigonometry

Formulas of spherical trigonometry are often needed to discuss differential cross sections and spectrometer apertures. These formulas are easily found from various handbooks, but we shall here derive the basic formulas in a way which emphasizes their symmetry properties.

Consider the spherical triangle with sides a, b, c and angles A, B, C as shown in Fig. B.1. Choosing the axes as in Fig. B.1, the unit vectors to the corners of the triangle are

$$\begin{aligned} e_1 &= (0, 0, 1) \\ e_2 &= (\sin c, 0, \cos c) \\ e_3 &= (\sin b \cos A, \sin b \sin A, \cos b) \end{aligned} \quad (B.1)$$

The law of sines is obtained by evaluating the volume of the parallelepiped spanned by the e_i :

$$e_1 \cdot e_2 \times e_3 = \sin b \sin c \sin A \quad (B.2)$$

But by cyclic symmetry this evidently equals $\sin c \sin a \sin B$ or $\sin a \sin b \sin C$, which leads to the law of sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \quad (B.3)$$

The law of cosines for sides is obtained by computing the scalar products between the e_i :

$$e_2 \cdot e_3 = \cos a = \cos b \cos c + \sin b \sin c \cos A \quad (B.4)$$

etc. cyclically.

In order to obtain further formulas in a symmetric fashion one has to consider the spherical triangle generated by the basis vectors $\hat{e}_i$ dual to the e_i :

$$\hat{e}_1 = \frac{e_2 \times e_3}{|e_2 \times e_3|}, \text{ etc. cyclically} \quad (\text{B.5})$$

By explicit computation from (B.1) and (B.5)

$$\begin{aligned} \hat{e}_1 &= (-\sin B \cos C, -\cos B, \sin B \sin C) \\ \hat{e}_2 &= (\sin A, -\cos A, 0) \\ \hat{e}_3 &= (0, 1, 0) \end{aligned} \quad (\text{B.6})$$

In evaluating (B.6) the y-component of $\hat{e}_1$ is obtained most simply from

$$\hat{e}_1 \cdot \hat{e}_3 = -\cos B, \text{ etc. cyclically} \quad (\text{B.7})$$

The volume of the parallelipiped generated by the $\hat{e}_i$ gives again the law of sines, but the scalar products $\hat{e}_i \cdot \hat{e}_j$ give the law of cosines for angles:

$$-\hat{e}_1 \cdot \hat{e}_2 = \cos C = \cos A \cos B - \sin A \sin B \cos c \quad (\text{B.8})$$

etc. cyclically.

If one of the angles of the triangle is 90° , the simplified formulas may be summarized in Napier's rule. Choosing $C = 90^\circ$ this rule says that, in the diagram of Fig. B.2, the cosine of any quantity equals

- a) the product of cotangents of adjacent quantities,
- b) the product of sines of non-adjacent quantities.

Appendix C

Three particle integration

The integral

$$I_3 = \int \frac{d^3 p_1 d^3 p_2 d^3 p_3}{8E_1 E_2 E_3} \delta^4(p_1 - p_1 - p_2 - p_3) |A|^2 \quad (C.1)$$

can be written in terms of five independent variables. In the absence of polarization, $|A|^2$ does not depend on the rotation around the beam axis. Integration over it gives 2π . The remaining integrals are written

$$I_3 = 2\pi \int_{a_4}^{b_4} dx_4 \int_{a_3}^{b_3} dx_3 \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} dx_1 W(x_i) |A|^2 \quad (C.2)$$

Several choices of x_i are listed in Table C.1 and the corresponding weight functions are in Table C.2 (Pirilä 1971). The function Δ_4 is written explicitly in Chapter V.

The limits for invariant combinations (t_1, t_2) , (s_1, t_1) and (s_2, t_2) can not be stated in a compact form and the reader is referred to Chapter V. Permuting the invariants in the cases I and J also gives such complicated limits and these are not written in the tables. Otherwise Table C.1 contains all the essentially different choices of variables in which the angles can vary freely.

	x_1	x_2	x_3	x_4	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
A	φ_a	$\cos\theta_{a1}$	$\cos\theta_{b3}^*$	s_1	0	2π	-1	1	-1	1	$(m_1+m_2)^2$	$(s-m_3)^2$
B	λ_3	$\cos\theta_{23}^{(12)}$	$\cos\theta_{b3}^*$	s_1				same as A				
C	φ_a	$\cos\theta_{a1}$	t_2	s_1	0	2π	-1	1	$\begin{cases} G(s, t_2, m_3^2, m_a^2, m_b^2, s_1) \leq 0 \\ s_1 \geq (m_1+m_2)^2 \end{cases}$			
D	λ_3	$\cos\theta_{23}^{(12)}$	t_2	s_1				same as C				
E	λ_3	$\cos\theta_{b3}^*$	s_2	s_1	0	2π	-1	1	$G(s, m_2^2, s_1, s_2, m_1^2, m_3^2) \leq 0$			
F	φ_a	$\cos\theta_{12}^{(a1)}$	t_2	t_1	0	2π	-1	1	see Chapter V			
G	φ_a	$\cos\theta_{b3}^*$	t_1	s_1	0	2π	-1	1	- " -			
H	λ_3	$\cos\theta_{12}^{(23)}$	t_2	s_2	0	2π	-1	1	- " -			
I	φ_a	t_1	t_2	s_1	0	2π	$G(s_1, t_1, m_2^2, m_a^2, t_2, m_1^2) \leq 0$	$\begin{cases} G(s, t_2, m_3^2, m_a^2, m_b^2, s_1) \leq 0 \\ s_1 \geq (m_1+m_2)^2 \end{cases}$				
J	λ_3	t_2	s_2	s_1	0	2π	$G(s_2, t_2, m_3^2, t_1, m_b^2, m_2^2) \leq 0$		$G(s, m_2^2, s_1, s_2, m_1^2, m_3^2) \leq 0$			
K	t_2	t_1	s_2	s_1			$\Delta_4 \leq 0$					

Table C.1.

$$A \quad \frac{\{\lambda(s, s_1, m_3^2) \lambda(s_1, m_1^2, m_2^2)\}^{1/2}}{64 s s_1}$$

B same as A

$$C \quad \frac{1}{32 s_1} \left\{ \frac{\lambda(s_1, m_1^2, m_2^2)}{\lambda(s, m_a^2, m_b^2)} \right\}^{1/2}$$

D same as C

$$E \quad \frac{1}{32 s}$$

$$F \quad \frac{1}{32 t_1} \left\{ \frac{\lambda(t_1, t_2, m_a^2) \lambda(t_1, m_a^2, m_1^2)}{\lambda(s, m_a^2, m_b^2) \lambda(s_1, t_1, m_a^2)} \right\}^{1/2}$$

$$G \quad \frac{1}{32 s} \left\{ \frac{\lambda(s, s_1, m_3^2)}{\lambda(s_1, t_2, m_a^2)} \right\}^{1/2}$$

$$H \quad \frac{1}{32 s_2} \left\{ \frac{\lambda(s, s_2, m_1^2) \lambda(s_2, m_2^2, m_3^2)}{\lambda(s, m_a^2, m_b^2) \lambda(s, s_1, m_3^2)} \right\}^{1/2}$$

$$I \quad \frac{1}{16 \left\{ \lambda(s, m_a^2, m_b^2) \lambda(s_1, t_2, m_a^2) \right\}^{1/2}}$$

$$J \quad \frac{1}{16 \left\{ \lambda(s, m_a^2, m_b^2) \lambda(s, s_1, m_3^2) \right\}^{1/2}}$$

$$K \quad \frac{1}{32 \left\{ \lambda(s, m_a^2, m_b^2) (-\Delta_4) \right\}^{1/2}}$$

Table C.2.

Answers to the Exercises

II.1. Length of the accelerator seen from by the electron $\approx (3\text{km}) \times \log\left(\frac{E}{m_e}\right) \times m_e / (E - m_e) \approx 81 \text{ cm}$.

II.3. a) $\gamma \leq (1+\epsilon)/(1-\epsilon)$, $v \leq 2\sqrt{\epsilon}$

b) $v \geq 1/(1+\epsilon)$, $\gamma \geq 1/\sqrt{2\epsilon}$.

To an accuracy of 1 % the approximation

$E = m + p^2/2m$ can be used if $p \leq 0.2 m$, $E = p$ if $p \geq 7 m$.

II.4. 10.0 ns, 10.1 ns, 13.7 ns.

II.5. a) $a = \gamma(c\gamma', \gamma'v + \gamma v')$, where $\gamma' = \gamma^3 v v' / c^2$ and the prime denotes a derivative with respect to time, b) That $a \cdot u = 0$ is obtained by differentiating $u \cdot u = c^2$, c) $a \cdot a = \gamma^6 (v')^2$.

II.6. $v^{\text{CM}} = 0.952$, $\gamma^{\text{CM}} = 3.26$, $p_z^* = -1.46 \text{ GeV}/c$,
 $E^* = 2.65 \text{ GeV}$.

II.7. In the target system the momentum vector lies on the hyperboloid $p_z^2 - (\gamma^{\text{CM}} v^{\text{CM}})^2 (p_x^2 + p_y^2) = (m_p \gamma^{\text{CM}} v^{\text{CM}})^2$.
The maximum value in the CMS is $p^* \leq \frac{1}{2} \sqrt{s - 4m^2}$.

Answers to the Exercises (continued)

III.1. a) GeV^{2-n} , b) GeV^{2n-4}

III.2. $I(E) = (2\pi m)^{3n/2} E^{3n/2-1} / \Gamma(3n/2)$

III.3. $S = \frac{5}{2}n + n \log\left(\frac{V/n}{\lambda^3}\right)$, where $\lambda^2 = 3\pi n/mE$
(Sackur-Tetrode's formula)

III.4. $Z(T,V,n) = V^n \left(\frac{mT}{2\pi}\right)^{3n/2}$

Figure captions

Fig. I.1.1. Particle decay and particle collision.

Fig. I.1.2. Example of a distribution in an energy variable.

Fig. II.1.1. Frames defining a Lorentz transformation.

Fig. II.1.2. Transformation with arbitrary direction of the velocity.

Fig. II.1.3. Longitudinal and parallel components of $\underline{x}$.

Fig. II.2.1. Transformation of velocities.

Fig. II.7.1. Definitions of various Lorentz systems.

Fig. II.8.1. Definition of the Cartesian and polar coordinates of $\underline{p}$.

Fig. III.4.1. The unitarity relation.

Fig. IV.1.1. The scattering angle θ_1 .

Fig. IV.1.2. The opening angle θ_{12} .

Fig. IV.1.3. Distributions in the opening angle θ_{12} of a $\gamma\gamma$ -final state for some values of the velocity v of the decaying system.

Fig. IV.2.1. The CM momenta and scattering angle.

Fig. IV.2.2. Mandelstam variables.

Fig. IV.2.3. Momenta in s-, t-, and u-channels.

Fig. IV.3.1. The Mandelstam plane.

Fig. IV.3.2. Physical regions for the process $\pi^0(135) + \omega(784) \rightarrow \pi^+(140) + \pi^-(140)$, and the crossed channels. The masses are related as $m_2 > m_3 = m_4 > m_1$; $m_2 > m_1 + m_3 + m_4$; $m_1 m_2 > m_3 m_4$. Decay is possible.

Fig. IV.3.3. The process $K(494) + N(938) \rightarrow \rho(760) + Y^*(1660)$, with $m_4 > m_2 > m_3 > m_1$; $m_1 m_4 > m_2 m_3$; $m_1^2 + m_4^2 > m_2^2 + m_3^2$. No decay.

Fig. IV.3.4. The process $\gamma + N(938) \rightarrow \pi(140) + N^*(1236)$ with $m_1 = 0$, $m_4 > m_2 > m_3$; $m_4 > m_2 + m_3$. Decay is possible.

Fig. IV.3.5. The process $\gamma + N(938) \rightarrow \rho(760) + N^*(1236)$ with $m_1 = 0$; $m_4 > m_2 > m_3$; $m_2 - m_3 < m_4 < m_2 + m_3$. No decay.

Fig. IV.3.6. The process $\gamma + N(938) \rightarrow \gamma + N^*(1236)$ with $m_4 \neq m_2$; $m_1 = m_3 = 0$. Decay is possible.

Fig. IV.3.7. The process $N + N \rightarrow N + N$ with $m_1 = m_2 = m_3 = m_4$. No decay.

Fig. IV.3.8. Chew-Low plots for the system $\pi^+ \pi^- \pi^0$ in the reaction $K^- p \rightarrow \pi^+ \pi^- \pi^0 \Lambda$.

Fig. V.2.1. Extremal points of the Dalitz plot.

Fig. V.2.2. Limits of s_{13} for given s_{23} .

Fig. V.2.3. Dalitz plot at large energy.

Fig. V.2.4. Change in the plot with energy increasing.

- Fig. V.3.1. The process $2 \rightarrow 3$.
- Fig. V.3.2. Invariants for the process.
- Fig. V.3.3. Quasi two particle scattering processes.
- Fig. V.3.4. Intersections of the surfaces $s_2 = \text{const.}$ and $t_1 = \text{const.}$ with the p_{1x}, p_{1z} plane at $p_a = 30$. All units GeV.
- Fig. V.3.5. Intersection of the mass constraint ellipsoid with p_{3x}, p_{3z} plane for the case $pp \rightarrow p\pi^0 p$, $s_2 = 10$, $t_1 = -0.4$.
- Fig. V.5.1. Dalitz plot of the reaction $pp \rightarrow p\pi^+ n$ at 5.5 GeV.
- Fig. V.5.2. Boundaries of $t_1 t_2$ plots for values of s and the masses denoted in the figures.
- Fig. V.5.3. The curves $s_{12}^+(t_1 t_2) = s_{11}^+(t_2)$ with all branches included for a reaction with the masses, in arbitrary units, $m_1 = m_2 = 3$, $m_3 = m_4 = m_5 = 1$, and for (a) $s = 50$, (b) $s = 37$, and (c) $s = 36$ = threshold energy squared.
- Fig. V.5.4. 2016 events of the reaction with $m_1 = m_2 = 3$, $m_3 = m_4 = m_5 = 1$, and $s = 50$ distributed according to phase space.

Fig. VI.1.1. Multiperipheral process

Fig. VI.1.2. A cascade decay

Fig. VI.2.1. Multiparticle process via one
compound system

Fig. VI.2.2. Tree graph for multiple decay

Fig. VI.2.3. Invariants of the basic process $2 \rightarrow 2$

Fig. VI.5.1. Diagram for calculation of the effective
mass distribution

Fig. VI.5.2. Two-particle mass distribution of the
 $E\pi$ -system of the reactions $K^-N \rightarrow K\bar{E}(n\pi)$
at 3.4 GeV/c computed from phase space

Fig. VI.5.3. Three-particle mass distribution of the
 $E\pi\pi$ -system of $K^-N \rightarrow K\bar{E}(n\pi)$ at 3.4 GeV/c

Fig. VI.5.4. Distribution in the opening angle between
two pions in $K \rightarrow 3\pi$ computed from (5.11)

Fig. X.3.1. Monte Carlo integration of a function $f(x)$ over the interval $(0,1)$: r_k is a random point, m is the exact value of the integral.

Fig. X.3.2. Illustrating the convergence of Monte Carlo integration. The function to be integrated is $f(x) = x$; the curves show how the normalized probability densities of the Monte Carlo estimates are peaked closer and closer to the correct value 0.5 as N increases.

Fig. X.5.1. Cascade-type choice of variables for the process $p_a + p_b \rightarrow p_1 + \dots + p_n$.

Fig. X.5.2. Choice of variables at one of the vertices in Fig. X.5.1.

Fig. X.5.3. Domain of integration over the two masses M_3 and M_2 for $N = 4$.

Fig. X.5.4. Example of an output of a Monte Carlo program. Details are explained in text.

Fig. X.6.1 Behaviour of the root $\beta(E)$ for the masses noted in the figure. The NR and ER limits are also shown.

Fig. X.6.2 Behaviour of the error of the zeroth and first-order approximations as a function of multiplicity in the ER and NR limits.

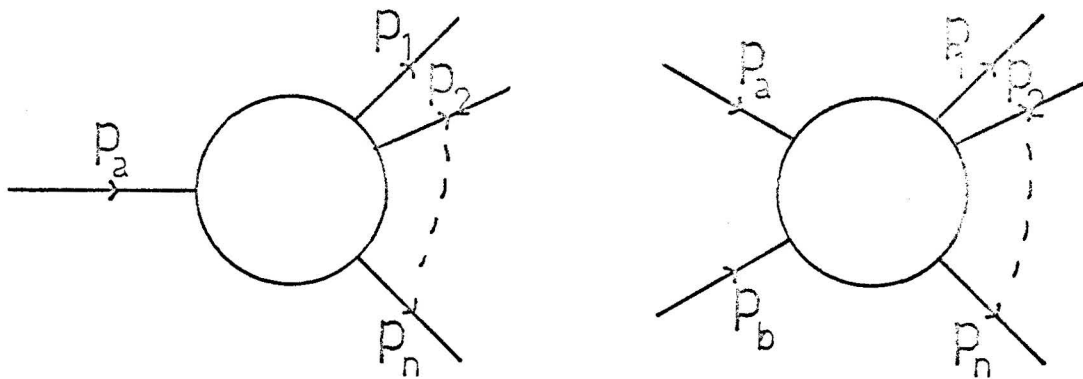


Fig I.1.1.

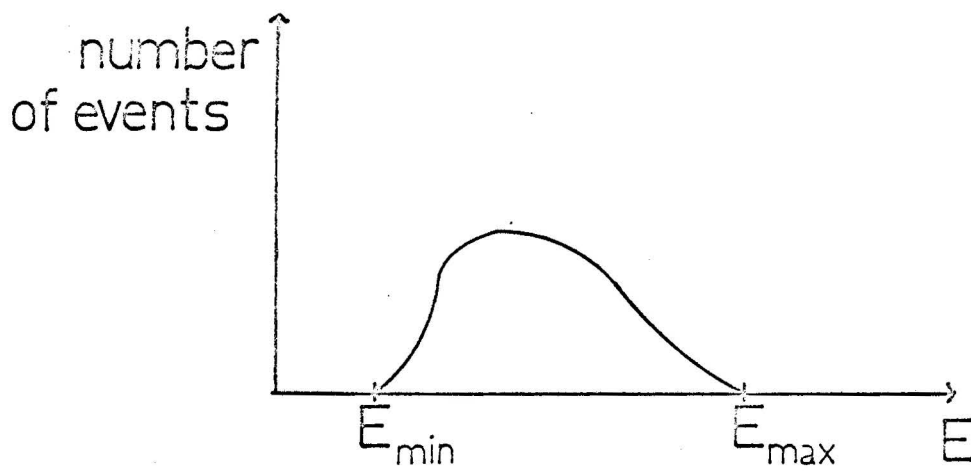


Fig I.1.2.

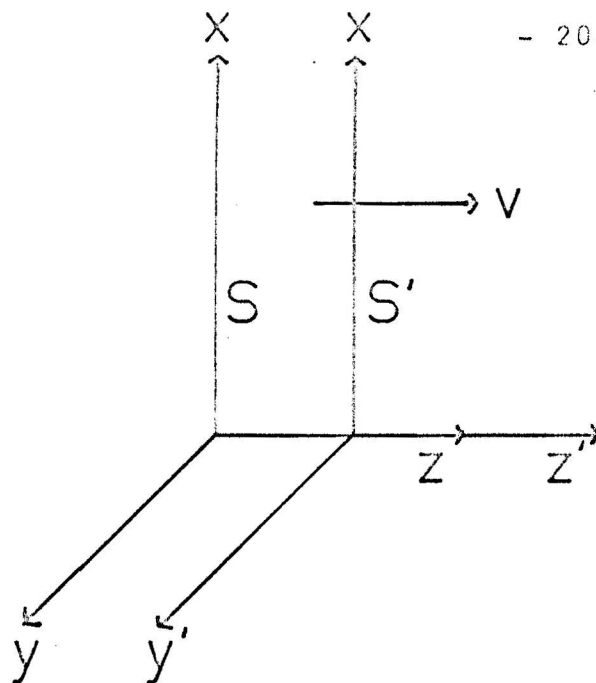


Fig II.1.1.

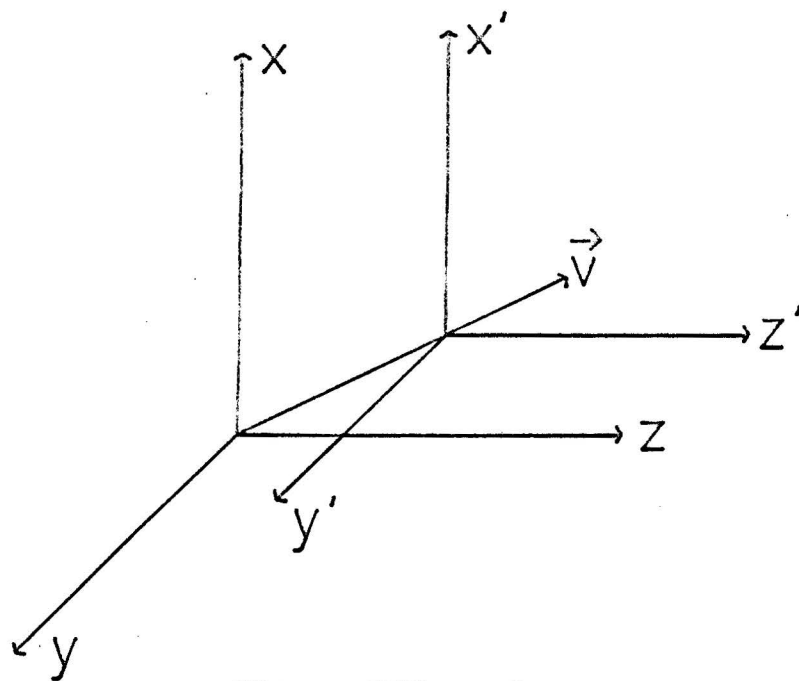


Fig II.1.2.

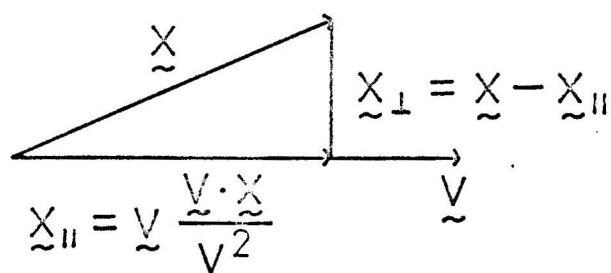


Fig. II.1.3.

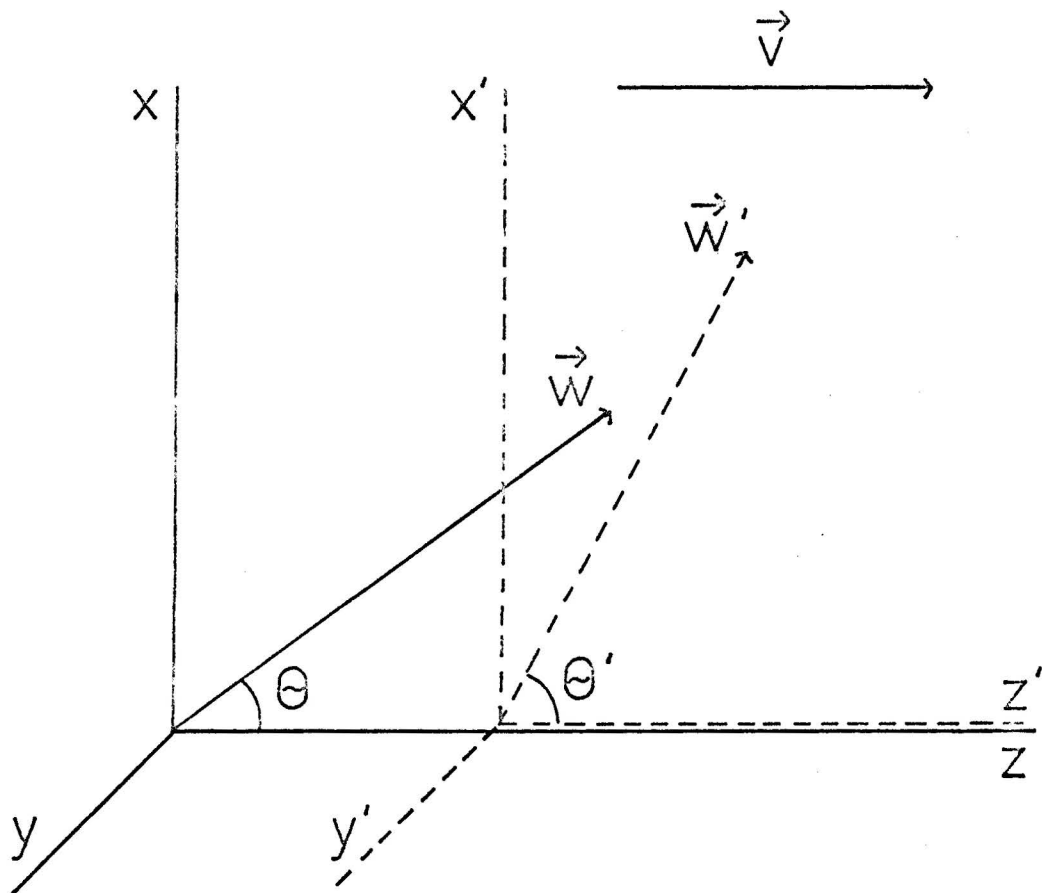


Fig II.2.1.

Center of momentum
system (CMS):

$$\vec{p}_a^* = -\vec{p}_b^*$$

Target system (TS):

$$\vec{p}_a^T \rightarrow \vec{p}_b^T = 0$$

Beam system (BS):

$$\vec{p}_a^B = 0 \quad \vec{p}_b^B$$

Colliding beam
system (CBS):

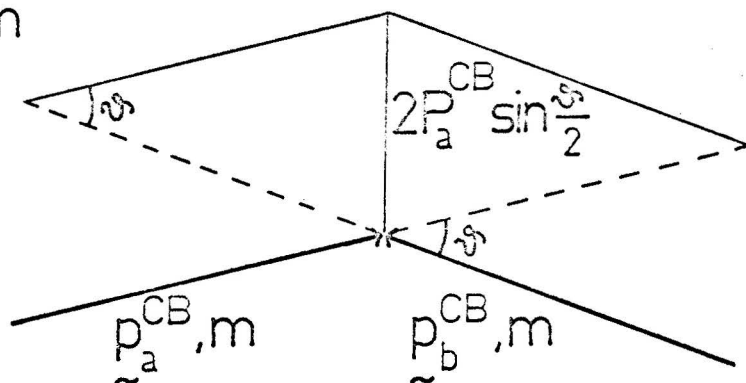


Fig II.7.1.

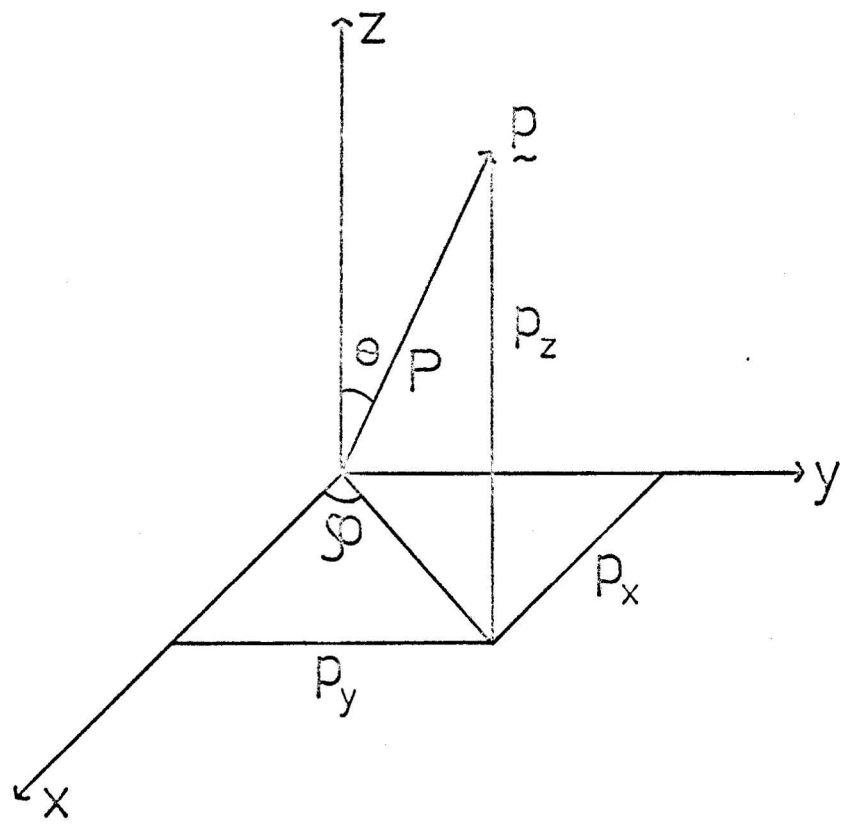


Fig II.8.1.

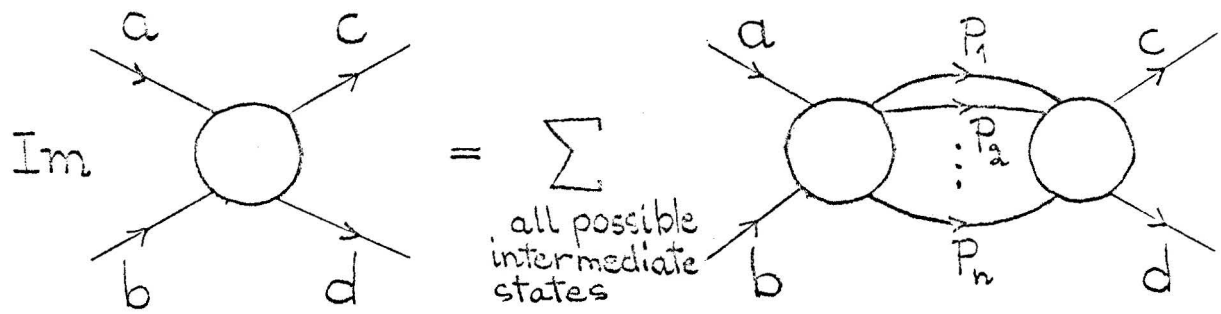


Fig. III.4.1.

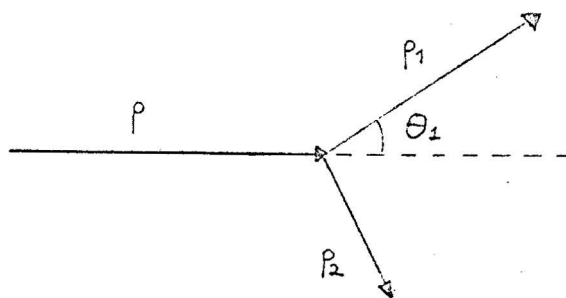


Fig. IV.1.1

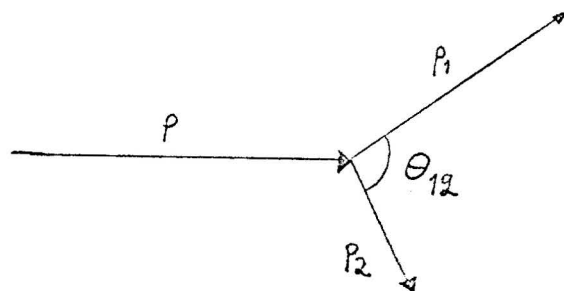


Fig. IV.1.2

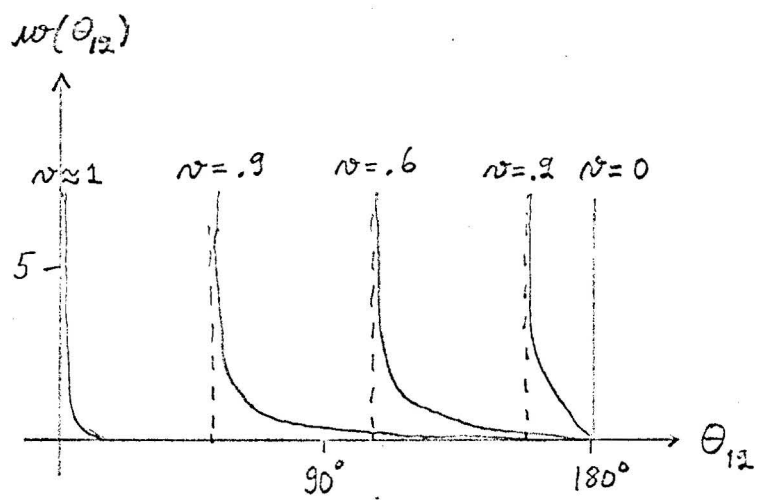


Fig. IV.1.3

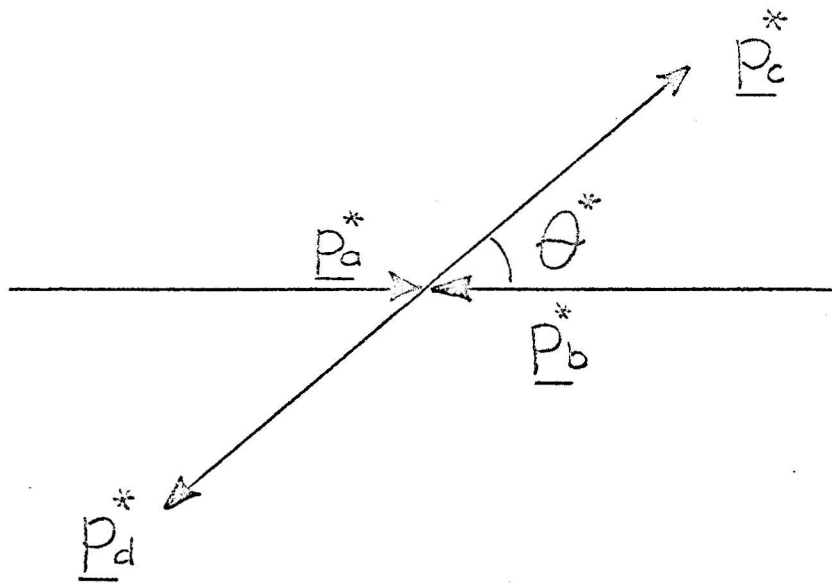


Fig. IV 2.1.

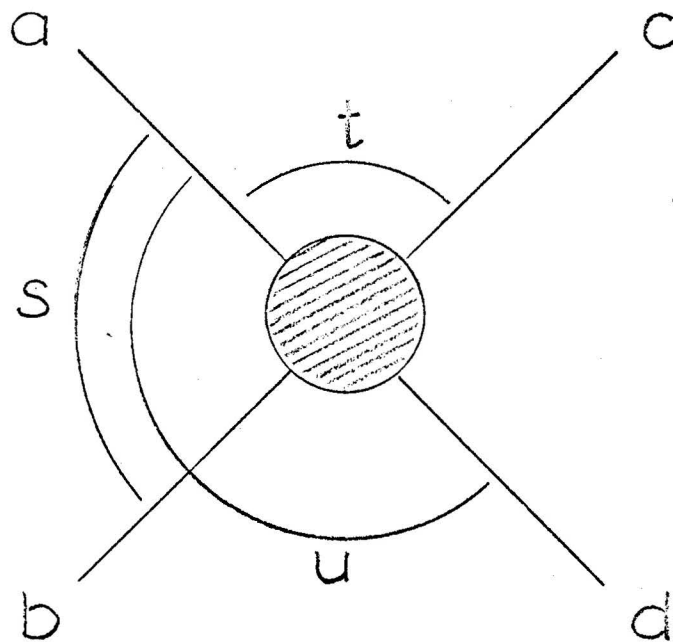
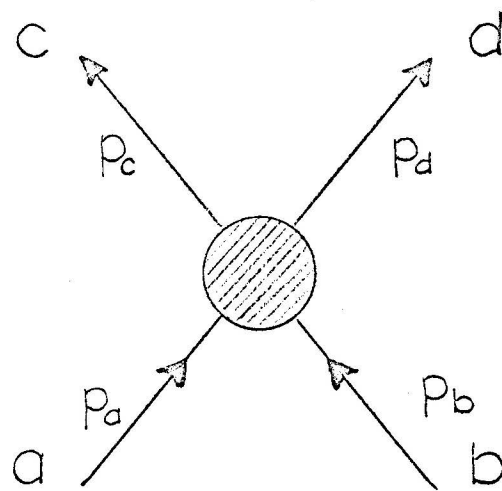
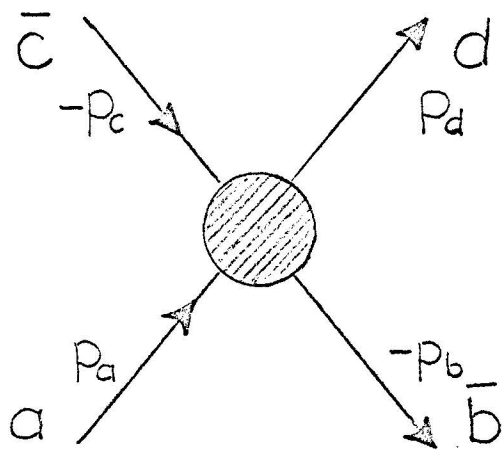


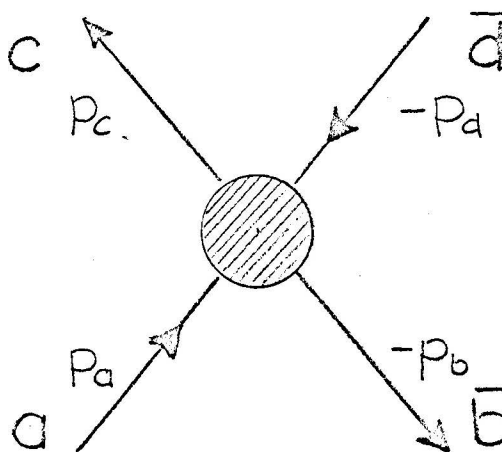
Fig. IV 2.2.



s



t



u

Fig. IV 2.3.

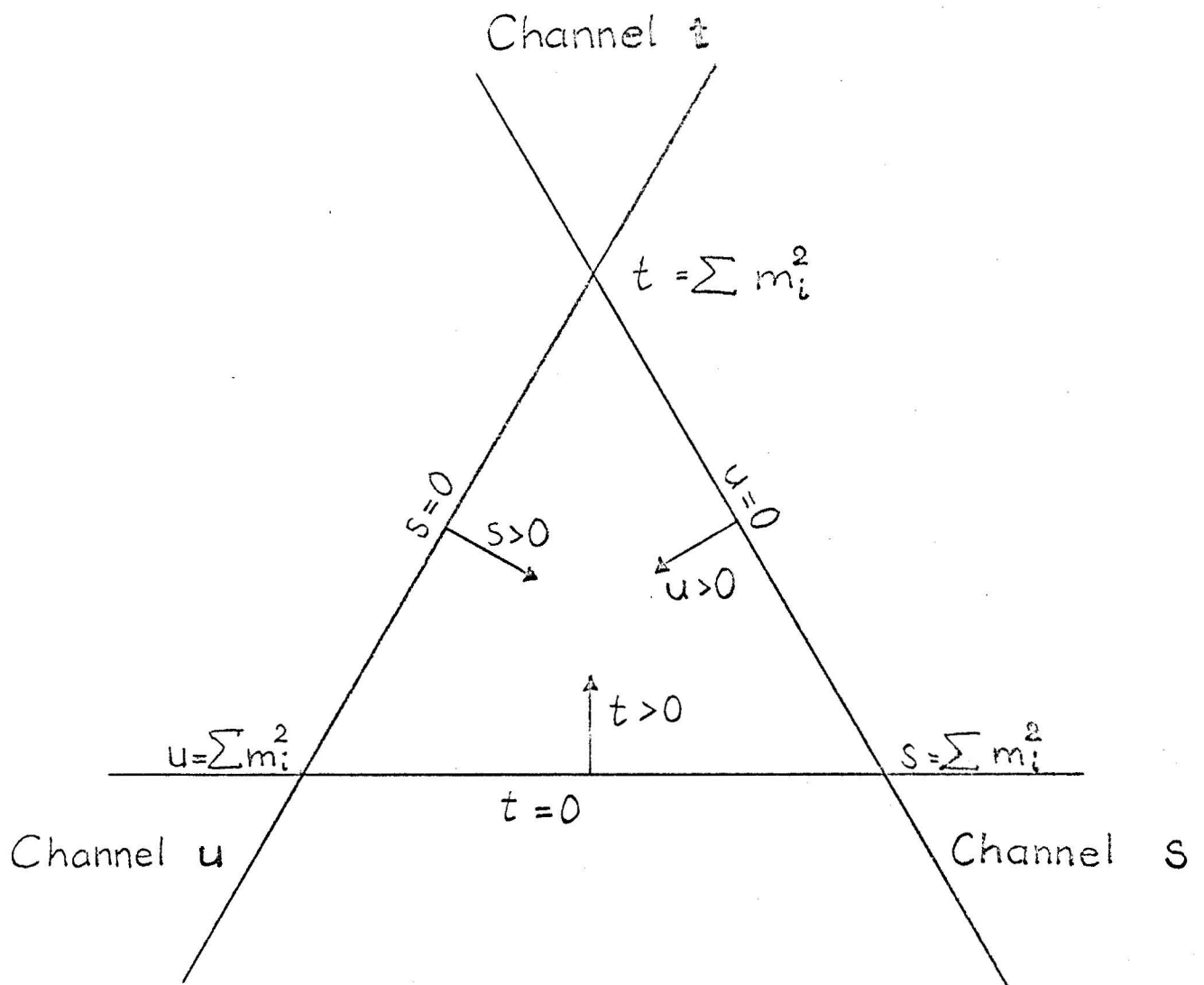


Fig. IV 3.1.

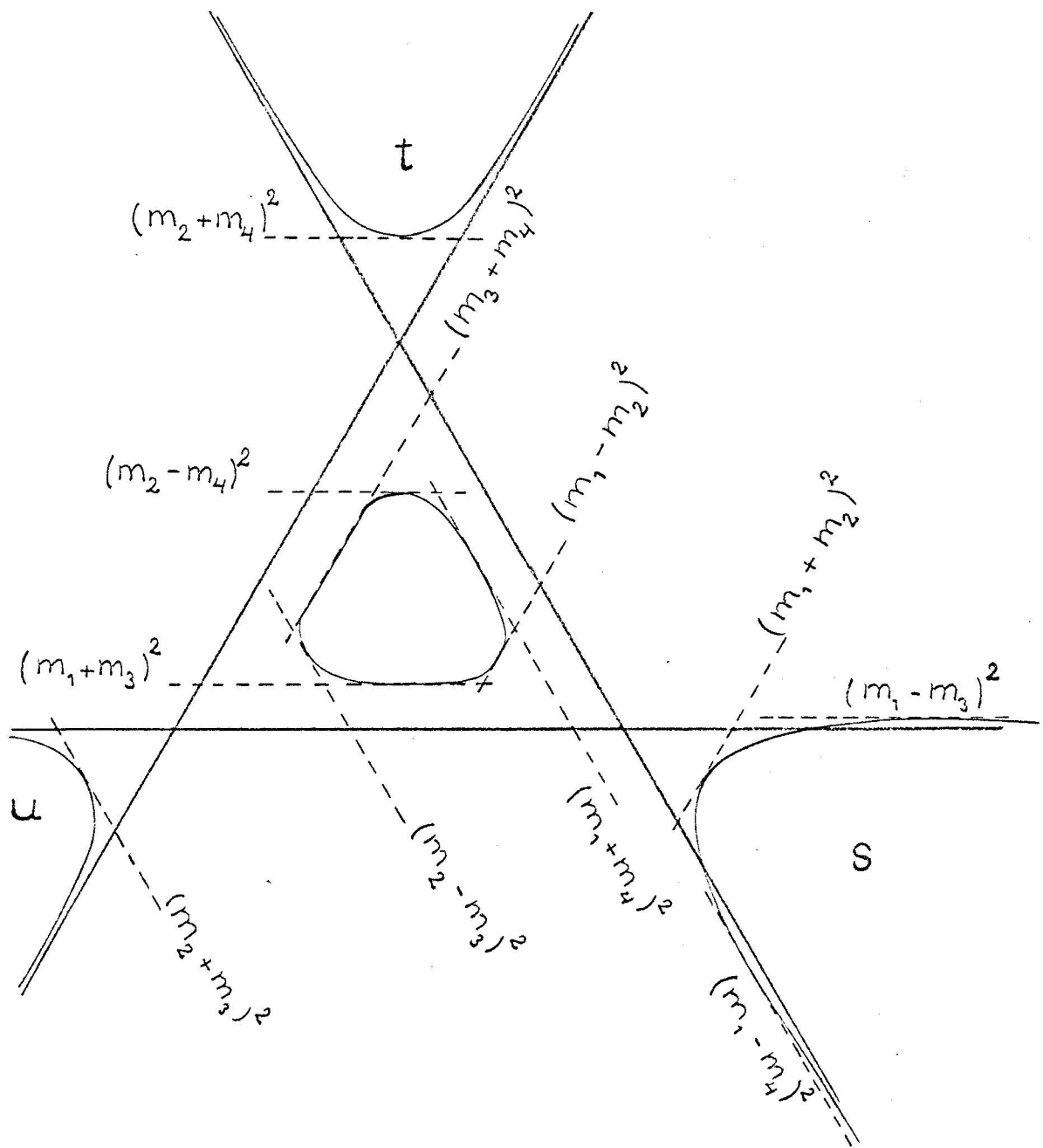


Fig. IV 3.2.

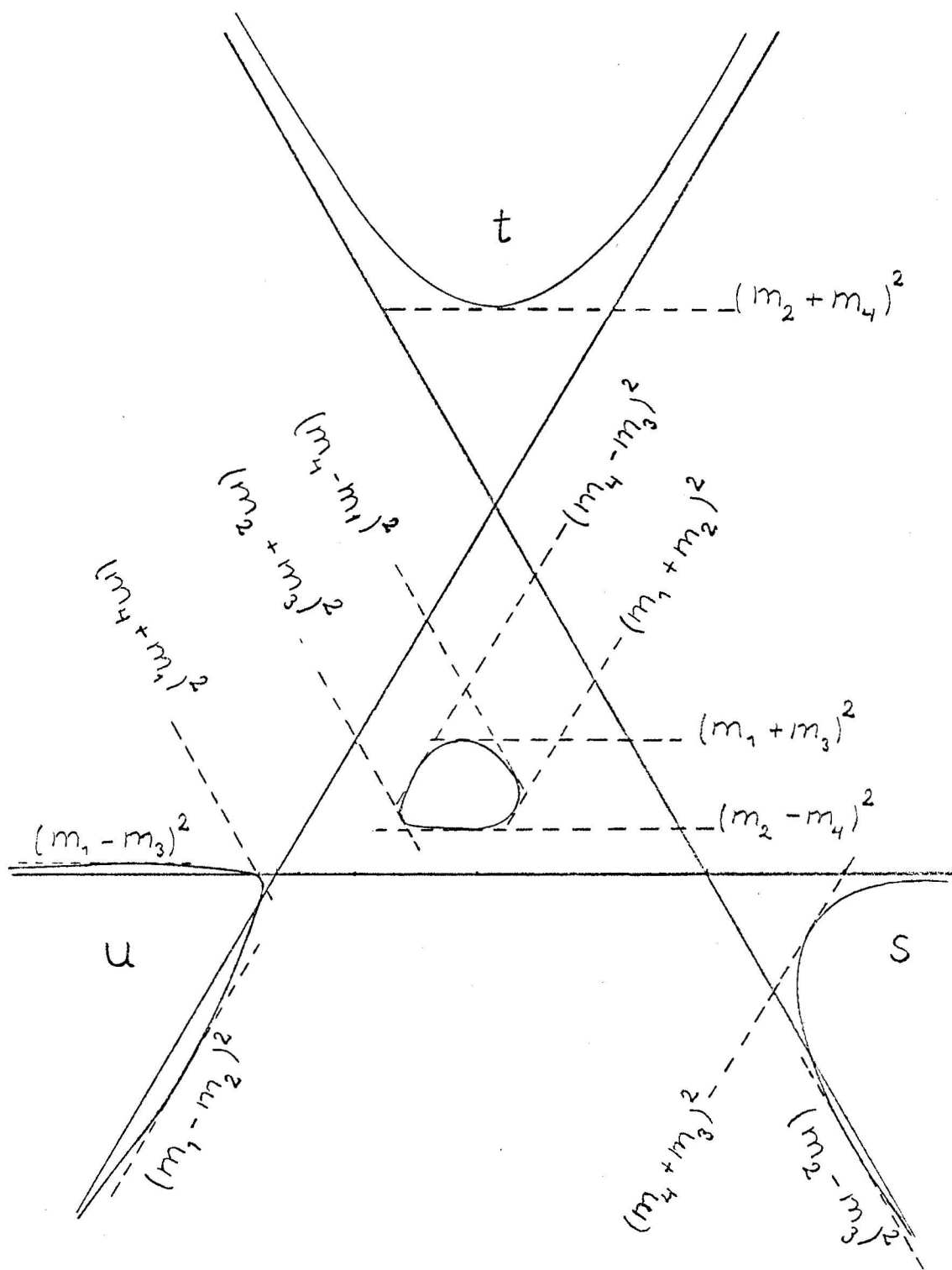


Fig. IV 3.3.

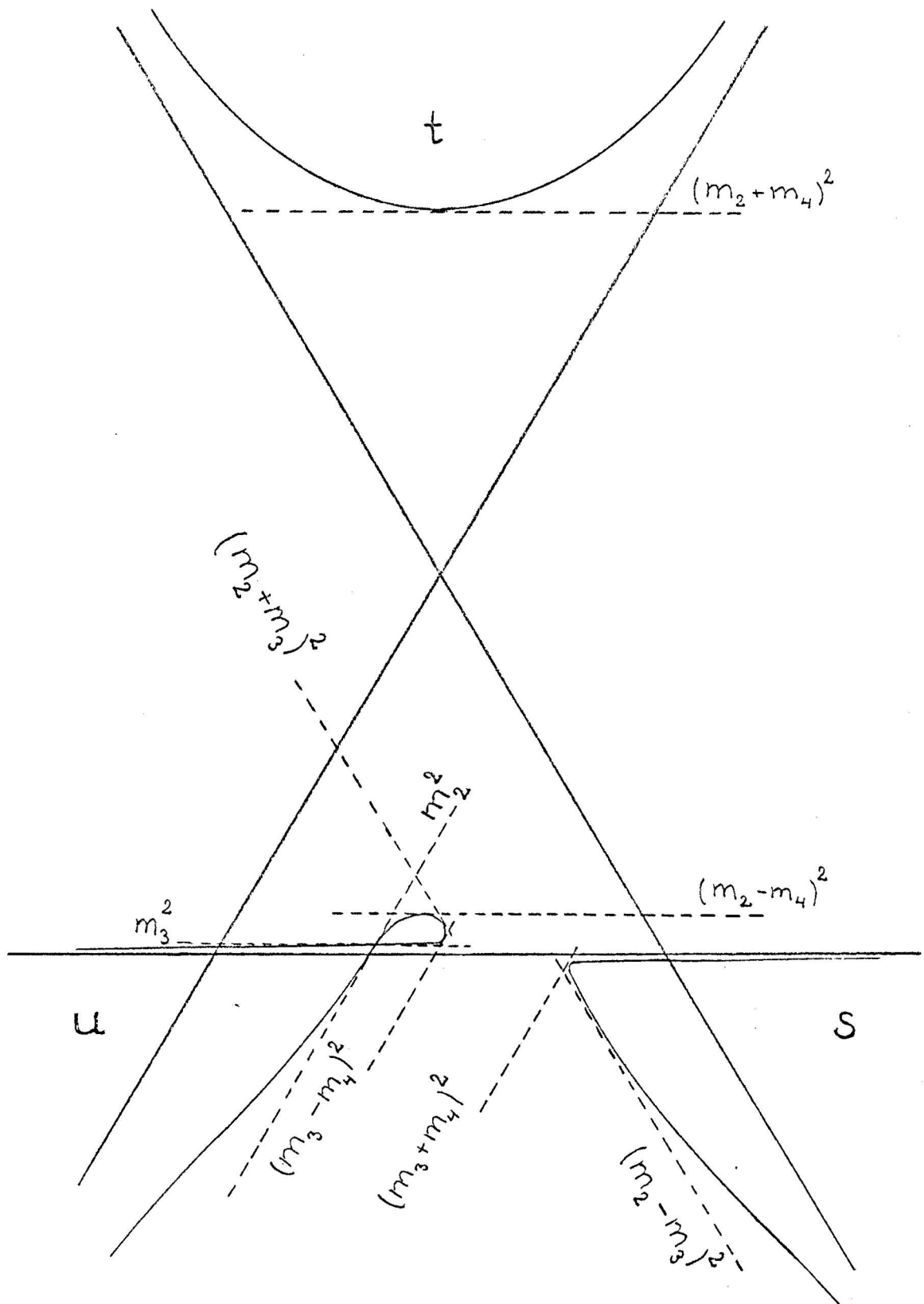


Fig. IV 3.4.

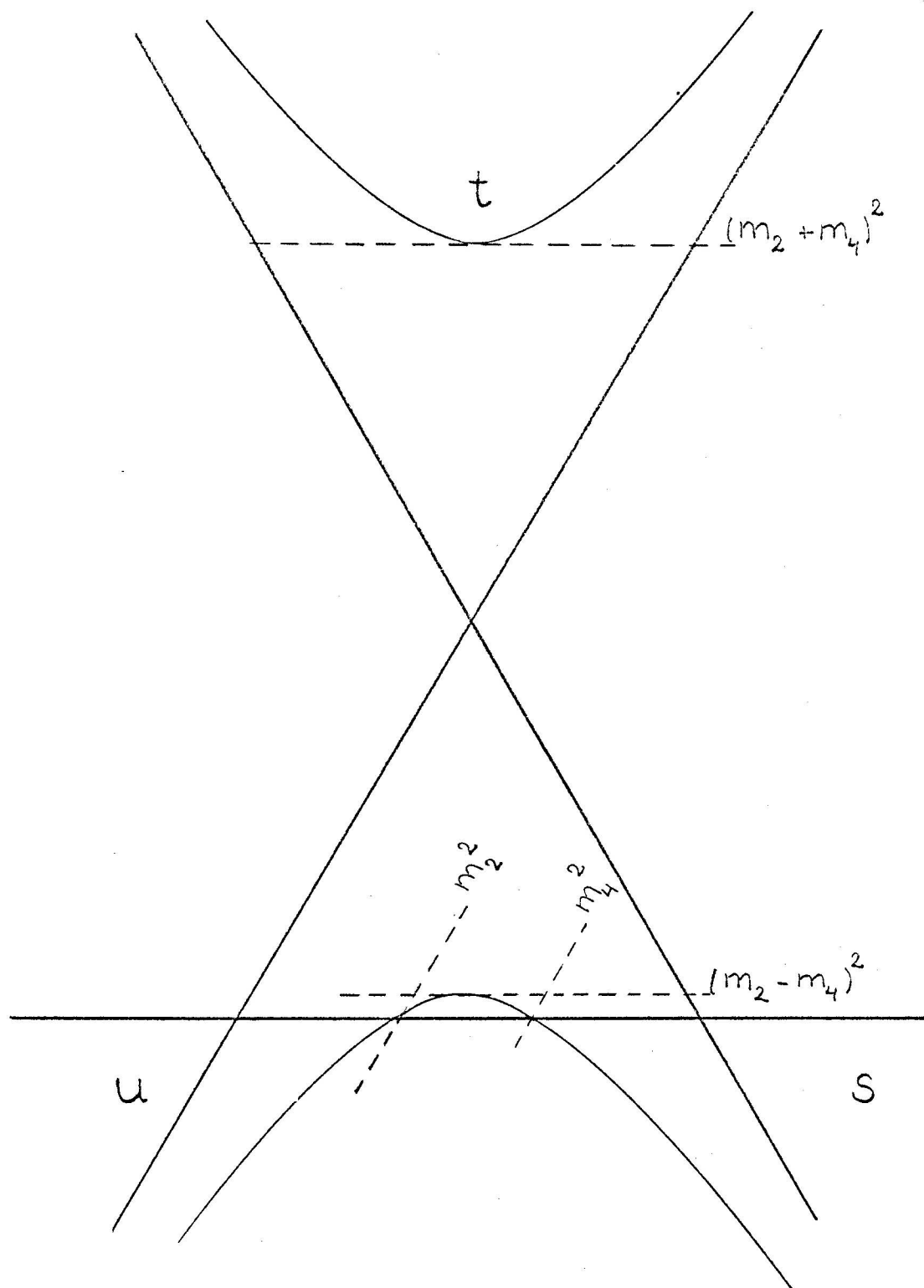


Fig. IV 3.6.

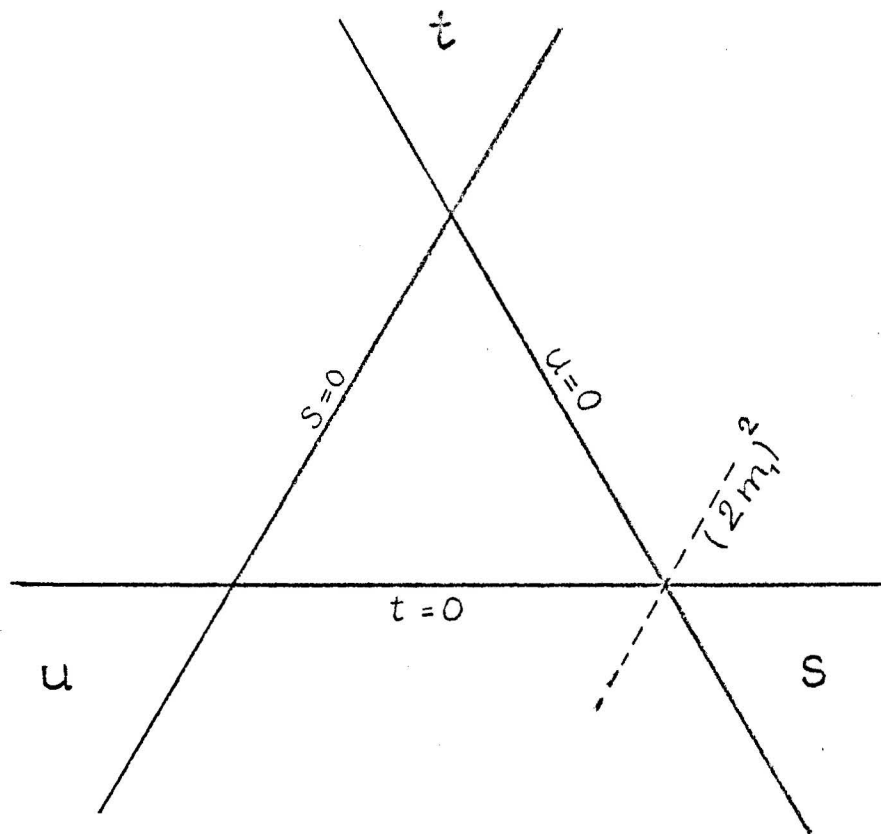


Fig. IV 3.7.

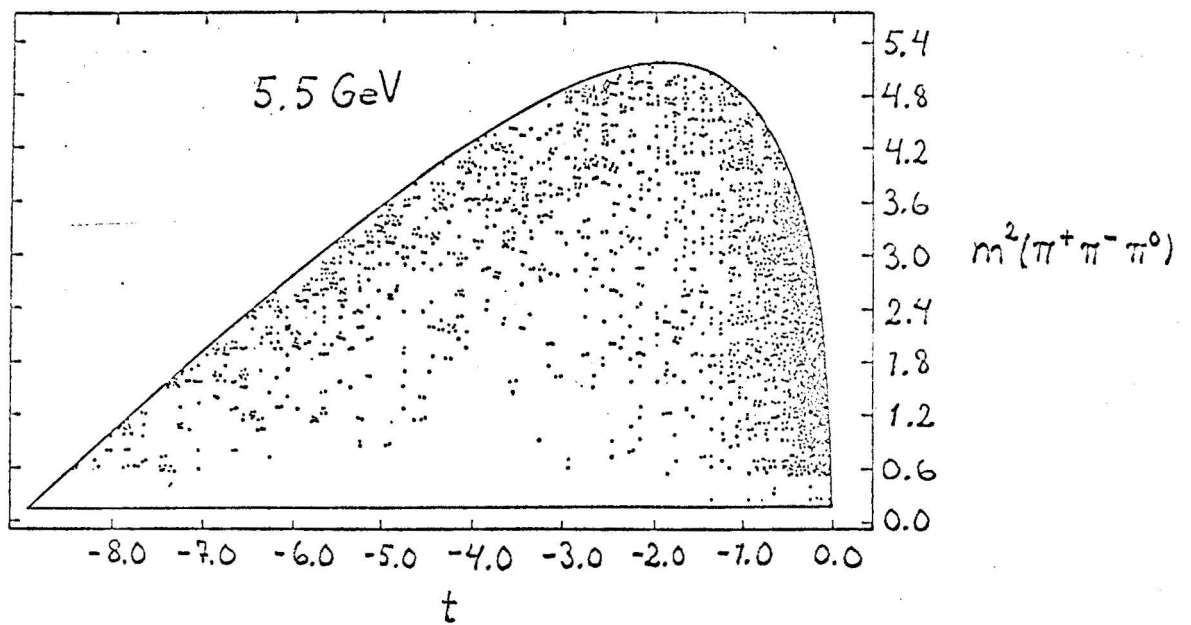
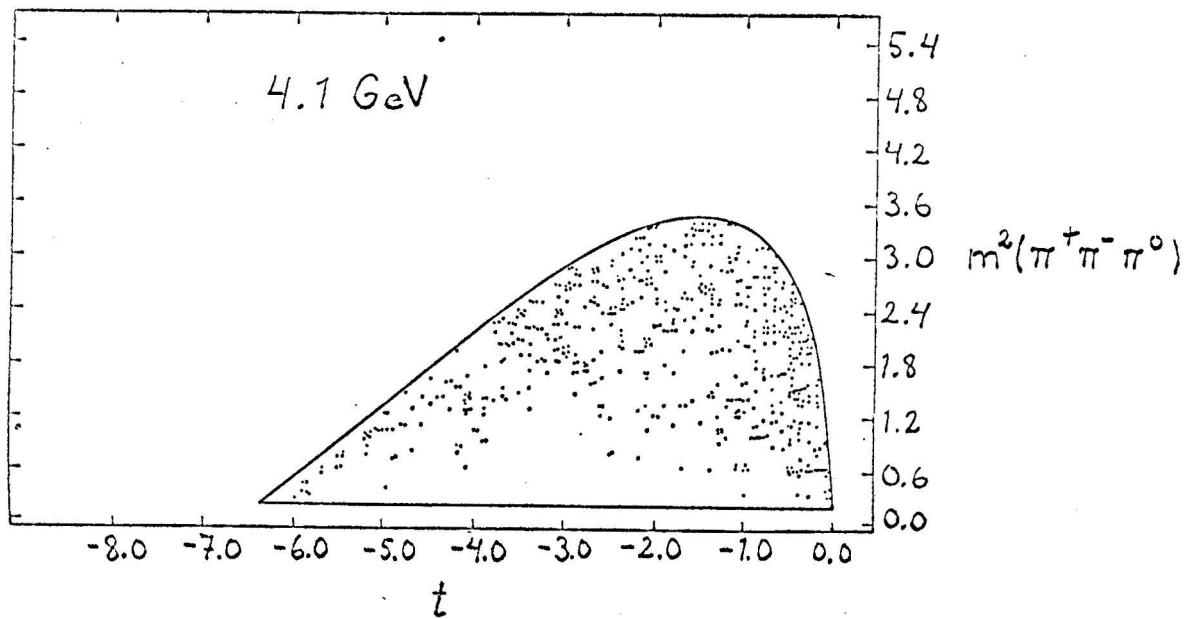


Fig. IV 3.8.

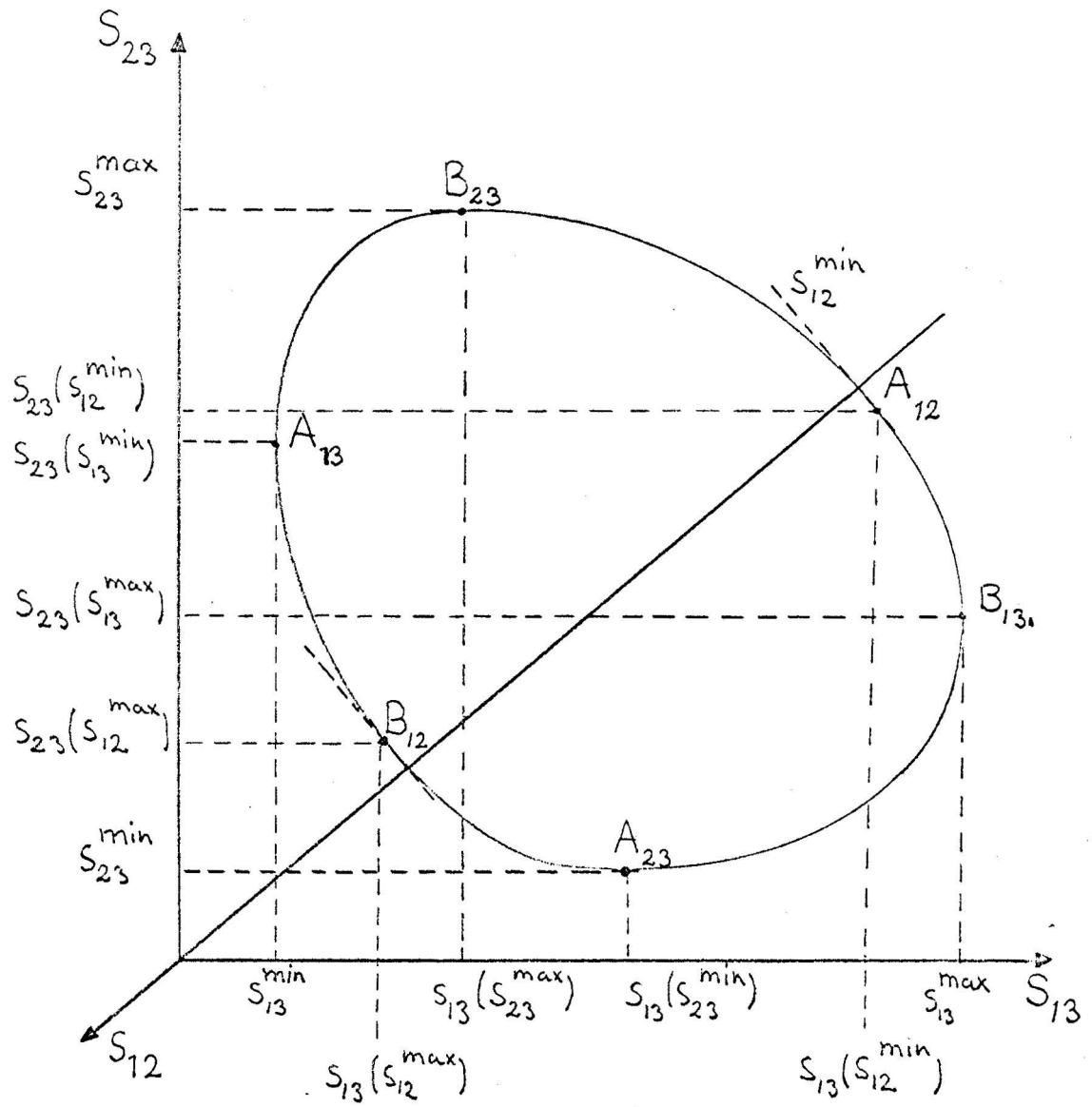


Fig. V 2.1.

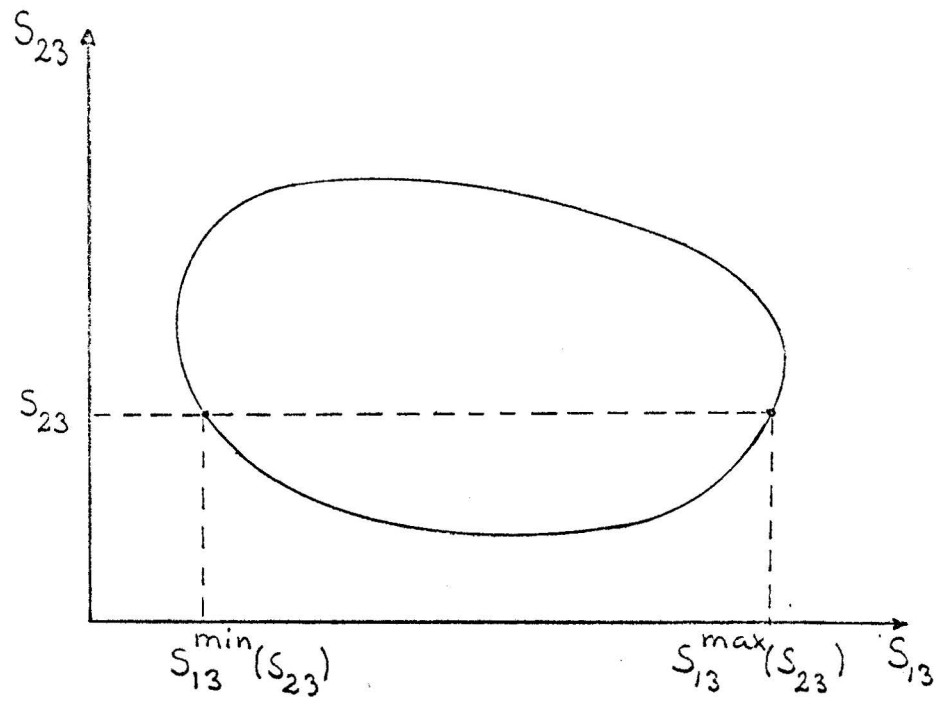


Fig. V 2.2.

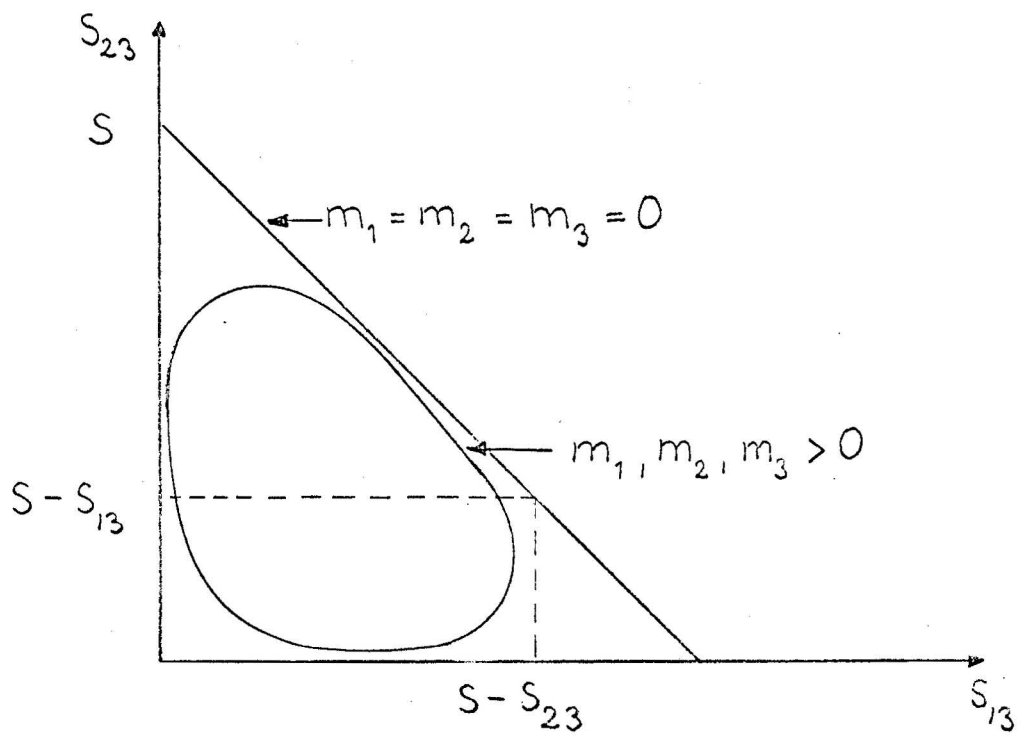


Fig. V 2.3.

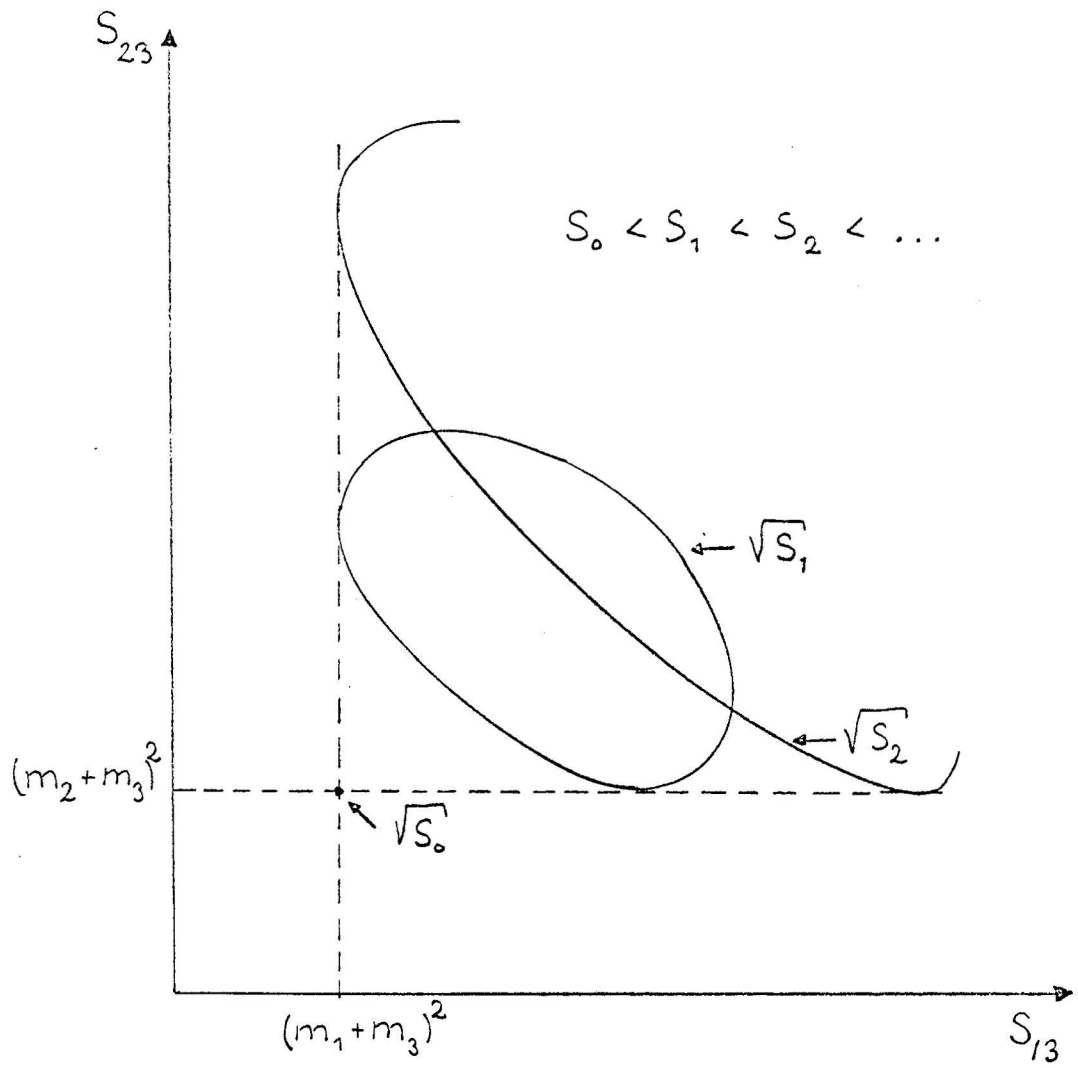


Fig. V 2. 4.

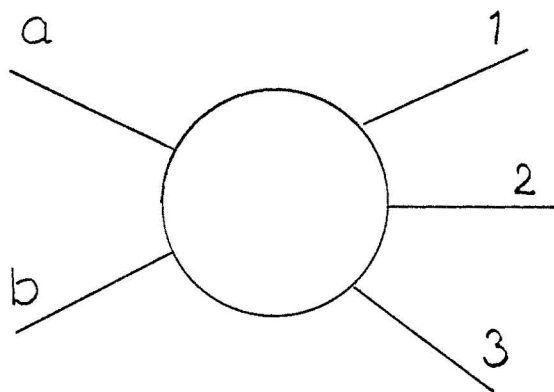


Fig. ∇ 3.1.

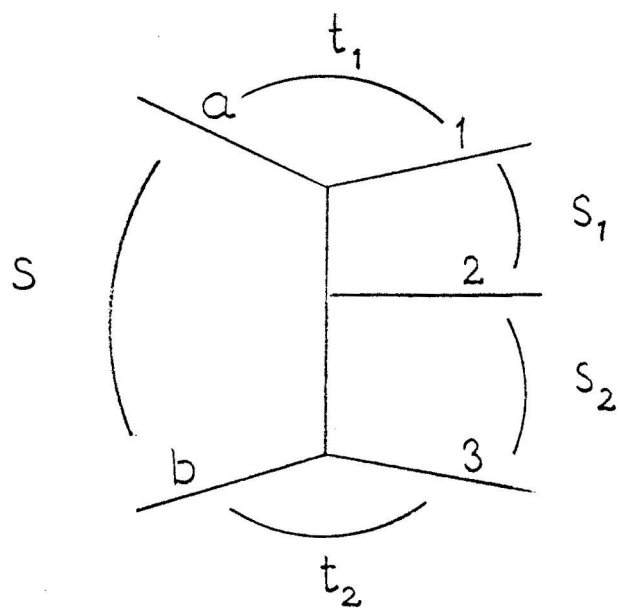
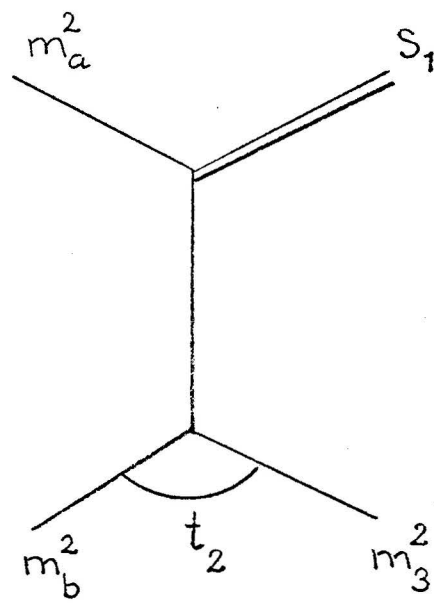
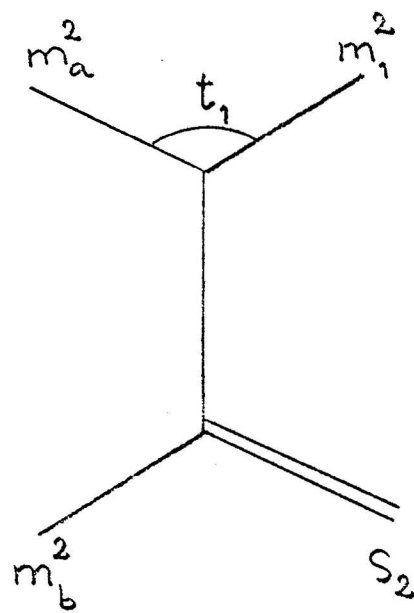


Fig. ∇ 3.2.



(a)



(b)

Fig. V 3.3.

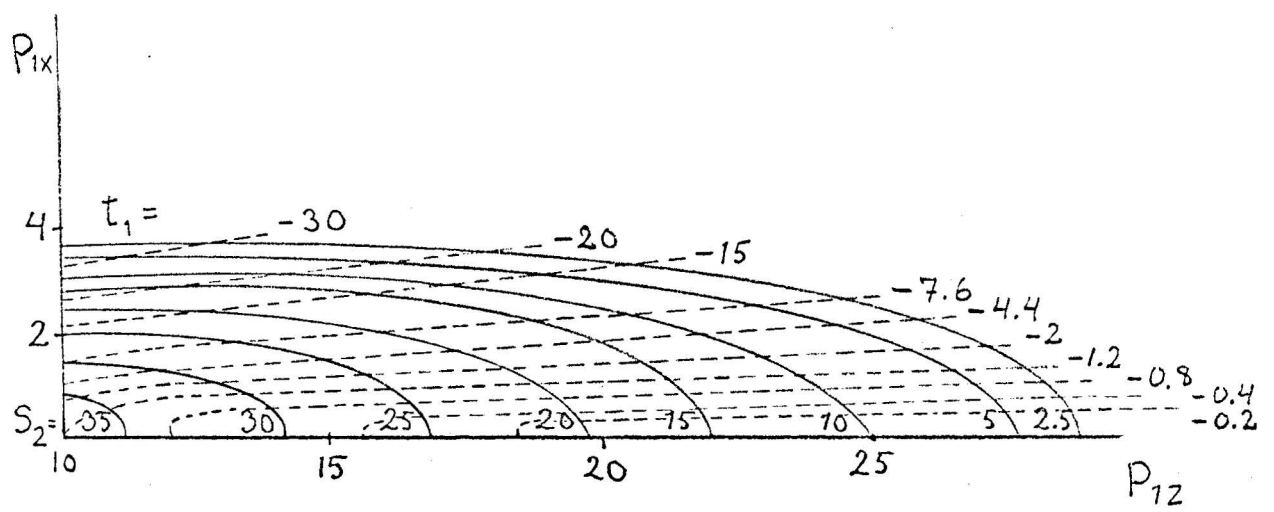


Fig. V 3.4.

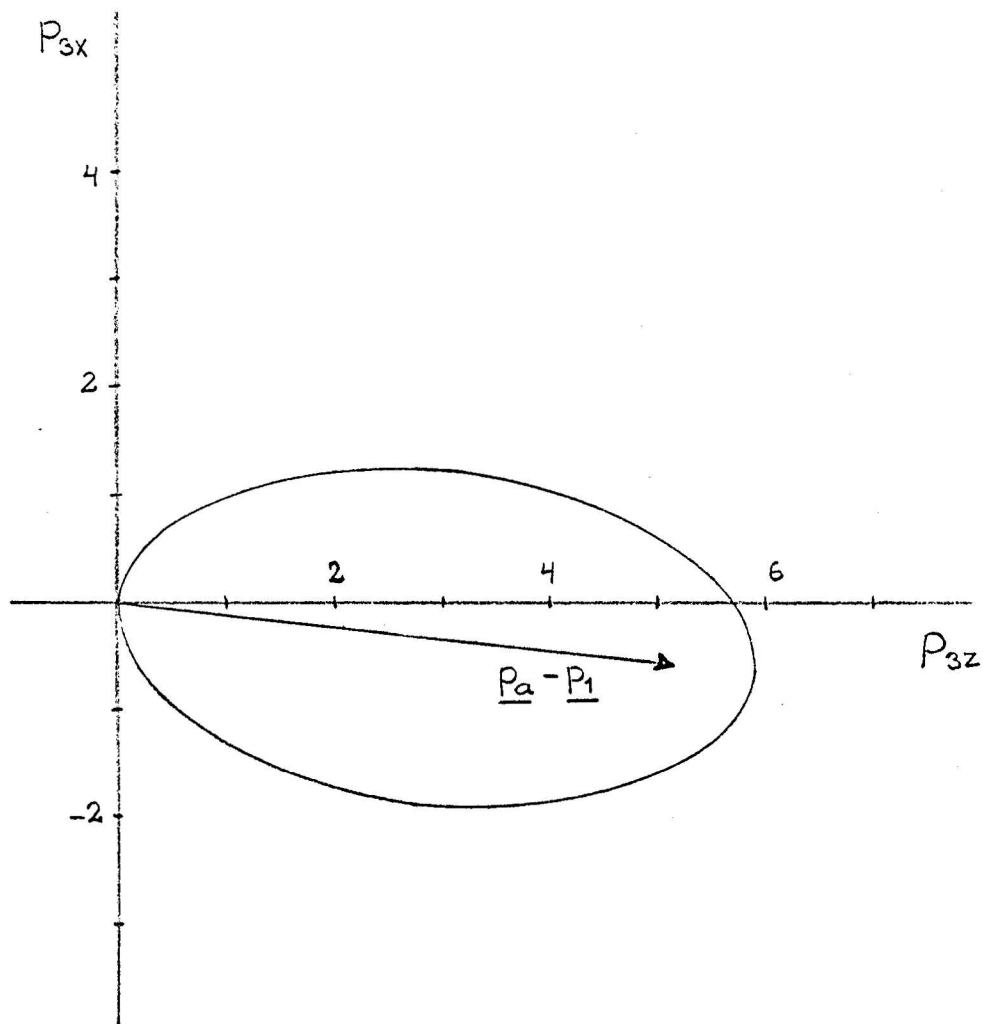


Fig. V 3.5.

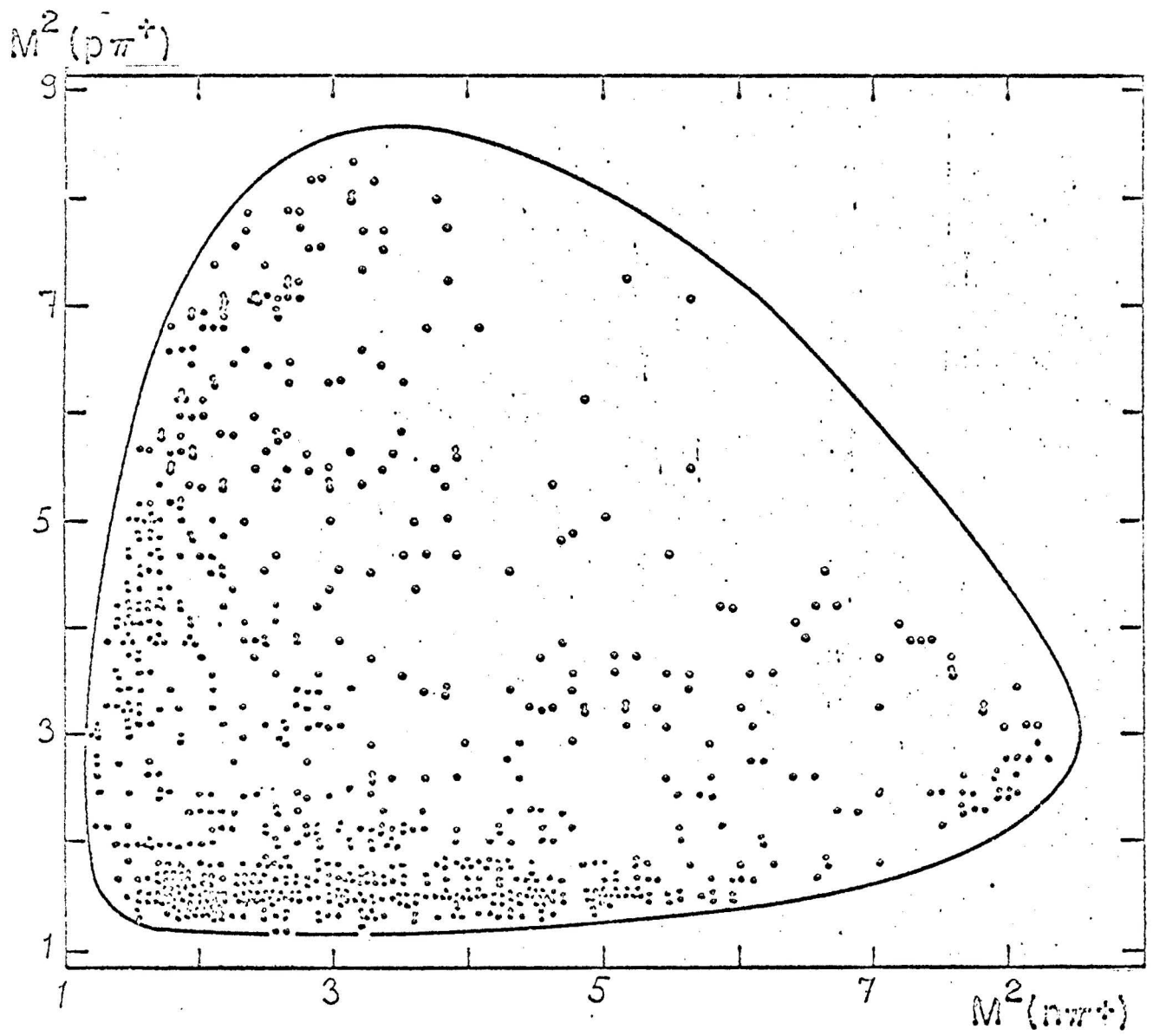
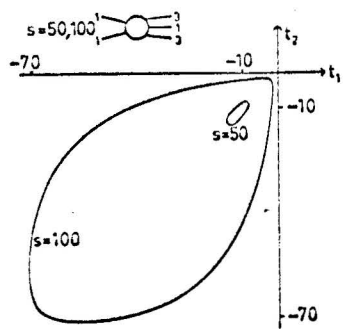
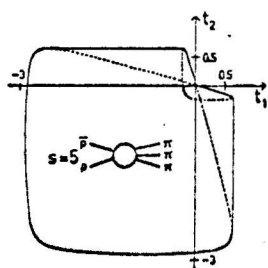


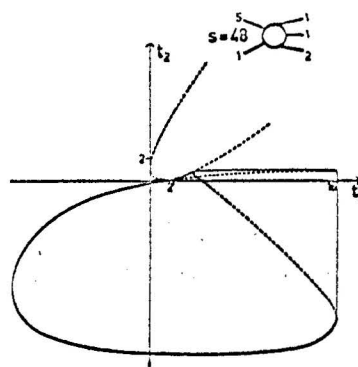
Fig V 5.1.



(a)



(b)



(c)

Fig. V 5.2.

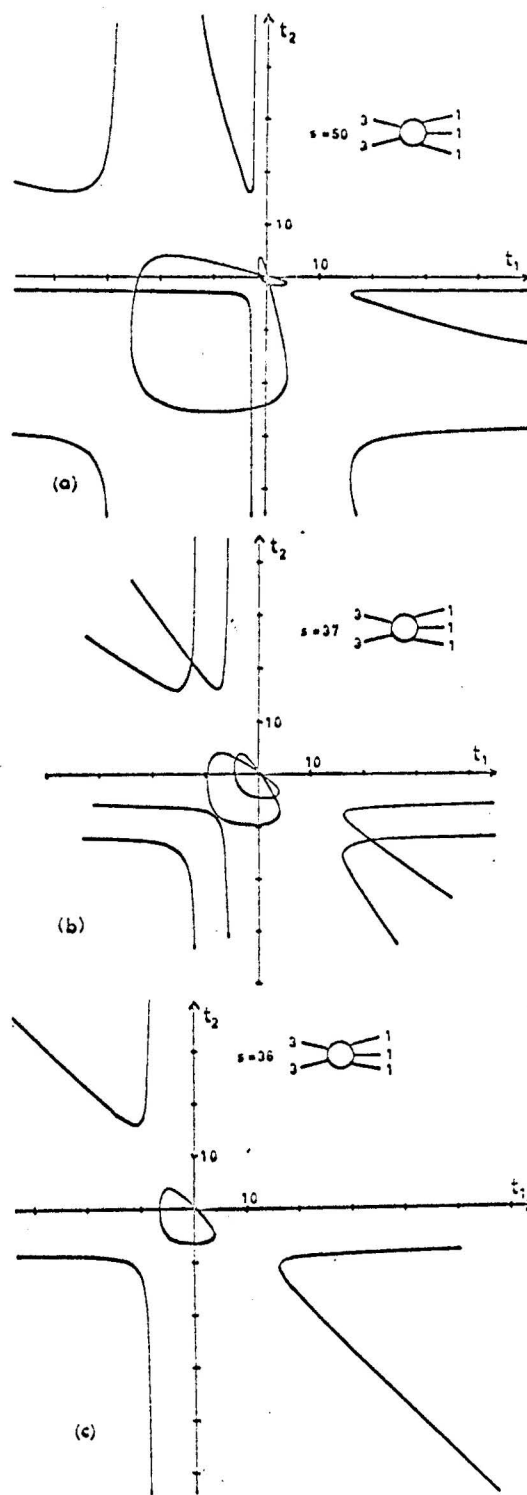


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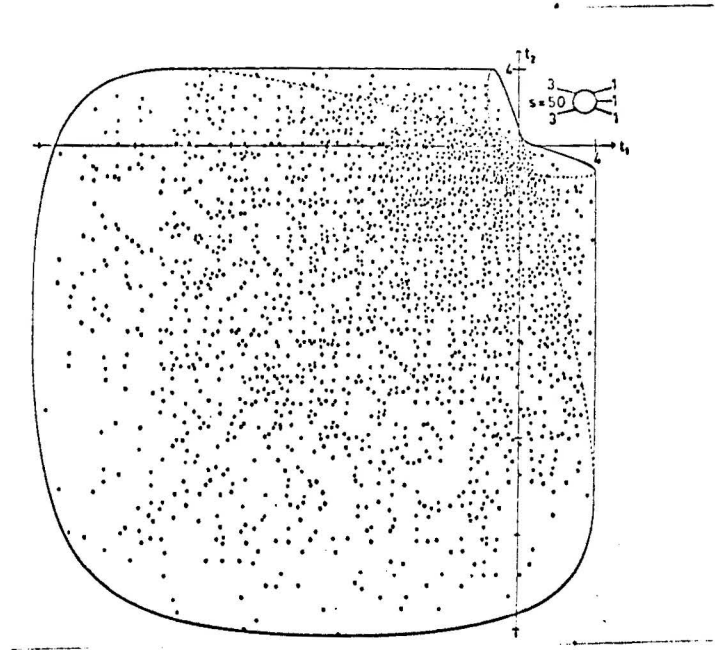


Fig. V 5.4.

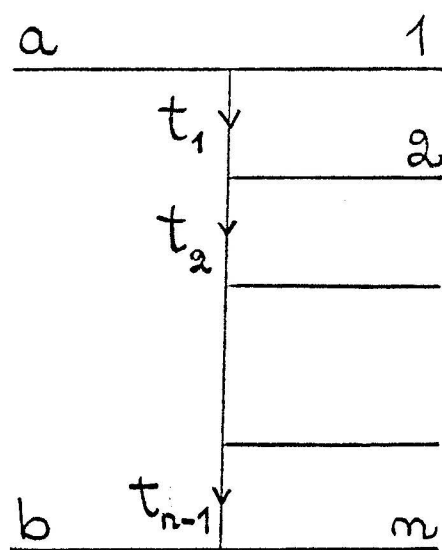


Fig. VI.1.1.

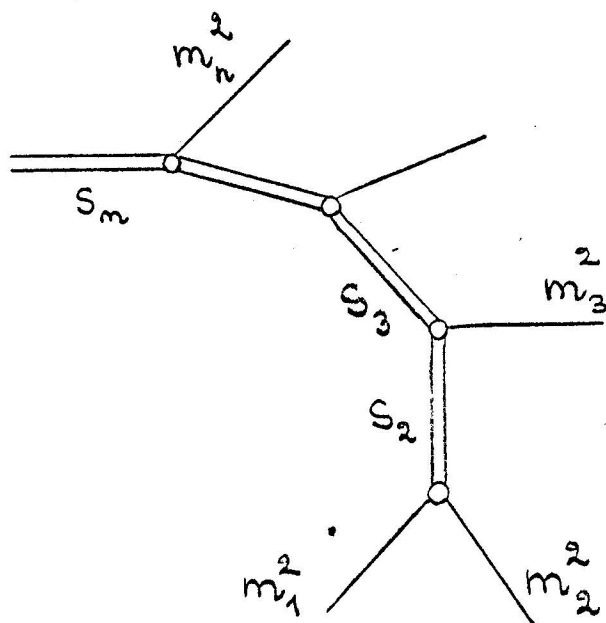


Fig. VI.1.2.

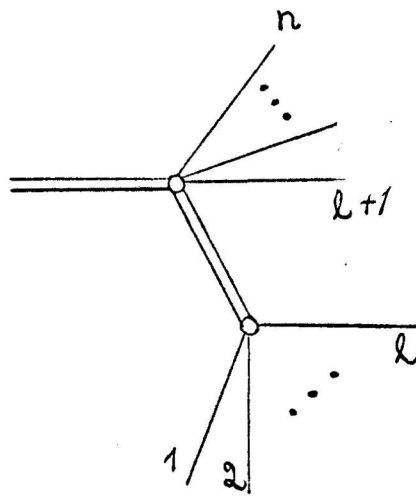


Fig. VI.2.1.

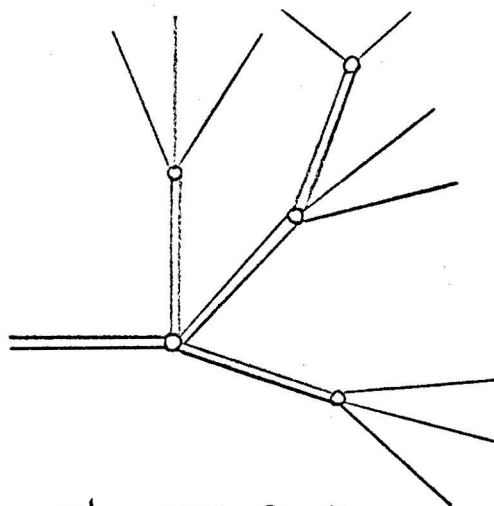


Fig. VI.2.2.

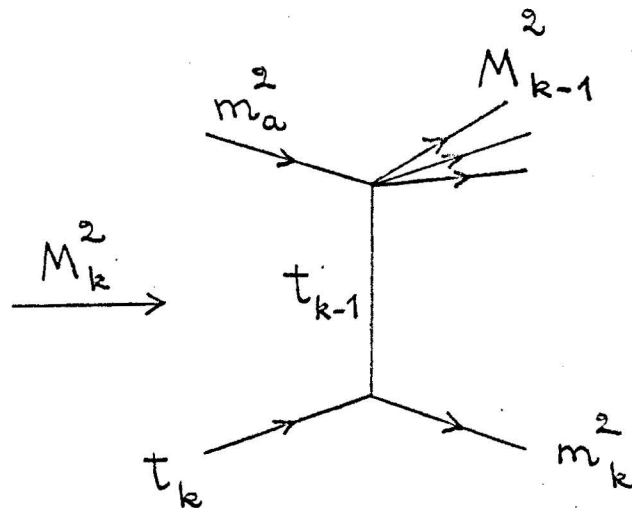


Fig. VI.2.3.

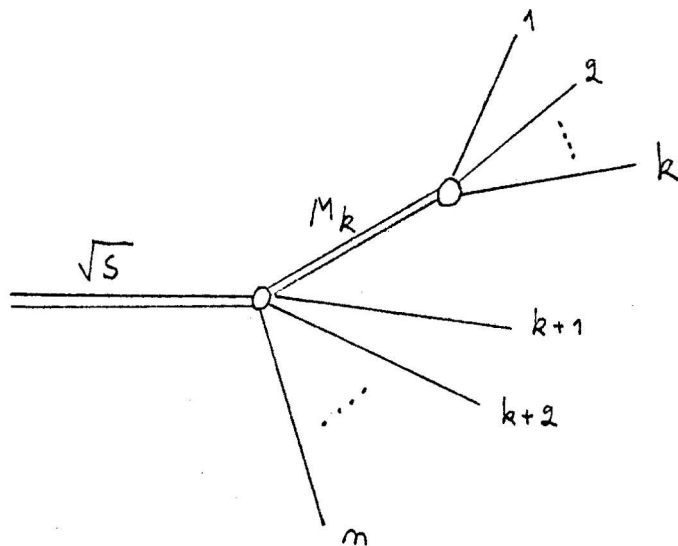


Fig. VI. 5. 1.

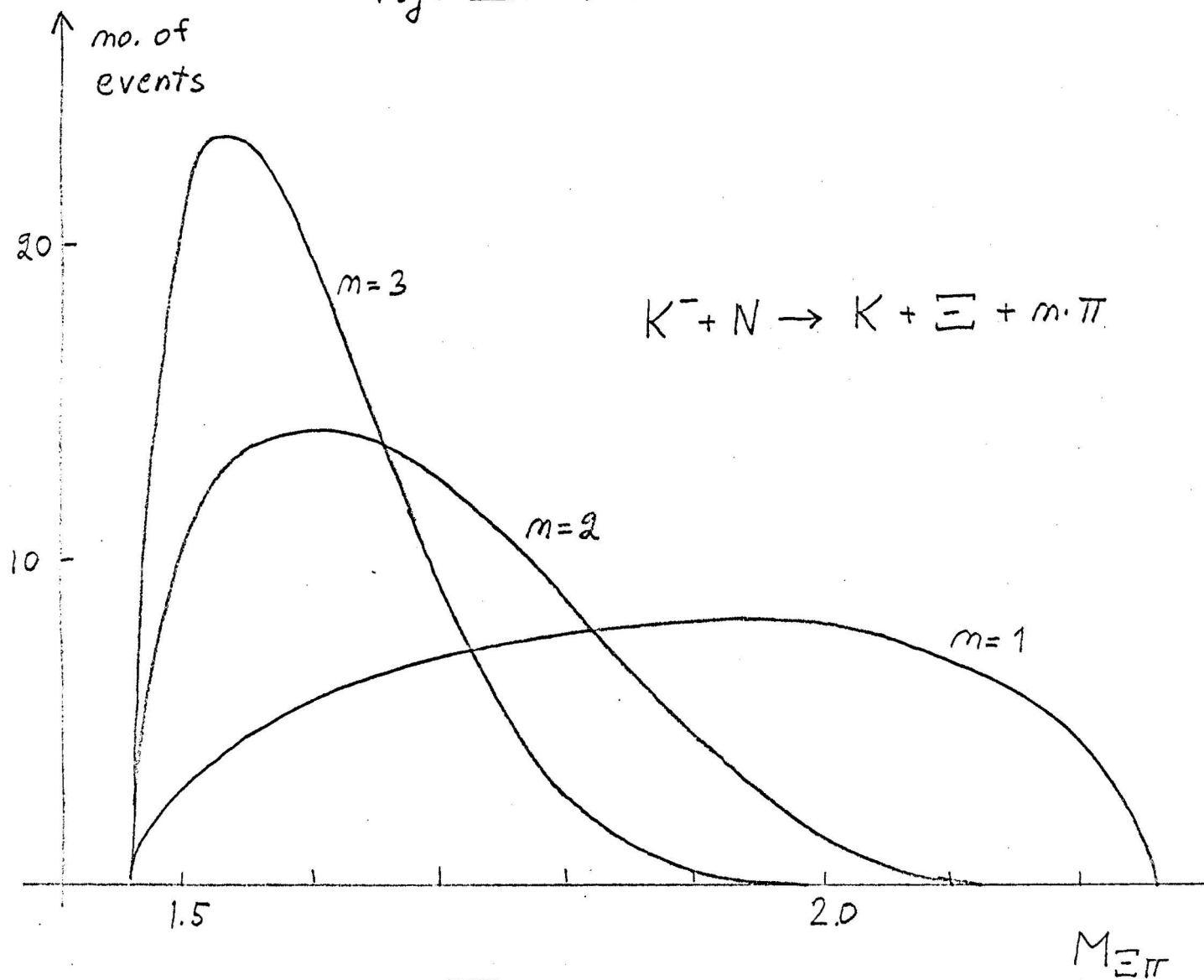


Fig. VI. 5. 2

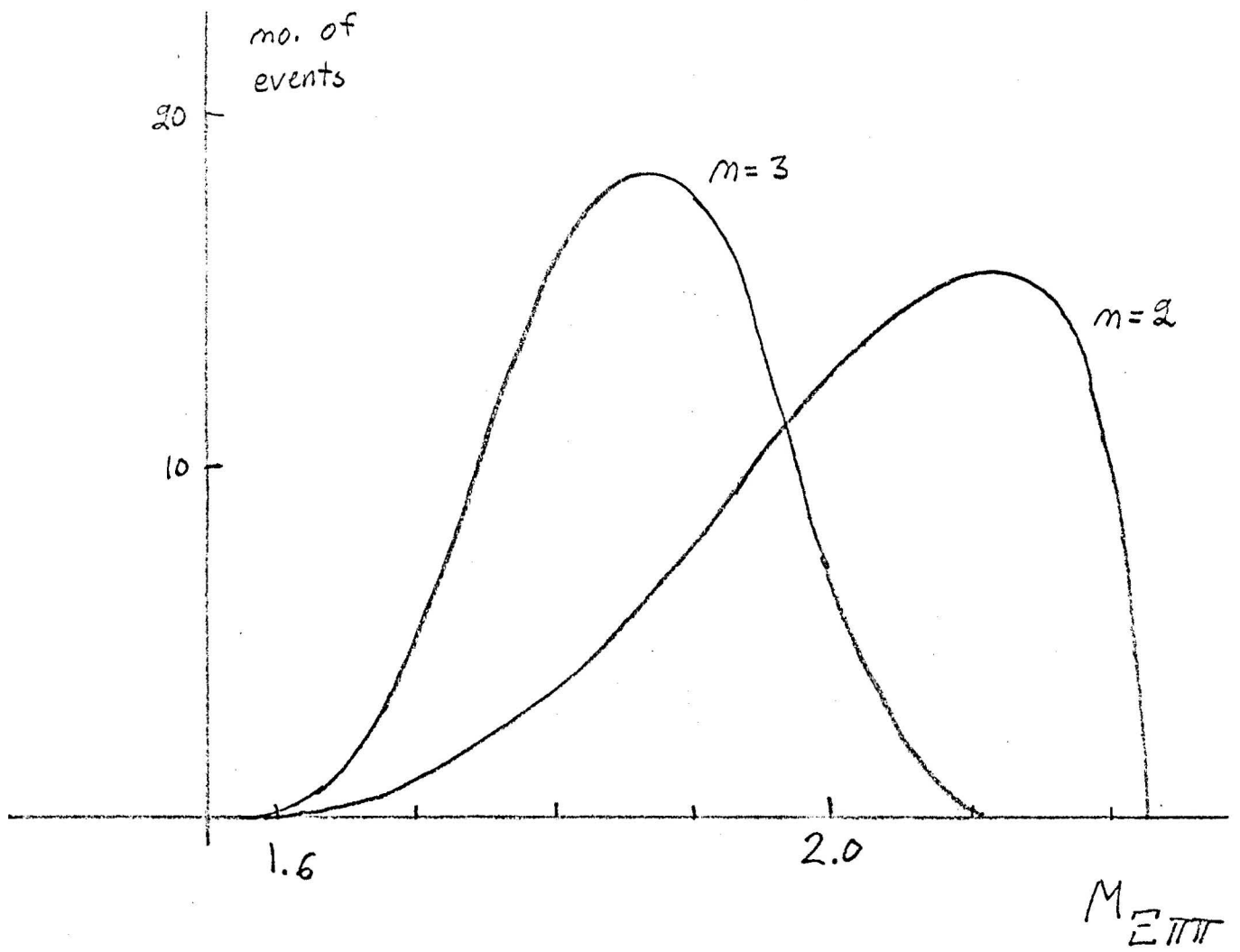


Fig. VI. 5.3

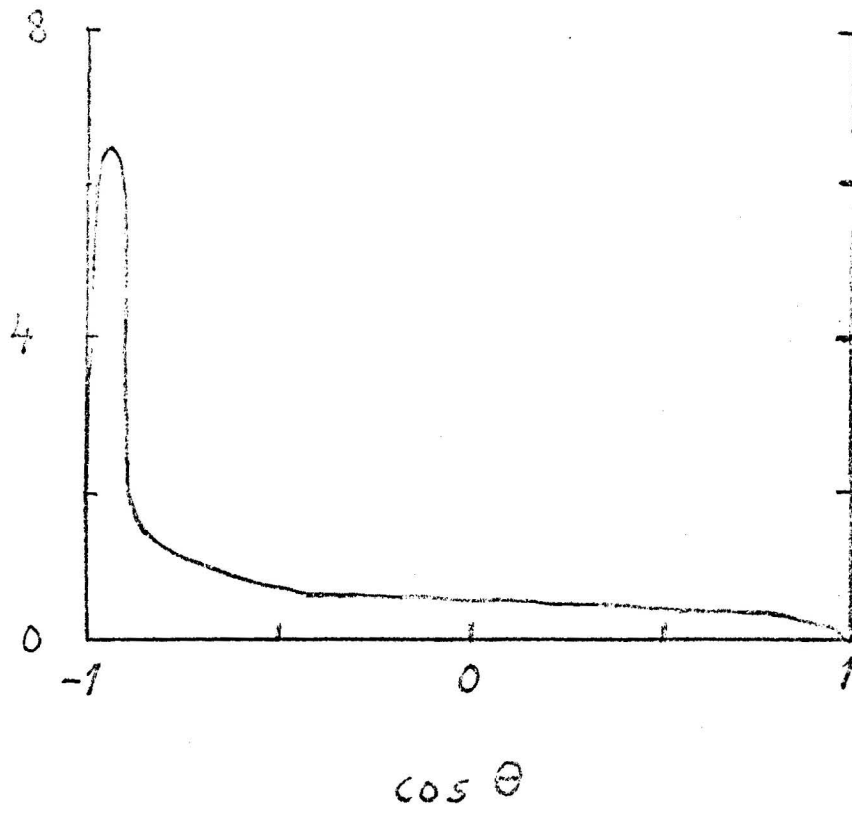


Fig. VI. 5. 4

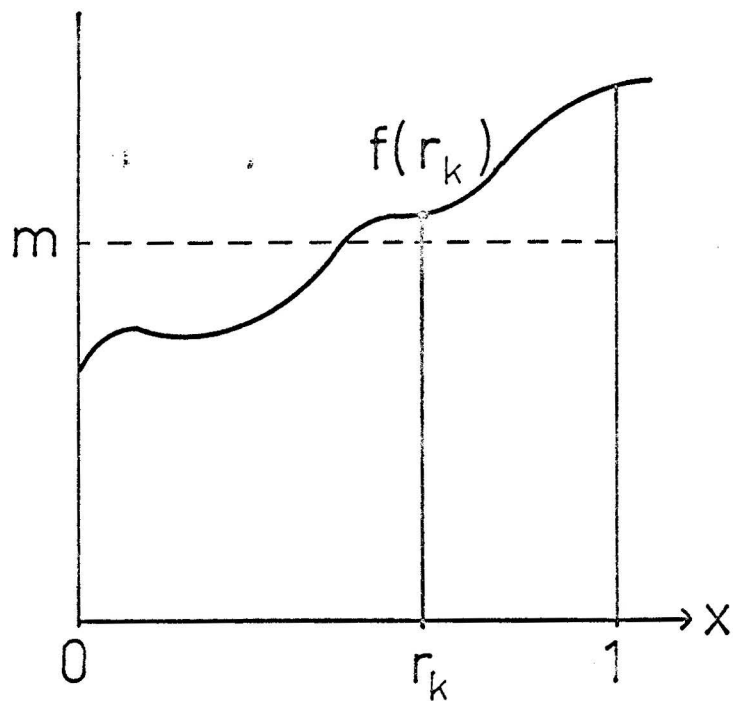


Fig.X.3.1.

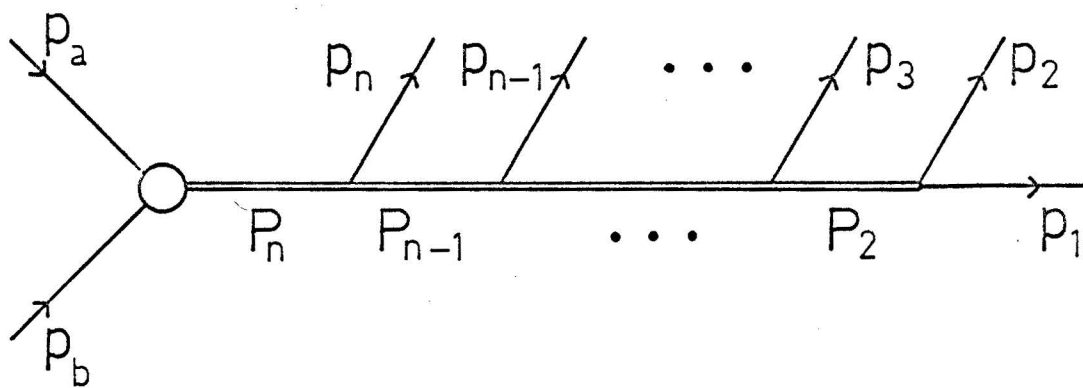


Fig.X.5.1.

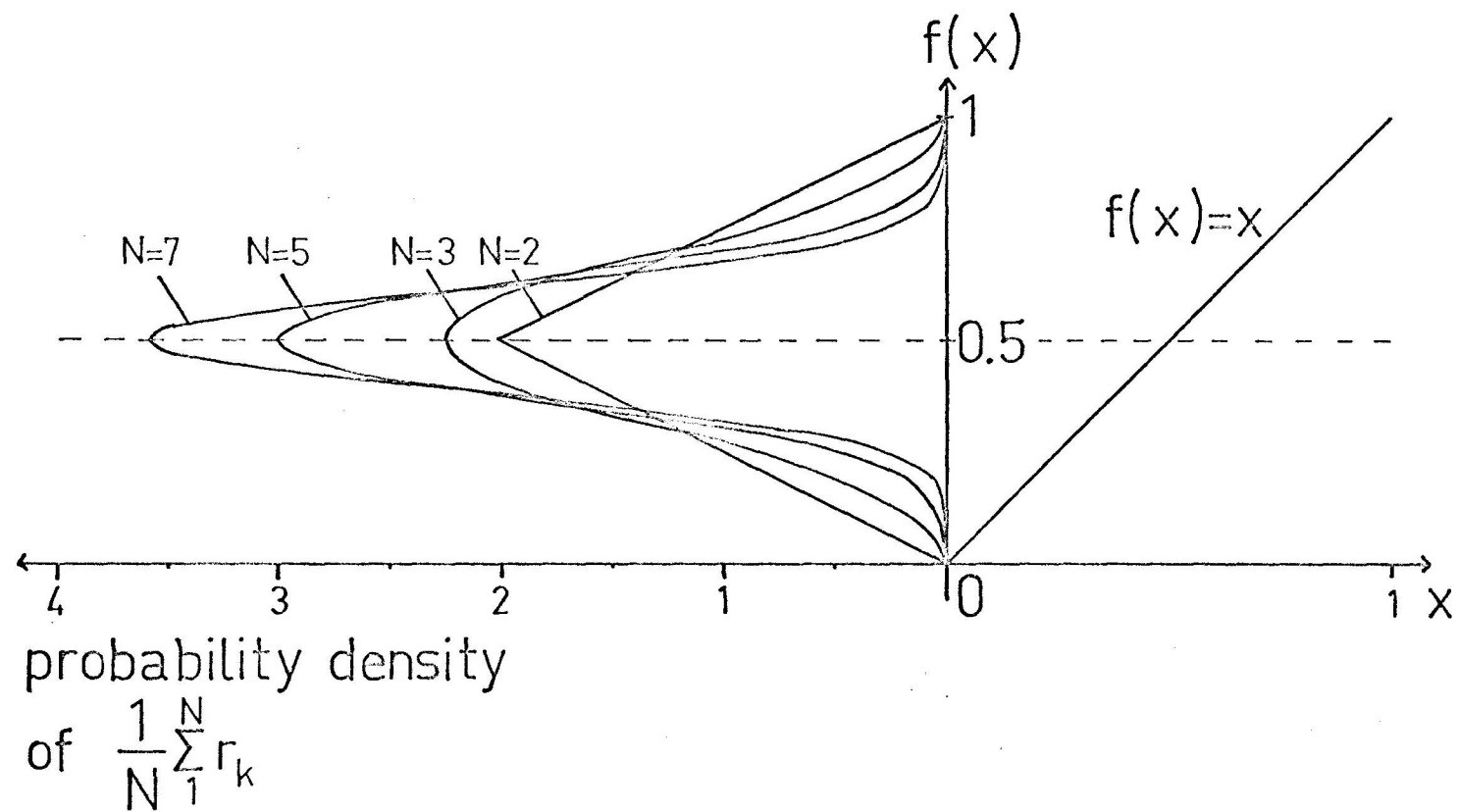


Fig. X.3.2.

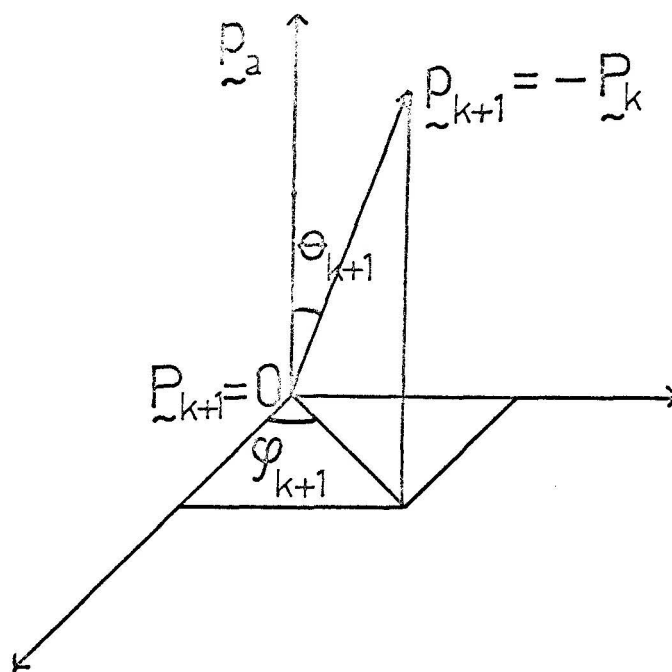


Fig. X. 5.2.

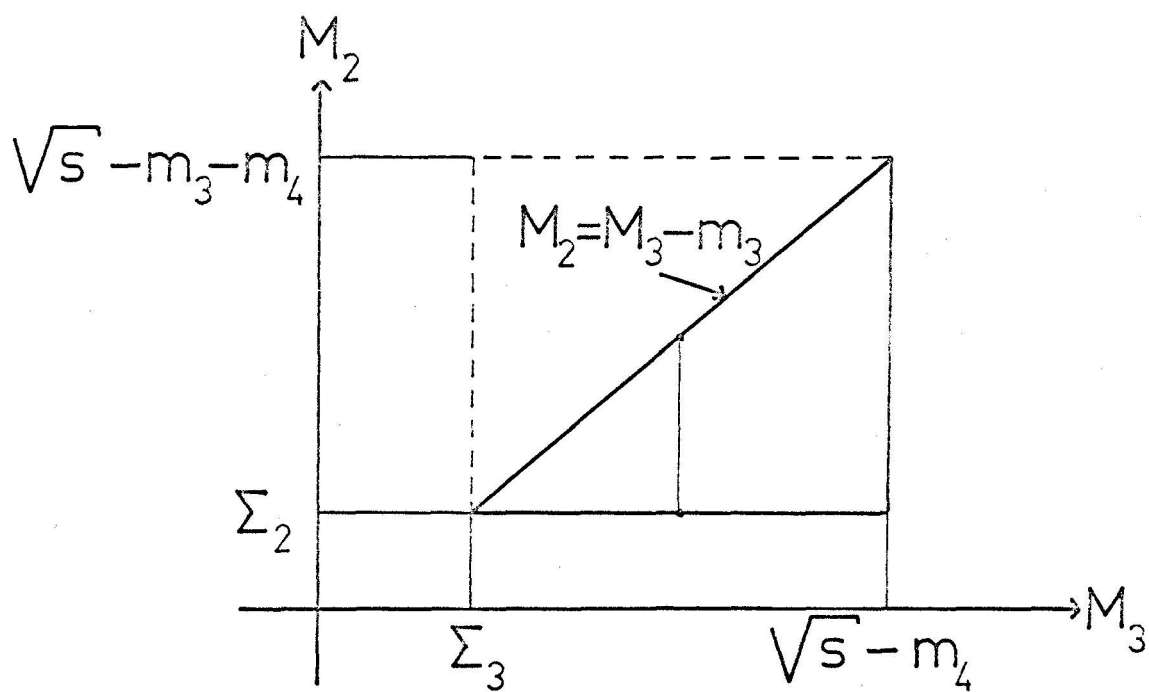


Fig. X. 5.3.

DATE 093070

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SUM OF WEIGHTS= 0.2045E 03
NORMALIZATION= 0.2045E 03
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BELOW 0.400E 00 0.000000E 00

0.400E 00	0.450E 00	0.000010	0.000002
0.450E 00	0.500E 00	0.000188	0.000027
0.500E 00	0.550E 00	0.000672	0.000090
0.550E 00	0.600E 00	0.001427	0.000188
0.600E 00	0.650E 00	0.002672	0.000347
0.650E 00	0.700E 00	0.004170	0.000544
0.700E 00	0.750E 00	0.006181	0.000800
0.750E 00	0.800E 00	0.008274	0.001091
0.800E 00	0.850E 00	0.011150	0.001490
0.850E 00	0.900E 00	0.015139	0.001961
0.900E 00	0.950E 00	0.019084	0.002495
0.950E 00	0.100E 01	0.022910	0.002986
0.100E 01	0.105E 01	0.029205	0.003777
0.105E 01	0.110E 01	0.034610	0.004448
0.110E 01	0.115E 01	0.038260	0.004979
0.115E 01	0.120E 01	0.047815	0.006170
0.120E 01	0.125E 01	0.050668	0.006585
0.125E 01	0.130E 01	0.056789	0.007362
0.130E 01	0.135E 01	0.065556	0.008434
0.135E 01	0.140E 01	0.063991	0.008284
0.140E 01	0.145E 01	0.063872	0.008479
0.145E 01	0.150E 01	0.066047	0.008627
0.150E 01	0.155E 01	0.067757	0.008968
0.155E 01	0.160E 01	0.065095	0.008506
0.160E 01	0.165E 01	0.060844	0.008050
0.165E 01	0.170E 01	0.055539	0.007281
0.170E 01	0.175E 01	0.050037	0.006558
0.175E 01	0.180E 01	0.040050	0.005240
0.180E 01	0.185E 01	0.027760	0.003642
0.185E 01	0.190E 01	0.017356	0.002296
0.190E 01	0.195E 01	0.006491	0.000910
0.195E 01	0.200E 01	0.000360	0.000107

[illegible]

ABOVE 0.200E 01 0.000000E 00

- 238 -

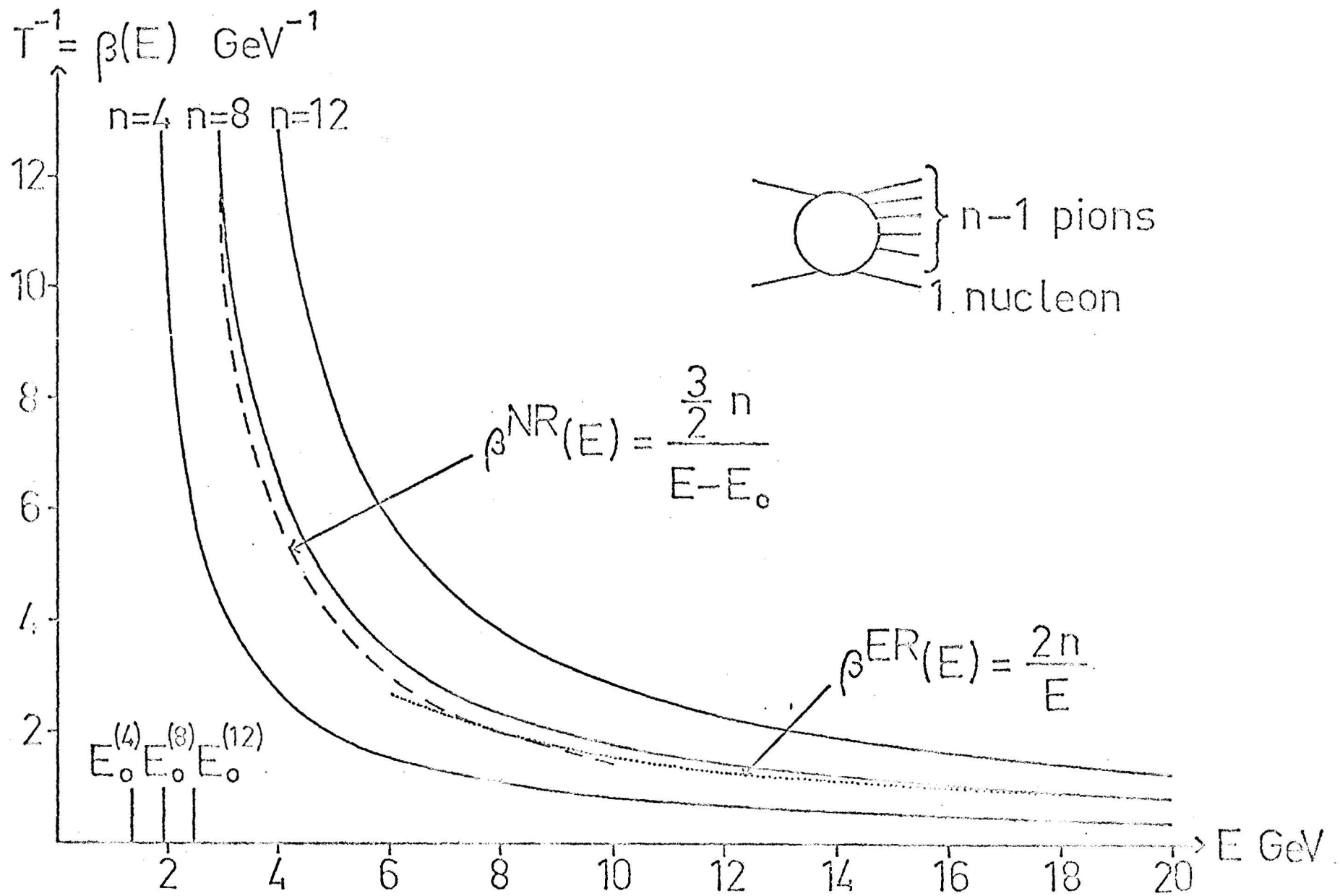


Fig. X. 6. 1

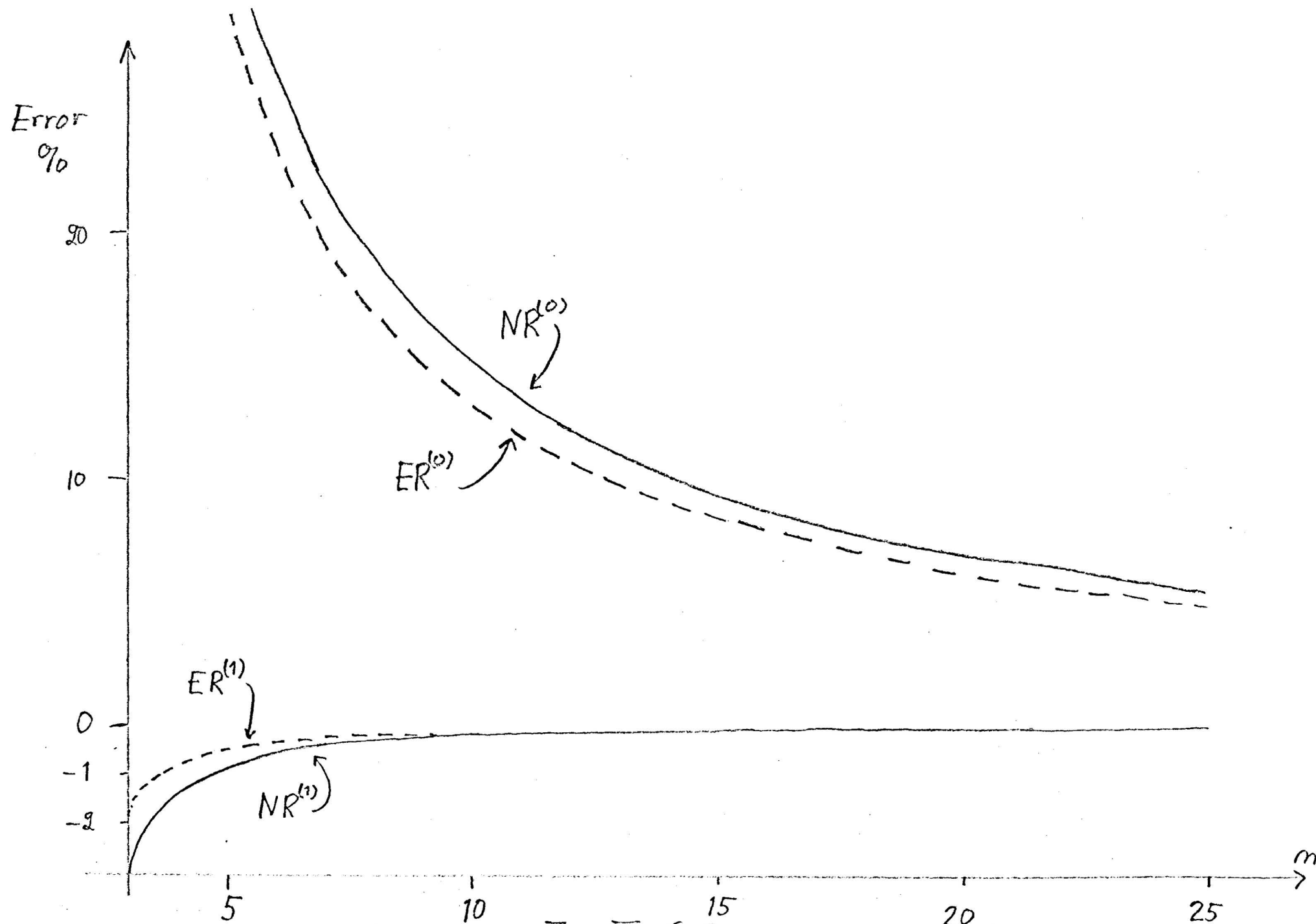


Fig. X.6.2

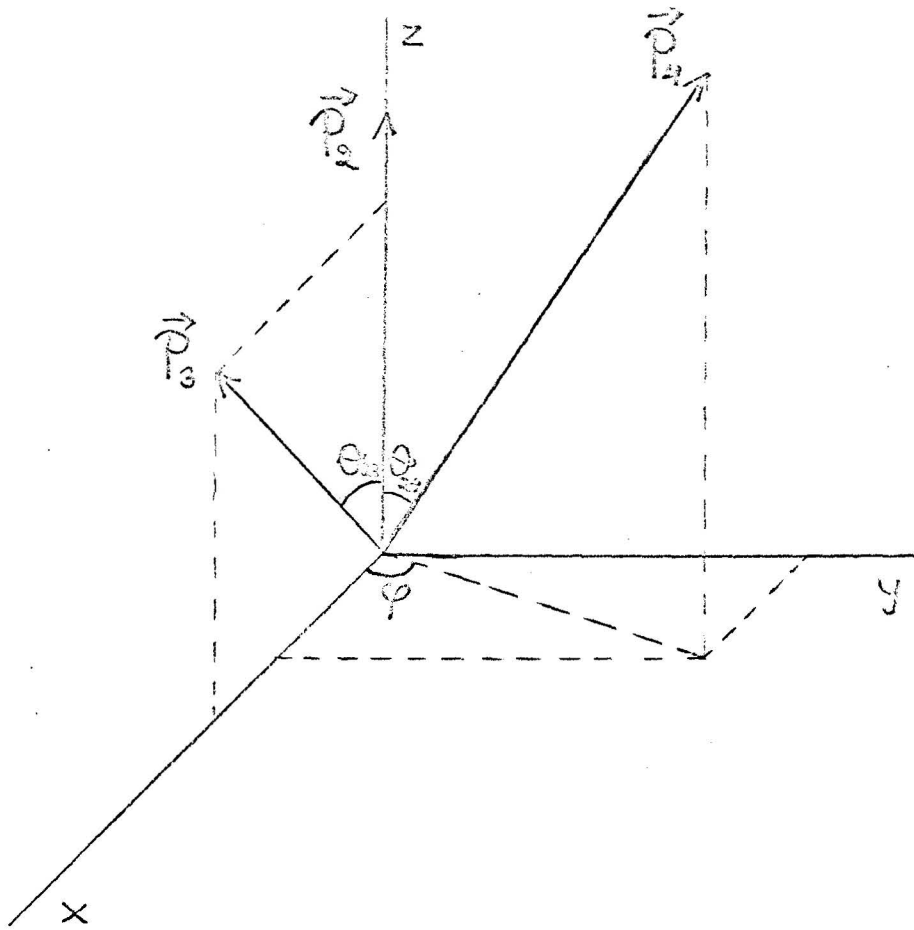


Fig. A.1.

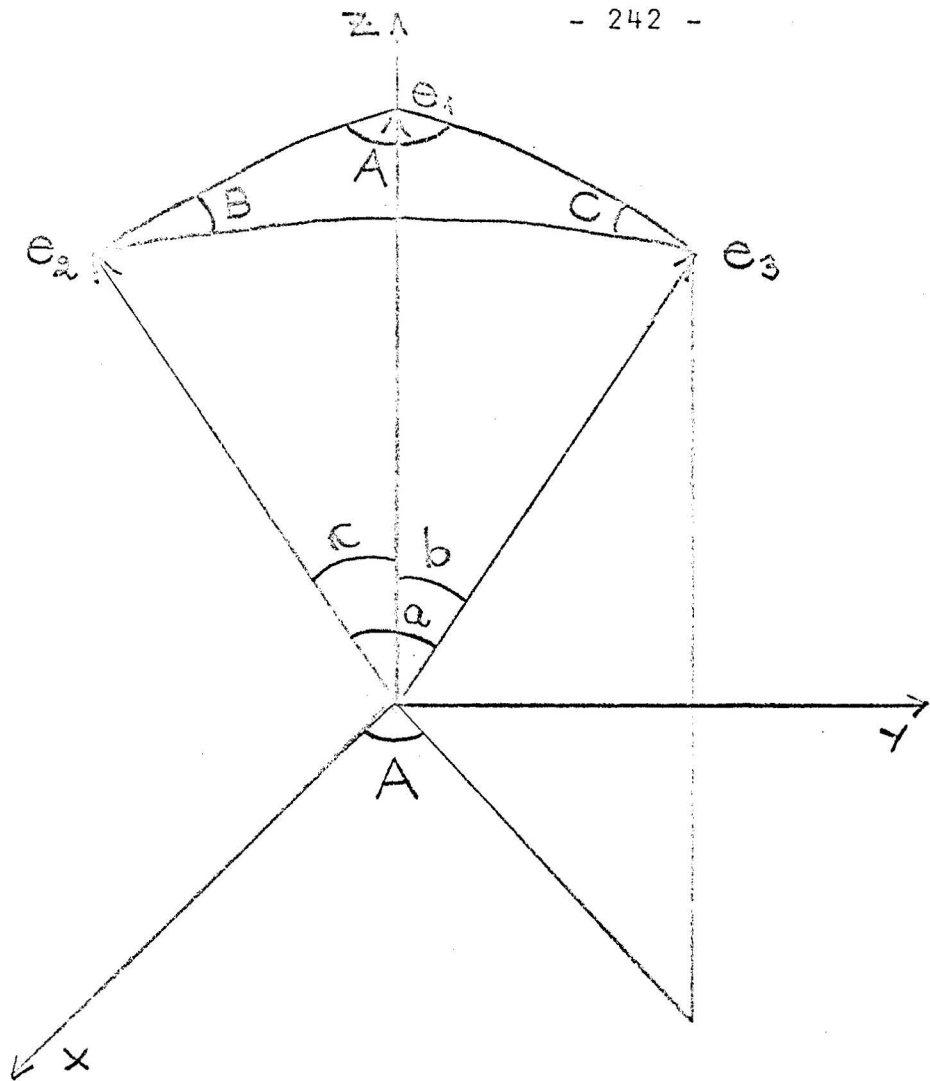


Fig. B.1.

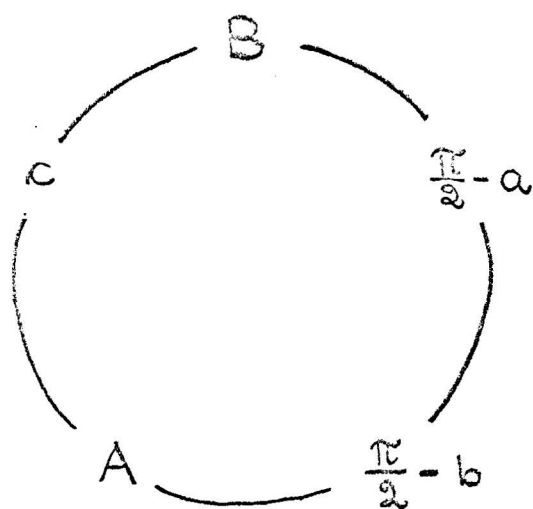


Fig. B.2.

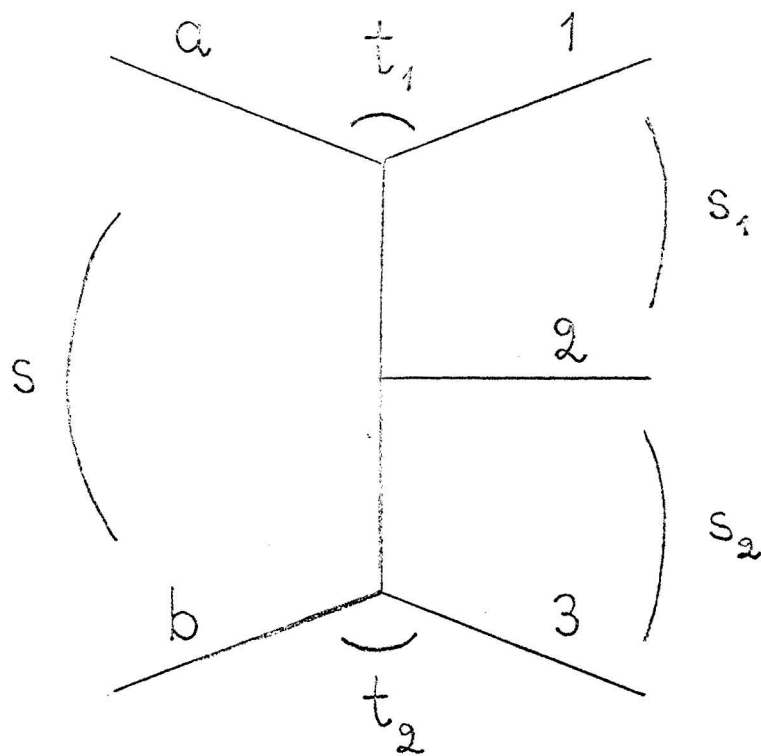


Fig. C. 1.