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A simple algorithm for polarized parton evolution

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We present an algorithm to include the correlation between the production and decay planes of gluons in a parton-shower simulation. The technique is based on identifying the charge currents responsible for the creation and annihilation of the vector field. It is applicable in both the hard-collinear and the soft wide-angle region. As a function of the number of particles, the algorithm scales linear in computing time and memory. We demonstrate agreement with fixed-order perturbative calculations in the relevant kinematical limits, and present a new observable that can be used to probe correlations beyond current-current interactions.

I. INTRODUCTION

Parton showers have been an integral part of the collider physics program over the past four decades [1–3]. They serve as computational tools to implement the evolution of particles from high to low resolution scales, and are essential for modeling the structure of the hadronic jets observed in experiments. They are needed to calibrate the response of particle detectors and infer information about the hard interactions that are used to probe the underlying theory of nature. In the context of previous measurements at the Fermilab Tevatron and the CERN Large Hadron Collider (LHC), the precision of parton showers has traditionally played an important, yet somewhat secondary role. While it was possible to probe leading higher-order corrections, such as the functional form of the scale of running coupling effects [4], and the two-loop effective coupling [5], it was difficult to resolve the precise radiation pattern at the fully differential level, mainly because hadronization effects and detector capabilities were limiting the resolution. This situation might change at the high-luminosity LHC, and is expected to change dramatically at a potential Future Circular Collider (FCC) [6–8]. Despite a small dynamic range, the high statistics at the Tera-Z option of the FCC will allow us to probe parton evolution in great detail. In addition, an improved understanding of the perturbative QCD effects implemented by the parton shower will allow for improved hadronization modeling, which is an important aspect in the determination of the Higgs couplings at high precision. At the LHC, a large dynamic range will allow us to further probe QCD evolution by measuring the detailed substructure of jets at highest energy and highest experimental precision [9–12]. The advances in experimental techniques will generally call for an increased precision in the Monte-Carlo simulation of jet substructure through parton showers.

Among the many aspects of parton shower algorithms [13–63], the possibility to include polarization effects has been one of the earliest under investigation [19–24]. Its importance becomes clear when considering the analogue in classical electrodynamics: To maximize the radio signal received on a dipole antenna, the receiver should be co-polarized with the transmitter [64]. This is because the polarization of the propagating electric field is linear in the case of a transmitting dipole, and the direction is determined by the orientation of the emitter. Such elementary correlations are essential to any gauge theory, hence they should be respected by every simulation program. In recent years, the topic has received renewed interest [59–63]. The commonly employed algorithm to implement polarization effects is the Shatz-Collins-Knowles method [20–24]. It is based on expressing the spin-dependent evolution of a particle in the hard collinear limit through spin-density matrices that are determined from the spin-dependent decay matrix elements. One of the algorithm’s main drawbacks is that it requires modifications for soft wide-angle gluon emission [21, 63]. Here, we propose a different, and much simpler approach that does not have this undesirable feature. Making use of a recently developed technique for the decomposition of the QCD splitting functions [65], we formulate a correlation algorithm in terms of the most basic polarization information in the gluon field, which stems from the orientation of the emitting QCD color dipole. We prove that, in the limit of zero gluon virtuality, this is sufficient to account for all correlations in matrix elements with up to three additional partons, and that corrections in matrix elements with four or more additional partons are strongly sub-leading. These subleading effects are true quantum interferences at $\mathcal{O}(\alpha_s^3)$. Including them in the simulation only becomes necessary once generic 3-loop accuracy must be achieved in the evolution. Based on this knowledge, we propose an observable that probes the co-polarization of the emitting and absorbing QCD dipole antennae. This observable can be used to search for polarization effects beyond those described by splitting functions up to $\mathcal{O}(\alpha_s^2)$.

The manuscript is organized as follows: Section II reviews the computation of the spin-dependent leading-order splitting functions, and derives the correlation algorithm. Section III presents a validation of the implementation in the Alaric parton shower [50] and discusses observable consequences. In Sec. IV we discuss potential future developments.

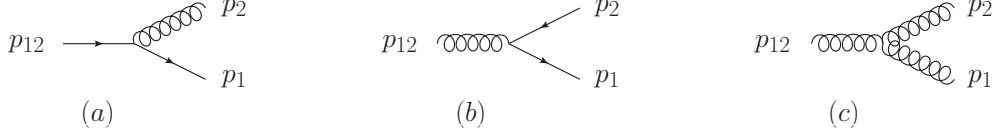


FIG. 1. Diagrams leading to the quark-to-quark-gluon splitting function (a), the gluon-to-quark-antiquark splitting function (b), and the gluon-to-gluon-gluon splitting function (c) at tree level. The labels indicate the particle momenta referred to in the main text.

II. POLARIZED PARTON EVOLUTION

In this section, we derive a polarization correlation algorithm based on the Lorentz structure of the splitting functions that define the branching probabilities for a parton shower. It will be helpful to recompute those splitting functions with a focus on their structure in terms of charge currents [65, 66]. This will be the topic of Sec. II A. In Sec. II B the new expressions will be used to formulate the algorithm for the parton shower simulation.

A. Splitting functions

In order to obtain a compact form of the splitting functions, it is convenient to use a Sudakov decomposition of the momenta [67]. We would, however like to obtain expressions that are sufficiently general to be used with various parton-shower recoil schemes. This can be achieved in terms of a generic decomposition using forward and generalized transverse momenta:

$$p_1^\mu = z_1 p_{12}^\mu - \tilde{p}_{1,2}^\mu, \quad \text{and} \quad p_2^\mu = z_2 p_{12}^\mu + \tilde{p}_{1,2}^\mu. \quad (1)$$

The individual components of this decomposition are given by

$$z_1 = \frac{p_1 \bar{n}}{p_{12} \bar{n}}, \quad z_2 = \frac{p_2 \bar{n}}{p_{12} \bar{n}}, \quad \text{and} \quad \tilde{p}_{1,2}^\mu = z_1 p_2^\mu - z_2 p_1^\mu, \quad (2)$$

where $p_{12} = p_1 + p_2$, and where \bar{n} is a light-like auxiliary vector, which is linearly independent of the collinear direction. In this formulation we can accommodate a broad class of kinematics mappings, two of which are discussed in App. A. In particular, we can show that certain components of the splitting functions take particularly convenient forms in certain recoil schemes.

1. Quark-to-quark splitting function

We begin the discussion with the simple case of the quark splitting function. It can be determined from the 2-particle fermion current, which corresponds to the diagram in Fig. 1(a). Here we consider massless partons only. The result is [65]

$$P_{q \rightarrow q}^{ss'}(p_1, p_2) = \delta^{ss'} \frac{C_F}{p_{12}^2} \frac{\text{Tr}[\not{p}_{12} \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_{12}]}{\text{Tr}[\not{p}_{12}]} d_{\mu\nu}(p_2, \bar{n}) = \delta^{ss'} P_{\bar{q} \rightarrow \bar{q}}(p_1, p_2) + P_{q \rightarrow q}^{ss'(\text{f})}(p_1, p_2), \quad (3)$$

where $p_{12} = p_1 + p_2$, and where the gluon polarization tensor, $d^{\mu\nu}$, has been defined in a light-like axial gauge in order to limit the number of Feynman diagrams contributing in the collinear limit [68–71]

$$d^{\mu\nu}(p, \bar{n}) = -g^{\mu\nu} + \frac{p^\mu \bar{n}^\nu + p^\nu \bar{n}^\mu}{p \bar{n}}. \quad (4)$$

Note that the gauge vector \bar{n}^μ does not have to be the same as the auxiliary vector used to define the splitting kinematics, but we will choose the two to be identical for convenience.

Following Ref. [66], the spin-averaged splitting functions are written as $\langle P_{q \rightarrow X}(1, \dots, m) \rangle = \delta_{ss'} P_{q \rightarrow X}^{ss'}(1, \dots, m)/2$. They are diagonal in the spin of the quark, hence we will only list their spin-averaged form in the following. The individual components are the scalar splitting function and the purely fermionic term derived in Ref. [65]

$$P_{\bar{q} \rightarrow \bar{q}}(p_1, p_2) = C_F \frac{2z_1}{z_2}, \quad \text{and} \quad \langle P_{q \rightarrow q}^{(\text{f})}(p_1, p_2) \rangle = C_F (1 - \varepsilon) z_2. \quad (5)$$

From Eq. (3) it is clear that, at leading power, the quark spin is unaffected by parton evolution in the collinear limit. In the soft limit, the spin information of the radiating particle is always unaffected. In order to implement polarization information in the parton shower, we therefore only need to understand the gluon polarization dependence of Eq. (3). The corresponding splitting tensor is given by

$$\langle P_{q \rightarrow q}^{\mu\nu}(p_1, p_2) \rangle = \frac{C_F}{p_{12}^2} \frac{\text{Tr}[\not{n} \not{p}_{12} \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_{12}]}{\text{Tr}[\not{n} \not{p}_{12}]} = P_{\bar{q} \rightarrow \bar{q}}^{\mu\nu}(p_1, p_2) + \langle P_{q \rightarrow q}^{(f)\mu\nu}(p_1, p_2) \rangle, \quad (6)$$

with the scalar and purely fermionic components

$$P_{\bar{q} \rightarrow \bar{q}}^{\mu\nu}(p_1, p_2) = \frac{C_F}{2} p_{12}^2 S^\mu(p_1, p_2) S^\nu(p_1, p_2), \quad \text{and} \quad \langle P_{q \rightarrow q}^{(f)\mu\nu}(p_1, p_2) \rangle = \frac{C_F}{2} \left[-g^{\mu\nu} z_2 - \frac{p_2^\mu p_2^\nu}{p_{12}^2} \right]. \quad (7)$$

The scalar current, S^μ , plays a central role in our algorithm. It is given by

$$S^\mu(p_1, p_2) = \frac{(2p_1 + p_2)^\mu}{p_{12}^2}. \quad (8)$$

The explicit expressions for vertex factors resulting from this current in specific kinematics are listed in App. A 1.

2. Gluon-to-quark splitting function

The gluon-to-quark splitting function is determined by the Feynman diagram in Fig. 1(b). The algebraic expression is obtained using the methods of Ref. [66], and is given by

$$\begin{aligned} P_{g \rightarrow q}^{\mu\nu}(p_1, p_2) &= \frac{T_R}{2p_{12}^2} d_\rho^\mu(p_{12}, \bar{n}) \text{Tr}[\not{p}_1 \gamma^\rho \not{p}_2 \gamma^\sigma] d_\sigma^\nu(p_{12}, \bar{n}) \\ &= T_R \delta_{ss'} \left[d^{\mu\nu}(p_{12}, \bar{n}) - p_{12}^2 D^\mu(p_1, p_2) D^\nu(p_1, p_2) \right]. \end{aligned} \quad (9)$$

The decay current, D^μ , is defined as

$$D^\mu(p_1, p_2) = d^{\mu\nu}(-p_{12}, \bar{n}) S_\nu(p_1, -p_{12}) = \frac{d_\nu^\mu(p_{12}, \bar{n})}{p_{12}^2} (p_1 - p_2)^\nu. \quad (10)$$

We compute the explicit expressions for certain kinematics mappings in App. A 2. Using a standard Sudakov parametrization [67], taking the collinear limit, and summing over quark spins, we can write Eq. (9) in the familiar form of the spin-dependent DGLAP splitting kernel [65, 66]. However, acknowledging the fact that Eq. (10) describes the absorption of the gluon on a scalar dipole radiator, the structure of the splitting tensor is more obvious in the form of Eq. (9), and we will therefore use this expression.

3. Gluon-to-gluon splitting function

The gluon-to-gluon splitting function is the most complicated of the $1 \rightarrow 2$ splittings, due to the fact that there are three vector particles that interact with each other. We find

$$P_{g \rightarrow g, \mu\nu}^{\rho\sigma, \alpha\beta}(p_1, p_2) = \frac{C_A}{2p_{12}^2} d_{\mu\lambda}(p_{12}, \bar{n}) \Gamma^{\rho\alpha\lambda}(p_1, p_2) \Gamma^{\sigma\beta\tau}(p_1, p_2) d_{\nu\tau}(p_{12}, \bar{n}), \quad (11)$$

where $\Gamma^{\mu\nu\rho}(p, q) = g^{\mu\nu}(p - q)^\rho + g^{\nu\rho}(2q + p)^\mu - g^{\rho\mu}(2p + q)^\nu$ defines the Lorentz structure of the three-gluon vertex. The indices α and β (ρ and σ) refer to the final-state gluon with momentum p_2 (p_1), while the indices μ and ν refer to the initial-state gluon, which carries momentum p_{12} . We separate the resulting splitting tensor into a symmetric and an interference part

$$P_{g \rightarrow g, \mu\nu}^{\rho\sigma, \alpha\beta}(p_1, p_2) = P_{g \rightarrow g, \mu\nu, (s)}^{\rho\sigma, \alpha\beta}(p_1, p_2) + P_{g \rightarrow g, \mu\nu, (i)}^{\rho\sigma, \alpha\beta}(p_1, p_2) + P_{g \rightarrow g, \nu\mu, (i)}^{\sigma\rho, \beta\alpha}(p_1, p_2). \quad (12)$$

The symmetric part reads

$$P_{g \rightarrow g, \mu\nu, (s)}^{\rho\sigma, \alpha\beta}(p_1, p_2) = \frac{C_A}{2} p_{12}^2 \left[S^\alpha(p_1, p_2) S^\beta(p_1, p_2) d_\mu^\rho(p_{12}, \bar{n}) d_\nu^\sigma(p_{12}, \bar{n}) \right. \\ \left. + S^\rho(p_2, p_1) S^\sigma(p_2, p_1) d_\mu^\alpha(p_{12}, \bar{n}) d_\nu^\beta(p_{12}, \bar{n}) \right. \\ \left. + D_\mu(p_1, p_2) D_\nu(p_1, p_2) g^{\rho\alpha} g^{\sigma\beta} \right]. \quad (13)$$

while the asymmetric part is given by

$$P_{g \rightarrow g, \mu\nu, (i)}^{\rho\sigma, \alpha\beta}(p_1, p_2) = \frac{C_A}{2} p_{12}^2 \left[-S^\rho(p_2, p_1) S^\beta(p_1, p_2) d_\mu^\alpha(p_{12}, \bar{n}) d_\nu^\sigma(p_{12}, \bar{n}) \right. \\ \left. + S^\rho(p_2, p_1) D_\nu(p_1, p_2) d_\mu^\alpha(p_{12}, \bar{n}) g^{\sigma\beta} - S^\beta(p_1, p_2) D_\mu(p_1, p_2) d_\nu^\sigma(p_{12}, \bar{n}) g^{\rho\alpha} \right]. \quad (14)$$

We show in App. B that this interference contribution either vanishes identically, or gives a strongly sub-leading correction, which only appears in configurations with four or more gluons in the final state. This can be seen as an extension of the algorithm proposed in Ref. [19] to the case where $\langle \delta P \rangle$ is identically zero. It is well known that four-gluon production also induces complicated sub-leading color structures [46, 72], which have hitherto escaped implementation in standard parton showers. We therefore find it justified to neglect the highly suppressed polarization effects from Eq. (14) in our algorithm and leave them to be implemented by matrix-element matching or amplitude evolution. We note that it is possible to systematically add these corrections, for example at fixed order through existing matching and merging methods [73–97]. However, as they only appear at $\mathcal{O}(\alpha_s^3)$ and higher, it is not necessary to include them in the evolution itself, unless the remainder of the parton shower matches this precision.

To conclude this section, we note that Eqs. (9) and (12) are independent of the kinematics parametrization, as the invariants have not yet been expressed in terms of transverse momenta, and the light-cone momentum fractions are defined unambiguously by Eq. (2).

4. Structure of the result

Comparing Eqs. (6), (9) and (13), we find that most components of the splitting functions can be written as the scalar radiation and decay vertices in Eqs. (8) and (10). The only other components are the fermionic gluon production vertex, $\langle P_{q \rightarrow q}^{(f)\mu\nu} \rangle$, and the $d^{\mu\nu}$ component of the gluon-to-quark splitting function. We therefore find precisely the pattern anticipated in the introductory discussion: Correlations between gluon production and decay are determined by simple dipole-dipole interactions that reflect the enhanced probability to absorb energy from the vector field if the receiving antenna is co-polarized with the emitter. Let us study this pattern in a bit more detail, using the most intricate example of the gluon splitting function. Equation (13) consists of three terms, each of which trivially factorizes into a component acting on the amplitude and one acting on the complex conjugate amplitude:

$$P_{g \rightarrow g, \mu\nu, (s)}^{\rho\sigma, \alpha\beta}(p_1, p_2) = \frac{C_A}{C_F} \left[P_{\bar{q} \rightarrow \bar{q}}^{\alpha\beta}(p_1, p_2) d_\mu^\rho(p_{12}, \bar{n}) d_\nu^\sigma(p_{12}, \bar{n}) + P_{\bar{q} \rightarrow \bar{q}}^{\rho\sigma}(p_2, p_1) d_\mu^\alpha(p_{12}, \bar{n}) d_\nu^\beta(p_{12}, \bar{n}) \right] \\ + \frac{C_A}{2} p_{12}^2 D_\mu(p_1, p_2) D_\nu(p_1, p_2) g^{\rho\alpha} g^{\sigma\beta}. \quad (15)$$

After contraction with $g_{\alpha\beta} g_{\rho\sigma}$ we find that the first (second) term describes the classical emission of gluon two (one) from gluon one (two). The last term describes the absorption of the incoming gluon on the QCD dipole formed by the two outgoing gluons. This indicates that the correct polarization correlations will be implemented by an algorithm based on the following ideas:

- For radiation vertices, described by $P_{\bar{q} \rightarrow \bar{q}}^{\mu\nu}(p_i, p_j)$, store the polarization vector of the radiating QCD antenna. (We will discuss this vector in more detail in Sec. II A 5). Transfer the polarization information for the incoming parton with label (ij) to the outgoing parton with label i .
- For decay vertices, described by $D^\mu(p_i, p_j)$, form the polarization vector, D^μ , and contract it with the polarization vector from the production vertex. Square the result, corresponding to the fact that one correlator exists for the amplitude, and one for the complex conjugate amplitude. Reweight the splitting probability accordingly.
- For fermionic production vertices, do not store polarization information, corresponding to the fact that $\langle P_{q \rightarrow q}^{(f)\mu\nu}(p_i, p_j) \rangle$ consists of a metric tensor, which will implement the trace of the decay tensor, and a term that is power suppressed in the strongly ordered limit.

- For the term proportional to $d^{\mu\nu}$ in the fermionic decay of a gluon, implement the standard spin-averaged component of the splitting function.

5. The gluon production current

In the previous section, we made use of the fact that, in the collinear limit, the polarization vector for the production of a gluon is defined by a single scalar radiation vertex of the form of Eq. (8). The extension beyond the limit is obtained with the help of the techniques in [98, 99]. In this method, the one-gluon current for a QCD multipole reads

$$\mathbf{J}^\mu(\{p\}; q) = ig_s \sum_i \hat{\mathbf{T}}_i S^\mu(p_i, q). \quad (16)$$

Here, $\hat{\mathbf{T}}$, are the charge operators, which are defined as $(\hat{\mathbf{T}}_i^c)_{ab} = T_{ab}^c$ for quarks, $(\hat{\mathbf{T}}_i^c)_{ab} = -T_{ba}^c$ for anti-quarks, and $(\hat{\mathbf{T}}_i^c)_{ab} = if^{acb}$ for gluons. Charge conservation implies that $\sum_i \hat{\mathbf{T}}_i = 0$. If we work in the color-flow representation [100, 101], we can use the relation $2if^{abc} T_{ij}^a T_{kl}^b = T_{kj}^c \delta_{il} - T_{il}^c \delta_{jk}$ to replace the charge operator of an adjoint with the two charge operators of a bi-fundamental. This means simply that each gluon acts as both a QCD charge and an anti-charge. The scalar current can then be written as a double-sum over charge dipoles

$$\mathbf{J}^\mu(\{p\}; q) = ig_s \frac{1}{n_q + n_g} \sum_{i \in q, g} \sum_{k \in \bar{q}, \bar{g}} \left[\hat{\mathbf{T}}_i S^\mu(p_i, q) + \hat{\mathbf{T}}_k S^\mu(p_k, q) \right], \quad (17)$$

where n_q and n_g are the number of quarks and gluons in the multipole, and where g (\bar{g}) in the sums indicates that only the charge (anti-charge) of the gluon is considered in the definition of $\hat{\mathbf{T}}_i$ ($\hat{\mathbf{T}}_k$). In the leading-color approximation, the sums collapse to

$$\mathbf{J}^\mu(\{p\}; q) = ig_s \sum_{i \in q, g} \hat{\mathbf{T}}_i J_{i\bar{i}}^\mu(q) \quad \text{where} \quad J_{i\bar{i}}^\mu(q) = S^\mu(p_i, q) - S^\mu(p_{\bar{i}}, q). \quad (18)$$

Here the index \bar{i} labels the parton which is color-connected to parton i . At leading color, we can thus organize the calculation such that there is always a relative minus sign between the scalar radiator for the charge and the anti-charge. Dipole currents of this form obey the naive Ward identity, and hence we find that the net effect of including the complete QCD multipole in the splitting probability is that $S_\nu(p, q) d^{\mu\nu}(q, \bar{n})$ occurring in the current correlators (see also App. A 1) is replaced by $J_{i\bar{i}}^\mu(q)$ in Eq. (18). This will form the basis for the assignment of polarization vectors in gluon production in our algorithm.

B. The correlation algorithm

Using the ingredients above, we can formulate a Monte-Carlo algorithm that allows us to easily implement polarization correlated evolution into any parton shower that respects color coherence. We assume that parton branching probabilities are separated into their scalar components and collinear remainders. The details of this decomposition can be found, for example in Refs. [50–52]. We note that it can straightforwardly be applied to any type of parton shower algorithm, including dipole showers and antenna showers.

The correlation algorithm can be formulated as follows:

1. If a gluon is emitted using the scalar splitting function, store the polarization vector of the emitting color dipole. For a dipole that is formed by partons i and j , the vector is given by the normalized scalar current¹ $j_p^\mu(p_i, p_j; q) = J_{ij}^\mu(q) / \sqrt{-J_{ij}^\nu(q) J_{ij, \nu}(q)}$. If gluon emission is described by the fermionic component of the $q \rightarrow qg$ splitting function, do not store a polarization vector.
2. If a gluon undergoes a splitting according to the $\tilde{p}_{i,j}$ -dependent part of the $g \rightarrow gg$ or $g \rightarrow q\bar{q}$ splitting function:

¹ Note that we assume massless partons for simplicity. The extension to massive partons is straightforward and only involves replacing the scalar currents with their massive equivalent.

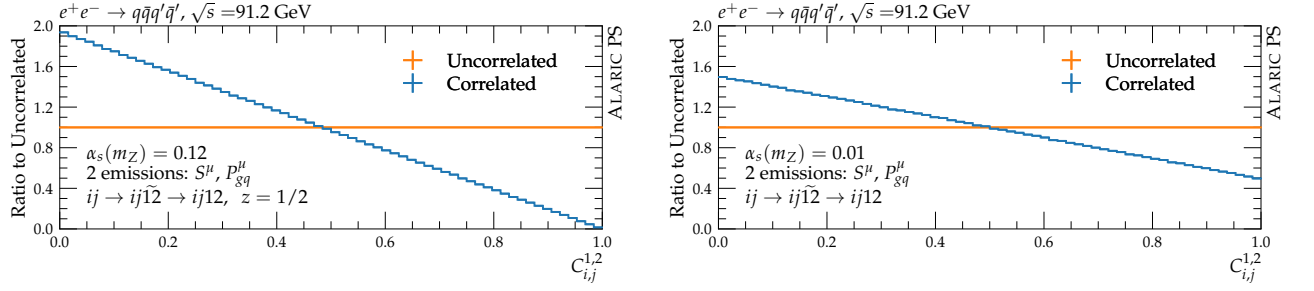


FIG. 2. The current-current correlator $C_{ij}^{1,2}$ for the initial quark pair (labeled i and j) and the emitted quark pair (labeled 1 and 2) in a splitting sequence $ij \rightarrow ij12 \rightarrow ij12$ where the intermediate gluon decays to quarks. Polarization correlated evolution is compared to uncorrelated evolution. The left panel shows the result at $z = 1/2$ in the second splitting. The right panel shows the complete result at $\alpha_s(m_Z) = 0.01$.

- If a polarization vector is defined for this gluon, contract it with the polarization vector for the decay, which is given by $j_d^\mu(p_i, p_j) = \tilde{p}_{i,j}^\mu / \sqrt{-\tilde{p}_{i,j}^\nu \tilde{p}_{i,j}^\nu}$. Reweight the splitting probability accordingly. If the splitting is a $g \rightarrow gg$ transition, mark the two outgoing gluons as correlated, but do not define polarization vectors for their production.
 - If no polarization vector is defined for this gluon, define the polarization vector for its correlated partner as the polarization vector of the decay, which is given by $j_d^\mu(p_i, p_j) = \tilde{p}_{i,j}^\mu / \sqrt{-\tilde{p}_{i,j}^\nu \tilde{p}_{i,j}^\nu}$.
3. If a gluon emits another gluon, transfer the polarization vector and the information on the correlation partner from the incoming particle to the emitter.

We note that this technique is much easier to implement than the Shatz-Collins-Knowles algorithm, yet it provides most of the correlations that can occur in multi-parton amplitudes, including the acausal quantum effects emulated by Step 2. It is straightforward to see that the computing time and memory associated with the evaluation of the correlations is constant for each parton, therefore it scales linearly with the number of particles in the final state. It is the most efficient scaling possible for such an algorithm.

III. VALIDATION AND PHENOMENOLOGY

In this section, we will present the validation of our new algorithm, using some standard observables as well as a dedicated observable which is defined with the help of the quantities in Sec. II B. Let us first introduce this observable. Based on the results in Sec. II A 4, we find that the correlations between production and decay of a gluon are described by the current-current correlator $C_{i,j}^{1,2} = (j_p^\mu(p_i, p_j; p_{12}) j_{d,\mu}(p_1, p_2))^2$, which evaluates to

$$C_{i,j}^{1,2} = \frac{s_{i12}s_{j12}}{4s_{ij}s_{12}} \frac{(z_1 + z_2)^2}{z_1 z_2} \left[1 + \frac{s_{12}}{s_{ij}} \frac{(s_{i1} - s_{j1} + s_{i2} - s_{j2})^2}{s_{i12}s_{j12}} \right]^{-1} \left(\frac{s_{i1} - s_{i2}}{2s_{i12}} - \frac{s_{j1} - s_{j2}}{2s_{j12}} \right)^2. \quad (19)$$

In the double-soft limit, $p_1 \rightarrow \lambda p_1$, $p_2 \rightarrow \lambda p_2$, $\lambda \rightarrow 0$, this reduces to the azimuthal angle defined in Ref. [55], while in the collinear limit, it reduces to the azimuthal angle of the squeezed correlator limit defined in Ref. [102].² We therefore propose $C_{i,j}^{1,2}$ as an observable to test polarization effects. While Eq. (19) is difficult to implement experimentally, it provides a theoretical tool to cleanly assess the various possible correlations that occur in gluon production and subsequent decay processes.

Our tests will be performed with the Alaric final-state parton shower [50], using the separation of kinematics mappings detailed in Ref. [51]³. As discussed in App. A, in particular Sec. A 2, the splitting kinematics used for the collinear remainder functions is particularly suited to describe the polarization effects in gluon decays. We set $C_F = (N_c^2 - 1)/(2N_c) = 4/3$ and $C_A = 3$. All quarks are considered massless. The running coupling is evaluated at two loop accuracy. We investigate $e^+e^- \rightarrow q\bar{q}$ at a center-of-mass energy of $\sqrt{s} = 91.2$ GeV.

² Ref. [102] claims that the angular correlation pattern in the squeezed limit of the energy correlator provides a probe of the quantum structure of jet evolution. The discussion in Sec. II and App. B shows that this claim only holds if one is able to resolve the impact of the finite remainder of higher-order virtual corrections, or the interference contributions of gluon splittings at $\mathcal{O}(\alpha_s^3)$ and beyond. The remaining tree-level effects are of classical nature.

³ The PyPy code for these tests can be found at <https://gitlab.com/shoeche/pyalaric/-/tree/pol>.

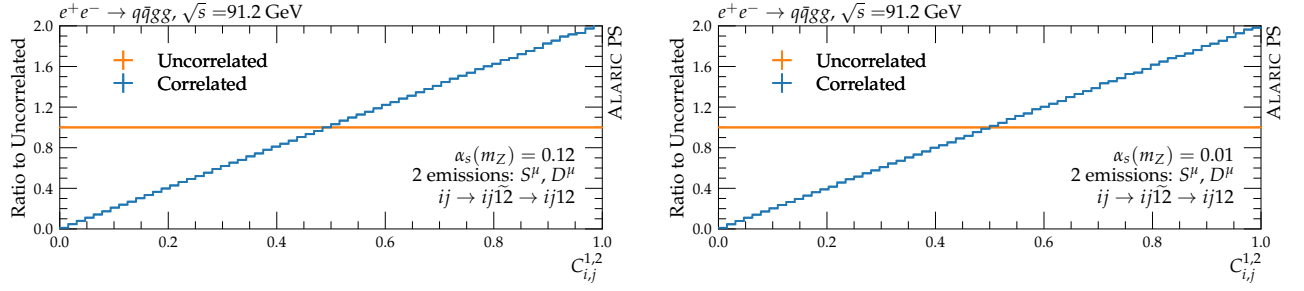


FIG. 3. The current-current correlator $C_{i,j}^{1,2}$ for the initial quark pair (labeled i and j) and the first two gluons (labeled 1 and 2) in a splitting sequence $ij \rightarrow ij\widetilde{12} \rightarrow ij12$. Polarization correlated evolution is compared to uncorrelated evolution, and the second splitting is restricted to a decay, described by Eq. (10). We set $\alpha_s(m_Z) = 0.12$ (left) and $\alpha_s(m_Z) = 0.01$ (right).

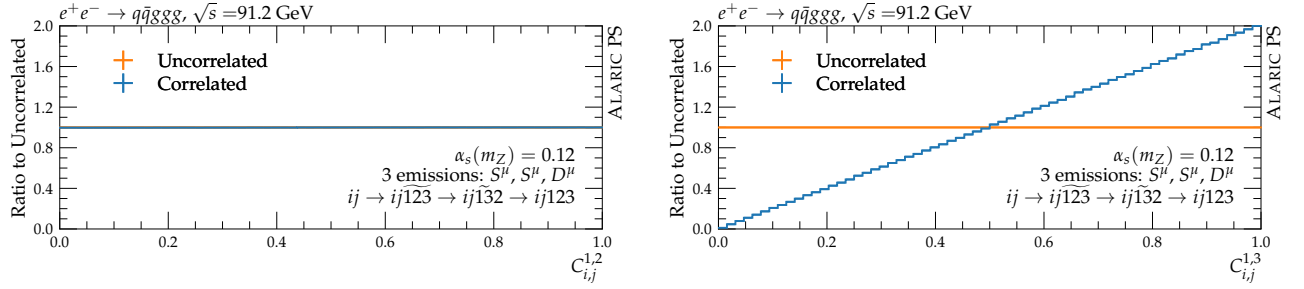


FIG. 4. The current-current correlators $C_{i,j}^{1,2}$ (left) and $C_{i,j}^{1,3}$ (right) in a splitting sequence $ij \rightarrow ij\widetilde{123} \rightarrow ij\widetilde{132} \rightarrow ij123$. Polarization correlated evolution is compared to uncorrelated evolution, and the second splitting is restricted to scalar radiation, described by Eq. (8), while the third is restricted to a decay, described by Eq. (10).

Figure 2 shows the current-current correlator $C_{i,j}^{1,2}$ for the initial quark pair (labeled i and j) and an emitted quark pair (labeled 1 and 2). The left panel shows the effect of polarized evolution for $z = 1/2$ in the second splitting. At this special kinematical point, polarization effects on the gluon-to-quark splitting function, Eq. (9), are maximal. The right panel shows the result for all values of z at $\alpha_s = 0.01$, where kinematical edge effects are highly suppressed, which otherwise affect the admixture of polarization-dependent and polarization-independent terms. In this case, the effect of the polarized evolution is reduced to 50%, in agreement with Eq. (9).

Figure 3 shows the current-current correlator $C_{i,j}^{1,2}$ for the initial quark pair (labeled i and j) and the first two gluons (labeled 1 and 2). We limit the evolution to only two emissions, in order to obtain a clean identification of the individual partons. The splitting sequence is $ij \rightarrow ij\widetilde{12} \rightarrow ij12$. We find the expected radiation pattern, i.e. that the polarization correlated evolution generates a linear dependence on $C_{i,j}^{1,2}$. The comparison between the left and the right figure shows that using the current correlator as an observable, it is not necessary to take the $\alpha_s \rightarrow 0$ limit conventionally used to assess the correctness of the parton shower in the region of vanishing recoil.

Figure 4 shows the two current-current correlators $C_{i,j}^{1,2}$ and $C_{i,j}^{1,3}$ in a configuration where we restrict the parton shower to three emissions, and we constrain the second branching to be a scalar emission, while the third is a decay. The splitting sequence is $ij \rightarrow ij\widetilde{123} \rightarrow ij\widetilde{132} \rightarrow ij123$. As expected, the prediction of $C_{i,j}^{1,2}$ in this case is flat, even in the correlated evolution algorithm, while $C_{i,j}^{1,3}$ has the behavior induced by $(J_{ij}^\mu(p_{13})D_\mu(p_1, p_3))^2$. This validates in particular the correct implementation of Step 3 of the correlation algorithm in Sec. II B. Note that small deviations from the ideal pattern can occur due to kinematical effects that stem from the radiation of the second gluon off the first. However, these effects are practically invisible for only one intermediate emission, even at $\alpha_s(m_Z) = 0.12$.

As a final test of the correct implementation of the correlation algorithm, we investigate in Fig. 5 the current-current correlators $C_{1,3}^{2,4}$ and $C_{i,j}^{1,2}$ in a configuration where we restrict the number of splittings to four, and constrain the second, third and fourth branching to the scalar decay in Eq. (8) in such a way that the splitting sequence is given by $ij \rightarrow ij\widetilde{1234} \rightarrow ij\widetilde{1324} \rightarrow ij1234 \rightarrow ij1234$. We find that the correlation between the decay planes of the two intermediate gluons 13 and 24 is described nearly perfectly by $(D_\mu(p_1, p_3)D^\mu(p_2, p_4))^2$, even for a standard value of α_s . This validates the correct implementation of both aspects of Step 2 of the correlation algorithm in Sec. II B. It is interesting to note that $C_{i,j}^{1,2}$ still shows a strong correlation of gluons 1 and 2, which stems from the initial gluon

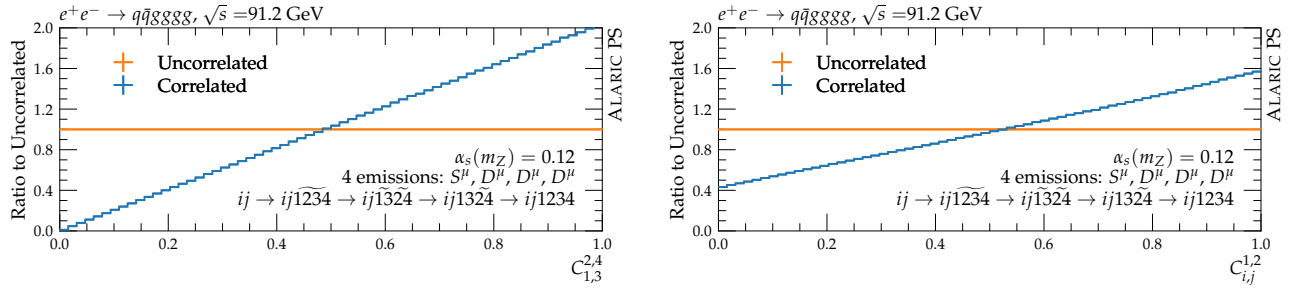


FIG. 5. The current-current correlators $C_{1,3}^{2,4}$ (left) and $C_{i,j}^{1,2}$ (right) in events with four branchings, where the second, third and fourth splitting are restricted to a decay described by Eq. (10), and we enforce the splitting sequence $ij \rightarrow ij\widetilde{1234} \rightarrow ij\widetilde{1324} \rightarrow ij\widetilde{1324} \rightarrow ij\widetilde{1234}$. Decay probabilities are enhanced by a factor 100 to increase statistics.

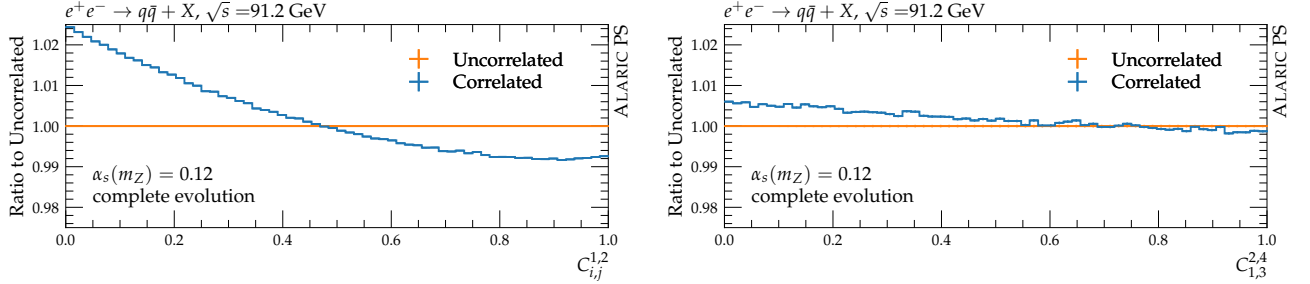


FIG. 6. The current-current correlators $C_{i,j}^{1,2}$ (left) and $C_{1,3}^{2,4}$ (right) in events generated with the full parton-shower evolution.

decay, $ij\widetilde{1234} \rightarrow ij\widetilde{1324}$.

Figure 6 shows the current-current correlators $C_{i,j}^{1,2}$ and $C_{1,3}^{2,4}$ in events with the complete parton-shower evolution. The large correlation effects observed in Figs. 2 and 3 are strongly suppressed in this case, due to the much larger branching probability for scalar emissions. This effect has previously been observed [55]. It makes the measurement of polarization effects in jet evolution very challenging. Note in particular that the overall scale of correlation effects is at the same level as the expected perturbative scale uncertainties, and the hadronization uncertainties in a typical simulation at LEP energies. This was also pointed out in Ref. [19].

In Fig. 7 we show the rapidity dependence of the azimuthal radiation pattern that was investigated in Ref. [63]. We limit the evolution to two emissions. The azimuthal angle ψ_{12} is defined as the angle between the production and decay planes of the intermediate gluon generated in the first emission, using the emitting quark and the more energetic of the gluon decay products to define the leading directions. In the left panels, we show the radiation pattern generated by the parton shower in $e^+e^- \rightarrow q\bar{q}q'\bar{q}'$, on the right we show the radiation pattern in $e^+e^- \rightarrow q\bar{q}gg$, but only including the scalar gluon production current, and only the decay component on the third line of Eq. (13). In the lower left panel, we compare the parton-shower result to Eq. (B.6) of Ref. [103]. In the lower right panel, we compare it to $(J_{ij}^\mu(p_{12})D_\mu(p_1, p_2))^2$, where p_1 and p_2 are the momenta of the two gluons, and p_i and p_j are the momenta of the quarks. In both cases we find very good agreement.

In Ref. [63] it was observed that, depending on the precise implementation of the parton shower, the Shatz-Collins-Knowles algorithm for spin correlations may not capture the required azimuthal modulation at small rapidity. We observe no such effects in Fig. 7, confirming that our new algorithm respects the polarization information, independent of other phase-space variables. For this, it is important that the information on the production current is appropriately Lorentz transformed during any subsequent splitting.

IV. OUTLOOK

For dipole antennae, the angular correlation between emitter and receiver determines the power transfer through electromagnetic waves. Being a consequence of Maxwell's equations, the effect is omnipresent in gauge theories and also determines certain angular correlations that occur in QED and QCD particle evolution during highly energetic reactions at collider experiments. These polarization effects should be respected in any simulation of high-energy collisions by Monte-Carlo event generators. In this manuscript, we have proposed a novel algorithm to include

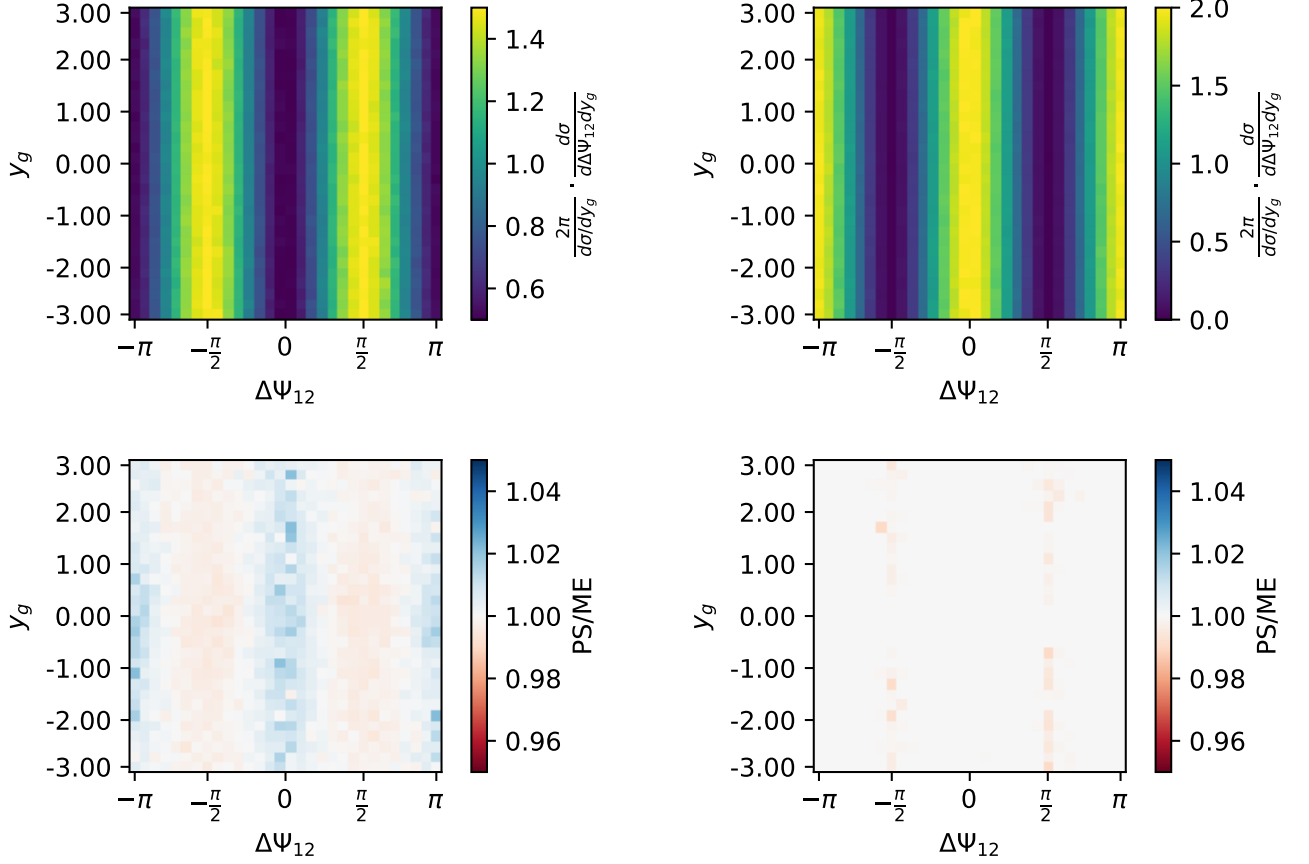


FIG. 7. Rapidity dependence of the azimuthal radiation pattern as defined in Ref. [63] at $\alpha_s = 0.01$. Left: $e^+e^- \rightarrow q\bar{q}q'\bar{q}'$, compared to Eq. (B.6) of Ref. [103]. Right: Decay component of $e^+e^- \rightarrow q\bar{q}gg$ compared to the product of Eqs. (18) and (10). See the main text for details.

polarization information in parton-showers. The method scales linearly in computing time and memory, and is thus one of the most efficient techniques available. Its true advantage does, however lie in its straightforward implementation and interpretation, and in the fact that the results provide a new observable to probe correlations beyond the angular radiation patterns typically tested in the context of polarized parton evolution. In the future, we will investigate the connection to the helicity formalism, and develop the matching to existing perturbative calculations for LHC and FCC physics.

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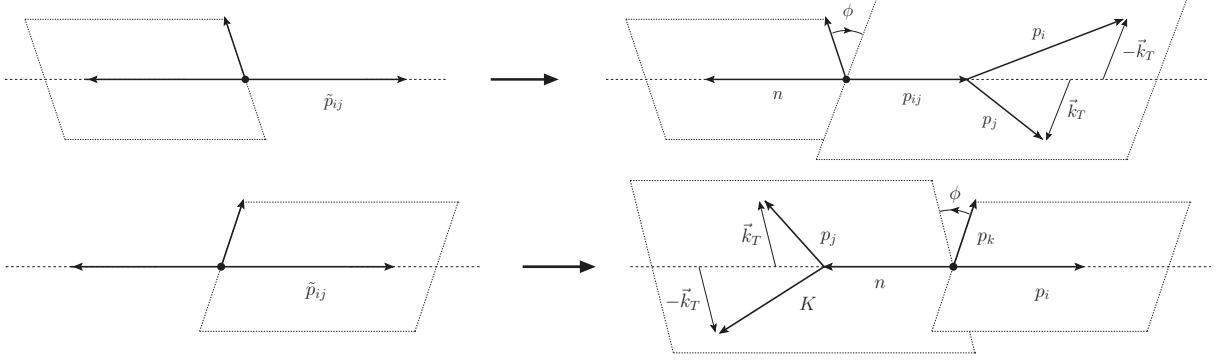


FIG. 8. Top: The splitting kinematics, described by Eq. (A1) and the choice $\alpha = 1$. The momentum n takes the recoil in the splitting, and the momentum p_k defines a reference direction for the azimuthal angle. Bottom: The radiation kinematics, described by Eq. (A1) and the choice $\alpha = 0$. The momentum K takes the recoil in the splitting, and the momentum p_k defines a reference direction for the azimuthal angle.

Appendix A: Kinematics

In order to obtain a concrete form of the splitting functions, one must define a Sudakov decomposition of the final-state momenta. Ideally, this parametrization will be designed to reflect the dynamics of the radiation and decay vertices. A generic decomposition in terms of forward and backward light-cone momenta is achieved as follows:

$$p_i^\mu = z_i \tilde{p}_{ij}^\mu + \frac{\alpha^2 k_\perp^2 + p_i^2}{z_i 2\tilde{p}_{ij}\bar{n}} \bar{n}^\mu - \alpha k_\perp^\mu, \quad \text{and} \quad p_j^\mu = z_j \tilde{p}_{ij}^\mu + \frac{k_\perp^2 + p_j^2}{z_j 2\tilde{p}_{ij}\bar{n}} \bar{n}^\mu + k_\perp^\mu, \quad (\text{A1})$$

where $k_\perp \tilde{p}_{ij} = k_\perp \bar{n} = 0$ and $k_\perp^2 = -k_\perp^2$. The light-cone momentum fractions satisfy $z_i + z_j = 1$ and are given by Eq. (2). The parameter α depends on the type of splitting: $\alpha = 1$ corresponds to k_\perp -symmetric splitting kinematics where final states are treated on equal footing, while $\alpha = 0$ corresponds to radiation kinematics, where particle i is identified. The two cases are sketched in Fig. 8. The vector n shown in the figure is related to \bar{n} through an on-shell shift. We define the forward light-cone direction as

$$\tilde{p}_{ij}^\mu = \frac{p_i^\mu + \alpha p_j^\mu}{z_i + \alpha z_j} - \frac{(p_i + \alpha p_j)^2}{z_i + \alpha z_j} \frac{\bar{n}^\mu}{2(p_i + \alpha p_j)\bar{n}} = p_{ij}^\mu - \frac{1 - \alpha}{z_i + \alpha z_j} \tilde{p}_{i,j}^\mu - \frac{(p_i + \alpha p_j)^2}{z_i + \alpha z_j} \frac{\bar{n}^\mu}{2(p_i + \alpha p_j)\bar{n}}, \quad (\text{A2})$$

The transverse three-momentum squared in a frame where \tilde{p}_{ij} and \bar{n} are back-to-back is related to the invariants by

$$k_\perp^2 = \frac{z_i z_j p_{ij}^2 \kappa_{i,j}^2}{(z_i + \alpha z_j)^2}, \quad \text{where} \quad \kappa_{i,j}^2 = 1 - \frac{z_j p_i^2 + z_i p_j^2}{z_i z_j p_{ij}^2}. \quad (\text{A3})$$

1. Radiation off a scalar

Here we compute the vertex factor associated with gluon radiation off the scalar vertex defined in Eq. (8). In the collinear limit, the produced gluon is associated with a polarization tensor that reflects either its propagation, or the sum over final-state helicities. The general expression for the product of a scalar emission vertex and this polarization sum is given by

$$S^\rho(p_i, p_j) d_\rho^\mu(p_j, \bar{n}) = \frac{1}{p_{ij}^2} \frac{2}{z_j} \left[\tilde{p}_{i,j}^\mu + (p_{ij}^2 - p_i^2) \frac{\bar{n}^\mu}{2p_{ij}\bar{n}} \right]. \quad (\text{A4})$$

It is interesting to note that the soft enhancement $1/z_j$ already appears at vertex level, which is a consequence of the gluon propagator in axial gauge, Eq. (4). Using the Sudakov decomposition, Eq. (A1) we find

$$S^\rho(p_i, p_j) d_\rho^\mu(p_j, \bar{n}) = \frac{1}{p_{ij}^2} \frac{2}{z_j} \left[(z_i + \alpha z_j) k_\perp^\mu + \left(\frac{z_i + \alpha z_j}{z_j} k_\perp^2 + \frac{1 + z_i}{2z_j} p_j^2 \right) \frac{\bar{n}^\mu}{p_{ij}\bar{n}} \right]. \quad (\text{A5})$$

In radiation kinematics, $\alpha = 0$, we obtain

$$S^\rho(p_i, p_j) d_\rho^\mu(p_j, \bar{n}) = \frac{1}{p_{ij}^2} \frac{2z_i}{z_j} \left[k_\perp^\mu + \left(k_\perp^2 + \frac{1+z_i}{2z_j} p_j^2 \right) \frac{\bar{n}^\mu}{p_j \bar{n}} \right]. \quad (\text{A6})$$

In the collinear limit, $k_\perp \rightarrow \lambda k_\perp$, and for massless final states, this is the expected polarized splitting amplitude

$$\lambda S^\rho(p_i, p_j) d_\rho^\mu(p_j, \bar{n}) \rightarrow 2 \sqrt{\frac{1}{p_{ij}^2} \frac{z_i}{z_j}} \frac{k_\perp^\mu}{k_\perp} + \mathcal{O}(\lambda). \quad (\text{A7})$$

2. Vector decay to scalars

Here we compute the vertex factor associated with the gluon decay in Eq. (10). We find the general expression

$$D^\mu(p_i, p_j) = \frac{2}{p_{ij}^2} \left[\tilde{p}_{i,j}^\mu + (p_i^2 - p_j^2) \frac{\bar{n}^\mu}{2p_{ij} \bar{n}} \right]. \quad (\text{A8})$$

Using the Sudakov decomposition in Eq. (A1), we obtain

$$D^\mu(p_i, p_j) = \frac{2}{p_{ij}^2} \left[(z_i + \alpha z_j) k_\perp^\mu + \left(\frac{z_i - \alpha z_j}{z_i + \alpha z_j} p_{ij}^2 - \frac{2(1-\alpha)}{z_i + \alpha z_j} (z_j p_i^2 + z_i p_j^2) \right) \frac{\bar{n}^\mu}{2p_{ij} \bar{n}} \right] \quad (\text{A9})$$

In splitting kinematics, $\alpha = 1$, Eq. (A9) reduces to an expression, which reflects the symmetry of the process

$$D^\mu(p_i, p_j) = \frac{2}{p_{ij}^2} \left[k_\perp^\mu + (z_i - z_j) \frac{p_{ij}^2}{2p_{ij} \bar{n}} \bar{n}^\mu \right]. \quad (\text{A10})$$

In the collinear limit, $k_\perp \rightarrow \lambda k_\perp$, and for massless final states, this is the expected polarized splitting amplitude

$$\lambda D^\mu(p_i, p_j) \rightarrow 2 \sqrt{\frac{z_i z_j}{p_{ij}^2}} \frac{k_\perp^\mu}{k_\perp} + \mathcal{O}(\lambda). \quad (\text{A11})$$

3. Polarization transfer

For gluon splitting functions we also need to understand how the polarization of the incoming gluon is affected when radiating a new gluon. The relevant expressions are given by the product of two polarization tensors with different momenta:

$$d^{\mu\rho}(p_{ij}, \bar{n}) d_\rho^\nu(p_i, \bar{n}) = -d^{\mu\nu}(p_{ij}, \bar{n}) + \left[\tilde{p}_{i,j}^\mu + (p_{ij}^2 + p_i^2 - p_j^2) \frac{\bar{n}^\mu}{2p_{ij} \bar{n}} \right] \frac{\bar{n}^\nu}{p_i \bar{n}}, \quad (\text{A12})$$

In the collinear limit, $k_\perp \rightarrow \lambda k_\perp$, and for massless final states, this simply gives the original polarization tensor

$$d^{\mu\rho}(p_{ij}, \bar{n}) d_\rho^\nu(p_i, \bar{n}) \rightarrow -d^{\mu\nu}(p_{ij}, \bar{n}) + \mathcal{O}(\lambda). \quad (\text{A13})$$

Appendix B: Interference contribution to the gluon splitting tensor

In order to form a complete amplitude for the production and evolution of a gluon, the splitting tensor in Eq. (11) must eventually be contracted with other Lorentz structures involving the polarization tensors of the gluons with momentum p_1 and p_2 . In the strongly ordered soft and collinear limit, and at leading power, these tensors become projectors, cf. App. A 3. Replacing $d^{\mu\nu}(p, \bar{n}) \rightarrow d^{\mu\rho}(p, \bar{n}) d_\rho^\nu(p, \bar{n})$, for each gluon, we can then associate one polarization tensor per gluon with Eq. (11). To simplify the corresponding result, we note that the vertex factors in Eq. (8) and (10) give

$$S_\nu(p_i, p_j) d^{\mu\nu}(p_j, \bar{n}) = \frac{2}{z_j} \tilde{p}_{i,j}^\mu + \dots, \quad D^\mu(p_i, p_j) = 2 \tilde{p}_{i,j}^\mu + \dots, \quad (\text{B1})$$

where the ellipses represent terms proportional to \bar{n}^μ that vanish upon multiplication with $d_\mu^\rho(p, \bar{n})$, independent of p . We use these identities to simplify Eq. (14) as follows

$$\begin{aligned} \tilde{P}_{g \rightarrow g, \mu\nu, (i)}^{\rho\sigma, \alpha\beta}(p_1, p_2) &= P_{g \rightarrow g, \mu\nu, (i)}^{\rho'\sigma', \alpha'\beta'}(p_1, p_2) d^{\alpha\alpha'}(p_2, \bar{n}) d^{\beta\beta'}(p_2, \bar{n}) d^{\rho\rho'}(p_1, \bar{n}) d^{\sigma\sigma'}(p_1, \bar{n}) \\ &\approx 2C_A p_{12}^2 \left[\tilde{p}_{1,2}^\rho d_\mu^\alpha(p_{12}, \bar{n}) \left(\frac{1}{z_1 z_2} \tilde{p}_{1,2}^\beta d_\nu^\sigma(p_{12}, \bar{n}) - \frac{1}{z_1} \tilde{p}_{1,2\nu} d^{\sigma\beta}(p_{12}, \bar{n}) \right) \right. \\ &\quad \left. - \frac{1}{z_2} \tilde{p}_{1,2}^\beta \tilde{p}_{1,2\mu} d_\nu^\sigma(p_{12}, \bar{n}) d^{\rho\alpha}(p_{12}, \bar{n}) \right]. \end{aligned} \quad (B2)$$

Let us now investigate the case where one of the final-state gluons does not decay. This means in particular that the gluon can undergo subsequent evolution, as long as it does not involve the current in Eq. (10), either in the amplitude, or its complex conjugate. Due to Eq. (4) being a projector in the strongly ordered limit, this corresponds to the trace

$$g_{\rho\sigma} \tilde{P}_{g \rightarrow g, \mu\nu, (i)}^{\rho\sigma, \alpha\beta}(p_1, p_2) \approx 2C_A p_{12}^2 \frac{1}{z_2} \left[\tilde{p}_{1,2\nu} d_\mu^\alpha(p_{12}, \bar{n}) - \tilde{p}_{1,2\mu} d_\nu^\alpha(p_{12}, \bar{n}) \right] \tilde{p}_{1,2}^\beta. \quad (B3)$$

We can use the fact that any physical gluon production tensor must be symmetric in the Lorentz indices μ and ν . This allows us to deduce that, in this case, the interference contribution vanishes at leading power [104]. Because the triple-gluon vertex is crossing invariant, the same is also true if the trace is taken over the Lorentz indices of the incoming gluon.

The only possibility for Eq. (B2) to contribute to the matrix element in the strongly ordered soft or collinear limit is therefore through a combination with D^μ , either in the amplitude or its conjugate. Note that both outgoing gluons must be combined with this tensor, otherwise the result will again correspond to the trace in Eq. (B3). This requires an initial QCD dipole plus four or more gluons or quarks / antiquarks. Equation (B1) as well as Eqs. (A11) and (A7) show that configurations which involve D^μ are not soft enhanced, and are therefore highly suppressed compared to the leading terms in the parton-shower approximation.

Neglecting $\tilde{P}_{g \rightarrow g, \mu\nu, (i)}^{\rho\sigma, \alpha\beta}$ in Eq. (12) can be seen as a variant of the factorization performed in Ref. [19], which also stated that the non-factorizing components of the gluon splitting function represent an acausal quantum correlation that cannot be accommodated in a classical branching algorithm. However, Ref. [19] is not based on the same form of the gluon splitting that we propose in Eq. (13), and their remainder function still includes a factorizable component. While this term is a quantum correction and can therefore not be described by a local hidden variable theory, it can still be simulated using Step 2 of our algorithm in Sec. II B. Using this improved proposal, Eq. (6) of Ref. [19] would vanish identically, which further reduces the impact of the unaccounted for quantum effects on observables.

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