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A New-Old Approach to Composite Scalars with Chiral Fermion Constituents

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We develop a dynamical, Lorentz invariant theory of composite scalars in configuration space consisting of chiral fermions, interacting by the perturbative exchange of a massive "gluon" of coupling g_0 and mass M_0^2 (the coloron model). The formalism is inspired by, but goes beyond, old ideas of Yukawa and the Nambu-Jona-Lasinio (NJL) model. It yields a non-pointlike internal wave-function of the bound state, $\phi(r)$, which satisfies a Schrödinger-Klein-Gordon (SKG) equation with eigenvalue μ^2. For super-critical coupling, $g_0 > g_{0c}$, we have $\mu^2 < 0$ leading to spontaneous symmetry breaking. The binding of chiral fermions is semiclassical, *not loop-level as in NJL* . The mass scale is determined by the interaction as in NJL. We mainly focus on the short-distance, large M_0^2 limit, yielding an NJL pointlike interaction, but the bound state internal wave-function, $\phi(\vec{r})$, remains spatially extended and dilutes $\phi(0)$. This leads to power-law suppression of the induced Yukawa and quartic couplings and requires radically less fine-tuning of a hierarchy than does the NJL model. We include a discussion of loop corrections of the theory. A realistic top condensation model appears possible.

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I. INTRODUCTION

There are many lines of investigation of composite models of bound states of chiral fermions that can be treated more-or-less analytically, including [1–5] to name a few. However, the Nambu–Jona-Lasinio model (NJL) [5] stands out as a concise and useful Lorentz invariant description of dynamical bound states of relativistic chiral fermions in quantum field theory.

The NJL model is fairly easy to implement, tying an underlying chirally invariant fermionic action to composite scalars and dynamical symmetry breaking. It operates at the quantum loop level, $\mathcal{O}(\hbar)$, and is most readily solved by using the renormalization group (RG) [6–9], and shares features with BCS superconductivity in the large N_c (fermion color) limit [10]. A large literature exists of successful applications to the chiral dynamics of QCD, [11–13], and it provided the first models of a composite Brout-Englert-Higgs (BEH) boson and related phenomena [8, 14–17].

There are, however, fundamental physical limitations of the NJL model: (1) the NJL model is an effective pointlike 4-fermion interaction associated with a "large" mass scale M_0; (2) the resulting bound states emerge as *pointlike* fields with mass $\mu^2 < M_0^2$; (3) in the NJL model the binding mechanism is entirely driven by quantum loop effects, while we see in nature that binding readily occurs semiclassically without quantum loops, such as the Hydrogen atom.

In the case of the Hydrogen atom, before turning on the Coulomb interaction, there are open scattering states involving free protons and electrons. As the interaction is adiabatically turned on the lowest energy scattering states flow to become the bound states, while most scattering states remain unbound. The dynamics is governed by the non-relativisitic, semiclassical Schrödinger equation [18] in an extended potential, leading to normalizable, yet spatially extended, wave-functions on the scale $(\alpha m_e)^{-1}$. The atom is described naturally in a configuration space picture. Quantum loop effects (such as the Lamb shift) are higher order corrections to this mostly semiclassical phenomenon.

In the NJL model the picture is substantially different. There is no semiclassical binding producing an extended bound state. Rather, the bound state is described by a *local, or pointlike,* effective field, $\Phi(x)$, with its constituents arising in quantum loops. The loops integrate out the fermions from the large mass scale of the interaction, M_0, down to an IR cut-off μ (e.g., $M_0 \sim 1$ GeV and $\mu \sim f_\pi \sim 100$ MeV in QCD). The discussion can be formulated in momentum space treated in the large N_{color} limit, where bound states appear as poles in the S-matrix upon summing leading-N_c fermion loop diagrams. With a large hierarchy $M_0/\mu \gg 1$ there are also large logarithms, and the solution is best handled by constructing the effective Lagrangian and using the renormalization group (RG). We summarize this procedure for the NJL model in Appendix A.

When we say "pointlike" we are referring to the explicit local form of $\Phi(x)$. In the Wilsonian RG picture the field at any scale $m < M_0$ can be viewed as *effectively* pointlike, on distance scales $r > m^{-1}$. However, $\Phi(x)$ has only the minimal dynamical degrees of freedom of a pointlike field. The NJL model is well suited to study the chiral dynamics of light quarks in QCD where "integrating out the fermions" mimics confinement and there are no free scattering states of massless chiral quarks at large distances. Any analogy with the Hydrogen atom does not apply.

In the case of chiral dynamics in a *non-confining theory*, however, one may ask what has become of semiclas-

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sical binding of chiral fermion pairs? Indeed, here there arises an apparent logical conflict in the NJL model if we tune the coupling constant near its "critical value." In this case the pointlike bound state becomes nearly massless, and at the critical coupling we have a conformal effective low energy theory. In configuration space, conformality implies a nearly scale invariant extended internal wave-function, $\phi(r) \propto 1/r$, (the limit of $\phi(r) \propto e^{-|\mu|r}/r$ for mass $|\mu| \ll M_0$). This extends to large distances, well beyond the range of the potential $\sim M_0^{-1}$. Hence we would expect that the near–critical bound state must always be an extended object, even if the interaction scale, M_0^{-1}, is a very short distance scale. We will argue that the NJL model, lacking an internal wave function, becomes misleading in this limit, and essentially fails.

To address and expand upon these issues, we presently explore a Lorentz invariant formulation in configuration space by introducing a fully dynamical color singlet composite bilocal field, $\Phi(x, y) \sim \overline{\psi}_R(x)\psi_L(y)$. This formulation contains the requisite internal wave-function with its own kinetic term. This is a "new-old" approach to compositeness as it hearkens back to Schrödinger [18] and relativistic generalizations of Yukawa [4]. However, we still lean heavily on the NJL model to provide intuition while extending it to a non-pointlike, bilocal field theory. Introducing $\Phi(x, y)$ has the immediate advantage of bosonizing the composite theory (we can then mostly bypass issues of Dirac operators until we do loops). Including $\phi(r)$ into the structure of the theory introduces new degrees of freedom and leads to interesting consequences.

Consider a pair of massless particles of 4-momenta p_1 and p_2, where $p_1^2 = p_2^2 = 0$, and a two-body "scattering state" consisting of plane waves, $\Phi(x, y) \sim \exp(ip_1x + ip_2y)$. We introduce "barycentric coordinates," $X = (x+y)/2$, and $r = (x - y)/2$, and corresponding momenta $P = p_1 + p_2$, $Q = p_1 - p_2$. Hence we can write $\Phi = \chi(X)\phi(r)$ where $\chi(X) = \exp(iPX)$ governs the "center of mass motion" and $\phi(r) = \exp(iQr)$ is the "internal wave-function" of the system. This factorization of Φ will be our ansatz for bound states.

Since $P_\mu Q^\mu = 0$, then in the rest frame we have $P = (P_0, \vec{0})$ and $Q = (0, \vec{q})$. Hence, the dependence upon the relative time r^0 drops out, and the internal wave-function is static $\phi(\vec{r})$. We construct the Hamiltonian for $\phi(\vec{r})$ including interactions. The extremalization of this Hamiltonian leads to the "Schrödinger-Klein-Gordon" (SKG) equation for $\phi(r)$, with eigenvalues μ^2 for the (mass)^2 of bound state solutions.

The most natural UV completion of the interaction of the NJL model is the "coloron model" [11, 15, 19]. The coloron is essentially a massive gluon, associated with an $SU(N_c)$ gauge theory broken to a global $SU(N_c)$, featuring an adjoint representation of massive gauge bosons of mass M_0. Integrating out the colorons (and removing relative time) yields a perturbative static Yukawa potential in the rest frame,

$$-g_0^2 N_c M_0 V_0(2r) \;\text{ where, }\; V_0(2r) = -\frac{e^{-2M_0 r}}{8\pi r}. \tag{1}$$

This is a non-pointlike generalization of the NJL interaction. The potential is enhanced by a factor of N_c colors, in analogy to the N_{Cooper} (Cooper pairs) enhancement of the Fröhlich interaction in the BCS theory of superconductivity [10], and arises here by the color singlet normalization of $\Phi(x, y)$. M_0 is the defining mass scale of the theory.

The bound state forms semiclassically. Even in a short-distance potential, (e.g., the large $M_0 > |\mu|$ limit of eq.(1)) this leads to a spatially extended internal bound state wave-function, $\phi(r)$. As g_0^2 approaches a critical value, $g_0^2 \to g_{0c}^2$ where $\mu^2 \to 0$, then $\phi(r) \to 1/r$ is conformal. For supercritical coupling $g_0^2 N_c$ the eigenvalue of the SKG equation, μ^2, *is always negative (tachyonic)* and a chiral vacuum instability and dynamical spontaneous symmetry breaking will therefore occur classically.

The most interesting result in this formalism is a significant wave-function spreading of $\phi(r)$, where the large distance part dominates the normalization of $\phi(r)$ and dilutes $\phi(0)$. There is an induced Yukawa coupling of the bound state to free fermions, $g_Y \propto \phi(0)$, which gives the *unbound* fermions their dynamical mass, and is therefore suppressed by the infrared dilution, as $\phi(0) \sim \sqrt{|\mu|/M_0}$. Hence, g_Y is then significantly smaller than the Yukawa coupling obtained in the NJL model, which runs only logarithmically.

Moreover, this has the direct consequence of reducing the sensitivity of the eigenvalue to the near critical value of the coupling constant, significantly reducing the degree of fine-tuning needed to have a hierarchy between M_0^2 and μ^2. We also compute loops in this formalism which depend upon g_Y, consistent with the dilution effect. The loops can generate the quartic coupling, $\lambda \propto g_Y^4 \ln(M_0/\mu)$ which is necessary to stabilize the vacuum in a tachyonic, supercritical solution. Again, the dilution effect leads to a smaller value of λ than expected in the NJL model.

This "infrared dilution" effect drastically contrasts the results from the NJL model where the RG running of the Yukawa coupling into the IR is logarithmic and comparatively slow. If, for example, this applies to the BEH boson, as in a top quark condensation scheme [15], then it can naturally arise as an extended composite object on the scale of the electroweak VEV, $v = 175$ GeV with $M_0 \sim 5$ to 10 TeV. The dilution of $\phi(0)$ effectively isolates the strong interaction at M_0 from the low energy physics and eliminates the need for drastic fine-tuning in these models, yielding fine-tuning at the few % level. The scale symmetry near criticality acts here as the custodial symmetry for the emergent low mass scale physics.

We give a lightning review of the familiar NJL model with pointlike potential treated in Appendix A, and discuss key aspects of it in the next section. We then formulate bilocal fields, and construct a model of an extended potential that arises from a single massive gluon exchange (the coloron model), suitably Fierz rearranged. The formalism leads to the SKG equation. We then analyze the induced Yukawa coupling, and induced quartic coupling (the latter from loops), and consider properties of the solutions to the SKG equation by variational methods. We show that the critical coupling in the semi-classical Yukawa potential case is almost identical to the critical coupling of the NJL model! The wave function spreading and power-law suppression of the Yukawa and quartic coupling emerges and the fermion loop renormalizations are consistent with the suppression and perturbative. Appendices contain detailed extensions of the discussion. Some precursory work to the present appears in [20]

II. BILOCAL THEORY OF COMPOSITE SCALARS

A. Key Aspects of the Nambu–Jona-Lasinio Model

Here we touch on key elements relevant to formulating a bilocal description of bound states. A detailed review of the RG approach to the NJL model is presented in Appendix A.

The NJL model [5] assumes chiral fermions, with N_c "colors." A chirally invariant $U(1)_L \times U(1)_R$ action takes the form:

$$S_{NJL} = \int d^4x \, \Big(i[\overline{\psi}_L(x) \partial\!\!\!/ \psi_L(x)] + i[\overline{\psi}_R(x) \partial\!\!\!/ \psi_R(x)] + \frac{g_0^2}{M_0^2} [\overline{\psi}_L(x)\psi_R(x)]\, [\overline{\psi}_R(x)\psi_L(x)] \Big). \tag{2}$$

Here the notation [..] implies color singlet combination.

We rewrite eq.(2) in an equivalent form by introducing a *local auxiliary field* $\Phi(x)$:

$$S_{NJL} = \int d^4x \, \Big(i[\overline{\psi}_L(x) \partial\!\!\!/ \psi_L(x)] + i[\overline{\psi}_R(x) \partial\!\!\!/ \psi_R(x)] - M_0^2 \Phi^\dagger(x)\Phi(x) + g_0[\overline{\psi}_L(x)\psi_R(x)]\Phi(x) + h.c. \Big). \tag{3}$$

The resulting "equation of motion" for Φ is:

$$M_0^2 \Phi(x) = g_0[\overline{\psi}_R(x)\psi_L(x)] \tag{4}$$

Substituting the $\Phi(x)$ equation back into eq.(3) reproduces the 4-fermion interaction of eq.(2). We emphasize that $\Phi(x)$ is a complex, local, or "pointlike," field. Generalizing the relation of eq.(4) is the starting point for our bilocal field theory.

Note that the induced Yukawa coupling, g_0, in eq.(3) is the same coupling as appears in the interaction of eq.(2). This implies that strong coupling required to induce chiral symmetry breaking in the NJL model will translate into a strong Yukawa coupling. In our bilocal theory this will not be the case.

Following Wilson, [6], we view eq.(3) as the action defined at the high energy (short-distance) scale $m \sim M_0$. We then integrate out the fermions to obtain the effective action for the composite field Φ at a mass scale of the low energy physics, $\mu \ll M_0$ [8, 9]. In leading N_c fermion loop approximation this yields:

$$S_\mu = \int d^4x \, \Big(i[\overline{\psi}_L \partial\!\!\!/ \psi_L] + i[\overline{\psi}_R \partial\!\!\!/ \psi_R] + Z\partial_\mu \Phi^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 + (g_0[\overline{\psi}_L\psi_R]\Phi(x) + h.c.) \Big). \tag{5}$$

where,

$$\mu^2 = M_0^2 - \frac{g_0^2 N_c}{8\pi^2} M_0^2, \qquad Z = \frac{g_0^2 N_c}{8\pi^2}\ln(M_0/\mu), \quad \lambda = \frac{g_0^4 N_c}{4\pi^2}\ln(M_0/\mu). \tag{6}$$

Here M_0^2 is the UV loop momentum cut-off, and we obtain induced kinetic and quartic interaction terms. The one-loop result can be improved by using the full renormalization group (RG) [8, 9].

It is a feature of the NJL model that apparently all of the fermions in the scale range M_0 to μ are integrated out to comprise a "tightly bound state." As seen in the formalism of Cornwall, Jackiw and Tomboulis (CJT) [3], however, one is integrating out the *quantum fluctuations* of the fermions, and we could introduce classical fermion sources to describe the residual unbound scattering state fermions.[1] Generally, in the RG approach, unbound fermions in the scale range M_0 to μ are simply assumed to remain in the action.

Note the behavior of the composite scalar boson mass, μ^2, of eq.(6) due to the $-N_c g^2 M_0^2/8\pi^2$ with UV cut-off M_0^2. The NJL model therefore has a critical value of its coupling defined by the vanishing of μ^2,

$$\frac{g_{0c}^2 N_c}{8\pi^2} = 1 \tag{7}$$

We can renormalize, $\Phi \to \sqrt{Z}^{-1}\Phi$, to obtain the full effective Lagrangian. The notable feature here is that the renormalized Yukawa and quartic couplings evolve logarithmically in the RG running mass m:

$$g_Y^2 = \frac{g_0^2}{Z} = \frac{8\pi^2}{N_c \ln(M_0/m)}, \quad \lambda_r = \frac{\lambda}{Z^2} = \frac{16\pi^2}{N_c \ln(M_0/m)}. \tag{8}$$

These are the solutions to the RG equations in the large N_c limit, [9], neglecting other interactions. This indicates that the couplings approach a Landau pole as $m \to M_0$, and we use the full RG equations with this boundary condition to obtain the low energy solution in the NJL model [8]. The evolution into the IR is then gradual, in particular, g_Y approaches an IR fixed point value [21]. In our present scheme, the value of g_Y at $m = M_0$ is determined by $\phi(0)$, is finite and power-law suppressed, hence the IR value can be below the fixed point.

[1] We are unaware of any formal treatment of the NJL model in the CJT formalism, which would be of interest, but it must dovetail with the RG formalism; in our own attempt we found this challenging.

For super-critical coupling, $g_0^2 > g_c^2$, we see that $\mu^2 < 0$ and there will be a chiral vacuum instability. The effective action, with the induced quartic $\sim \lambda_r(\Phi^\dagger\Phi)^2$ term, is then the usual sombrero potential. The chiral symmetry is spontaneously broken and Φ acquires a VEV, and produces Nambu–Goldstone bosons and a Higgs boson. The additional predictions of the NJL model are discussed Appendix A.

B. Semiclassical and Non-Point-like Generalization of the NJL Model

Consider a semiclassical approach to binding in a non-confining theory.[2] In the limit of shutting off the interaction a bound state is just a two-body scattering state, such as a product of a free electron and free proton wave-functions in the case of Hydrogen. For chiral fermions this can be described by a bilocal field $\Phi_B^A(x,y) = \overline{\psi}_R^A(x)\psi_{LB}(y)|_b$, where (A,B) arbitrary color and flavor indices and b denotes the subset of fermions that will comprise the bound state. We also have the remaining unbound free fermionic scattering states, that will remain free after the interaction is turned on (denoted by subscript f).

A superposition of the bound and free fermions can be written as, [3]

$$\overline{\psi}_R^A(x)\psi_{LB}(y) = \overline{\psi}_R^A(x)\psi_{LB}(y)_f + M^2\Phi_B^A(x,y) \quad (9)$$

The components are orthogonal,

$$\int_{xy} \overline{\psi}_R^A(x)\psi_{LB}(y)_f \Phi^{\dagger}{}_{B'}^{A'}(x,y) = 0 \quad (10)$$

Note we have implicitly defined Φ as a mass dimension-1 field, like a scalar, and the mass prefactor, M^2, should be viewed as part of the wave-function and will be elaborated below.

Eq.(9) has a formal similarity to the factorized auxiliary field expression in eq.(4). However, here $\Phi_B^A(x,y)$ is a distinct physical bilocal field and its kinetic term is not induced by loops, and will have a free field kinetic term. While the NJL auxiliary field is not present in the NJL theory when the interaction is turned off, the ingredients of the bound state are indeed present in the semi-classical scheme in the absence of the interaction.

[2] "Semiclassical" implies $\hbar = 0$ (no loops). The leading term in the $\hbar$ expansion of a quantum field theory is the sum over all tree-diagrams which is equivalent to a (Fredholm expansion) of a classical nonlinear field theory, expressed in terms of frequencies, ω, and wave vectors, $\vec{k}$. Terms of order $\hbar^N$ in the expansion contain N-loops.

[3] This can be viewed as a matrix element of the operator $\overline{\psi}_R(x)\psi_L(y)$ sandwiched between the vacuum, $\langle 0|$ and a superposition of quantum states, $|\Phi_B^A\rangle + |\overline{\psi}_R^A, \psi_{LB}\rangle$, where $|\Phi_B^A\rangle$ is a coherent state.

We'll presently restrict ourselves to a single flavor, hence a $U(1)_L \times U(1)_R$ flavor symmetry, and $(A,B) \to (i,j)$ are $SU(N_c)$ color indices (this can be readily extended to $G_L \times G_R$ flavor groups as in Appendix (B)).

In the coloron model, of the next section, we will see that only the color singlet field forms a bound state of a pair of chiral fermions. With $SU(N_c)$ color indices, (i,j), the field $\Phi_j^i(X,r)$ is a complex matrix that transforms as a product of $SU(N_c)$ representations, $\overline{N}_c \times N_c$, and therefore decomposes into a singlet plus an adjoint representation. We designate the color singlet bilocal field as Φ^0 and conventionally normalize it as,

$$\Phi_j^i(x,y) = \frac{1}{\sqrt{N_c}}\delta_j^i\Phi^0(x,y) \quad (11)$$

This normalization allows canonically normalized kinetic terms, $\mathrm{Tr}[\partial\Phi^\dagger\partial\Phi] = \partial\Phi^{0\dagger}\partial\Phi^0$. Note $\mathrm{Tr}\,\Phi = \Phi_i^i(x,y) = \sqrt{N_c}\Phi^0(x,y)$.[4] Since only the color singlet binds, we can rewrite eq.(9) containing free fields and the color singlet bound state of eq.(11),

$$\overline{\psi}_L^i(x)\psi_{jR}(y) \to \overline{\psi}_L^i(x)\psi_{jR}(y)_f + M^2\frac{\delta_j^i}{\sqrt{N_c}}\Phi^0(x,y). \quad (12)$$

C. The Coloron Model

The pointlike NJL model can be viewed as the limit of a physical theory with a bilocal interaction. The primary example is the "coloron model" [15, 16, 19] (the coloron idea stems from QCD applications of NJL [11] and the top condensation theory [15]). The coloron is a perturbative, massive gauge boson, analogue of the gluon, arising in a local $SU(N_c)$ gauge theory broken to a global $SU(N_c)$. We integrate out the massive coloron and keep the single particle exchange potential to define the model. This yields a bilocal current-current form.

$$S' = -g_0^2\int_{xy}[\overline{\psi}_L(x)\gamma_\mu T^A\psi_L(x)]D^{\mu\nu}(x-y)[\overline{\psi}_R(y)\gamma_\nu T^A\psi_R(y)] \quad (13)$$

where $T^A = T_i^{Aj}$ are generators of $SU(N_c)$, and color singlet combinations indicated in brackets [...]. We adopt a notational convenience for integrals as defined in Appendix E,

The coloron propagator in Feynman gauge is:

$$D_{\mu\nu}(x-y) = g_{\mu\nu}D_F(x-y)$$
$$D_F(x-y) = -\int \frac{1}{q^2 - M_0^2}e^{iq(x-y)}\frac{d^4q}{(2\pi)^4}. \quad (14)$$

[4] In the NJL model, where Φ has no bare kinetic term, the color normalization arises automatically at loop level together with the kinetic term.

A Fierz rearrangement of the interaction to leading order in $1/N_c$ leads to a potential:

$$S' = g_0^2 \int_{xy} [\overline{\psi}_L(x)\psi_R(y)] D_F(x-y) [\overline{\psi}_R(y)\psi_L(x)], \quad (15)$$

(see Appendix C for details, of the Fierz rearrangement). The above displayed term is the most attractive channel and leading in large N_c.

Hence, we replace the pointlike 4-fermion interaction with the non-pointlike S' of eq.(15). Note that if we suppress the q^2 term in the denominator of eq.(14) we have,

$$D_F(x-y) \to \frac{1}{M_0^2}\delta^4(x-y), \quad (16)$$

and we recover the pointlike NJL model interaction.

Now, substitute eq.(12) into eq.(15) to obtain,[5]

$$\begin{aligned} S' \to\ & g_0^2 \int_{xy} [\overline{\psi}_L(x)\psi_R(y)]_f D_F(x-y) [\overline{\psi}_R(y)\psi_L(x)]_f \\ & + g_0^2\sqrt{N_c} M^2 \int_{xy} [\overline{\psi}_L(x)\psi_R(y)]_f D_F(x-y)\ \Phi^0(x,y) + h.c. \\ & + g_0^2 N_c M^4 \int_{xy} \Phi^{0\dagger}(x,y)\ D_F(x-y)\ \Phi^0(x,y) \end{aligned} \quad (17)$$

The leading term S' of eq.(17) is the unbound 4-fermion scattering interaction and has the structure of the NJL interaction in the limit of eq.(16) and identifies g_0 as the analogue of the NJL coupling constant. The second term $\sim g_0^2\sqrt{N_c}[\psi^\dagger\psi]\Phi^0 + h.c.$ in eq.(17) has the form of the Yukawa interaction between the bound state Φ^0 and the free fermion scattering states.

Note the appearance of the color factors, $\sqrt{N_c}$ and N_c, in the second and third lines respectively. The third term, where from eq.(11) we have $|\operatorname{Tr}\Phi|^2 = N_c|\Phi^0|^2$, is the potential that creates the semiclassical bound state with the N_c enhancement.

The mass scale, M, is ultimately inherited by the bound state Φ from the coloron interaction. The interaction introduces the scale M_0 and it will be treated as a cut-off, as in the NJL model, hence $M < M_0$. The prefactor M in eq.(9) is *a priori* arbitrary, but we can swap M for a dimensionless parameter ϵ as:

$$M = \epsilon M_0 \quad (18)$$

ϵ should be viewed as part of the wave-function of Φ. In a variational calculation below, (Section (III C)), we will see that $\epsilon = 1$ extremalizes the SKG effective Hamiltonian in a bound state (with negative μ^2 eigenvalue). Also, $\epsilon = 0$ is the extremal value for the subcritical case. Hence, in the subcritical case Φ disappears and we are left with only unbound fermions. The reason is simple: ϵ rescales the coupling constant $\sim g_0^2\epsilon$ and the maximal value of the coupling is therefore $\epsilon \to 1$.

[5] This can equivalently be viewed as the matrix element of eq.(15) sandwiched between the quantum states, $|\Phi^0\rangle + |\overline{\psi}_R, \psi_L\rangle$.

D. Fake Chiral Instability: The Need for the Dynamical Internal Wave Function

Consider the pointlike limit of eq.(17) and the semiclassical fields in eq.(16), where we replace $\Phi^0(x,y) \to \Phi^0(x)$, and obtain,[6]

$$\begin{aligned} S' \to \int_x \Bigg(& \frac{g_0^2}{M_0^2}[\overline{\psi}_L\psi_R]_f[\overline{\psi}_R\psi_L]_f + \widetilde{M}^2\Phi^{0\dagger}\Phi^0 \\ & + g_0^2\epsilon\sqrt{N_c}([\overline{\psi}_L\psi_R]_f\Phi^0 + h.c.) \Bigg) \end{aligned} \quad (19)$$

where $\widetilde{M}^2 = g_0^2 N_c \epsilon^2 M_0^2$.

Eq.(19) contains a wrong-sign ("tachyonic") mass term, implying a potential, $-\widetilde{M}^2|\Phi|^2$. This appears to generate spontaneous symmetry breaking for any values of the underlying parameters M_0 and g_0 and the vacuum implodes. A chiral vacuum instability is apparently an immediate, large effect of introducing eq.(12)!

Such a conclusion is obviously physically incorrect. In naively replacing $\Phi(x,y)$ with $\Phi(x)$ we have neglected the kinetic term of the internal wave-function, $|\partial_r\Phi|^2$ where $r = (x-y)/2$. This opposes the instability like a repulsive interaction and will stabilize the vacuum in weak coupling. This is similar to the stabilization of the classical Hydrogen atom by the Schrödinger wave-function. A chiral instability can occur through competition of the repulsive internal wave-function kinetic term and the attractive potential, but will require a sufficiently large coupling, $g_0^2 > g_{0c}^2$, to drive it, and a quartic coupling to stabilize the vacuum.

We therefore must consider the internal dynamics of the non-pointlike bound state. We note that this in the spirit of ref.[22], but physically distinct, as the authors were arguing for a bare kinetic term of the factorized NJL model of eq.(4), while we are arguing for a bare kinetic term of the bilocal field $\Phi(x,y)$, including the internal coordinates $\sim x-y$.

E. Bilocal Free Fields

Consider a pair of massless particles of 4-momenta p_1 and p_2. We have $p_1^2 = p_2^2 = 0$. and we can have two-body plane waves, $\Phi(x,y) \sim \exp(ip_1x + ip_2y)$. We pass to the total momentum $P = (p_1 + p_2)$ and relative momentum $Q = (p_1 - p_2)$. The plane waves become $\exp(iPX + iQr)$ where we define "barycentric coordinates,"

$$\begin{aligned} X^\mu = \frac{x^\mu + y^\mu}{2}, \qquad & r^\mu = \frac{x^\mu - y^\mu}{2}, \\ \partial_x = \frac{1}{2}(\partial_X + \partial_r), \quad & \partial_y = \frac{1}{2}(\partial_X - \partial_r) \end{aligned} \quad (20)$$

[6] $\Phi(x)$ here *should not be confused with the factorized NJL interaction*, eq.(A2), where M_0^2 is a right-sign non-tachyonic mass. Here Φ is not a pure auxiliary field, but rather is physical.

Note that $P_\mu Q^\mu = p_1^2 - p_2^2 = 0$. This implies that there is always a rest frame in which $P = (P_0, 0)$ and $Q = (0, \vec{q})$. Hence, in the rest frame the dependence upon $\vec{X}$ and r^0 drops out. If the particles are constituents of a bound state then this is the rest frame of the composite particle.

To proceed we need the generalized kinetic term of $\Phi(x, y)$ viewed as a bilocal field with an internal wave-function coordinate, r^μ. A free particle scattering state, $\Phi(x, y)$, composed of massless particles, will satisfy [4],

$$\partial_x^2 \Phi(x, y) + \partial_y^2 \Phi(x, y) = 0 \tag{21}$$

or equivalently,

$$\frac{1}{2}\partial_X^2 \Phi'(X, r) + \frac{1}{2}\partial_r^2 \Phi'(X, r) = 0 \tag{22}$$

where $\Phi'(X, r) = \Phi(X - r, X + r)$. The bilocal field $\Phi(x, y) = \overline{\psi}_R(x)\psi_L(y)$ represents a "bosonization" of the pair of chiral fermions, as in chiral Lagrangians. The equations of motion follow from the square of the free particle Dirac equations, $(\partial\!\!\!/\,)_x^2 \psi_R(x) = 0$ and $(\partial\!\!\!/\,)_y^2 \psi_L(y) = 0$. Note that we have chosen to describe the separation as r which is the radius, where $2r = \rho \equiv (x - y)$ denotes the separation of the particles. The choice of r leads to more symmetrical expressions in r and X, and (somewhat) suppresses inconvenient factors of 2.

The ∂_x^2 and ∂_y^2 are, in principle, independent for free fields. We can also write an independent Lorentz invariant equation,

$$\frac{1}{2}\partial_x^2 \Phi - \frac{1}{2}\partial_y^2 \Phi = \frac{\partial}{\partial X^\mu}\frac{\partial}{\partial r_\mu}\Phi'(X, r) = 0 \tag{23}$$

This can be viewed as a Lorentz invariant constraint on the bilocal field. In Dirac constraint theory, in a Hamiltonian formalism [23], it is the "primary constraint." This constraint implies that in the rest system we have a stationary solution in which the "relative time," $2r^0 = \rho^0$, is removed, e.g.,

$$\Phi'(X, r) \to \Phi'(X^0, \vec{r}), \qquad \vec{P} = 0, \; r^0 = 0. \tag{24}$$

where $P_\mu = i\partial\Phi'/\partial X^\mu$ is the total 4-momentum of the state. This is "stationary" in the sense that the total 3-momentum of the system vanishes in the rest (barycentric) frame, $\vec{P} = 0$, and we have a wave-function, $\Phi'(t, \vec{r})$ where $t = X^0$, but depends only upon $\vec{r}$, with $r^0 = 0$.

We can view the equations of motion as arising from an action, subject to the constraint. With $M^2\Phi$ we choose the dimensionality of the field Φ to be that of a scalar field, i.e., $\sim$ (mass). We then write a dimensionless, free bilocal action,

$$S_K = \int_{xy} M^4 Z_0 \left(|\partial_x \Phi|^2 + |\partial_y \Phi|^2 \right) \tag{25}$$

The normalization factor Z_0 is defined below. In the barycentric coordinates $\Phi(x, y) = \Phi'(X, r)$ and this becomes,

$$S_K = \frac{1}{2} J \int_{Xr} Z_0 M^4 \left(|\partial_X \Phi'|^2 + |\partial_r \Phi'|^2 + \ldots \right) \tag{26}$$

where $J = |\partial(x, y)/\partial(X, r)| = 2^4$ is the Jacobian in passing from (x^μ, y^ν) to the barycentric coordinates (X^μ, r^ν). (Our abbreviated notation for integrals is defined in Appendix E).

We can formally implement the constraint by adding to the action a Lagrange multiplier, η, while preserving Lorentz invariance,

$$S_\eta = \int_{Xr} \eta M^4 \left| \mathrm{Tr}\, \frac{\partial \Phi'^\dagger}{\partial X^\mu} \frac{\partial \Phi'}{\partial r_\mu} \right|^2 \tag{27}$$

We will assume the Lagrange multiplier term is implicitly present in all the bilocal actions we write subsequently, and $\delta S_\eta / \delta \eta = 0$.

Following Yukawa [4] we consider, in the barycentric coordinates, a factorization of $\Phi'(X, r)$ as,

$$\sqrt{J/2}\; \Phi'(X, r) = \chi(X)\phi(r) \tag{28}$$

Obviously, an arbitrary function $\Phi(x, y)$ can be written in terms of $\Phi'(X, r)$ since $\Phi(x, y) = \Phi(X + r, X - r) \equiv \Phi'(X, r)$. However an arbitrary $\Phi(x, y)$ cannot generally be written in terms of a factorized pair as in eq.(28). An arbitrary $\Phi(x, y)$ can, however, be written as a sum over basis functions of the form, $\sum \beta^{ij} \chi_i(X)\phi_j(r)$.

The pure factorization assumption of eq.(28), however, can be viewed as an ansatz for the particular bilocal wave-function we expect for composite particles. Then $\chi(X)$ describes the center-of-mass motion of the system and $\phi(r)$ describes the internal wave-function.

The action eq.(26) with the factorized field becomes,

$$S = M^4 \int_{Xr} Z_0 \left(|\phi(r)|^2 |\partial_X \chi(X)|^2 + |\chi(X)|^2 |\partial_r \phi(r)|^2 \right) \tag{29}$$

In the rest-frame the constraint implies,

$$\frac{\partial}{\partial r^0}\Phi'(X, r) = \frac{\partial}{\partial \vec{X}}\Phi'(X, r) = 0 \tag{30}$$

In the rest-frame we have the total 3-momentum, $\vec{P} \sim \partial/\partial\vec{X} = 0$, and time is carried by X^0. The internal wave-function is independent of the relative time r^0 and has dependence only upon $\vec{r}$. Hence we can define in this frame,

$$\sqrt{J/2}\; \Phi'(X, r) = \chi(X^0)\phi(\vec{r}). \tag{31}$$

We can therefore integrate out r^0 and $\vec{X}$ (as defined in the rest frame),

$$\int d^3\vec{X} = V_3 \qquad \int dr^0 = T \tag{32}$$

We now choose the normalization factor, Z_0, as,

$$1 = M \int dr^0 Z_0 = Z_0 M T \tag{33}$$

Here Z_0 is defined in the rest system with a cut-off, $\int dr^0 \equiv T$.

While this may seem like a frame dependent constraint, we can in principle give a more detailed manifestly Lorentz invariant prescription for removal of the relative time integral in the kinetic term along the lines of Dirac's Hamiltonian constraint theory [23] (See [20] for the scalar field case that more directly leads to Z_0)

These parameters are intuitively related to the physical properties of the state. The spatial volume of the bound state is $\sim M^{-3} = \epsilon^{-3}M_0^{-3}$. We view M as part of the wave-function, and it is determined by extremalizing the Hamiltonian. We find that, for supercritical coupling, $\epsilon \to 1$ and $M = M_0$, as seen in a simple variational calculation below. For subcritical coupling the extremalization produces $\epsilon \to 0$, and indicates that $\phi(r)$ does not then involve the mass, M_0, but only momenta $\phi(\vec{q}\cdot\vec{r})$, and The volume of $\phi(r)$ then abruptly switches from M_0^{-3} to V_3, from a compact bound state to an extended plane wave. T in the bound state is essentially the transit time of the constituent pair in the core of the potential, and we therefore expect $T \sim M_0^{-1}$, and $Z_0 \sim 1$.

With eq.(33) the action in the rest frame becomes,

$$S_K = M^3 V_3 \int dX^0 d^3r \left(|\phi(r)|^2 |\partial_0\chi|^2 - |\chi|^2 |\vec{\partial}_r\phi(r)|^2 \right) \quad (34)$$

Notice only the spatial derivative terms appear in the ϕ kinetic term.

To have the usual canonical normalization of the χ field we therefore require the normalization of the dimensionless internal wave-function $\phi(\vec{r})$ to be

$$M^3 \int d^3r \, |\phi(\vec{r})|^2 = 1. \quad (35)$$

Hence we are choosing the internal wave-function, factor field $\phi(r)$, to be dimensionless.

As a trivial example, consider an internal wave-function with $\phi(\vec{r}) = N\exp(2i\vec{q}\cdot\vec{r})$. corresponding to a two body scattering state. The above normalization is then formally $M^3N^2\int d^3r = 1$ and the action then becomes,

$$S_K = V_3 \int dX^0 \left(|\partial_0\chi(X^0)|^2 - 4\vec{q}^{\,2}|\chi(X^0)|^2 \right) \quad (36)$$

This is a state described by $\chi(X)$ which satisfies the equation of motion,

$$\partial_0^2\chi + \mu^2\chi = 0 \qquad \mu^2 = 4\vec{q}^{\,2}, \quad (37)$$

a zero 3-momentum two body scattering state of invariant mass $2|\vec{q}| = \mu$, with conventional normalization[7]

$$\chi = \frac{1}{\sqrt{2\mu V_3}} e^{i\mu X^0} \quad (38)$$

We discuss the bilocal scattering states further in Appendix D. A key point in this example is that $\vec{q}^{\,2}$, which appears as an "invariant mass," $-q^2 = \mu^2$, is not really a mass at all, since the stress tensor trace remains zero for massless particles. A true mass, such as associated with a bound state of massless particles, requires a mass scale generated in the interaction and a nonzero stress tensor trace. In QCD this happens via the trace anomaly, which is $\propto \beta(g)/g$, and similarly in Coleman-Weinberg dynamical symmetry breaking [26]. See [27] for discussion of the role of the trace anomalies. In the present instance the mass scale M_0 could come indirectly from a $\propto \beta(g_0)/g_0$ or other trace anomaly in the coloron dynamics.

F. Turning on the Interaction

We now consider the interaction of eq.(17), particularly the last term. Here we have the full space-time dependence of the propagator $D_F(x^\mu - y^\mu) = D_F(2r^\mu)$. Hence the *quadratic action* of eq.(26) for $\Phi(X^0, \vec{r})$ in the rest frame becomes,

$$JM^3V_3 \int_{X^0\vec{r}} \left(\frac{1}{2}|\partial_{X^0}\Phi'|^2 - \frac{1}{2}|\partial_{\vec{r}}\Phi'|^2 \right.$$
$$\left. + JM^4V_3 \int dX^0 d^3r \, dr^0 \, g_0^2 N_c D(2r)|\Phi'|^2 \right) \quad (39)$$

(we consider the Yukawa and quartic interaction terms subsequently).

We again integrate over r^0 (the relative time), where $\Phi'(X^\mu, \vec{r})$ is constrained to have no dependence upon r^0. However, now in the third term we have,[8]

$$\int dr^0 D_F(2r) = -\int dr^0 \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - M_0^2} e^{i2q_\mu r^\mu}$$
$$= \frac{1}{2}\int \frac{d^3q}{(2\pi)^4} \frac{1}{\vec{q}^{\,2} + M_0^2} e^{i2q_\mu r^\mu} = -\frac{1}{2}V_0(2|\vec{r}|) \quad (40)$$

where the $\vec{q}$ momentum integral yields the familiar Yukawa potential,

$$V_0(\rho) = -\frac{e^{-M_0\rho}}{4\pi\rho} = -\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}; \qquad \rho = 2|\vec{r}| \quad (41)$$

In the limit of suppressing the $\vec{q}^{\,2}$ in the denominator of eq.(40) we obtain using $J = 2^4$, and $\delta^3(\vec{r}) = (4\pi r^2)^{-1}\delta(r)$:

$$V_0(2r) \to -\frac{1}{M_0^2}\delta^3(2\vec{r}) = -\frac{1}{2\pi J M_0^2 r^2}\delta(r) \quad (42)$$

From eq.(26), including color factors and introducing a quartic interaction (which is presently assumed to be

[7] The conventional free field normalization implies that the action reduces to the classical massive particle form, $\int P_\mu dx^\mu = \int \mu \, dX^0$.

[8] Note the argument of $D_F(2r^\mu)$ is the separation of the particles, $\rho^\mu = 2r^\mu$ which leads to the factor of $\frac{1}{2}$ in eq.(40), which joins the overall factor of $\frac{1}{2}$ in eq.(43).

$|\Phi|^4 \propto |\Phi(x,y)|^4$) the resulting action in the barycentric frame is,

$$S = \frac{1}{2}J\epsilon^3 M_0^3 V_3 \int dX^0 d^3r \Bigg(|\partial_{X^0}\Phi'|^2 - |\partial_{\vec{r}}\Phi'|^2 - g_0^2 N_c M V_0(2r)|\Phi'|^2 - \frac{\lambda_0}{2}|\Phi'|^4 + ... \Bigg) \quad (43)$$

Note that since eq.(43) is an action it contains $-(..)V_0(2r)$, while in eq.(41) we see that $V_0 = -(..)$ is defined with an intrinsic minus sign, hence this term is positive in the action and thus attractive in a Hamiltonian. Recall that with $\epsilon = 1$ we have $M = M_0$.

We emphasize the procedure of going to the rest frame is a "gauge choice" while the theory remains overall Lorentz ("gauge") invariant. Note that $J\epsilon^3 M_0^3$ is a measure of the inverse volume of the bound state. We can write the action (noting $M = \epsilon M_0$ and factors of 2 that come from the $\frac{1}{2}J$ normalization of eq.(28), and eq.(35)) in the format:

$$S = V_3 \int dX^0 \Bigg(|\partial_{X^0}\chi|^2 - |\chi|^2\mathcal{M}^2 - \frac{\tilde{\lambda}}{2}|\chi|^4 \Bigg)$$

$$\text{where,} \quad \mathcal{M}^2 = M^3 \int_{\vec{r}} \Bigg(|\partial_{\vec{r}}\phi|^2 + g_0^2 N_c M V_0(2r)|\phi|^2 \Bigg)$$

$$= M^3 \int_{\vec{r}} \Bigg(|\partial_{\vec{r}}\phi|^2 - g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}|\phi|^2 \Bigg) \quad (44)$$

We have assumed a simple form for the quartic term where $\tilde{\lambda}$ absorbs an integral over $\phi(r)^4$ and factors of J. The quartic interaction will be generated at loop level in Section IV.

$\mathcal{M}^2$ is the "internal Hamiltonian" that describes the internal dynamics of the bound state. $\mathcal{M}^2$ appears in the action, but contains no time derivatives since we have eliminated relative time. Therefore, including a minus sign, $\mathcal{M}^2$ is a Hamiltonian for the static internal wave-function $\phi(\vec{r})$.

Using $V_0(2r)$ of eq.(40) and extremalizing $\mathcal{M}^2$ gives the SKG equation and it's eigenvalue, μ^2:

$$-\Bigg(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \Bigg)\phi(r) - g_0^2 N_c M \frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi(r) \quad (45)$$

μ^2 is then the physical mass of the bound state, and the χ action in any frame is manifestly Lorentz invariant:

$$S = \int d^4X \Bigg(|\partial_X \chi(X)|^2 - |\chi(X)|^2\mu^2 - \frac{\tilde{\lambda}}{2}|\chi(X)|^4 \Bigg). \quad (46)$$

The Yukawa potential has a critical coupling, $g_0 = g_{0c}$, where the eigenvalue is then $\mu = 0$. For $g_0 > g_{0c}$ then $\mu^2 < 0$, and we have spontaneous symmetry breaking.

A central feature of our formalism is that the internal coordinate, $\vec{r}$, is always associated with the mass scale M_0 in the integrals,

$$\sim M_0^3 \int d^3r \; F(M_0\vec{r}) \quad (47)$$

with relevant factors of ϵ associated with ϕ. We can pass to a scale invariant internal coordinate variable, $u = M_0 r$, as in the case of the coloron model, as scale invariant Hamiltonian $\hat{\mathcal{H}}$,

$$\hat{\mathcal{H}} = \mathcal{M}^2/M_0^2 = \epsilon^3 \int d^3u \Bigg(|\partial_u\phi|^2 - g_0^2 N_c \epsilon \frac{e^{-2u}}{8\pi u}|\phi|^2 \Bigg) \quad (48)$$

The eigenvalue of the associated SKG equation is then the dimensionless $\hat{\mu}^2 = \mu^2/M_0^2$. Postulating eq.(48) as a general starting point, we can argue that the mass scale is entirely determined by the interaction, by passing back to $u = M_0 r$ to match the Yukawa interaction. In this way we make explicit that it is the Yukawa interaction that dictates the mass scale, M_0.

G. The Induced Bound State Yukawa Interaction

The Yukawa interaction of the bound state with the free scattering state fermions is now induced from the second term, S'_Y, in eq.(17). We have, noting eq.(31):

$$S'_Y = g_0^2\sqrt{N_c}M^2 \int_{xy} [\overline{\psi}_L(x)\psi_R(y)]_f D_F(x-y) \; \Phi^0(x,y) + h.c.$$

$$= \sqrt{2N_cJ}g_0^2 M_0^2 \times \int_{Xr} [\overline{\psi}_L(X+r)\psi_R(X-r)]_f D_F(2r) \; \chi(X)\phi(\vec{r}) + h.c. \quad (49)$$

Consider the pointlike limit of the potential, eq.(49)

$$S'_Y \to g_0^2\sqrt{N_c}M^2 \int_{xy} [\overline{\psi}_L(x)\psi_R(y)]_f \frac{\delta^4(x-y)}{M_0^2} \; \Phi^0(x,y) + h.c.$$

$$\to g_0^2\epsilon^2\sqrt{N_c}\int_x [\overline{\psi}_L(x)\psi_R(x)]_f \; \Phi^0(x,x) + h.c.$$

$$\to g_0^2\epsilon^2\sqrt{2N_c/J}\int_x [\overline{\psi}_L(x)\psi_R(x)]_f \; \chi(x)\phi(0) + h.c. \quad (50)$$

We therefore see that the induced Yukawa coupling in the pointlike limit to the field $\chi(x)$ is:

$$g_Y = \hat{g}_Y\phi(0) \qquad \hat{g}_Y \equiv g_0^2\epsilon^2\sqrt{2N_c/J} \quad (51)$$

This is a significant result and fundamentally different than the pointlike NJL result. We have taken the pointlike limit of the potential as in the NJL model, but obtain a result that is dependent crucially upon the internal wave-function $\propto \phi(0)$. The implication is that a strong coupling, g_0^2, can produce, in principle, a small Yukawa coupling if $\phi(0) \ll 1$. In the usual pointlike NJL model the induced Yukawa coupling runs to smaller values in the IR, but it does so only logarithmically via the RG. Here the behavior of $\phi(0)$ is a suppression of g_Y that is power-law near the critical coupling.

Alternatively, in terms of an extended potential, $V_0(2r)$,

$$S' \to -\sqrt{N_cJ/2}g_0^2M^2 \times \int_{X\vec{r}} [\overline{\psi}_L(X^+)\psi_R(X^-)]_f \chi(X)V(2r)\phi(\vec{r}) + h.c. \quad (52)$$

where $X^{\mu\pm} = (X^0, \vec{X} \pm \vec{r})$ and we have the induced Yukawa coupling

$$g_Y = -g_0^2\epsilon^2\sqrt{N_cJ/2}\; M_0^2 \int 4\pi r^2 dr V_0(2r)\phi(r) \quad (53)$$

We discuss some further issues of the induced Yukawa interaction in Appendix D 4.

III. THE SCHRÖDINGER-KLEIN-GORDON (SKG) EQUATION

While formally similar to the non-relativisitic Schrödinger equation, the SKG equation has key physical differences: (1) the potential has dimension $(\text{mass})^2$, rather than energy; (2) the eigenvalue describes resonances for positive μ^2; (3) "tachyons" occur for negative μ^2, which implies vacuum instability and spontaneous symmetry breaking. Mainly the SKG Hamiltonian is amenable to variational calculations as we show below. We presently give some examples of solutions and stress some subtleties. Much can be done to refine and extend this discussion. The solutions allow the computation of the induced Yukawa coupling of the bound state to free fermions, g_Y, via the wave-function at the origin, $\phi(0)$.

A negative eigenvalue of the Schrödinger equation defines our conventional view of a nonrelativistc bound state, however, in the relativistic case for a pair of chiral fermions the SKG equation implies a negative μ^2. This is, of course, the behavior of Σ-model in QCD and the BEH boson in the standard model and requires additional physics to stabilize the vacuum, such as quartic interactions. Hence, the general result is that a scalar bound state of massless chiral fermions in the symmetric (unbroken) phase must either be an unstable resonance (subcritical coupling and positive μ^2), which decays rapidly to its constituents, or tachyonic (supercritical coupling, negative μ^2) leading to a chiral instability of the vacuum.

From the action of eq.(44) the "internal Hamiltonian" $\mathcal{M}^2$ describes the $\phi(r)$ field, and since there is no time derivative for $\phi(r)$, $\mathcal{M}^2$ is just -1 times the ϕ action. The most negative eigenvalue occurs when $g_0^2\epsilon$ is maximal, $\epsilon = 1$ as seen below from a variational calculation.

A. Exact Criticality of the Yukawa Potential

The coloron model furnishes a direct UV completion of the NJL model. It leads to an SKG potential of the Yukawa form which has a critical coupling, $g_0^2 = g_{0c}^2$. The critical coupling is the nonzero value of g_0^2 for which the eigenvalue μ^2 is zero. We wish to determine g_{0c}^2.

The criticality of the Yukawa potential in the nonrelativistic Schrödinger equation is widely discussed in the literature in the context of "screening" (see [28] and references therein). The nonrelativistic Schrödinger equation $r = |\vec{r}|$ is:

$$-\nabla^2\psi - 2m_e\alpha\frac{e^{-\mu r}}{r}\psi = 2m_eE \quad (54)$$

with m_e the electron mass, and eigenvalue $E = 0$ occurs for a critical screening with $\mu = \mu_c$. A numerical analysis yields, [28],

$$\mu_c = 1.19061\;\alpha m_e. \quad (55)$$

For the spherical SKG equation in the coloron model eq.(45) we have from the Hamiltonian,

$$-\nabla^2\phi(r) - g_0^2N_cM_0\frac{e^{-2M_0|\vec{r}|}}{8\pi|\vec{r}|}\phi(r) = \mu^2\phi(r) \quad (56)$$

where we assume $\epsilon = 1$ as determined by a variational calculation below.

We can obtain the critical coloron model coupling constant by comparing, eq.(54) and eq.(56). We have,

$$\begin{aligned} 2m_e\alpha &\to g_0^2N_cM_0/8\pi \\ \mu_c &\to 2M_0 \end{aligned} \quad (57)$$

substituting into eq.(55), $2M_0 = 1.19061(g_0^2N_cM_0/16\pi)$ and therefore,

$$\left.\frac{g_0^2N_c}{8\pi^2}\right|_c = \frac{4}{(1.19061)\pi} = 1.06940 \quad (58)$$

By comparison, the loop level NJL critical value of eq.(A4) is,

$$\left.\frac{g_0^2N_c}{8\pi^2}\right|_{NJLc} = 1.00 \quad (59)$$

Hence, we see that the NJL quantum critical coupling has a remarkably similar numerical value to the classical critical coupling. (It is beyond the scope of the present paper to understand why these are not identically equal!)

Note that in what follows we will use a different definition of the coupling,

$$\kappa = \frac{g_0^2N_c}{4\pi} \qquad \kappa_c = 2\pi \quad (60)$$

where we quote the implied NJL critical value κ_c.

B. Rectangular Potential Well

Presently, we consider a generic potential, $W(r)$, to write,

$$\mathcal{M}^2 = M_0^3\int_{\vec{r}}\Big((\partial_r\phi(r))^2 + g_0^2N_cW(r)|\phi|^2\Big) \quad (61)$$

from which we obtain the SKG equation for a spherically symmetric wave-function with eigenvalue μ^2:

$$-\nabla^2\phi(r) + g_0^2N_cW(r)\phi = \mu^2\phi(r) \quad (62)$$

where $\nabla^2 = \partial_r^2 - (2/r)\partial_r$. As a warm-up exercise we turn to the rectangular potential well,

$$W(r) = -M_0^2\theta(R_0 - r) \tag{63}$$

With $\phi(r) = u(r)/r$, the SKG equation becomes (here $\epsilon = 1$),

$$-\frac{\partial^2}{\partial r^2}u(r) - g_0^2 N_c M_0^2 \theta(R_0 - r)u(r) = \mu^2 u(r). \tag{64}$$

For super-critical coupling, $g \gtrsim g_c$, we have a solution that is finite at $r = 0$, and exponentially attenuating at large r (hence normalizable),

$$u(r) = A\sin(kr)\theta(R_0 - r) + Be^{-|\mu|(r-R_0)}\theta(r - R_0) \tag{65}$$

where the eigenvalue is determined in the well as,

$$\mu^2 = k^2 - g_0^2 N_c M_0^2 \tag{66}$$

The matching boundary conditions of the field value and derivative at $r = R_0$ imply,

$$k^2\cot^2(kR_0) = k^2 - g_0^2 N_c M_0^2$$
$$\sin^2(kR_0) = \frac{k^2}{k^2 + |\mu|^2} \tag{67}$$

With critical (supercritical) binding we have $\mu^2 = 0$ ($\mu^2 < 0$). The critical value of $g = g_c$ therefore corresponds to, $k = \pi/2R_0$, and the critical coupling implies,

$$\frac{\kappa_c}{2\pi} \equiv \left.\frac{g_{0c}^2 N_c}{8\pi^2}\right|_c = \frac{1}{32M_0^2R_0^2} \tag{68}$$

We can use the square well as an approximation to the Yukawa potential. In order for this to reproduce the critical value of the NJL model, close to the classical Yukawa potential result, we require,

$$R_0 = \frac{1}{4\sqrt{2}}\, M_0^{-1} \tag{69}$$

This is a narrow rectangular potential approximation as we would expect from the e^{-2M_0}/r Yukawa form. We remark that a "frying pan" potential, in which $R_0 >\, > M_0^{-1}$, can have an arbitrarily small critical coupling. However, typical gauge boson exchange potentials will have $R_0 \sim 1/M_0$ as indicated by this result.

For the near critical coupling case $|\mu| \ll 1$ we see from eq.(67) that $\sin(kR_0) \to 1$, hence from eq.(65), $B \to A$, and therefore,

$$A^{-1}\phi(r) \approx \frac{1}{r}\theta(R_0 - r)\sin\left(\frac{\pi r}{2R_0}\right) + \frac{e^{-|\mu|(r-R_0)}}{r}\theta(r - R_0) \tag{70}$$

where A is the normalization of $\phi(r)$ and $\mu \approx 0$. We see that the critical solution $\mu = 0$ has a tail for large r, $\phi(r) \sim 1/r$. We include a small nonzero $|\mu|$ in the last term as an IR cut-off on the normalization integral. We can approximate the coupling by the critical value in the near critical case.

Recall that the normalization is defined by eq.(35),

$$1 = M_0^3\int_0^\infty 4\pi r^2|\phi(r)|^2 dr \approx 2\pi A^2 M_0^3\left(R_0 + |\mu|^{-1}\right) \tag{71}$$

We see that near criticality the normalization is dominated by the large distance tail,

$$A = \frac{M_0^{-3/2}|\mu|^{1/2}}{\sqrt{2\pi}} \tag{72}$$

The result for $\phi(0)$ is then

$$\phi(0) = \lim_{r\to 0}\frac{A}{r}\sin\left(\frac{\pi r}{2R_0}\right) = \frac{\sqrt{\pi}|\mu|^{1/2}}{\sqrt{8}M_0^{3/2}R_0} = \left(\frac{4\pi|\mu|}{M_0}\right)^{1/2} \tag{73}$$

where in the last term we inserted the NJL critical value $M_0R_0 = 1/4\sqrt{2}$.

The key observation is that $\phi(0) \sim (|\mu|/M_0)^{1/2}$ becomes small as we approach the critical value where $|\mu| \ll M_0$. This has the effect of suppressing the induced Yukawa coupling by a factor of $\phi(0)$ from eq.(51) (for $\epsilon = 1$):

$$g_Y = g_0^2\sqrt{N_c/8}\;\phi(0) \tag{74}$$

In contrast, the renormalized Yukawa coupling the NJL model diverges as a Landau pole at the scale M_0, and then evolves logarithmically, as in eq.(A8). Subsequently it can be matched onto the full RG evolution [8], which leads to the IR fixed point [21], hence in NJL model the RG evolution is slow. In the present semiclassical case the logarithmic evolution is replaced by the more rapid power-law evolution $\phi(0) \propto \sqrt{|\mu|/M_0}$.

It is instructive to derive the square well potential results in an approximation that we will develop in the next section on variational methods. We can set $R_0 = 1$ and we then assume $R_0M_0 = M_0 = 1/(4\sqrt{2}) \equiv f$ from eq.(69) for our approximation to the Yukawa potential. We write an approximate trial wave-function for $\phi(r)$, as:

$$\phi(r) = \frac{u(r)}{r}$$
$$u(r) = (r + Br^3)\theta(1 - r) + Ce^{-\mu(r-1)}\theta(r - 1) \tag{75}$$

Demanding the differentiability of $u(r)$ at $r = R_0$ (i.e., $u(r)$ is a C^1 function) we have,

$$B = -\frac{1+\mu}{3+\mu} \qquad C = \frac{2}{3+\mu} \tag{76}$$

and the Hamiltonian has dimension $1/R_0^2$, and in the field $u(r)$ is

$$\mathcal{M}^2 = 4\pi A^2 f^3\int_0^\infty dr\left(|\partial_r u(r)|^2 + g_0^2 N_c W(r)|u(r)|^2\right)$$
$$= 4\pi A^2 f^3\left(\frac{8}{15} + \frac{2}{45}\mu - g^2 N f^2\left(\frac{68}{315} - \frac{64}{945}\mu\right)\right) \tag{77}$$

where $W(r) = -f^2\theta(1 - r)$ (note, the formal difference with the $\mathcal{M}^2(\phi(r))$ expression of eq.(61) is a total derivative that integrates to zero). The normalization is,

$$A^2 = \left(4\pi f^3\int_0^\infty dr|u(r)|^2\right)^{-1} = \frac{9\mu}{8\pi f^3} - \frac{12\mu^2}{35f^3\pi} + \mathcal{O}(\mu^3) \tag{78}$$

The normalized expectation value of the Hamiltonian is obtained, restoring factors of $R_0^{-1} = M_0/f$, and $\mu \to \mu R_0 = f\mu/M_0$,

$$\mathcal{M}^2 = 4\pi A^2 f\left(\frac{8}{15}M_0^2 + \frac{2}{45}f\mu M_0 - g^2 N f^2\left(\frac{68}{315}M_0^2 - \frac{64}{945}f\mu M_0\right)\right) = \mu M_0\left(\frac{48\sqrt{2}}{5} + \frac{\mu}{5M_0} - \frac{17\sqrt{2}}{35}\pi\kappa + \frac{4\mu}{105M_0}\pi\kappa\right) \quad (79)$$

where $\kappa = g^2N/4\pi$. Note the suppression of $\mathcal{M}^2$, which would be $\sim M_0^2$ without the tail of the wave-function, but has now become $\sim \mu M_0$ due to the infrared dilution.

The critical coupling corresponds to $\mu = 0$ and implies

$$\kappa = \frac{336}{17\pi} = 6.2913, \qquad g^2N/8\pi^2 = 1.0013, \quad (80)$$

in excellent agreement with the NJL result (which was an input of 1.000 to define f; this calculation essentially amounts to replacing the $\sin(kr)$ by the first two terms in its series expansion).

Another consideration is the fine-tuning for a hierarchy, $\mu/M_0 << 1$. The result for $\mathcal{M}^2$ is the eigenvalue $-|\mu|^2$. Therefore we have, keeping the leading terms in μ^2 in the Hamiltonian,

$$-\mu^2 \approx \mu M_0\left(\frac{48\sqrt{2}}{5} - \frac{17\sqrt{2}}{35}\pi\kappa\right) \quad (81)$$

hence the linear relation:

$$\kappa = 6.2913 + 0.90499\frac{\mu}{M_0} \quad (82)$$

This means that fine-tuning to achieve a small value of μ/M_0 is highly suppressed, $\delta\kappa/\kappa \sim \mu/M_0$ as opposed to μ^2/M_0^2 without the dilution. This is due to the approximate the linearity of this relationship.

C. Variational Calculations

A solution to the SKG equation for the eigenvalue can be computed approximately in a variational calculation. In using variational methods it is important that the ansatz for the field configuration be a continuous function of the field value and its first derivative (a C^1 function). Discontinuities in the kinetic term lead to unwanted spikes in the Hamiltonian and affect the energetics. It is also important, if possible, to use known properties of the solution for the asymptotics.

We demonstrate this with a crude approximation in the present section that shows $\epsilon = 1$ is the extremal solution for a bound state. We then illustrate a refined "spline" approximation that obtains precise (and interesting) results in Section III D.

First we recall $M = \epsilon M_0$, as the mass scale in the ansatz, reintroducing the parameter ϵ. As in the NJL model, we view this as a UV cut-off theory where the largest mass scale is the coloron mass M_0, the largest mass scale at which the static potential approximation is applicable. Hence we require $\epsilon \leq 1$. ϵ is seen to multiply the underlying coupling constant, $\tilde{g}_0^2 = \epsilon g_0^2$. The largest value of the "effective coupling," $\tilde{g}_0^2$, is therefore g_0^2, hence $\epsilon = 1$ implies the smallest possible critical value of the underlying coupling g_0^2. However, we can view ϵ as part of the wave-function ansatz, and allow the variational calculation of the bound state mass to produce $\epsilon = 1$ to minimize the Hamiltonian, $\mathcal{M}^2$, of eq.(44) with the Yukawa potential of eq.(41).

We consider a simple example of assuming an ansatz that is a Hydrogenic wave-function, $\widetilde{\phi}(r) = Ae^{-Mr}$, with $M = \epsilon M_0$ and ϵ as the variational parameter, and M_0 is the scale of the Yukawa potential. This cannot be an accurate description near criticality where the eigenvalue μ^2 is small because it lacks the large distance tail $\propto e^{-\mu r}/r$ for small μ (we'll include that subsequently). However, this gives a rough approximation and establishes $\epsilon \to 1$ dynamically.

The normalization of the ansatz is defined in eq.(35),

$$1 = 4\pi A^2\epsilon^3 M_0^3\int_0^\infty e^{-2\epsilon M_0 r}r^2 dr \qquad A^2 = \frac{1}{\pi} \quad (83)$$

The Hamiltonian $\mathcal{M}^2$ of eq.(44) with $M = \epsilon M_0$ is

$$\mathcal{M}^2 = \epsilon^3 M_0^3\int_{\vec{r}}\left(|\partial_{\vec{r}}\phi|^2 + g_0^2 N_c\epsilon M_0 V_0(2r)|\phi|^2\right) \quad (84)$$

where $V_0(2r) = -e^{-2M_0 r}/8\pi r$ as in eq.(41), with the fixed coloron mass M_0 (no ϵ factor is present in $V_0(2r)$!). We then compute the eigenvalue $\mu^2 = \mathcal{M}^2$ as a function of ϵ and κ:

$$\begin{aligned}\mathcal{M}^2 &= A^2\epsilon^3 M_0^3\int_{\vec{r}}\left(|\partial_{\vec{r}}\phi|^2 - \frac{\kappa\epsilon M_0}{2}\frac{e^{-2M_0 r}}{r}|\phi|^2\right)\\ &= 4\epsilon^3 M_0^3\int_0^\infty\left(\epsilon^2 M_0^2 e^{-2\epsilon M_0 r} - \frac{\kappa\epsilon M_0}{2}\frac{e^{-2(1+\epsilon)M_0 r}}{r}\right)r^2 dr\\ &= M_0^2\left(\epsilon^2 - \frac{\kappa\epsilon^4}{2(1+\epsilon)^2}\right); \qquad \kappa = \frac{N_c g_0^2}{4\pi}\end{aligned} \quad (85)$$

In Fig.(1) we plot a family of curves of $\mathcal{M}^2 = \mu^2$ for various values of κ as function of ϵ. We see that the extremal (smallest) value for positive μ^2 is $\mu^2 = 0$ and occurs for any $\kappa < 8$. On the other hand, for $\kappa > 8$ the extremal (most negative) value of $\mu^2 = \mathcal{M}^2$ occurs when $\epsilon \to 1$. Hence, the critical coupling is $\kappa = 8$ and we have

$$\frac{\kappa}{2\pi} = \frac{g_{0c}^2 N_c}{8\pi^2} = 1.27, \quad (86)$$

compared to the NJL critical value 1.00.

A negative eigenvalue $\mu^2 = \mathcal{M}^2$ is determined where the corresponding $\kappa > 8$ curve intersects the vertical line at $\epsilon = 1$, that is,

$$\mu^2 = M_0^2\left(1 - \frac{\kappa}{8}\right). \quad (87)$$

Since for subcritical, $\kappa < 8$, the eigenvalue $\mu^2 = 0$, we see that is a second order phase transition behavior in $|\mu|$ vs κ.

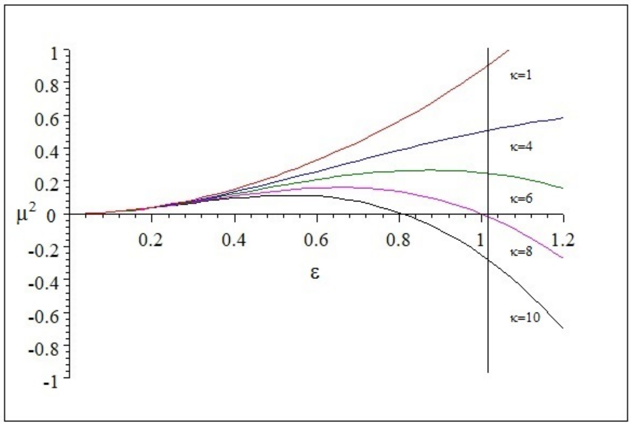

FIG. 1: $\mathcal{M}^2 = \mu^2$ of eq.(84) is plotted vs. ϵ, with $M_0 = 1$ for values of $\kappa = (1, 4, 6, 8, 10)$. The critical coupling is the value of κ for which a massless bound state occurs, $\mathcal{M}^2 = 0$, hence where the $\kappa = 8$ curve intersects with the $\epsilon = 1$ vertical line. For supercritical $\kappa > \kappa_c$ we have $\epsilon = 1$ and the eigenvalues μ^2 are negative; For subcritical $\kappa < \kappa_c$ we have $\epsilon = 0$ and the eigenvalues μ^2 are positive. Resonances can exist in scattering states with eigenvalues $\vec{q}^{\ 2}$ as in eq.(36) with plane wave volume normalization.

The variational ansatz isn't too far off, however, this gives a false value for the normalized trial wave-function at the origin for critical coupling,

$$\phi(0) = \frac{1}{\sqrt{\pi}} \tag{88}$$

The reason is, of course, that this variational calculation truncates the $1/r$ tail at large r, which significantly affects the normalized $\phi(0)$. We now do a refined calculation that demonstrates the effects of the large distance tail of $\phi(r)$.

D. Spline Approximation

We can improve the method by constructing an ansatz for $\phi(r)$ that implements the short distance $e^{-M_0 r}$, but approximates the correct asymptotic form, $\sim e^{-\mu r}/r$, at large r.

For μ small (near critical coupling) this is approximately $1/r$, and we then have a convenient "spline" (power + exponential) for $\phi(r)$,

$$\phi(r) = A\left(e^{-M_0 r}\theta(1 - M_0 r) + \frac{e^{-1}}{M_0 r}\theta(M_0 r - 1)\right) \tag{89}$$

where the step function is $\theta(x) = 1$ ($= 0$) for $x > 0$ (< 0). The spline is differentiable (C^1) which avoids the discontinuity in the value of the function, as well as $\partial\phi \sim \delta(r - M)$ in its first derivative, which would lead to "energy spikes" in the kinetic term.

With this definition of $\phi(r)$, however, we have an infrared divergent normalization and kinetic term integrals which require a cut-off with a small mass $|\mu|$. The cut-off is equivalent to redefining the second term in eq.(89) as $\rightarrow (e^{-1}/M_0 r)\theta(M_0 r - 1)\theta(1 - |\mu| r)$, which allows the integrals to run from 0 to ∞. The cut-off at $|\mu|^{-1}$ imitates the desired true asymptotic form, $\sim e^{-|\mu| r}/r$, and maintaining the continuity $\phi(r)$ at $r = M_0^{-1}$.

Now, however, we have introduced a discontinuity at $r = |\mu|^{-1}$ and an unwanted $\delta(r - |\mu|^{-1})$ in the kinetic term. To remedy this we can simply extend the spline with a pure decaying exponential, as:

$$\phi(r) = A\Bigg(e^{-M_0 r}\theta(1 - M_0 r) + \frac{e^{-1}}{M_0 r}\theta(M_0 r - 1)\theta(1 - |\mu| r) + \frac{|\mu|}{M} e^{-|\mu| r}\theta(|\mu| r - 1)\Bigg) \tag{90}$$

This is now a differentiable function with value and derivatives matching at $Mr = 1$ an $|\mu| r = 1$. and we have eliminated the IR divergence and can use this for any value of $|\mu|$.

We have experimented with several ansatze and prefer the simplicity of eq.(90). We find splining to an asymptotic function $\sim e^{-|\mu| r}/r$ leads to more cumbersome integrals. Note the discontinuity at the origin, $\partial_r \theta(r) e^{-M_0 r} \sim \delta(r)$, receives a factor of zero in the integrals due to the $r^2 dr$, hence causes no discontinuity problem at $r = 0$. One might worry that this wave-function is not C^2 and cannot evidently satisfy the equation of motion, however this is a variational ansatz and none of it satisfies the equation of motion, yet it can produce reasonably reliable Hamiltonian expectation values.

The normalization of the trial wave-function of eq.(90) must satisfy eq.(35), defined by A given by,

$$1 = 4\pi A^2 M_0^3 \Bigg(\int_0^{M_0^{-1}} e^{-2M_0 r} r^2 dr + \int_{M_0^{-1}}^{|\mu|^{-1}} \left(\frac{e^{-1}}{Mr}\right)^2 r^2 dr + \int_{|\mu|^{-1}}^{\infty} \left(\frac{|\mu|}{M}\right)^2 e^{-2|\mu| r} r^2 dr\Bigg) \tag{91}$$

hence, defining $x = |\mu|/M_0$ and $M_0 = 1$ we have

$$A = \frac{1}{\sqrt{\pi}}\left(1 + 9e^{-2}\left(\frac{1}{x} - 1\right)\right)^{-1/2} \tag{92}$$

We see this is dominated by the tail of the wave-function for small $|\mu|$ as:

$$A \approx \frac{e}{3\sqrt{\pi}}\left(\frac{|\mu|}{M_0}\right)^{1/2} \tag{93}$$

We then compute $\mathcal{M}^2$ as (recall $\kappa = g_0^2 N_c/4\pi$):

$$\mathcal{M}^2 = 4\pi M_0^3\left(\int_{\vec{r}}\left(|\partial_{\vec{r}}\phi|^2 - \frac{\kappa M_0}{2}\frac{e^{-2M_0 r}}{r}|\phi|^2\right)\right.$$
$$= 4\pi A^2 M_0^3\left(\int_0^{M_0^{-1}}\left(M_0^2 e^{-2M_0 r} - \frac{\kappa M_0}{2}\frac{e^{-4M_0 r}}{r}\right)r^2 dr\right.$$
$$+\int_{M_0^{-1}}^{|\mu|^{-1}}\left(\frac{e^{-1}}{Mr}\right)^2\left(\frac{1}{r^2} - \frac{\kappa M_0}{2}\frac{e^{-2M_0 r}}{r}\right)r^2 dr$$
$$\left.+\int_{|\mu|^{-1}}^{\infty}\left(\frac{|\mu|}{M}e^{-\mu r}\right)^2\left(\mu^2 - \frac{\kappa M_0 e^{-2M_0 r}}{2r}\right)\right)r^2 dr\right) \quad (94)$$

We obtain the result:

$$\mathcal{M}^2 = \pi A^2\left(1 - e^{-2}(1-x) - \kappa\left(2e^{-2}(\mathrm{Ei}(1,2) - \mathrm{Ei}(1,2/x))\right.\right.$$
$$\left.\left.+\frac{1}{8}\left(1-e^{-4}\right) + \frac{1}{2}\frac{3x^2+2x}{(1+x)^2}e^{-2(1+x)/x}\right)\right) \quad (95)$$

where the exponential integral is $\mathrm{Ei}(x) = \int_{-\infty}^{x}(e^t/t)dt$.

We proceed to compute $\mathcal{M}^2(\kappa)$. We are interested only in negative eigenvalues for the compact wave-function. We input a trial value of $|\mu|/M_0$ (where we set $M_0 = 1$) and compute $\mathcal{M}^2 = \mu^2$ for given values of κ. This leads to the family of curves seen in Fig.(2).

The eigenvalue $\mu^2 = -|\mu|^2 = \mathcal{M}^2$ is produced by this formula. However, we have used as input $|\mu|/M_0$ to define the large distance tail of the wave-function. The resulting output $\mu^2 = -|\mu|^2$ *must self-consistently match the input value* $|\mu|$. Fig.(3) shows $\mathcal{M}^2$ with $M_0 = 1$ plotted for given values of κ vs the input $|\mu|$ with $M_0 = 1$. Self-consistent solutions occur where the given κ curve intersects a μ^2 curve (thick curve Fig.(3)). The intersections implicitly determine κ for any consistent value of input $|\mu|$.

The logic of the plots is that we are seeking the value of κ that best yields a self-consistent solution for given μ (as in the critical coupling determination where we sought the best value of κ that yields $\mu = 0$). This then gives us the value of μ for a given input value of κ. In Fig.(4) we plot the values of κ vs. the corresponding consistent value of input $|\mu|$. We now find the critical coupling, but with $\kappa_c = 6.82$, hence $g_0^2 N_c/8\pi^2 = 1.082$, slightly larger than the NJL result, and very close to the exact Yukawa potential result $g_0^2 N_c/8\pi^2 = 1.06940$ of eq.(58). The tail-spline calculation has significantly improved the precision determination of the variational calculation of the critical behavior.

Note that, to an excellent approximation we now find a linear relation of κ and eigenvalue μ as

$$\kappa = 6.8197 + 10.693|\mu|/M_0 \quad (96)$$

Likewise, $\phi(0)$ is then determined by A in eq.(92) to an excellent approximation in the limit eq.(93).

If we again demand, $g_Y = 1$, as in a top quark condensation model, then from eqs.(51,93),

$$g_Y = 1 = \hat{g}_Y\phi(0) = \sqrt{\frac{N_c}{8}\left(\frac{8\pi^2}{N_c}\right)\frac{e}{3\sqrt{\pi}}\left(\frac{|\mu|}{M_0}\right)^{1/2}} \quad (97)$$

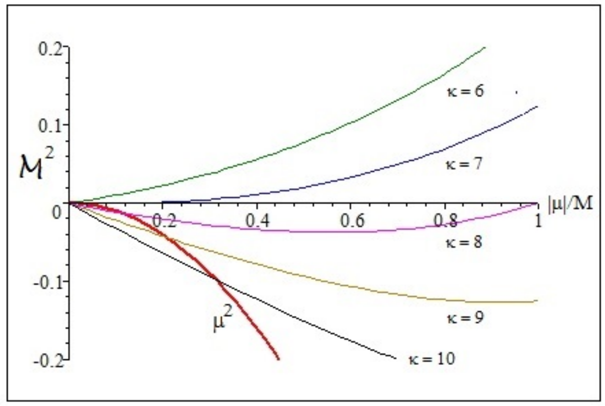

FIG. 2: $\mathcal{M}^2$ of eq.(95) with eq.(92) is plotted vs. μ/M_0 for $M_0 = 1$, for values of $\kappa = (6, 7, 8, 9, 10)$. The thick (red) curve is the eigenvalue μ^2. Consistent solutions occur where the $-|\mu|^2$ curve intersects the $\mu^2 = \mathcal{M}^2$ curves for given value of κ. The critical coupling is the smallest value of κ for which these curves do not intersect, $\approx \kappa = 6.8198$. For smaller values we have no solution with negative μ^2.

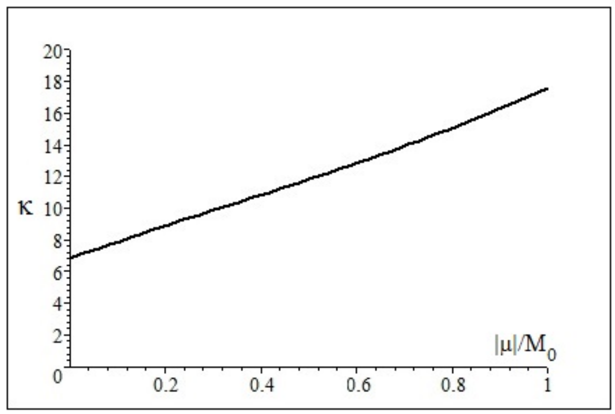

FIG. 3: The value of the coupling $\kappa = g_0^2 N_c/4\pi$ vs the bound state mass $|\mu|$ which consistently matches the trial input value of μ/M_0 to the eigenvalue $\mu^2 = \mathcal{M}^2$ (for negative μ^2). The result is fit by eq.(96, $\kappa = 6.8197 + 10.693|\mu|/M_0$. This implies that the fine-tuning of a hierarchy is significantly reduced as $\delta\kappa/\kappa \sim |\mu|/M_0$.

hence

$$\frac{|\mu|}{M_0} = \left(\frac{N_c}{8\pi^2}\right)^2\left(\frac{8}{N_c}\right)\frac{9\pi}{e^2} = 2.6\times 10^{-2} \quad (98)$$

with $N_c = 3$ and the normalization at the origin. In a top condensation model with $|\mu| = 88$ GeV this would imply a coloron mass scale of order $M_0 \sim 6$ TeV. Due to the linear relationship between κ and $|\mu|/M_0$ we see the degree of fine-tuning a hierarchy is of order $\delta\kappa/\kappa \sim |\mu|/M_0 \sim 1.4\%$. This is an astonishing improvement over the old NJL based top condensate theory [8], an indicates a possible top quark coloron in the ~ 5–10 TeV range.

IV. FERMION LOOPS

Presently we focus on the induced Yukawa interaction of eq.(52):

$$S' \to -\sqrt{N_c J/2} g_0^2 M_0^2 \times \int_{X\vec{r}} [\overline{\psi}_L(X^+)\psi_R(X^-)]_f \chi(X) U(r) + h.c. \tag{99}$$

where $X^{\mu\pm} = (X^0, \vec{X} \pm \vec{r})$ and we define the combination,

$$U(\vec{r}) = V(2\vec{r})\phi(\vec{r}) \tag{100}$$

This is the analogue of the fermion loop in the NJL model of figure(4) as in the leading large-N_c calculation. The Yukawa coupling involves $\phi(r)$ to the free fermion bilinear, $[\overline{\psi}_L(X^+)\psi_R(X^-)]$, via the potential $V(2r)$.

The meaning of the "free fermions" now presents a subtlety: a free fermion bilinear can itself be viewed as a bilocal field describing a scattering state, $[\overline{\psi}_L(X^+)\psi_R(X^-)] \sim \Omega(X,r)$ and this would be a solution to the SKG equation with positive eigenvalue, q^2, and must therefore be orthogonal to $\Phi(X,r)$:

$$\int_{X\vec{r}} \Phi(X,r)^\dagger \Omega(X,r) = 0 \tag{101}$$

This in turn implies a nontrivial constraint on the basis functions that comprise the free fermion states, and thus the Feynman propagator. Essentially, we have extracted the particular combination of free fermion field basis functions that comprise Φ from the complete set of basis functions that would normally appear in an arbitrary $\Omega(X,r)$.

For the present calculation, however, we can consider Φ to be a near critical bound state with an extended $\phi(r)$. This serves to dilute the loop by a factor of $\phi(0)^2$. We would expect that the orthogonality projection of Φ out of the set of free field basis functions would lead to a further correction of order $\phi(0)^4$. We will therefore approximate the free fermions by plane wave Dirac fields and do the loop as we would with the usual free Feynman propagator.

The free field Feynman fermion propagator then takes the usual form,

$$S_F(x) = i\int \frac{d^4\ell}{(2\pi)^4}\frac{\not{\ell}}{\ell^2} e^{i\ell_\mu x^\mu} \tag{102}$$

and the left handed projection operator is $\mathcal{P}_5 = (1-\gamma^5)/2$ (we follow conventions of [31]).

The fermion loop integral becomes,

$$I = F M_0^4 \int \frac{d^4\ell}{(2\pi)^4} d^3r d^3r' \frac{\ell\cdot(\ell+P)}{\ell^2(\ell+P)^2} U(\vec{r})U^\dagger(\vec{r}\,') e^{2i\vec{\ell}\cdot(\vec{r}-\vec{r}\,')} \tag{103}$$

where we have the combinatorial overall factor,

$$F = -2JN_c g_0^4 N_c^2 \int d^4X |\chi_0|^2 \tag{104}$$

Note the factor of $g_0^4 N_c^2$ from the internal wave-function interaction, eq.(52), the additional factor of N_c from the loop.

A. $P = 0$ Result

For simplicity first consider $P = 0$, and do the ℓ_0 integral by residues:

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2-\mu^2+i\epsilon} = \frac{i}{2}\int\frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{(\vec{\ell}^2+\mu^2)^{1/2}} \tag{105}$$

to obtain,

$$\begin{aligned} I &= \frac{i}{2} F M_0^4 \int d^3r d^3r' \int_\mu^{M_0} \frac{d^3\vec{\ell}}{(2\pi)^3}\frac{1}{|\vec{\ell}|} U(\vec{r})U^\dagger(\vec{r}\,') e^{2i\vec{\ell}\cdot(\vec{r}-\vec{r}\,')} \\ &= \frac{i}{32\pi^2} F M_0^4 \int d^3r d^3r' U(\vec{r})U^\dagger(\vec{r}\,') \times \\ &\quad \times \left[\frac{\sin^2(M_0|\vec{r}-\vec{r}\,'|) - \sin^2(\mu|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|^2}\right]. \end{aligned} \tag{106}$$

where we introduced the $\vec{\ell}$ integral with UV cut-off M_0 and IR cut-off μ.

The result of eq.(106) is general. We can now take the pointlike limit for the potential as in eq.(41) to compare to the NJL model:

$$V_0(2r) \to -\frac{1}{M_0^2}\delta^3(2\vec{r}) = -\frac{2}{JM_0^2}\delta^3(\vec{r}) \tag{107}$$

to obtain:

$$\begin{aligned} I &= i\frac{N_c}{8\pi^2}|\phi(0)|^2(M_0^2-\mu^2)(2g_0^4 N_c/J)\int d^4X|\chi_0|^2 \\ &= i\frac{\hat{g}_Y^2|\phi(0)|^2 N_c}{8\pi^2}(M_0^2-\mu^2)\int d^4X|\chi_0|^2 \end{aligned} \tag{108}$$

Recall the induced Yukawa coupling of eq.(51),

$$g_Y = g_0^2\sqrt{2N_c/J}\phi(0) = \hat{g}_Y\phi(0) \tag{109}$$

to obtain,

$$I = i g_Y^2 N_c \int d^4X |\chi_0|^2 \frac{M_0^2-\mu^2}{8\pi^2} \tag{110}$$

We have recovered the usual momentum space results of eq.(A5) as in [8].

Note the key difference with the NJL model is that, in NJL we would have $g_Y = g_0$, which is very large near criticality. However, presently we have $g_Y = \hat{g}_Y\phi(0) \equiv g_0^2\sqrt{2N_c/J}\phi(0)$ and g_Y is significantly diluted near criticality by $\phi(0)$.

B. P Dependence

Keeping the P dependence to $\mathcal{O}(P^2)$ from eq.(103) yields, $I = I_0 + I_P$ and we use

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell\cdot(\ell+P)}{\ell^2(\ell+P)^2} = \frac{i}{2}\int\frac{d^3\vec{\ell}}{(2\pi)^3}\left(\frac{1}{|\vec{\ell}|}+\frac{P^2}{2\hat{\ell}^3}\right) \tag{111}$$

hence

$$I_P = \frac{i}{8} F M_0^4 \int d^3r d^3r' \int_\mu^{M_0} \frac{d|\vec{\ell}|}{4\pi^2}\frac{U(\vec{r})U^\dagger(\vec{r}\,')\sin(2|\vec{\ell}||\vec{r}-\vec{r}\,'|)}{|\vec{\ell}|^2|\vec{r}-\vec{r}\,'|}\frac{P^2}{2} \tag{112}$$

We use an approximate result for the log-divergent integral,

$$\int_{\mu}^{M_0} \frac{\sin(2xR)}{x^2 R} dx$$

$$\approx \frac{\sin(2\mu R)}{\mu R} - 2(\gamma + \ln(2\mu R) - R^2\mu^2) + O\left(\frac{1}{M_0}, \mu^3\right) \quad (113)$$

to obtain,

$$I_P = \frac{i}{32\pi^2} F M_0^4 \int d^3r d^3r' U(\vec{r}) \mathrm{U}^\dagger(\vec{r}') \frac{P^2}{2}$$
$$\left(\frac{\sin(2\mu|\vec{r}-\vec{r}'|)}{\mu|\vec{r}-\vec{r}'|} - 2\ln(2\mu e^\gamma|\vec{r}-\vec{r}'|) - 2\mu^2|\vec{r}-\vec{r}'|^2\right)$$
$$-(\mu \to M_0) + \mathcal{O}(1/M_0, \mu^3) \quad (114)$$

Taking the pointlike limit gives,

$$I_P = i\frac{\hat{g}_Y^2|\phi(0)|^2 N_c}{8\pi^2} P^2 \int d^4X |\chi_0|^2 \ln\left(\frac{2M_0}{\mu} e^\gamma\right)$$
$$= i\frac{g_Y^2 N_c}{8\pi^2} P^2 \int d^4X |\chi_0|^2 \ln\left(\frac{2M_0}{\mu} e^\gamma\right) \quad (115)$$

where the log structure matches the result of eq.(A5). Hence the wave-function renormalization constant of the Φ field is

$$Z = 1 + \frac{g_Y^2 N_c}{8\pi^2}\left(\ln(M_0/\mu) + c\right) \quad (116)$$

in agreement with the NJL loop result, eq.(A5) (the difference is the additive constant in the limit $g_Y \to 0$). Hence this loop calculation, which drives the entire kinetic term structure of the NJL model, is now a perturbative correction in our semiclassical scheme.

We remark that here we see something noteworthy: In the NJL model the limit $|\vec{r}-\vec{r}'| < M_0^{-1}$ is inconsistent with treating M_0 as a momentum space cut-off. That is, if we insist upon momentum scales above M_0 to be disallowed then in configuration space we must require $|\vec{r}-\vec{r}'| \gtrsim M_0^{-1}$. This informs us that the usual NJL model assumption of a cut-off theory at scale M_0 is potentially inconsistent.

C. Quartic Interaction

As in the NJL model, the fermion loops will induce a quartic interaction. A full calculation in configuration space is tedious but leads to results similar to the following in the pointlike potential limit.

The loop for a quartic interaction has four bilocal vertices and takes the form:

$$\lambda = -\frac{1}{2} F_4 \int \frac{d^4\hat{\ell}}{(2\pi)^4} \frac{1}{\hat{\ell}^4} \prod_{i=1}^{4} d^3r_i \; |U(\vec{r}_i)|^4 e^{2i\vec{\hat{\ell}}\cdot\vec{r}_i} \quad (117)$$

where

$$\int d^4r \; e^{2i\vec{p}_1\cdot\vec{r}} D(2r)\phi(\vec{r}) = -\frac{1}{2}\int d^3r \; e^{2i\vec{p}_1\cdot\vec{r}} V_0(2\vec{r})\phi(\vec{r})$$
$$F_4 = \left(N_c(\sqrt{2N_c J} g_0^2 M_0^2)^4 \int d^4X |\chi_0|^4\right) \quad (118)$$

We integrate $\hat{\ell}^2$ from μ^2 to M_0^2 as IR and UV cut-offs (we could equally well include μ^2 in the propagator denominator for the IR cu-off with similar results). The pointlike limit for the potential is as above,

$$V_0(2r) \to -\frac{2}{JM_0^2}\delta^3(\vec{r}) \quad (119)$$

and obtain

$$\frac{\lambda}{2} = F_4\left(\frac{1}{JM_0^2}\right)^4 |\phi(0)|^4 \int_{\mu^2}^{M_0^2} \frac{d^4\hat{\ell}}{(2\pi)^4}\frac{1}{\hat{\ell}^4} \int d^4X|\chi_0|^4$$
$$= \frac{N_c}{8\pi^2}\hat{g}_Y^4|\phi(0)|^4\left(\ln\left(\frac{M_0}{\mu}\right) + \mathcal{O}\left(\frac{\mu^2}{M_0^2}\right)\right)\int d^4X|\chi_0|^4 \quad (120)$$

The log evolution matches the result for the pointlike NJL case of eq.(A5), with $g_Y = \hat{g}_Y\phi(0)$,

$$\lambda \approx \frac{N_c g_Y^4}{4\pi^2}\ln\left(\frac{M_0}{\mu}\right) + \text{constant}. \quad (121)$$

V. SPONTANEOUS SYMMETRY BREAKING

With super-critical coupling, $g_0 > g_{0c}$, the bilocal field $\Phi(X,r)$ has a negative $(\text{mass})^2$ eigenvalue (tachyonic), μ^2 with a well-defined localized wave-function. In the region external to the potential (forbidden zone) the field is exponentially damped. At exact criticality there is a $1/r$ tail that switches to exponential damping for $g_0 > g_{0c}$. The supercritical solutions are localized and normalizable over the entire space $\vec{r}$, but with $\mu^2 < 0$ they would lead to exponential runaway in time of the field $\chi(X^0)$, and must be stabilized, typically with a $\sim \lambda|\Phi|^4 \to \lambda|\chi|^4$ interaction.

We then treat the supercritical case as resulting in spontaneous symmetry breaking. The spontaneous symmetry broken phase is then a configuration where the field $\Phi(X,r) = \chi(X)\phi(r)$, and where $\chi(X)$ develops a vacuum expectation value (VEV), while $\phi(r)$ remains a localized wave-function satisfying the SKG equation. . We then obtain the "sombrero potential",

$$V(\chi) = -|\mu|^2|\chi|^2 + \frac{\lambda}{2}|\chi|^4 \quad (122)$$

The field χ develops a VEV, $\langle\chi\rangle = v = |\mu|/\sqrt{\lambda}$.

The external scattering state "free" fermions, $\psi_f^a(X)$, will then acquire mass through the Yukawa interaction, the second term in eq.(17), $\sim g_Y v[\overline{\psi}_L\psi_R]_f + h.c.$. In our treatment the internal fermion pair belongs to $\phi(r)$ which interacts with itself through the SKG equation and loop induced quartic interaction. It then drives the *external free fermions* to acquire mass. At leading order there is no "back reaction" on the internal fermions that bind to comprise $\phi(r)$. We segregated the free external fermions from the bound state wave-function, Φ, in eq.(17), so while Φ forms a VEV as described above, and the scattering state fermions independently acquire mass as spectators, the internal dynamics are as described by Φ_0 as in eq.(17).

In a more general large-distance potential we may consider possible feedback on the r dependence of $\phi(r)$, rather than the pointlike limit of the potential where things depend only upon $\phi(0)$. The simplest large-distance sombrero potential might be modeled as,

$$S = \int'_{X\vec{r}} \left(|\phi|^2 \left| \frac{\partial \chi}{\partial X} \right|^2 - |\chi|^2 (|\nabla_r \phi|^2 + g^2 N_c M V(r) |\phi(r)|^2) \right.$$
$$\left. - |\chi|^4 \frac{\hat{\lambda}}{2} |\phi(\vec{r})|^4 \right) \tag{123}$$

In the simplest case of a perturbatively small $\hat{\lambda}$ we expect the eigensolution of ϕ to be essentially unaffected

$$\int'_r \left(-|\partial_{\vec{r}} \phi|^2 + g^2 N_c M V(r) |\phi(\vec{r})|^2 \right) \approx -\mu^2 \tag{124}$$

The effective quartic coupling is then further renormalized by

$$|\chi|^4 \int'_r \frac{\hat{\lambda}}{2} |\phi(\vec{r})|^4 = |\chi|^4 \frac{\widetilde{\lambda}}{2} \tag{125}$$

Then χ develops a VEV in the usual way:

$$\langle |\chi|^2 \rangle = |\mu^2| / \widetilde{\lambda} = v^2 \tag{126}$$

This is consequence of $\phi(\vec{r})$ remaining localized in its potential

However, for general $\widetilde{\lambda}$, possibly large (as in a non-linear sigma model), the situation is potentially more complicated. The VEV is determined by joint integro-differential equations for constant χ

$$0 = -\int'_r \left(-|\partial_{\vec{r}} \phi|^2 + g^2 N_c M V(r) |\phi(\vec{r})|^2 + \widetilde{\lambda} |\chi|^2 | |\phi(\vec{r})|^4 \right)$$
$$0 = -\nabla_r^2 \phi - g^2 N_c M V(r) \phi(r) - \widetilde{\lambda} |\chi|^2 |\phi(r)|^2 \phi(r) \tag{127}$$

If we can can solve the second local equation then the global one follows, but we see that $\phi(r)$ cannot become constant in a potential $V(r)$ which has r dependence! While perturbative solutions maintain locality in $\phi(\vec{r})$, it is unclear what solutions exist to the latter equation for non-perturbative λ.

VI. SUMMARY AND CONCLUSIONS

Bilocal fields, introduced by Yukawa [4], provided a starting point for a theory of correlated pairs of fermions in a Lorentz invariant action. Our formalism, inspired by the NJL model [5], is a semiclassical theory of bound states and yields a sensible physical picture. The introduction of the bilocal field, $\Phi(x, y) = \overline{\psi}_L(x) \psi_R(y)$, is a bosonization of the fermion pair, and simplifies many aspects of the formalism. We differ from Yukawa in the treatment of "relative time." The physical mass scale for a bound state is inherited from the interaction. Our present construction leads to a number of novel results and the following is a synopsis.

To describe a scalar bound state we can write a factorized bilocal field ansatz, $\Phi = \chi(X)\phi(r)$, where $\chi(X)$ is normal pointlike field that describes the center of mass motion, and $\phi(r)$ is the internal wave-function that describes the structure of the bound state. In the rest frame, the relative time, r^0, disappears and can be integrated out. The "internal wave-function" $\phi(\vec{r})$ is then a static function of the constituent's separation, $2\vec{r}$, and $\chi(X) \to \chi(X^0)$ has only time dependence.

We consider the coloron model [11][15][16], which is a single boson exchange of a massive gluon, of mass M_0, in leading order of large N_c. This leads in the rest frame to a Hamiltonian for the internal wave function with a static Yukawa potential:

$$\mathcal{M}^2 = M_0^3 \int d^3 r \left(|\partial_{\vec{r}} \phi|^2 - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi |\vec{r}|} |\phi|^2 \right) \tag{128}$$

Here $\phi(r)$ is normalized as,

$$M_0^3 \int d^3 r \ |\phi(\vec{r})|^2 = 1 \tag{129}$$

The potential is semiclassically enhanced by a factor of N_c, the number of colors, by the color singlet normalization of $\Phi(x, y)$.

By variation of $\mathcal{M}^2$ we obtain the Schrödinger-Klein-Gordon equation and it's eigenvalue, μ^2:

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \phi(r) - g_0^2 N_c M_0 \frac{e^{-2M_0|\vec{r}|}}{8\pi |\vec{r}|} \phi(r) = \mu^2 \phi(r) \tag{130}$$

The eigenvalue, μ^2, is then the physical *squared mass* of the bound state, $\chi(X)$, field in any frame. Here "binding" represents a negative μ^2 and a chiral instability of the vacuum.

We also obtain an induced Yukawa coupling of the bound state to external unbound fermions, $g_Y = g_0^2 \phi(0) \sqrt{N_c/8}$. The main feature is that $g_Y \propto \phi(0)$ with interesting consequences.

We use the Hamiltonian directly in variational calculations of $\phi(r)$. First, we establish that the critical value of our Yukawa potential, i.e. the value $g_0^2 = g_{0c}^2$ for which $\mu^2 = 0$, is equivalent to that determined in the literature for screened Coulomb potentials, (which are of the Yukawa form). To high precision, [28] we find, remarkably, that this corresponds very closely to the critical value in the NJL model obtained at loop level:

$$\left. \frac{g_{0c}^2 N_c}{8\pi^2} \right|_{screened} = 1.06940 \qquad \left. \frac{g_{0c}^2 N_c}{8\pi^2} \right|_{NJL} = 1.00. \tag{131}$$

Moreover, in a variational calculation, using a splined-wave-function, we obtain in the present formalism:

$$\left. \frac{g_0^2 N_c}{8\pi^2} \right|_{present} = 1.082 \tag{132}$$

in good agreement with the screened Coulomb result. The quantitative agreement with the loop level NJL model is striking and we are unaware of it's being previously noted.

For subcritical coupling there are unstable resonances decaying into their constituents, which are non-normalizable solutions of the SKG equation with incoming and outgoing radiative tails. Supercritical coupling, $g_0^2 > g_{0c}^2$, implies a bound state with a negative μ^2 eigenvalue, and therefore spontaneous chiral symmetry breaking must occur.

We mainly study the large M_0 limit in which the Yukawa potential approaches a $\delta^3(\vec{r})$ potential. While this becomes the NJL potential, nonetheless, in this limit $\phi(r)$ remains a spatially extended field and the theory remains non-pointlike. This is a major difference with the NJL model.

Our most important results have to do with $\phi(r)$ near criticality. As we approach critical coupling $g_0^2 \to g_{0c}^2$ the $\phi(r)$ becomes scale invariant, and we have for $r \gg M_0^{-1}$,

$$\phi(r) \sim \frac{Ae^{-|\mu|r}}{r} \to \frac{A}{r} \qquad |\mu| \ll M_0 \tag{133}$$

where A is the normalization constant. The normalization of $\phi(r)$ is then dominated by the long distance tail. As $r \to 0$, and $|\mu| < M_0$, the $\phi(r)$ solutions tend to a constant, $\phi(0)$. This is suppressed as $\phi(0) \propto (|\mu|/M_0)^{1/2}$.

This "infrared dilution effect" of $\phi(0) \ll 1$ has significant implications on the results of the theory that are quite different than those of the NJL model. The induced Yukawa coupling is $g_Y \propto \phi(0)$. This means that, though $g_0^2 \gg 1$, the value of $g_Y \ll g_0$ dilutes quickly to small values. In contrast, in the NJL model the value of g_Y is suppressed, but only logarithmically $g_Y \sim 1/\ln(M_0/\mu)$ at leading N_c fermion loop level, and evolves relatively slowly with scale into the IR, even if the full renormalization group is applied [8][21]. In the present semiclassical scheme, the suppression is fast and power-law, $\sim (|\mu|/M_0)^{1/2}$. This decouples the strong dynamics underlying the bound state at short distances, making it effectively perturbative at low energies. The "custodial symmetry" of this dilution is scale invariance, as we approach the critical coupling.

For example, applying the NJL model in top condensation models [8][15][16], then $\mu^2 = -(88)^2$ GeV^2, the Lagrangian BEH mass of the standard model, and we would typically require $|\mu|/M_0 \sim 10^{-15} - 10^{-19}$ to get g_Y of order unity (g_Y never reaches unity and tends to the IR fixed point value [21]). In the present semiclassical scheme, owing to the $\phi(0)$ dilution, one can achieve $g_Y = 1$ with $M_0 \sim 6$ TeV!

Moreover, the critical behavior is significantly modified. Typically, as in the NJL model, the critical behavior would go as a second order phase transition where,

$$\mu^2 = M_0^2 - \frac{g_0^2}{g_{0c}^2} M_0^2 \tag{134}$$

This implies significant fine-tuning to obtain a hierarchy, at the level of $(|\mu|/M_0)^2 \sim 10^{-28}$, or more, in the NJL top condensation scheme. However, in the present framework the *rhs* of eq.(134) is renormalized by $\phi^2(0)$, and we obtain,

$$\mu^2 = \phi^2(0)\left(M_0^2 - \frac{g_0^2}{g_{0c}^2} M_0^2\right) = |\mu| M_0 \left(1 - \frac{g_0^2}{g_{0c}^2}\right) \tag{135}$$

Thus we have a linear relationship between g_0^2 and $|\mu|$,

$$\frac{|\mu|}{M_0} = \left(\frac{g_0^2}{g_{0c}^2} - 1\right), \qquad g_0^2 > g_{0c}^2. \tag{136}$$

This implies that fine-tuning a hierarchy is now significantly reduced to $\delta g_0^2/g_0^2 \sim \mu/M_0 \sim 10^{-2}$... a few % with $M_0 \sim 6$ TeV!

Also, in the NJL model of top condensation the value of the quartic coupling is determined by the renormalization group with "compositeness boundary conditions" and is generally too large (this is problematic for many theories of a composite BEH boson). However, owing to the suppression of g_Y the quartic coupling is now generated in loops as in eq.(121) and found to be close to the standard model result.

We think this bodes well for a renaissance of the top condensation/topcolor scheme, or perhaps other constituent models of a composite BEH boson. We will revisit this elsewhere [34]. We believe the most important challenge to the LHC is to ascertain whether or not the BEH boson is pointlike or an extended object (e.g. showing deviations from the standard model, particularly in 3rd generation processes, or perhaps by way of tools such as [33] and others).

In summary, our key result is that, near criticality, the NJL model fails, while a semiclassical theory contains additional degrees of freedom, i.e., an internal wave-function $\phi(r)$, and the major low energy results are significantly modified. Near criticality the low energy $\phi(r)$ is approximately dynamically scale invariant, and scale symmetry is acting as the custodial symmetry of the physics of a low mass bound state of chiral fermions. Our results may be of some general interest to practitioners of the NJL model in QCD, e.g., [11],[12], etc. For example we think it would be interesting to apply this formalism to heavy-light systems as in [13].

We remark that the transition from unbound to bound state is associated with the internal wave-function becoming a "compact" solution to the SKG equation. Just below critical coupling, $g_0 < g_{0c}$, the eigenfunctions at large r are two body scattering states, such as $\sim e^{iqr}/r$, requiring space-volume, V_3, normalization, $\sim 1/\sqrt{V_3}$. For $g_0 > g_{0c}$ the internal bound state eigenfunction discontinuously becomes compact and normalizable $\sim e^{-\mu r}/r$. This transition is non-analytic in momentum space, but intuitive, in configuration space as reflected in the eigenvalue flow from subcritical to critical coupling.

Weinberg has emphasized the non-analyticity of the transition from unbound to bound state in momentum space as an outstanding challenge to bound state formalism [35]. Tree level scattering amplitudes for unbound pairs are perturbative, and can be approximated to any desired order by a *finite number* of Feynman diagrams, but generate no bound state. A bound state requires summing an *infinite number* of loop diagrams to generate the bound state pole. In practice this involves a choice of particular subset of diagrams to sum, e.g., in a Bethe-Salpeter equation or a fermion loop "bubble sum" approximation to the NJL model. However, at this stage the procedure can become non-systematic (except in a well defined subset, e.g., large-N_c fermion loop approximation in NJL). Therefore, the usual diagrammatic expansion in the coupling has a discontinuity, a non-analytic behavior, as binding commences.

This non-analyticity in momentum space traces in configuration space to the transition from non-normalizable scattering state wave-functions to the normalizable bound state wave-functions. In the present formalism the transition is a first order phase transition in $\epsilon = 0$ to $\epsilon = 1$ as g_0^2 crosses from subcritical to critical. Our present configuration space framework therefore offers, perhaps, a more intuitive view of the bound state in field theory. We can envision more applications of this approach.

Appendix A: Review of the Point-like Nambu–Jona-Lasinio Model

The Nambu–Jona-Lasinio model (NJL) [5] is the simplest field theory of a composite scalar boson, consisting of chiral fermions. An effective *pointlike* bound state emerges from an assumed *pointlike* 4-fermion interaction. We begin with a lightning review of the modern solution of the NJL model using concepts of the renormalization group.

We assume chiral fermions, each with N_c "colors." A non-confining, chiral invariant $U(1)_L \times U(1)_R$ action, then takes the form:

$$S_{NJL} = \int d^4x \left(i[\overline{\psi}_L(x)\partial\!\!\!/\,\psi_L(x)] + i[\overline{\psi}_R(x)\partial\!\!\!/\,\psi_R(x)] + \frac{g_0^2}{M_0^2}[\overline{\psi}_L(x)\psi_R(x)]\,[\overline{\psi}_R(x)\psi_L(x)]\right). \quad (A1)$$

Here $\psi_L = (1-\gamma_5)\psi/2$, $\psi_R = (1+\gamma_5)\psi/2$, and we write color singlet combinations in brackets [...] as $\overline{\psi}^i_L(x)\psi_{iR}(x) \equiv [\overline{\psi}_L(x)\psi_R(x)]$. This can be readily generalized to a $G_L \times G_R$ chiral symmetry which we do explicitly in the section on currents below.

We can rewrite eq.(A1) in an equivalent form by introducing the local color singlet auxiliary field $\Phi(x)$:

$$S_{NJL} = \int d^4x \left(i[\overline{\psi}_L(x)\partial\!\!\!/\,\psi_L(x)] + i[\overline{\psi}_R(x)\partial\!\!\!/\,\psi_R(x)] - M_0^2\Phi^\dagger(x)\Phi(x) + g_0[\overline{\psi}_L(x)\psi_R(x)]\Phi(x) + h.c.\right). \quad (A2)$$

The resulting "equation of motion" for Φ is:

$$M_0^2\Phi(x) = g_0[\overline{\psi}_R(x)\psi_L(x)] \quad (A3)$$

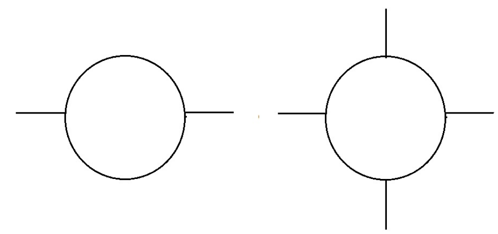

FIG. 4: Diagrams contributing to the pointlike NJL model effective Lagrangian, eqs.(A7). External lines are Φ and internal lines are fermions ψ.

Using the Φ equation of motion in eq.(A2) reproduces the 4-fermion interaction of eq.(A1).

Note that the induced (unrenormalized) Yukawa coupling g_0 in eq.(A2) is the same coupling as appears in the interaction of eq.(A1). In the semiclassical treatment of this paper this relationship is significantly modified as in eq.(51).

Following Wilson, [6], we view eq.(A2) as the action defined at the high energy (short-distance) scale $m \sim M$. We then integrate out the fermions to obtain the effective action for the composite field Φ at a lower scale $m \ll M$. The calculation in the large-N_c limit is discussed in detail in [8, 9]. The leading N_c fermion loop yields the result for the Φ terms in the action at a new scale μ:

$$S_\mu = \int d^4x \left(i[\overline{\psi}_L\partial\!\!\!/\,\psi_L] + i[\overline{\psi}_R\partial\!\!\!/\,\psi_R] + Z\partial_\mu\Phi^\dagger\partial^\mu\Phi - \mu^2\Phi^\dagger\Phi - \frac{\lambda}{2}(\Phi^\dagger\Phi)^2 + g_0[\overline{\psi}_L\psi_R]\Phi(x) + h.c.\right). \quad (A4)$$

where the diagrams of Fig.(1) yield,

$$\mu^2 = M_0^2 - \frac{N_c g_0^2}{8\pi^2}M_0^2,$$
$$Z = \frac{N_c g_0^2}{8\pi^2}\ln(M_0/\mu), \quad \lambda = \frac{N_c g_0^4}{4\pi^2}\ln(M_0/\mu). \quad (A5)$$

Here M_0^2 is the UV loop momentum cut-off, and we include the induced kinetic and quartic interaction terms. The one-loop result can be improved by using the full renormalization group (RG) [8, 9]. Hence the NJL model is driven by fermion loops, which are $\propto \hbar$ intrinsically quantum effects.

Note the behavior of the composite scalar boson mass, μ^2, of eq.(A5) in the UV. The $-N_c g_0^2 M_0^2/4\pi^2$ term arises from the negative quadratic divergence in the loop involving the pair (ψ_R, ψ_L) of Fig.(1), with UV cut-off M_0^2. Therefore, the NJL model has a critical value of its coupling defined by the cancellation of the large M_0^2 terms,

$$g_{0c}^2 = \frac{8\pi^2}{N_c} \quad (A6)$$

Here we have m as the running RG mass, and is the lower limit of the loop integrals. This can in principle be small in logs and neglected in the quadratically divergent loops. At the critical coupling the mass of the bound state is then $\mu^2 = 0$.

We can renormalize, $\Phi \to \sqrt{Z}^{-1}\Phi$, hence:

$$S_\mu = \int d^4x\, \left(i\overline{\psi}^a_L \partial\!\!\!/ \psi_{aL} + i\overline{\psi}^a_R \partial\!\!\!/ \psi_{aR} + \partial_\mu \Phi^\dagger \partial^\mu \Phi \right.$$
$$\left. -\mu_r^2 \Phi^\dagger \Phi - \frac{\lambda_r}{2}(\Phi^\dagger\Phi)^2 + g_Y \overline{\psi}^a_L \psi_{aR}\Phi(x) + h.c.\right). \quad \text{(A7)}$$

where,

$$\mu_r^2 = \frac{1}{Z}\left(M_0^2 - \frac{N_c g^2}{8\pi^2}(M_0^2)\right)$$
$$g_Y^2 = \frac{g^2}{Z} = \frac{8\pi^2}{N_c \ln(M_0/m)}, \quad \lambda_r = \frac{\lambda}{Z^2} = \frac{16\pi^2}{N_c \ln(M_0/m)}. \quad \text{(A8)}$$

For super-critical coupling, $g_0^2 > g_{0c}^2$, we see that $\mu_r^2 < 0$ and there will be a chiral vacuum instability. The effective action, with the induced quartic $\sim \lambda_r(\Phi^\dagger\Phi)^2$ term, is then the usual sombrero potential. The chiral symmetry is spontaneously broken, the field Φ acquires a VEV,

$$\langle \Phi \rangle = v = \frac{\mu_r}{\sqrt{\lambda_r}}, \quad \text{(A9)}$$

and the chiral fermions acquire mass,

$$m_f = g_Y v \quad \text{(A10)}$$

The physical radial mode ("Higgs" boson), defined as $\sqrt{2}Re(\Phi)$, has a mass m_h given by,

$$m_h^2 = 2\lambda v^2. \quad \text{(A11)}$$

The Nambu-Goldstone mode, $Im(\Phi)$, is massless. Hence we see that the NJL model yields a prediction for the radial mode

$$m_h = \frac{\sqrt{2\lambda_r}}{g_Y} m_f = 2m_f. \quad \text{(A12)}$$

The effective action also generates Nambu-Goldstone bosons that also emerge as pointlike bound states.

Fine-tuning of $g_0^2 \to g_{0c}^2$ can be done to attempt to create a hierarchy, $|\mu^2| \ll M_0^2$. In that case we appeal to the behavior of the renormalized couplings as $\mu \to M_0$. We see that both g_Y and λ_r diverge in the ratio $\lambda_r/g_Y^2 \to 2$. This can be used as a boundary condition on the full RG evolution of g_Y and λ_r including gauge fields and scalar interactions themselves.

The NJL model is useful in QCD applications and supercritical coupling. However, in the present semiclassical treatment of binding we claim that the results of the critical coupling limit of the NJL model are incorrect. The model does not include the internal wave-function, $\phi(r)$, which behaves as $\phi(r) \propto e^{-\mu r}/r \sim 1/r$ in the limit, and dilutes $g_Y \propto \phi(0) \sim \sqrt{\mu M_0}$ by a power-law.

Appendix B: Currents

Under a $G_L \times G_R$ chiral symmetry transformation we have for free fermions,

$$\psi_L(y) \to G_L \psi_L(y)$$
$$\psi_R(y) \to G_R \psi_R(y) \quad \text{(B1)}$$

where dotted indices refer to G_L and undotted to G_R. Therefore the free fermion field theory has chiral currents,

$$j^A_{L\mu} = [\overline{\psi}_L(x)\gamma_\mu T^A_L \psi_L(x)]$$
$$j^A_{R\mu} = [\overline{\psi}^a_R(x)\gamma_\mu T^A_R \psi_R(x)] \quad \text{(B2)}$$

where [...] means we have contracted color indices and T_L (T_R) is the generator of G_L (G_R). These groups have corresponding currents in the composite two body field Φ. The defining equation for $\Phi(x,y)$ implies its chiral symmetry properties under $G_L \times G_R$,

$$M_0^2 \Phi(x,y) \to G_L \Phi(x,y) G_R^\dagger \quad \text{(B3)}$$

Conserved currents provide a connection between the normalization of the fundamental constituent fields and the composite fields. The current normalization of Φ is equivalent to it's kinetic term normalization. The values of the associated charges lock the fundamental fermion fields, $\psi_R(x)$ and $\psi_L(y)$ to the composite field $\Phi(x,y)$. The matching of the composite to the constituent currents can be made exact for scalar constituents, which we do in [20]. The matching will also hold for fermions as operator constraints in the two body sector of the Fock space of states, as is the case for chiral Lagrangians in general.

It is useful to focus on the *global* $U(1)_L \times U(1)_R$ subgroup which s present for any $G_L \times G_R$ generalized chiral symmetry group. With the two currents,

$$j_{L\mu} = [\overline{\psi}_L(x)\gamma_\mu \psi_L(x)] \qquad j_{R\mu} = [\overline{\psi}_R(x)\gamma_\mu \psi_R(x)] \quad \text{(B4)}$$

We have the vector current $j = j_L + j_R$ and an axial vector $j^5 = j_L - j_R$, corresponding to $U_V(1) \times U_A(1)$

The symmetries act upon the $\Phi(x,y)$ as a local gauge transformation of the form,

$$\Phi(x,y) \to U_R^\dagger(x)\Phi(x,y)U_L(y) \quad \text{(B5)}$$

where in principle the gauge rotations are different at x and $y \neq x$. In this way we can introduce chiral gauge fields. If, however, the gauge transformation is global, then the U's are independent of x, y. In barycentric coordinates we have $U(X+r) = U(X-r) = U(X)$ hence in the factorization form $\Phi'(X,r) \sim \chi(X)\phi(r)$ we can assign the global symmetry representation to χ and treat $\phi(r)$ as a scalar. However, we require local transformations to generate Noether currents. Hence,

$$\Phi(x,y) \to e^{-i\theta_L(x}\Phi(x,y),$$
$$\Phi'(X,r) \to e^{i\theta_R(y)}\Phi(x,y) \quad \text{(B6)}$$

generates the bilocal currents from eq.(25),

$$J_{L\mu}(x,y) = iZM^4\Phi^\dagger(x,y))\frac{\overleftrightarrow{\partial}}{\partial x^\mu}\Phi(x,y)$$
$$J_{R\mu}(X,r) = iZM^4\Phi^\dagger(x,y))\frac{\overleftrightarrow{\partial}}{\partial y^\mu}\Phi(x,y) \quad \text{(B7)}$$

Likewise, jumping to the $\Phi'(X,r)$ representation and the bilocal action eq.(29), we generate two Noether currents corresponding to $U^+(1)\times U^-(1)$, via the local transformations,[9]

$$\Phi'(X,r)\to e^{i\theta^+(X)}\Phi'(X,r),$$
$$\Phi'(X,r)\to e^{i\theta^-(r)}\Phi'(X,r) \tag{B10}$$

The bilocal currents are,

$$J^+_\mu(X,r) = iZ'M^4[\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X)]\phi^\dagger(r)\phi(r)$$
$$J^-_\mu(X,r) = iZ'M^4[\phi^\dagger(r)\frac{\overleftrightarrow{\partial}}{\partial r^\mu}\phi(r)]\chi^\dagger(X)\chi(X) \tag{B11}$$

These can be integrated to form,

$$J^+_\mu(X) = iZM^4[\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X)]\int_r\phi^\dagger(r)\phi(r)$$
$$J^-_\mu(r) = iZM^4[\phi^\dagger(r)\frac{\overleftrightarrow{\partial}}{\partial r^\mu}\phi(r)]\int_X\chi^\dagger(X)\chi(X) \tag{B12}$$

The charge corresponding to J^+ counts the number of ψ_L plus ψ_R particles and we can therefore match J^+_μ to the underlying $j_V = j_1 + j_2$ vector current. With eq.(33) and eq.(35) we have

$$1 = ZM^4\int d^4r|\phi(r)|^2 = M^3\int d^3r|\phi(r)|^2 \tag{B13}$$

and the $U^+(1)$ current becomes,

$$J_\mu(X) = i\chi^\dagger(X)\frac{\overleftrightarrow{\partial}}{\partial X^\mu}\chi(X) \tag{B14}$$

If we consider a pointlike field $\Phi(X)$ then we lose the independent $U_A(1) = j_R - j_L$ transformation, i.e. $e^{i\theta(r)}$ is meaningless for the local field, and $\Phi(X)$ can only represent a single $U_V(1)$ transformation. We are using the term "(+)" current because $\Phi\to e^{i\theta(X)}\Phi(X) = e^{i\theta(X)}\Phi(X)$ can correspond to either a vectorlike (electromagnetic) gauge transformation or a $U^5(1)$ axial transformation. The concept of parity arises in the bosonic chiral Lagrangian only when we include couplings to chiral fermions. E.g., with a coupling to fermions, $\sim\overline{\psi}_L\Phi\psi_R$ the $U^+(1)$ becomes the axial transformation $\psi_L\to e^{-i\theta}\psi_L$, $\psi_R\to e^{i\theta}\psi_R$. In the pointlike limit where Φ is invariant under $\Phi\to e^{i\theta(X)}\Phi(X)e^{-i\theta(X)}$ the corresponding $U^-(1)$ current is zero. However, the bilocal case has a nonzero $U^-(1)$ current.

However, if we have a static $\phi(\vec{r})$ then the axial charge is zero,

$$J^-_0(X,r) = iZ[\phi^\dagger(\vec{r})\frac{\overleftrightarrow{\partial}}{\partial r^0}\phi(\vec{r})]\int_X\chi^\dagger(X)\chi(X) = 0 \tag{B15}$$

Note that we can define a phase in analogy to a would be $U_5(1)$ Nambu-Goldstone boson, such as the η',

$$\phi(r) = e^{i\eta'(r^0)/f}\phi(\vec{r}) \tag{B16}$$

and the static condition implies in the rest frame,

$$\partial^0_r\eta'(r^0) = 0 \qquad \eta' = \theta = \text{constant}. \tag{B17}$$

This is suggestive of a mechanism to elevate the mass of the η' such as instantons and the η' is therefore a constant at the minimum of a deep potential. While any other Nambu-Goldstone boson, such as the π, can have zero momentum but nonzero time dependence, the η' cannot be a Nambu-Goldstone boson if the constraint is applied.

Appendix C: Colorons and Fierz Rearrangement

The coloron model is a massive, perturbative gluon field, B^μ_A, in an $SU(N_c)$ gauge theory broken to the diagonal global $SU(N_c)$. We assume fermions couple through vector color currents and a coloron mass M_0, with relevant Lagrangian terms:

$$-g_0\overline{\psi}(x)\gamma_\mu T^A\psi B^\mu_A + M_0^2B^\mu_AB_{A\mu} \tag{C1}$$

Integrating out the coloron yields the interaction,

$$S' = \frac{1}{2}\int_{xy}[-ig_0\overline{\psi}\gamma_\mu T^A\psi]_x\langle TB^\mu_A(x)B^\nu_B(y)\rangle[-ig_0\overline{\psi}\gamma_\nu T^B\psi]_y$$
$$= -\frac{g_0^2}{2}\int_{xy}[\overline{\psi}(x)\gamma_\mu T^A\psi(x)]D^{\mu\nu}(x-y)[\overline{\psi}(y)\gamma_\nu T^A\psi(y)] \tag{C2}$$

where we note that $iD^{\mu\nu}(x-y)\delta^{AB} = \langle T\ B^A_\mu(x)B^B_\nu(y)\rangle$ and we strip off a factor of $+i$ for the action. In Feynman gauge [31],

$$D_{\mu\nu} = g_{\mu\nu}D_F(x-y)$$
$$D_F(x-y) = -\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x-y)}}{q^2-M_0^2+i\epsilon} \tag{C3}$$

We are interested in chiral fermions and write $\psi = \psi_L + \psi_R$ with chiral projections

$$\psi_L = \frac{1-\gamma^5}{2}\psi \qquad \psi_R = \frac{1+\gamma^5}{2}\psi \tag{C4}$$

and the interaction of interest becomes the cross-term of L and R currents in eq.(13):

$$S' = -g_0^2\int_{xy}[\overline{\psi}_L(x)\gamma_\mu T^A\psi_L(x)]D^{\mu\nu}(x-y)[\overline{\psi}_R(y)\gamma_\nu T^A\psi_R(y)] \tag{C5}$$

[9] Note that the constraint takes the form

$$|\Omega|^2 = \left(\frac{\partial\chi^\dagger(X)}{\partial X^\mu}\frac{\partial\phi(r)}{\partial r^\mu}\phi^\dagger(r)\chi(X) + h.c.\right)^2 = 0 \tag{B8}$$

The constraint is invariant under the global $U(1)_L\times U(1)_R$ transformations. Under the local transformation it generates

$$\left(i\frac{\partial\theta^+(X)}{\partial X^\mu}\frac{\partial\phi(r)}{\partial r_\mu}\phi^\dagger(r)|\chi(X)|^2 + h.c.\right)\Omega + h.c. = 0$$
$$\left(-i\frac{\partial\chi^\dagger(X)}{\partial X^\mu}\frac{\partial\theta^-(r)}{\partial r_\mu}|\phi(r)|^2\chi(X) + h.c.\right)\Omega + h.c. = 0 \tag{B9}$$

where [...] denotes color summed indices, e.g.

$$[\overline{\psi}_L\gamma_\mu\psi_L] \equiv \overline{\psi}^i_L\gamma_\mu\psi_{iL}, \quad [\overline{\psi}_L\gamma_\mu T^A\psi_L] \equiv \overline{\psi}^i_L\gamma_\mu T^{A}{}^j_i\psi_{jL}. \quad \text{(C6)}$$

The interaction can be Fierz transposed. We define the operators for generic fermions, ψ_i, sequentially ordered as (1234):

$$\begin{aligned} &\mathcal{O}_1 = \overline{\psi}_1\psi_2\overline{\psi}_3\psi_4, \quad \mathcal{O}_2 = \overline{\psi}_1\gamma_\mu\psi_2\overline{\psi}_3\gamma^\mu\psi_4 \\ &\mathcal{O}_3 = \overline{\psi}_1\sigma_{\mu\nu}\psi_2\overline{\psi}_3\sigma^{\mu\nu}\psi_4 \\ &\mathcal{O}_4 = \overline{\psi}_1\gamma^5\gamma_\mu\psi_2\overline{\psi}_3\gamma^5\gamma^\mu\psi_4, \quad \mathcal{O}_5 = \overline{\psi}_1\gamma^5\psi_2\overline{\psi}_3\gamma^5\psi_4 \end{aligned} \quad \text{(C7)}$$

and define operators $\overline{\mathcal{O}}_i$ identical to the above but reordered as (1432). Note that we can allow any given fermion to have chiral projection, e.g. $\psi_1 \to \psi_{1L}$, etc. We then have the Fierz identity:

$$\mathcal{O}_i = \sum_j M_{ij}\overline{\mathcal{O}}_j \quad \text{(C8)}$$

where the matrix is,

$$M_{ij} = -\frac{1}{8}\begin{pmatrix} 2 & 2 & 1 & -2 & 2 \\ 8 & -4 & 0 & -4 & -8 \\ 24 & 0 & -4 & 0 & 24 \\ -8 & -4 & 0 & -4 & 8 \\ 2 & -2 & 1 & 2 & 2 \end{pmatrix} \quad \text{(C9)}$$

The overall minus sign is due to assumed anticommutation property of field operators. We also have the "color Fierz" identity

$$\sum_A T^{Ai}{}_j T^{Ak}{}_\ell = \frac{1}{2}\left(\delta^i_\ell\delta^k_j - \frac{1}{N_c}\delta^i_j\delta^k_\ell\right) \quad \text{(C10)}$$

In particular we see from the second line of eq.(C9) that

$$[\overline{\psi}_L\gamma_\mu T^A\psi_L][\overline{\psi}_R(y)\gamma_\nu T^A\psi_R] = -[\overline{\psi}_L\psi_R][\overline{\psi}_R(y)\psi_L] \quad \text{(C11)}$$

This yields to the form of the interaction of eq.(15), to leading order in $1/N_c$,

$$S' = g_0^2\int_{xy}[\overline{\psi}_L(x)\psi_R(y)]D_F(x-y)[\overline{\psi}_R(y)\psi_L(x)], \quad \text{(C12)}$$

If we now take the δ-function limit of D_F eq.(16) we have

$$D_F(x-y) \to \frac{1}{M_0^2}\delta^4(x-y), \quad \text{(C13)}$$

and our interaction becomes the 4-fermion NJL interaction of eq.(A1):

$$S' = \frac{g_0^2}{M_0^2}\int_x[\overline{\psi}_L(x)\psi_R(x)][\overline{\psi}_R(x)\psi_L(x)], \quad \text{(C14)}$$

The ingredients of the Fierz rearrangement used here for Dirac matrices and color can be found in the Appendix of ref.([32]).

Appendix D: Notes on Scattering States

1. Bilocal Scattering States

For subcritical coupling the SKG eigenvalue, μ^2, is positive and the large r solution becomes $u(r) \sim a\exp(i\mu r) + b\exp(-i\mu r)$. This is a steady state sum of incoming and out going waves and represent the formation and decay of a resonance. This is an open scattering state description by a bilocal field of the resonance in the barycentric frame, $\Phi'(X,r) = \chi(X)u(r)/r$.

In the weak coupling limit, $g_0 << g_{0c}$, resonances appear in scattering states, centered at positive invariant $(\text{mass})^2$, $\mu^2 = \mu^2 = k^2 - g_0^2M_0^2$ with $\cos(\mu R_0) = 0$. The full spatial solution for $\Phi'(X,r)$ will then be a wavefunction with an an extended tail for large r, consisting of incoming production and outgoing decay modes. As we increase the coupling μ^2 decreases and approaches the scale invariant critical value.

A scattering state is non-compact in $\vec{r}$ and technically non-normalizable due to the external spherical scattering waves. For a narrow resonance we can define an effective finite radius, $r \sim R_0$, of the bound state as a cut-off. For example, we might imagine something like a BEH boson composed of massless top quarks in the symmetric phase of the standard model, with a large positive μ^2. Such an object would therefore be a resonance in the $\bar{t}t$ scattering amplitude with a width $\Gamma \propto \mu$. The decay width can be estimated by computing the classical power in the outgoing wave divided by μ.

As a simple example consider a rectangular $(\text{mass})^2$ potential well,

$$V_0(2r) = -g^2M_0^2\theta(R_0 - r) \quad \text{(D1)}$$

We focus upon s-wave scattering and write the large r the form

$$\chi = e^{-i\mu t}, \;\; \phi(r) = \frac{u(r)}{r}, \;\; u(r) = \sin(kr + \delta) \quad \text{(D2)}$$

where δ is the phase shift.

For the attractive well $\mu^2 = k'^2 + V_0 = k^2$, $k'^2 = \mu^2 + g_0^2M_0^2$, with interior solution,

$$\phi_{int}(r) = \frac{A\sin(k'r)}{r} \qquad k'^2 + V_0 = \mu^2 \quad \text{(D3)}$$

and exterior solution,

$$\phi_{int}(r) = \frac{A\sin(kr+\delta)}{r} \qquad k^2 = \mu^2 \quad \text{(D4)}$$

Matching:

$$k'\cot(k'R_0) = k\cot(kR_0 + \delta) \quad \text{(D5)}$$

hence,

$$\tan\delta = \frac{k\tan(k'R_0) - k'\tan(kR_0)}{k' + k\tan(kR_0)\tan(k'R_0)} \quad \text{(D6)}$$

For small $kR_0 << 1$ we have the scattering length,

$$a_0 \approx -\frac{\tan\delta_0}{k} \approx -\frac{\delta}{k} = -R_0\left(\frac{\tan k'R_0}{k'R_0} - 1\right) \quad \text{(D7)}$$

Total crossection, $\sim 4\pi\sin^2(\delta)/k^2$.

We can solve for A

$$|A|^2 = \left(1 + \left(\frac{g_0^2 M_0^2}{\mu^2}\right)\cos^2(R_0\sqrt{\mu^2 + g^2 M_0^2})\right)^{-1} \quad \text{(D8)}$$

Resonances occur for maxima of $|A|^2$, these are approximately the vanishing of $\cos(k' R_0)$ or $k' R_0 = (n+1/2)\pi$. We have $k'^2 - g_0^2 M_0^2 = \mu^2$.

We can approximate the resonance by the lump contained within the potential, which we normalize to unity as per our formalism for $\phi(r)$ This can then be use to compute Yukawa coupling g_Y. We then have the decay width of a scalar particle of mass μ into chiral fermions. This yields,

$$\Gamma = \frac{g_Y^2 N_c}{16\pi}\mu \quad \text{(D9)}$$

2. General Notes and Kinematics

For free particles of 4-momenta $p_{i\mu}$ we see that Φ describes a scattering state,

$$\Phi(x,y) = \exp(iP_\mu X^\mu + iQ_\mu r^\mu) \quad \text{(D10)}$$

where,

$$P_\mu = p_{1\mu} + p_{2\mu} \qquad Q_\mu = p_{1\mu} - p_{2\mu} \quad \text{(D11)}$$

For massive particles, $p_1^2 = p_2^2 = \mu^2$, and Φ then satisfies

$$\left(\frac{\partial^2}{\partial X^\mu \partial X_\mu} + \frac{\partial^2}{\partial r^\mu \partial r_\mu} + 4\mu^2\right)\Phi'(X,r) = 0 \quad \text{(D12)}$$

and the constraint is then,

$$\frac{\partial^2}{\partial X^\mu \partial r_\mu}\Phi'(X,r) = 0 \to P_\mu Q^\mu = 0 \quad \text{(D13)}$$

and in the rest frame, $P_\mu = (2\mu, 0, 0, 0)$, hence $P_\mu Q^\mu = 0$ implies $Q = (0, 2\vec{q})$. Hence $\Phi(X, \vec{r})$ is independent of r^0 and the evolution of the system is described by the single time variable, X^0.

In the rest frame we have,

$$Q^2 = (p_1 - p_2)^2 = -(\vec{p}_1 - \vec{p}_2)^2 = -4\vec{q}^{\,2} \quad \text{(D14)}$$

Hence, from eqs.(D12,D13)

$$\frac{\partial^2}{\partial X^\mu \partial X_\mu}\Phi(X,\vec{r}) = 4(\mu^2 + \vec{q}^{\,2})\Phi(X,\vec{r}) \quad \text{(D15)}$$

Therefore, Φ has continuum of invariant "masses" $m_{\vec{q}^{\,2}}^2 = 4(\mu^2 + \vec{q}^{\,2})$ This is an "unparticle," as in [30].

We factorize $\Phi = \chi(X)\phi(r)$ and the factor field $\phi(r)$ then satisfies the static SKG equation which generates the eigenvalue $4(\mu^2 + \vec{q}^{\,2})$,

$$-\nabla_{\vec{r}}^2\phi(\vec{r}) + 4\mu^2\phi(\vec{r}) = 4(\mu^2 + \vec{q}^{\,2})\phi(\vec{r}) \quad \text{(D16)}$$

and the solutions of the factorized field $\phi(r)$ are static, box normalized, plane waves,

$$\phi(\vec{r}) = \frac{1}{\sqrt{V}}\exp(2i\vec{q}\cdot\vec{r}) \quad \text{(D17)}$$

$\chi(X)$ then satisfies the KG equation with $X^0 = t$ and $\vec{X} = 0$,

$$\partial_t^2\chi(t) + 4(\mu^2 + \vec{q}^{\,2})\chi(t) = 0 \qquad t = X^0 \quad \text{(D18)}$$

3. More on the Removal of Relative Time

With secondary Lagrange multiplier constraints added to the action, we can define a timelike unit vector, ω^μ,

$$\eta_1\left|i\Phi^\dagger\frac{\partial}{\partial X^\mu}\Phi - P_\mu\Phi^\dagger\Phi\right|^2 + \eta_2\left|\omega_\mu\sqrt{P^\rho P_\rho} - P_\mu\right|^2 \quad \text{(D19)}$$

We then define

$$Z_0 = \delta(M\omega_\mu r^\mu). \quad \text{(D20)}$$

The δ-function removes $\int dr^0$ in the rest frame but maintains manifest Lorentz invariance.

The normalizer, Z_0, is needed for composite fields. Fields are not directly observable, but their charges and current are. For the composite fields, which may describe bound states, we need charges that match those of the constituents and, e.g., that count the number of bound states in a given quantum state. Z_0 normalizes these charges, seen made more precisely below where we discuss the charges and currents in the bilocal field theory, (see Appendix B , for the discussion in the case of scalar field bilocal theory in [20]). Note that Z_0 appears only in the kinetic terms, where the currents arise, and is not part of the interaction.

4. Comments on the Induced Yukawa Interaction

It is useful to consider the kinematics of the induced Yukawa interaction.

In the rest frame, suppose we have the decay of a resonant state of mass μ to a pair of free massless fermions. Then $\chi(X^0) \sim e^{i\mu X^0}$, and fermions with $\overline{\psi}_L(X+r) \sim \exp(ip_1(X+r))$ and $\psi_R(X-r) \sim \exp(ip_2(X-r))$. The integral over X yields energy and momentum conservation as usual:

$$\mu = p_{10} + p_{20}; \quad 0 = \vec{p}_1 + \vec{p}_2; \quad \text{hence,} \;\; p_{10} = p_{20} \quad \text{(D21)}$$

We then have the integral over r^0 in eq.(40),

$$\int d^4r\, e^{2i\vec{p}_1\cdot\vec{r}} D(2r)\phi(\vec{r}) = -\frac{1}{2}\int d^3r\, e^{2i\vec{p}_1\cdot\vec{r}} V_0(2\vec{r})\phi(\vec{r}) \quad \text{(D22)}$$

Here we see that momentum conservation implies $p_{10} = p_{20}$, hence $e^{i(p_{10}-p_{20})r^0} = 1$, which allows us to integrate out r^0 over $D(2r)$ with the static $\phi(\vec{r})$ as before. The remaining integral over $2\vec{r}$ is the Fourier transform of $V_0(2\vec{r})\phi(\vec{r})$ with the 3-momentum difference $\vec{p}_1 - \vec{p}_2 = 2\vec{p}_1$ flowing through the extended vertex. If the mass scale associated with $V_0(2\vec{r})\phi(\vec{r})$ is large, i.e. $M_0^2 \gg \mu^2$, then we can reliably replace this with,

$$\int d^3r\, e^{2i\vec{p}_1\cdot\vec{r}} V_0(2\vec{r})\phi(\vec{r}) \sim \int d^3r\, V_0(2\vec{r})\phi(\vec{r}) + \mathcal{O}\left(\frac{\vec{p}_1^{\,2}}{M_0^2}\right) \quad \text{(D23)}$$

An interesting feature is that the decay amplitude of a resonance depends only upon the part of the wavefunction localized within the potential.

On the other hand we can suppose that the field χ has developed a VEV $\langle\chi\rangle = f_0$ (which is the case with supercritical coupling as we discuss below). Then the internal wave-function, $\phi(\vec{r})$, is a localized solution of the SKG equation. In this case the free fermion mass is spontaneously generated by the Yukawa coupling to the VEV.

In this case we have fermions of zero 3-momentum, an incoming fermion, $\sim \exp(ip_1(X+r))$ and outgoing fermion $\sim \exp(-ip_2(X-r))$. Now the d^4X integral yields $p_1^0 = p_2^0 = \mu$, where μ is the induced fermion mass, and

$$\int d^4r\; e^{2i\mu r^0} D(2r)\phi(\vec{r}) = -\frac{1}{2}\int d^3r\; V_0'(2\vec{r},\mu)\phi(\vec{r}) \quad \text{(D24)}$$

where now the potential is slightly distorted,

$$V_0'(2\vec{r},\mu) = -\frac{M' e^{-2M'|\vec{r}|}}{8\pi|\vec{r}|} \qquad M'^2 = M_0^2 - \mu^2 \quad \text{(D25)}$$

Assuming $\mu \ll M_0$ we have, $V_0'(2\vec{r},\mu) \approx V_0(2\vec{r}) + \mathcal{O}(\mu^2/M_0^2)$.

Appendix E: Summary of Notation

Barycentric coordinates:

$$X = \frac{1}{2}(x+y) \qquad \rho = (x-y) \qquad r = \frac{1}{2}(x-y)$$
$$\partial_x = \frac{1}{2}(\partial_X + \partial_r) \quad \partial_y = \frac{1}{2}(\partial_X - \partial_r) \quad \text{(E1)}$$

Integrals:

$$\int_{u\dots v} = \int d^4u..d^4v; \qquad \int_{\vec{x}\dots\vec{y}} = \int d^3x..d^3y$$
$$\int_{u\dots v;\vec{x}\dots\vec{y}} = \int d^4u..d^4v\; d^3x...d^3y; \quad \int d^4x d^4y = J\int d^4X d^4r$$
$$\text{Jacobian, } J = (2)^4: \quad \int d^4x d^4y \left(|\partial_x\phi|^2 + |\partial_y\phi|^2 - \mu^2|\phi|^2\right)$$
$$= J\int d^4X d^4r\left(\frac{1}{2}|\partial_X\phi|^2 + \frac{1}{2}|\partial_r\phi|^2 - \mu^2|\phi|^2\right) \quad \text{(E2)}$$

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