

The Non-Relativistic Effective Field Theory Of Dark Matter-Electron Interactions

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Electronic excitations in atomic, molecular, and crystal targets are at the forefront of the ongoing search for light, sub-GeV dark matter (DM). In many light DM-electron interactions the energy and momentum deposited is much smaller than the electron mass, motivating a non-relativistic (NR) description of the electron. Thus, for any target, light DM-electron phenomenology relies on understanding the interactions between the DM and electron in the NR limit. In this work we derive the NR effective field theory (EFT) of general DM-electron interactions from a top-down perspective, starting from general high-energy DM-electron interaction Lagrangians. This provides an explicit connection between high-energy theories and their low-energy phenomenology in electron excitation based experiments. Furthermore, we derive Feynman rules for the DM-electron NR EFT, allowing observables to be computed diagrammatically, which can systematically explain the presence of in-medium screening effects in general DM models. We use these Feynman rules to compute absorption, scattering, and dark Thomson scattering rates for a wide variety of high-energy DM models.

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I. INTRODUCTION

Recent years have seen an explosion of ideas motivating light dark matter (DM) candidates with sub-GeV mass. In addition to familiar thermal freeze-out, this mass range accommodates multiple new production mechanisms, for example, modified freeze-out [1–6], asymmetric DM [7–9], freeze-in [10–13], misalignment [14–17] and inflationary production [18]. Such light DM evades conventional nuclear-recoil direct detection searches since the energy deposited rapidly vanishes for DM lighter than the target nuclei. This limitation has inspired the development of next generation direct detection experiments, many at the interface of condensed matter and particle physics – see Refs. [19–21] for recent reviews.

To efficiently probe light DM in a direct detection experiment, the target and DM must be kinematically matched, such that the target responds to the energy deposited during an interaction with DM. This energy deposition ω varies dramatically depending on the interaction:

- **Absorption:** if a DM particle of mass m is absorbed into the target, the energy deposition is $\omega \sim m$.
- **Elastic Scatter:** if DM scatters elastically with the target, $\omega \lesssim mv^2$, where $v \sim 10^{-3}$ is the local DM velocity.
- **Inelastic Process:** if the incoming and outgoing dark states differ in mass, e.g., inelastic DM [22] or “dark-Compton” scattering [23], ω may lie between the absorption and elastic scattering regimes, $mv^2 \lesssim \omega \lesssim m$.

Therefore, to extend direct detection sensitivity down to MeV scale masses via scattering, or eV scale masses via absorption, the target must have \sim eV energy levels. Since this is the typical energy scale of electronic excitations in most materials, these systems are kinematically well-matched for sub-GeV DM direct detection.

There is currently a large experimental effort devoted to probing DM induced electronic excitations in a variety of targets. Electrons in liquid Xe or Ar can be ionized in the XENON [24, 25] and DarkSide [26] experiments, respectively, if DM deposits $\omega \gtrsim 10$ eV. Electrons can also be excited across the band gap in semiconducting Si or Ge targets with eV scale energy deposits in the CDEX [27, 28], DAMIC [29], EDELWEISS [30], SENSEI [31], and SuperCDMS [32, 33] experiments.

Furthermore, there are several new ideas for exploiting electronic transitions in various other materials. In the \sim few-eV energy range, scintillators [34, 35], dielectrics [36–38], carbon nanotubes [39, 40], molecular targets [41–43], and quantum dots [44] have all been proposed as promising targets. There are also many ideas for exploiting meV-scale electronic excitations to extend sensitivity to even lower DM masses. Targets such as graphene [45–47], Dirac materials [48–50], spin-orbit coupled materials [51, 52], narrow gap semiconductors [53], superconductors [54–58], and doped semiconductors [59] have been recently studied.

Given the unknown nature of DM, and the rapidly expanding experimental program, it is crucial to understand how a general DM candidate can interact with a given target, and several aspects of this problem have been studied in the recent literature. Elastic electron scattering in atomic [60–62] and crystal [62–67] targets has been well studied in the context of DM models with a dark photon-like mediator. Elastic electron scattering in more general DM models has been studied using a “bottom-up” approach, which enumerates possible interactions in the non-relativistic (NR) interaction Hamiltonian of the DM–electron system [46, 47, 68–72]. In complementary analyses, it has also been shown that some DM–electron scattering rates can be related to the target dielectric function [36, 38], and other generalized susceptibilities using linear response theory [73].

Additionally there has been similar progress towards understanding DM absorption into electronic excitations, where initial work primarily studied axion and dark photon absorption in atomic and isotropic crystal targets [41, 48–

50, 56, 57, 74–79]. It has also been shown that the DM absorption rate, in some DM models, can be related to the target dielectric function [74, 76]. More recent work has extended the absorption calculation to scalar DM models in crystal targets [57], electron-coupled axion DM in spin-polarized targets [80], photon-coupled axion DM in magnetized targets [81], and electric and magnetic dipole-coupled DM in both atomic and crystal targets [82]. First principles calculations have also been performed for axion and dark photon DM absorption into Dirac materials [48–50]. In addition to scattering and absorption, a dark Compton-like process was shown to be important for heavier axion and dark photon DM candidates [23].

In this paper we develop a framework for calculating DM-electron interaction rates in *any* target material, given *any* high-energy DM-electron interaction. The key to this generalization is the NR effective field theory (EFT) describing the DM-electron interactions at energies and momenta below the electron mass. While the kinematics of different interaction processes (absorption, scattering, inelastic processes . . .) can vary dramatically, they all deposit momentum \mathbf{q} and energy ω well below the electron mass ($|\mathbf{q}|, \omega \ll m_e$), and can therefore be described with the NR EFT of DM-electron interactions. Moreover, all of these processes induce transitions between the electronic states of the target, so all observables of interest can be related to matrix elements of the form

$$\mathcal{M} \sim \int d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \psi_F^\dagger(\mathbf{x}) \mathcal{O}_{\text{NR}} \psi_I(\mathbf{x}), \quad (1)$$

where $\psi_I(\mathbf{x}), \psi_F(\mathbf{x})$ are initial and final state NR electronic wavefunctions, and \mathcal{O}_{NR} is an NR interaction operator. The NR EFT Lagrangian then determines the operators that appear in the transition matrix elements (and their relative weights), while the target electronic structure determines the wavefunctions. Therefore, interaction rates written in terms of the matrix elements from Eq. (1) can be used for any target, and the NR EFT Lagrangian specifies which NR interaction operators, \mathcal{O}_{NR} , contribute to the NR matrix element.

An additional benefit of the Lagrangian formalism for the DM-electron NR EFT is it allows observables to be computed using Feynman diagrams. Given the DM-electron NR EFT Lagrangian, we derive Feynman rules (Sec. IV) for the NR electron propagator, three-point and four-point vertices, and some commonly appearing loop diagrams (built from the previous vertex and propagator Feynman rules). These Feynman rules depend on the transition matrix elements in Eq. (1), and include the target-dependent wavefunctions for the relevant observable. This diagrammatic approach is useful because it allows complicated results to be built from simple primitive rules, and can be used to identify new observables. Furthermore, it clarifies when screening effects are present, which manifests as the cancellation between diagrams which can be easily overlooked in a tree level calculation.

To determine the NR EFT of DM-electron interactions we adopt a “top-down” approach: starting from a high-energy theory defined at energies above m_e , we match on to the low-energy NR EFT by finding a map between the high-energy and low-energy electron fields. This is sometimes referred to as “integrating out the positron,” since the low-energy theory will only contain two components of the high-energy, four-component electron field. This top-down approach is beneficial because it provides an explicit connection between high-energy model building and low-energy DM-electron phenomenology. This approach can be contrasted with the other EFTs of electron [68–70], nuclear recoil [83–87], and collective excitation scattering [88] because it is only an EFT of the electron degrees of freedom; we do *not* assume that the DM kinematics are non-relativistic. The EFT approach here applies whenever $\omega, |\mathbf{q}| \ll m_e$, and no further approximations are required on the DM side of the calculation.

This paper is organized as follows: in Sec. II we derive the NR QED Lagrangian to order m_e^{-2} using the Foldy-Wouthuysen method, which provides the mapping between the high-energy electron field, involved in high-energy DM-electron interactions, and the two component, low-energy electron field, involved in the low-energy DM-electron interactions. In Sec. III we use this mapping to connect a general high-energy DM-electron interaction Lagrangian

to the low-energy, NR EFT of DM-electron interactions. In Sec. IV we derive the Feynman rules for this NR EFT, allowing any observable to be computed diagrammatically. In Secs. V, VI, and VII we use the Feynman rules derived in Sec. IV to compute absorption, elastic scattering, and dark Thomson scattering (the low-energy limit of the dark Compton scattering process in Ref. [23]), respectively, for a wide variety of DM models. App. A contains a summary of the NR EFT interaction Lagrangians and their corresponding Feynman rules. App. B contains a detailed derivation of the in-medium photon propagator which is needed whenever screening effects are relevant.

II. NON-RELATIVISTIC QED

Our main goal is to derive the NR EFT of DM-electron interactions. That is, we want to find an interaction Lagrangian coupling the DM fields to the electron field ψ in the NR limit. To do this we must first understand how ψ is related to the high-energy electron spinor Ψ , which appears in the high-energy DM-electron interaction Lagrangian. To leading order in the DM-electron coupling, the mapping of $\Psi \rightarrow \psi$ is independent of the high-energy DM-electron interaction, and is found by taking the NR limit of QED. Therefore, we begin by deriving the NR QED Lagrangian, $\mathcal{L}_{\text{QED}}^{\text{NR}}$, which describes the dynamics of ψ . While this has been developed before in a variety of different contexts (see, for example, Refs. [57, 82, 89]), here we present a pedagogical derivation for completeness, focusing on the method best suited for the calculations in Sec. III. This approach is “top-down,” as our starting point is the Dirac Lagrangian and we derive NR QED by taking its low energy limit to a fixed order in m_e . This approach contrasts with previous “bottom-up” calculations (see for example Ref. [89]), which focus on symmetry principles to build a basis of operators in the NR limit.

A. Framing the Problem: Diagonalizing QED

To motivate our approach consider the Dirac Lagrangian

$$\mathcal{L}_{\text{QED}} = \bar{\Psi} (i\gamma^\mu D_\mu - m_e) \Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (2)$$

where Ψ is the usual four-component electron Dirac spinor, ψ_1, ψ_2 are two-component spinors, $D_\mu = \partial_\mu + ieA_\mu$ is the gauge covariant derivative, $e = -|e|$ is the electron charge, and A_μ is the photon field. We will find it convenient to work in the Dirac basis for the gamma matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3)$$

where σ^i are the Pauli matrices. In the zero-momentum ($\partial_i \rightarrow 0$), non-interacting ($A_\mu \rightarrow 0$) limit, the QED Lagrangian in Eq. (2) can be written

$$\mathcal{L}_{\text{QED}} = \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \begin{pmatrix} i\partial_t - m_e & 0 \\ 0 & i\partial_t + m_e \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (4)$$

where ψ_1 can now be identified as the two-component electron field within Ψ , since its equation of motion in this limit is

$$\frac{\delta \mathcal{L}_{\text{QED}}}{\delta \psi_1^\dagger} = (i\partial_t - m_e) \psi_1 = 0 \quad \implies \quad \psi_1 \propto e^{-im_e t}, \quad (5)$$

and we see that ψ_1 has positive frequency m_e . Furthermore, by re-phasing the four-component field according to

$$\Psi \rightarrow e^{-im_e t} \Psi, \quad (6)$$

the QED Lagrangian in the zero-momentum, non-interacting limit, Eq. (4), can be simplified to yield

$$\mathcal{L}_{\text{QED}} = \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \begin{pmatrix} i\partial_t & 0 \\ 0 & i\partial_t + 2m_e \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (7)$$

explicitly demonstrating that the electron field ψ_1 satisfies the free NR Schrödinger equation, $i\partial_t\psi_1 = 0$. Since ψ_1, ψ_2 have no interactions, ψ_2 decouples from the theory in this zero-momentum, non-interacting limit.

However, once momentum dependence and interactions are restored, the full QED Lagrangian in Eq. (2) contains off-diagonal terms that couple ψ_1 to ψ_2

$$\mathcal{L}_{\text{QED}} = \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \begin{pmatrix} iD_t & i\sigma^i D_i \\ i\sigma^i D_i & iD_t + 2m_e \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (8)$$

so developing the NR EFT of QED at energy scales below m_e reduces to the task of integrating out ψ_2 (which is considered “heavy”) to remove these off-diagonal interactions. A common approach to eliminating these interactions is to substitute for ψ_2 using its equation of motion

$$\psi_2 = -\frac{1}{iD_t + 2m_e} (i\sigma^i D_i \psi_1), \quad (9)$$

and then expand in powers of $1/m_e$ to derive NR QED at a given order. While this approach is conceptually simple, it introduces technical complexity since at every order in $1/m_e$ terms with additional time derivatives appear, which alter the kinetic term for ψ_1 . These additional terms can be removed with further field redefinitions, but determining the appropriate form for such transformations is generically non-trivial and calls for a more systematic treatment, which we describe below.

B. Foldy-Wouthuysen Transformations in QED

An alternative technique for removing the ψ_1, ψ_2 interactions in Eq. (8) is known as the Foldy-Wouthuysen (FW) method [90–94], which perturbatively diagonalizes the Lagrangian with successive field redefinitions. This approach is tantamount to determining a set of n Hermitian operators $\{X_0, \dots, X_{n-1}\}$ for which the field redefinition

$$\Psi \rightarrow U_n \Psi \equiv \left[\exp\left(-i\frac{X_0}{m_e}\right) \dots \exp\left(-i\frac{X_{n-1}}{m_e^n}\right) \right] \begin{pmatrix} \psi \\ \psi_H \end{pmatrix}, \quad (10)$$

diagonalizes the QED Lagrangian in Eq. (8) to order m_e^{-n+1} , where U_n is understood to be the operator in the square brackets, and the post-field redefinition Ψ is written in terms of the electron field ψ and the heavy field ψ_H . To be explicit, after the field redefinition in Eq. (10) there will be no terms in the QED Lagrangian which contract ψ with ψ_H at order m_e^{-n+1} or lower; they may only exist at order m_e^{-n} or higher.

For future convenience in determining the X_i diagonalization operators, we introduce the following operator classification:

- **Even Operators:** An *even* operator is diagonal in the Dirac basis, and commutes with γ^0 in any basis; if $[E, \gamma^0] = 0$ then E is even. Upon contraction with Dirac spinors an even operator does not mix components in the Dirac basis, so $\Psi^\dagger E \Psi \rightarrow \psi^\dagger(\dots)\psi + \psi_H^\dagger(\dots)\psi_H$, where $\Psi = (\psi \ \psi_H)^T$. As an example, γ^0 is an even operator.

- **Odd Operators:** An *odd* operator is off-diagonal in the Dirac basis, and anti-commutes with γ^0 in any basis, so if $\{O, \gamma^0\} = 0$ then O is an odd operator. Upon contraction with Dirac spinors, an odd operator mixes components in the Dirac basis, so that $\Psi^\dagger O \Psi \rightarrow \psi^\dagger(\dots)\psi_H + \psi_H^\dagger(\dots)\psi$. As an example, γ^i is an odd operator.

These definitions follow the standard rules of even and odd quantities: an even operator multiplied by an even (odd) operator is even (odd), and an odd operator multiplied by an odd operator is even. In this language diagonalizing the QED Lagrangian to order m_e^{-n} amounts to removing all terms containing an odd operator to order m_e^{-n} . Additionally we define the projection operators

$$P_+ = \frac{1 + \gamma^0}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_- = \frac{1 - \gamma^0}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (11)$$

which have simple interpretations in the Dirac basis: P_+ projects out the upper component of a spinor, and P_- projects out the lower component. Note that both P_\pm are even operators. Lastly, since they will appear frequently, we define special even and odd operators

$$\mathcal{O}_E \equiv iD_t, \quad \mathcal{O}_O \equiv i\gamma^0\gamma^i D_i, \quad (12)$$

so that the QED Lagrangian in Eq. (8) is

$$\mathcal{L}_{\text{QED}} = \Psi^\dagger (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) \Psi. \quad (13)$$

To illustrate the procedure we will explicitly find X_0 and X_1 , and then in Sec. II C provide the algorithm for finding the others.

Expanding QED to First Order

We begin by diagonalizing the QED Lagrangian in Eq. (13) to order m_e^0 by removing the odd operator, \mathcal{O}_O , present at order m_e^0 . Substituting the $n = 1$ FW transformation from Eq. (10) in to Eq. (13)

$$\begin{aligned} \mathcal{L}_{\text{QED}} &\rightarrow \Psi^\dagger \left[U_1^\dagger (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) U_1 \right] \Psi \\ &= \Psi^\dagger \left[e^{iX_0/m_e} (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) e^{-iX_0/m_e} \right] \Psi \\ &\approx \Psi^\dagger (\mathcal{O}_E + \mathcal{O}_O + [iX_0, 2P_-]) \Psi, \end{aligned} \quad (14)$$

where in the last line we have only kept terms to order m_e^0 . The operator X_0 is defined by requiring that it removes all the odd terms at order m_e^0 , which means removing \mathcal{O}_O in Eq. (14). Since \mathcal{O}_O is odd and P_- is even, X_0 must be odd, so its commutator with P_- can be computed as $[iX_0, 2P_-] = -i[X_0, \gamma^0] = 2i\gamma^0 X_0$. Thus, demanding that X_0 cancels \mathcal{O}_O implies

$$[iX_0, 2P_-] = 2i\gamma^0 X_0 = -\mathcal{O}_O \quad \implies \quad X_0 = -\frac{1}{2}\gamma^i D_i. \quad (15)$$

The NR QED Lagrangian, which contains only the ψ degrees of freedom, can then be extracted to order m_e^0 by inserting P_+ projection operators, $\Psi \rightarrow P_+ \Psi$ after the $n = 1$ FW transformation has been applied

$$\mathcal{L}_{\text{QED}}^{\text{NR}} \approx \Psi^\dagger P_+ \left[\hat{U}_1^\dagger (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) U_1 \right] P_+ \Psi \approx \psi^\dagger iD_t \psi. \quad (16)$$

Note that we did not need to know an explicit form for X_0 to find $\mathcal{L}_{\text{QED}}^{\text{NR}}$ to order m_e^0 ; we just needed to remove the odd operators from the Lagrangian at order m_e^0 . For example, if we had simply removed \mathcal{O}_O from the Lagrangian in Eq. (13) we would have found the NR QED Lagrangian to order m_e^0 . By repeating this procedure with additional operators in Eq. (10) we can systematically obtain a diagonal Lagrangian to any order in $1/m_e$.

Expanding QED to Second Order

We now repeat the steps above using X_0 from Eq. (15) to determine both X_1 and the NR QED Lagrangian to order m_e^{-1} . Applying the $n = 2$ FW transformation from Eq. (10) yields

$$\Psi \rightarrow U_2 \Psi = \left[\exp\left(-i\frac{X_0}{m_e}\right) \exp\left(-i\frac{X_1}{m_e^2}\right) \right] \Psi, \quad (17)$$

and we insert this into Eq. (13) to obtain

$$\mathcal{L}_{\text{QED}} \rightarrow \Psi^\dagger \left[e^{iX_1/m_e^2} e^{iX_0/m_e} (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) e^{-iX_0/m_e} e^{-iX_1/m_e^2} \right] \Psi. \quad (18)$$

Expanding the exponentials to order m_e^{-1} with the standard Baker-Campbell-Hausdorff (BCH) formula yields

$$\mathcal{L}_{\text{QED}} \approx \Psi^\dagger \left\{ 2m_e P_- + \mathcal{O}_E + \frac{1}{m_e} \left([iX_0, \mathcal{O}_E] + [iX_0, \mathcal{O}_O] + \frac{1}{2} [iX_0, [iX_0, 2P_-]] + [iX_1, 2P_-] \right) \right\} \Psi, \quad (19)$$

and X_1 is determined by requiring that there are no odd terms at order m_e^{-1} in Eq. (19). Since P_\pm are even and X_0 is odd, the only odd term that needs to be eliminated from Eq. (19) is $[iX_0, \mathcal{O}_E]/m_e$. Therefore we must choose X_1 such that $i[X_1, 2P_-] = -[iX_0, \mathcal{O}_E]$. Again using the fact that X_1 is odd, $i[X_1, 2P_-] = 2i\gamma^0 X_1$, and therefore

$$X_1 = -\frac{i}{2}\gamma^0 (-[iX_0, \mathcal{O}_E]) = \frac{e}{4}\gamma^0 \gamma^i F_{0i}, \quad (20)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor satisfying $[D_\mu, D_\nu] = ieF_{\mu\nu}$.

To identify the NR QED Lagrangian to order m_e^{-1} we again explicitly project out the upper component of Ψ in Eq. (19) with the substitution $\Psi \rightarrow P_+ \Psi$ yielding

$$\mathcal{L}_{\text{QED}}^{\text{NR}} \approx \Psi^\dagger P_+ \left\{ 2m_e P_- + \mathcal{O}_E + \frac{1}{m_e} \left([iX_0, \mathcal{O}_O] + \frac{1}{2} [iX_0, [iX_0, 2P_-]] \right) \right\} P_+ \Psi, \quad (21)$$

which can be further simplified using X_0 from Eq. (15) to yield

$$\mathcal{L}_{\text{QED}}^{\text{NR}} \approx \psi^\dagger \left[iD_t + \frac{1}{2m_e} (\sigma^i \sigma^j D_i D_j) \right] \psi. \quad (22)$$

As expected, the NR QED Lagrangian to order m_e^{-1} renders an equation of motion for ψ which is simply the Schrödinger equation for a spin- $\frac{1}{2}$ charged particle.

C. Summary: Expanding QED to Arbitrary Order

The procedure for finding X_{n-1} and the NR QED Lagrangian to order $m_e^{-(n-1)}$ should now be clear:

1. **Rephase:** Using the known expressions for $\{X_0, \dots, X_{n-2}\}$, determined in the previous iterations of this process, rephase Ψ with the n^{th} order FW transformation

$$\Psi \rightarrow U_n \Psi \equiv \left[\exp\left(-i\frac{X_0}{m_e}\right) \dots \exp\left(-i\frac{X_{n-1}}{m_e^n}\right) \right] \Psi \quad (23)$$

where the X_{n-1} operator is initially treated as an unknown quantity. As we found above, the first two X operators can be written

$$X_0 = -\frac{1}{2}\gamma^i D_i \quad , \quad X_1 = \frac{e}{4}\gamma^0\gamma^i F_{0i} . \quad (24)$$

2. **Expand:** Insert Eq. (23) into the QED Lagrangian

$$\mathcal{L}_{\text{QED}} \rightarrow \Psi^\dagger [U_n^\dagger (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) U_n] \Psi , \quad (25)$$

and use the BCH formula to expand the exponentials to order $m_e^{-(n-1)}$. Identify all odd operators at order $m_e^{-(n-1)}$ and remove them with a suitable definition for X_{n-1} .

3. **Project:** Replace $\Psi \rightarrow P_+ \Psi$, to extract only the electron degrees of freedom, ψ , from the full Dirac spinor Ψ , and expand to order m_e^{-n+1}

$$\mathcal{L}_{\text{QED}}^{\text{NR}} = \Psi^\dagger P_+ U_n^\dagger (\mathcal{O}_E + \mathcal{O}_O + 2m_e P_-) U_n P_+ \Psi \approx \psi^\dagger (\dots) \psi , \quad (26)$$

which results in the desired expression: the NR QED Lagrangian in terms of ψ to order m_e^{-n+1} .

Using this procedure we can find the NR QED Lagrangian to order m_e^{-2} . Note that we do not need to explicitly solve for X_2 as long as we require that it remove all odd operators at order m_e^{-2} . Following this procedure we obtain

$$\mathcal{L}_{\text{QED}}^{\text{NR}} \approx \psi^\dagger \left\{ iD_t + \frac{1}{2m_e} (-D^i D_i + e\sigma^i B_i) + \frac{e}{8m_e^2} [\sigma^i \sigma^j (E_i D_j - E_j D_i - (\partial_i E_j))] \right\} \psi \quad (27)$$

where $E^i = -F^{0i}$, and $B^i = -\epsilon^{ijk} F_{jk}/2$ ($F_{ij} = \epsilon_{ijk} B^k$) are the electric and magnetic fields, respectively.

III. NON-RELATIVISTIC DARK MATTER-ELECTRON INTERACTIONS

We now introduce interactions between the four-component electron spinor Ψ and an arbitrary set of dark fields (either the DM itself or a mediator particle). In the ultraviolet (UV), at energies above the electron mass, the general interaction Lagrangian can be written

$$\mathcal{L}_{\text{int}}^{\text{UV}} = \bar{\Psi} \mathcal{O}_{\text{UV}} \Psi , \quad (28)$$

where \mathcal{O}_{UV} contains the dark fields and has a Lorentz structure that contracts with Ψ . For example, if a dark boson ϕ has a Yukawa interaction with electrons, we would write

$$\mathcal{L}_{\text{int}}^{\text{UV}} = g \phi \bar{\Psi} \Psi \quad , \quad \mathcal{O}_{\text{UV}} = g \phi \quad , \quad (29)$$

where g is a coupling constant. Alternatively, for a four-Fermi interaction with a dark fermion χ , we would write

$$\mathcal{L}_{\text{int}}^{\text{UV}} = \frac{1}{\Lambda^2} [\bar{\chi} \gamma^\mu \chi] \bar{\Psi} \gamma_\mu \Psi \quad , \quad \mathcal{O}_{\text{UV}} = \frac{1}{\Lambda^2} [\bar{\chi} \gamma^\mu \chi] \gamma_\mu , \quad (30)$$

where it is understood that $[\bar{\chi} \gamma^\mu \chi]$ is a scalar quantity, and therefore Ψ only contracts with the γ_μ outside the brackets.

In this section, our goal is to determine the NR limit of Eq. (28) by applying the Foldy-Wouthuysen transformation from Sec. II and then express the result in a basis of NR operators that act on the two-component NR electron field ψ . Schematically, we identify the mapping between high-energy and low-energy interactions as

$$\mathcal{L}_{\text{int}}^{\text{UV}} = \bar{\Psi} \mathcal{O}_{\text{UV}} \Psi \quad \longrightarrow \quad \mathcal{L}_{\text{int}}^{\text{NR}} = \psi^\dagger \mathcal{O}_{\text{NR}} \psi , \quad (31)$$

where \mathcal{O}_{NR} is the NR equivalent of \mathcal{O}_{UV} and will be written as a linear combination of NR basis operators.

A. The NR Interaction Lagrangian

To derive the NR interaction Lagrangian, we utilize the relation between Ψ and ψ derived in Sec. II. Thus, we apply the unitary FW transformation U_n in Eq. (10) on Ψ , whose upper and lower components may then be identified as the ψ and the heavy field ψ_H

$$\mathcal{L}_{\text{int}}^{\text{NR}} = \bar{\Psi} \mathcal{O}_{\text{UV}} \Psi \longrightarrow \begin{pmatrix} \psi^\dagger & \psi_H^\dagger \end{pmatrix} \left[\hat{U}_n^\dagger \gamma^0 \mathcal{O}_{\text{UV}} \hat{U}_n \right] \begin{pmatrix} \psi \\ \psi_H \end{pmatrix}. \quad (32)$$

As we found in Sec. II, the resulting NR interaction Lagrangian is extracted by projecting out the heavy field ψ_H with the explicit substitution $\Psi \rightarrow P_+ \Psi = (\psi \ 0)^T$ to obtain¹

$$\mathcal{L}_{\text{int}}^{\text{NR}} = \Psi^\dagger (P_+ U_n^\dagger \gamma^0 \mathcal{O}_{\text{UV}} U_n P_+) \Psi = \begin{pmatrix} \psi^\dagger & 0 \end{pmatrix} \left[U_n^\dagger \gamma^0 \mathcal{O}_{\text{UV}} U_n \right] \begin{pmatrix} \psi \\ 0 \end{pmatrix} = \psi^\dagger (\text{Tr} [P_+ U_n^\dagger \gamma^0 \mathcal{O}_{\text{UV}} U_n]) \psi, \quad (33)$$

where the trace is over the 2×2 blocks of the 4×4 Dirac matrix structure. From the definition in Eq. (31), the NR DM-electron interaction operator \mathcal{O}_{NR} can then be identified from Eq. (33) as

$$\mathcal{O}_{\text{NR}} = \text{Tr} [P_+ U_n^\dagger \gamma^0 \mathcal{O}_{\text{UV}} U_n]. \quad (34)$$

To only keep terms to order m_e^{-2} , we use the FW operator from Eq. (10)

$$U_2 = \exp\left(-i \frac{X_0}{m_e}\right) \exp\left(-i \frac{X_1}{m_e^2}\right), \quad (35)$$

where X_0, X_1 are defined in Eq. (24). Inserting this expression into Eq. (34), the explicit NR DM-electron interaction operator can be written

$$\mathcal{O}_{\text{NR}} \approx \text{Tr} \left[P_+ \left(\mathcal{O}_{\text{UV}} + \frac{i}{2m_e} \{\gamma^i D_i, \mathcal{O}_{\text{UV}}\} - \frac{1}{8m_e^2} \{\gamma^i D_i, \{\gamma^j D_j, \mathcal{O}_{\text{UV}}\}\} - \frac{ie}{4m_e^2} \{\gamma^0 \gamma^i F_{0i}, \mathcal{O}_{\text{UV}}\} \right) \right]. \quad (36)$$

Note that Eq. (36) is general, and does not assume anything about the nature or the multiplicity of the dark fields in \mathcal{O}_{UV} , as long as all higher dimension operators in the high-energy theory above m_e are suppressed by energy scales much greater than m_e . In the remainder of this section, we will specialize to interactions that are linear in the dark sector fields, while retaining full generality in the Lorentz structure in \mathcal{O}_{UV} .

B. A Representative Interaction Lagrangian

Using the main result of Sec. III A, Eq. (36), any (field-dependent) \mathcal{O}_{UV} can be related to a corresponding NR operator \mathcal{O}_{NR} to order m_e^{-2} . Our goal now is to decompose the latter as a linear combination of NR basis operators. However, since the physics of the dark sector is currently unknown, there is a vast multiplicity of possible fields and Lorentz structures that could be included in \mathcal{O}_{UV} . Therefore, in this subsection we identify a representative class of UV operators that couple linearly to Ψ , and map these onto a basis of NR operators that act on ψ .

¹ Naively the expression in Eq. (32) is concerning because a generic \mathcal{O}_{UV} operator will introduce off-diagonal terms, e.g., $\psi^\dagger(\dots)\psi_H$, that survive even after the FW transformation has been applied. However, since the FW transformation removes the QED interactions between ψ and ψ_H , any interactions introduced from further integrating out the heavy field ψ_H will be quadratic in the DM coupling (and higher order in $1/m_e$). Since such couplings are typically very small these extra terms are therefore suppressed. These terms may be kept by including \mathcal{O}_{UV} in the QED Lagrangian and diagonalizing the QED+DM Lagrangian together with the methods presented in Sec. II.

Representative UV Interaction

To illustrate the utility of Eq. (36) in developing the NR EFT, consider the UV interaction Lagrangian

$$\mathcal{L}_{\text{int}}^{\text{UV}} = y_s \phi \bar{\Psi} \Psi + i y_p \phi \bar{\Psi} \gamma^5 \Psi + g_v V_\mu \bar{\Psi} \gamma^\mu \Psi + g_a V_\mu \bar{\Psi} \gamma^\mu \gamma^5 \Psi + \frac{d_M}{2} V_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi, \quad (37)$$

where y_s, y_p, g_v, g_a and d_M are constant coefficients, ϕ is a dark scalar field, V_μ is a dark vector field, and $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the dark field strength tensor. The corresponding form of \mathcal{O}_{UV} in Eq. (28) can be written

$$\mathcal{O}_{\text{UV}} = (y_s + i y_p \gamma^5) \phi + (g_v \gamma^\mu + g_a \gamma^\mu \gamma^5) V_\mu + \frac{d_M}{2} \sigma^{\mu\nu} V_{\mu\nu}, \quad (38)$$

which contains the most general structure of matrices that can contract with Ψ .²

This collection of possible interaction operators can describe a surprisingly versatile range of physics. For example, if only y_s, y_p are non-zero and ϕ is the DM this is the most general dimension-four electron-scalar DM interaction. Similarly if only g_v, g_a are non-zero and V_μ is the DM, this is the most general dimension-four electron-vector DM interaction. If only d_M is non-zero, the electron has a “dark” magnetic dipole moment with respect to V_μ , and it was recently shown that V_μ can achieve the observed DM abundance through the UV freeze-in mechanisms [11]. The operator in Eq. (38) is also useful even when ϕ or V are not the DM; for example if they are mediator particles that couple the electron to the dark sector. A similarly general Lagrangian could be written down for a fermionic DM particle χ (replacing Ψ in Eq. (37)) which would enumerate all possible dimension-four interactions between the dark sector and electron, and more due to the inclusion of the dark magnetic dipole term. The NR limit of \mathcal{O}_{UV} will be useful in understanding the physics of all these models.

Note that ϕ and V_μ in Eq. (38) need not be fundamental and can represent operators built from multiple dark fields and/or their derivatives. For example, replacing $V_\mu \rightarrow \partial_\mu a$, where a is an axion-like field, does not change the NR limit calculation; one can simply substitute V_μ with $\partial_\mu a$ at the end to recover axion-electron phenomenology [80]. The same reasoning also applies to contact interactions built from products of dark fields. For example, the replacement $V^\mu \rightarrow \bar{\chi} \gamma^\mu \chi$, where χ is a dark fermion, also does not change the NR limit calculation. Although studying these alternative operators are beyond the scope of this paper, a thorough investigation may be useful for future work.

A Basis Of NR Electron Operators

In principle, Eq. (36) alone suffices to define the NR EFT for the UV interaction in Eq. (37); by directly substituting \mathcal{O}_{UV} from Eq. (38) into Eq. (37), it is straightforward to determine \mathcal{O}_{NR} as a function of the couplings y_s, y_p, g_v, g_a and d_M . However, if this NR expansion is not executed carefully, it can yield a complicated and unintuitive collection of terms.

Therefore we introduce a simple basis of 8 *dimensionless* operators $\hat{\mathcal{O}}_\ell$, with $\ell \in \{1, \dots, 8\}$, whose linear combinations generate all possible NR operators that will act on ψ to order m_e^{-2} . These objects which arise in the NR expansion are built out of $\{1, \nabla^i, \sigma^i, \Phi\}$, where Φ is the background potential on the electrons in the target material (for example, the electric potential sourced by the ions in a crystal).³ To order m_e^{-2} , this basis of operators acting on ψ can be organized as follows:

² Recall that the 16 matrices $\{1, \gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}\}$ form a basis for all other 4×4 Dirac matrices [95].

³ In principle we could also include contributions from a background magnetic field, which would result in additional terms in this basis set. However, for most targets this is not relevant, so we omit these for simplicity.

- **Derivative-Only Operators:** There are three basis elements which only depend on derivatives that act on ψ :

$$\hat{\mathcal{O}}_1 = 1 \quad , \quad [\hat{\mathcal{O}}_2]^i = \frac{\nabla^i}{m_e} \quad , \quad [\hat{\mathcal{O}}_3]^{ij} = \frac{\nabla^i \nabla^j}{m_e^2} \quad , \quad (39)$$

which we have normalized with powers of m_e to make the $\hat{\mathcal{O}}$ dimensionless. Note that ∇^i has dimension one, which limits the number of terms that can arise at this order.

- **Spin-Dependent Operators:** We also have spin-dependent basis elements proportional to Pauli matrices:

$$[\hat{\mathcal{O}}_4]^i = \boldsymbol{\sigma}^i \quad , \quad [\hat{\mathcal{O}}_5]^{ij} = \frac{\boldsymbol{\sigma}^i \nabla^j}{m_e} \quad , \quad [\hat{\mathcal{O}}_6]^{ijk} = \frac{\boldsymbol{\sigma}^i \nabla^j \nabla^k}{m_e^2} \quad . \quad (40)$$

Any term with two Pauli matrices may be reduced to a term with one using the identity $\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j = \delta^{ij} + i\epsilon^{ijk} \boldsymbol{\sigma}^k$.

- **Background-Dependent Operators:** We also have two operators that depend on the background electric potential field Φ :

$$[\hat{\mathcal{O}}_7]^i = \frac{e(\nabla^i \Phi)}{m_e^2} \quad , \quad [\hat{\mathcal{O}}_8]^{ij} = \frac{e \boldsymbol{\sigma}^i (\nabla^j \Phi)}{m_e^2} \quad , \quad (41)$$

where the brackets around $(\nabla^i \Phi)$ indicate that the enclosed gradient only acts on Φ . Note that Φ has dimension one, which also limits the terms that can appear at this order; operators containing $(\nabla \nabla \Phi)$ or $(\nabla \Phi)^2$ terms are higher order.

With the eight dimensionless operators in Eqs. (39)-(41), any \mathcal{O}_{NR} can be schematically decomposed as

$$\mathcal{O}_{\text{NR}} = \sum_{\ell=1}^8 (\dots) \hat{\mathcal{O}}_\ell \quad , \quad (42)$$

where we have suppressed the spin and spatial indices on $\hat{\mathcal{O}}_\ell$, and the expression (\dots) represents any other combination of fields which do not act on ψ or contract with its spinor indices, such that $\psi^\dagger \mathcal{O}_{\text{NR}} \psi = \sum_\ell (\dots) (\psi^\dagger \hat{\mathcal{O}}_\ell \psi)$. Note that in our convention, \mathcal{O}_{NR} has mass dimension 1 to match \mathcal{O}_{UV} , which differs from the dimensionless basis operators $\hat{\mathcal{O}}_\ell$.

Expanding The Field Content In \mathcal{O}_{NR}

We now turn to the problem of classifying all the field-dependent terms that can appear inside the (\dots) brackets of Eq. (42). In general, these expressions are combinations of dark sector fields $\{\phi, V^\mu\}$, the QED photon field A^μ , and derivatives acting on these.⁴ However, even with only the UV operators in Eq. (38), extracting the equivalent NR operators to order m_e^{-2} in Eq. (36) yields a large multiplicity of terms, so unlike the $\hat{\mathcal{O}}$ basis operators that act on ψ , a simple enumeration of all possible fields and derivatives combinations is unwieldy.

To manage this we introduce the dimensionless (field-independent) coefficients C to expand any combination of fields and derivatives that may appear in the (\dots) brackets of Eq. (42)

$$\begin{aligned} \mathcal{O}_{\text{NR}} = \sum_{\ell=1}^8 \left[& C_{\phi, \ell} \phi + C_{(\nabla \phi), \ell} \frac{(\nabla \phi)}{m_e} + C_{V^0, \ell} V^0 + C_{\mathbf{V}, \ell} \mathbf{V} + C_{(\nabla \mathbf{V}), \ell} \frac{(\nabla \mathbf{V})}{m_e} \right. \\ & \left. + C_{\phi A^0, \ell} \frac{\phi A^0}{m_e} + C_{\phi \mathbf{A}, \ell} \frac{\phi \mathbf{A}}{m_e} + C_{V^0 \mathbf{A}, \ell} \frac{V^0 \mathbf{A}}{m_e} + C_{(\nabla \phi) \mathbf{A}, \ell} \frac{(\nabla \phi) \mathbf{A}}{m_e^2} + \dots \right] \hat{\mathcal{O}}_\ell \quad , \quad (43) \end{aligned}$$

⁴ Note that the photon field A^μ represents quantized electromagnetic field excitations and is treated as distinct from the background field Φ , which appears in the $\hat{\mathcal{O}}_\ell$ basis elements and corresponds to the background expectation value $\Phi = \langle A^0 \rangle$.

where we have listed an exemplary set of field operators, and decomposed the four-dimensional fields into their 0 and spatial components explicitly: $A^\mu = (A^0, \mathbf{A})$, $V^\mu = (V^0, \mathbf{V})$. Each term in the brackets in Eq. (43) is normalized with the appropriate power of m_e to ensure that the overall mass dimension is one to match \mathcal{O}_{NR} . Each coefficient C in Eq. (43) has two subscripts separated by a comma:

- **Left Subscript:** Matches the field content it multiplies in Eq. (43). For example, $C_{(\nabla\phi),\dots}$ multiplies $(\nabla\phi)$.
- **Right Subscript:** Matches the subscript of the multiplied basis operator $\hat{\mathcal{O}}_\ell$.

Furthermore note that we have suppressed all Cartesian indices in Eq. (43). These may be added back in by matching the indices on the field operators and $\hat{\mathcal{O}}_\ell$ with C indices. To illustrate this, we expand Eq. (43) further and keep all indices explicit

$$\begin{aligned} \mathcal{O}_{\text{NR}} = & \left(C_{\phi,1} \phi + [C_{(\nabla\phi),1}]^i \frac{(\nabla^i \phi)}{m_e} + C_{V^0,1} V^0 + [C_{\mathbf{V},1}]^i \mathbf{V}^i + [C_{(\nabla\mathbf{V}),1}]^{ij} \frac{(\nabla^i \mathbf{V}^j)}{m_e} \right) \hat{\mathcal{O}}_1 \\ & + \left([C_{\phi A^0,2}]^k \frac{\phi A^0}{m_e} + [C_{\phi \mathbf{A},2}]^{i,k} \frac{\phi \mathbf{A}^i}{m_e} + [C_{V^0 \mathbf{A},2}]^{i,k} \frac{V^0 \mathbf{A}^i}{m_e} + [C_{(\nabla\phi) \mathbf{A},2}]^{ij,k} \frac{(\nabla^i \phi) \mathbf{A}^j}{m_e^2} \right) [\hat{\mathcal{O}}_2]^k + \dots, \end{aligned} \quad (44)$$

where the superscripts on the C 's in Eq. (44) follow a similar convention as the subscripts

- **Left Superscript:** Matches the indices of the field content (ij in Eq. (44)).
- **Right Superscript:** Matches the indices of $\hat{\mathcal{O}}_\ell$ (k in Eq. (44)).

Lastly, note that *all Cartesian indices are raised* in Eq. (44); the lowered components of a vector v are related by $v_i = \eta_{ij} v^j$, where $\eta^{ij} = -\delta^{ij}$ are the spatial components of the metric tensor with mostly negative signature.

C. Matching The Representative UV-NR Interactions

With this notation our goal of taking NR limit crystallizes: after substituting the general \mathcal{O}_{UV} in Eq. (38) into Eq. (36), find the C 's as functions of y_s, y_p, g_v , and g_a ; we present analogous expressions for d_M in App. A. This defines the NR EFT of DM-electron interactions to order m_e^{-2} determined by the high-energy Lagrangian in Eq. (37). This is a straightforward, albeit tedious, task which is greatly simplified with the help of FeynCalc [96–100]. The results for the scalar coupled models (y_s, y_p non-zero) and the vector coupled models (g_v, g_a non-zero) are shown in Tables I and II, respectively.

To further elucidate the interactions that arise from the C 's in Tables I and II we can separate out the scalar and vector contributions to \mathcal{O}_{NR} as

$$\mathcal{O}_{\text{NR}} = \mathcal{O}_{\text{NR}}^\phi + \mathcal{O}_{\text{NR}}^{V^0} + \mathcal{O}_{\text{NR}}^{\mathbf{V}}. \quad (45)$$

All non-zero entries in the tables can be written as terms in either $\mathcal{O}_{\text{NR}}^\phi$, $\mathcal{O}_{\text{NR}}^{V^0}$ or $\mathcal{O}_{\text{NR}}^{\mathbf{V}}$. For the scalar coefficients, this decomposition yields

$$\begin{aligned} \mathcal{O}_{\text{NR}}^\phi = & \sum_{\ell=1}^8 \left[C_{\phi,\ell} \phi + C_{(\nabla\phi),\ell} \frac{(\nabla\phi)}{m_e} + C_{\phi \mathbf{A},\ell} \frac{\phi \mathbf{A}}{m_e} + C_{(\nabla\nabla\phi),\ell} \frac{(\nabla\nabla\phi)}{m_e^2} + C_{\phi(\partial_t \mathbf{A}),\ell} \frac{\phi(\partial_t \mathbf{A})}{m_e^2} \right. \\ & \left. + C_{\phi(\nabla A^0),\ell} \frac{\phi(\nabla A^0)}{m_e^2} + C_{(\nabla\phi) \mathbf{A},\ell} \frac{(\nabla\phi) \mathbf{A}}{m_e^2} + C_{\phi(\nabla \mathbf{A}),\ell} \frac{\phi(\nabla \mathbf{A})}{m_e^2} + C_{\phi \mathbf{A} \mathbf{A},\ell} \frac{\phi \mathbf{A} \mathbf{A}}{m_e^2} \right] \hat{\mathcal{O}}_\ell, \end{aligned} \quad (46)$$

NR EFT Lagrangian Coefficients, $C_{(\text{fields}),(\text{basis operator})}$

	$\hat{\mathcal{O}}_1$	$[\hat{\mathcal{O}}_4]^i$	$[\hat{\mathcal{O}}_2]^i$	$[\hat{\mathcal{O}}_5]^{ij}$	$[\hat{\mathcal{O}}_3]^{ij}$	$[\hat{\mathcal{O}}_8]^{ij}$
	1	σ^i	$\frac{\nabla^i}{m_e}$	$\frac{\sigma^i \nabla^j}{m_e}$	$\frac{\nabla^i \nabla^j}{m_e^2}$	$\frac{e\sigma^i (\nabla^j \Phi)}{m_e^2}$
ϕ	y_s	0	0	0	$y_s \frac{\delta^{ij}}{2}$	$-y_p \frac{\delta^{ij}}{2}$
$\frac{(\nabla^a \phi)}{m_e}$	0	$-y_p \frac{\delta^{ai}}{2}$	$y_s \frac{\delta^{ai}}{2}$	$y_s \left[\frac{-i\epsilon^{aij}}{4} \right]$		
$\frac{\phi \mathbf{A}^a}{m_e}$	0	0	$-iey_s \delta^{ai}$	0		
$\frac{(\nabla^a \nabla^b \phi)}{m_e^2}$	$y_s \frac{\delta^{ab}}{8}$	0				
$\frac{\phi (\partial_t \mathbf{A}^a)}{m_e^2}$	0	$-ey_p \frac{\delta^{ai}}{2}$				
$\frac{\phi (\nabla^a A^0)}{m_e^2}$	0	$-ey_p \frac{\delta^{ai}}{2}$				
$\frac{(\nabla^a \phi) \mathbf{A}^b}{m_e^2}$	$ey_s \left[\frac{-i\delta^{ab}}{2} \right]$	$ey_s \frac{\epsilon^{abi}}{4}$				
$\frac{\phi (\nabla^a \mathbf{A}^b)}{m_e^2}$	$ey_s \left[\frac{-i\delta^{ab}}{2} \right]$	$ey_s \frac{\epsilon^{abi}}{2}$				
$\frac{\phi \mathbf{A}^a \mathbf{A}^b}{m_e^2}$	$-e^2 y_s \frac{\delta^{ab}}{2}$	0				

TABLE I. The C coefficients of the NR interaction Lagrangian derived from the terms with a dark scalar field, ϕ , in the representative UV Lagrangian (Eq. (37) with y_s, y_p non-zero) to order m_e^{-2} . Each entry is a C with a left subscript matching the fields to the left, and right subscript matching the basis operator subscript to the top. Superscripts on C match the corresponding field content and basis operator. For example, $[C_{(\nabla \nabla \phi),1}]^{ij} = y_s \delta^{ij}/8$ contributes a term in the NR interaction Lagrangian written abstractly as $\mathcal{L}_{\text{int}}^{\text{NR}} \supset C_{(\nabla \nabla \phi),1} [(\nabla \nabla \phi)/m_e^2][\psi^\dagger \hat{\mathcal{O}}_1 \psi]$, and concretely as $\mathcal{L}_{\text{int}}^{\text{NR}} \supset [C_{(\nabla \nabla \phi),1}]^{ij} (\nabla^i \nabla^j \phi)(\psi^\dagger \psi)/m_e^2 = y_s (\nabla^2 \phi) \psi^\dagger \psi / 8 m_e^2$. The basis operators and field content are ordered by powers of $1/m_e$; entries farther right contribute terms higher order in electron velocity, and entries farther down contribute terms higher order in incoming/outgoing field four-momenta. Grayed entries contribute terms in the NR interaction Lagrangian at order m_e^{-3} and above, and are therefore neglected here; columns for $\hat{\mathcal{O}}_6, \hat{\mathcal{O}}_7$ are ignored because all entries are zero for this set of UV interactions.

The corresponding expression for vector time component V^0 is

$$\mathcal{O}_{\text{NR}}^{V^0} = \sum_{\ell=1}^8 \left[C_{V^0,\ell} V^0 + C_{(\nabla V^0),\ell} \frac{(\nabla V^0)}{m_e} + C_{V^0 \mathbf{A},\ell} \frac{V^0 \mathbf{A}}{m_e} + C_{(\nabla \nabla V^0),\ell} \frac{(\nabla \nabla V^0)}{m_e^2} + C_{(\nabla V^0) \mathbf{A},\ell} \frac{(\nabla V^0) \mathbf{A}}{m_e^2} \right] \hat{\mathcal{O}}_\ell, \quad (47)$$

NR EFT Lagrangian Coefficients, $C_{(\text{fields}),(\text{basis operator})}$

	$\hat{\mathcal{O}}_1$	$[\hat{\mathcal{O}}_4]^i$	$[\hat{\mathcal{O}}_2]^i$	$[\hat{\mathcal{O}}_5]^{ij}$	$[\hat{\mathcal{O}}_6]^{ijk}$	$[\hat{\mathcal{O}}_8]^{ij}$
	1	σ^i	$\frac{\nabla^i}{m_e}$	$\frac{\sigma^i \nabla^j}{m_e}$	$\frac{\sigma^i \nabla^j \nabla^k}{m_e^2}$	$\frac{e\sigma^i (\nabla^j \Phi)}{m_e^2}$
V^0	g_v	0	0	$-ig_a \delta^{ij}$	0	0
\mathbf{V}^a	0	$-g_a \delta^{ai}$	$ig_v \delta^{ai}$	0	$\frac{g_a}{2} H_1^{aijk}$	$-\frac{g_v}{2} \epsilon^{aij}$
$\frac{\nabla^a V^0}{m_e}$	0	$-\frac{ig_a}{2} \delta^{ai}$	0	$\frac{ig_v}{4} \epsilon^{aij}$		
$\frac{\nabla^a \mathbf{V}^b}{m_e}$	$\frac{ig_v}{2} \delta^{ab}$	$-\frac{g_v}{2} \epsilon^{abi}$	$\frac{ig_a}{4} \epsilon^{abi}$	$\frac{g_a}{4} H_2^{abij}$		
$\frac{V^0 \mathbf{A}^a}{m_e}$	0	$-eg_a \delta^{ai}$	0	0		
$\frac{\mathbf{V}^a \mathbf{A}^b}{m_e}$	$eg_v \delta^{ab}$	0	0	$-\frac{ieg_a}{2} H_2^{abji}$		
$\frac{\nabla^a \nabla^b V^0}{m_e^2}$	$\frac{g_v}{8} \delta^{ab}$	0				
$\frac{\nabla^a \nabla^b \mathbf{V}^c}{m_e^2}$	0	$-\frac{g_a}{8} \delta^{ab} \delta^{ci}$				
$\frac{(\nabla^a V^0) \mathbf{A}^b}{m_e^2}$	0	$-\frac{eg_v}{4} \epsilon^{abi}$				
$\frac{(\nabla^a \mathbf{V}^b) \mathbf{A}^c}{m_e^2}$	$\frac{eg_a}{4} \epsilon^{abc}$	$-\frac{ieg_a}{4} H_2^{abci}$				
$\frac{\mathbf{V}^a (\partial_t \mathbf{A}^b)}{m_e^2}$	0	$\frac{eg_v}{2} \epsilon^{abi}$				
$\frac{\mathbf{V}^a (\nabla^b \mathbf{A}^c)}{m_e^2}$	$-\frac{eg_a}{2} \epsilon^{abc}$	$-\frac{ieg_a}{4} H_2^{abci}$				
$\frac{\mathbf{V}^a (\nabla^b A^0)}{m_e^2}$	0	$\frac{eg_v}{2} \epsilon^{abi}$				
$\frac{\mathbf{V}^a \mathbf{A}^b \mathbf{A}^c}{m_e^2}$	0	$-\frac{e^2 g_a}{2} H_1^{abci}$				

TABLE II. The C coefficients of the NR interaction Lagrangian derived from the terms with a dark vector field, V , in the representative UV Lagrangian (Eq. (37) with g_v, g_a non-zero) to order m_e^{-2} . Table entries (C 's) follow the same convention as in Table I, and is discussed in detail in Sec. III. Grayed entries contribute terms in the NR interaction Lagrangian at order m_e^{-3} and above, and are therefore neglected here. Columns for $\hat{\mathcal{O}}_3, \hat{\mathcal{O}}_7$ are ignored because all entries are zero. Furthermore, to help with presentation we define some four index tensors, $H_1^{ijkl} \equiv \delta^{ik} \delta^{jl} - \delta^{ij} \delta^{kl}$, $H_2^{ijkl} \equiv \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} - 2\delta^{il} \delta^{jk}$.

and finally, the analogous field expansion for the vector spatial components \mathbf{V} is

$$\begin{aligned} \mathcal{O}_{\text{NR}}^{\mathbf{V}} = \sum_{\ell=1}^8 & \left[C_{\mathbf{V},\ell} \mathbf{V} + C_{(\nabla\mathbf{V}),\ell} \frac{(\nabla\mathbf{V})}{m_e} + C_{\mathbf{V}\mathbf{A},\ell} \frac{\mathbf{V}\mathbf{A}}{m_e} + C_{(\nabla\nabla\mathbf{V}),\ell} \frac{(\nabla\nabla\mathbf{V})}{m_e^2} + C_{(\nabla\mathbf{V})\mathbf{A},\ell} \frac{(\nabla\mathbf{V})\mathbf{A}}{m_e^2} \right. \\ & \left. + C_{\mathbf{V}(\partial_t\mathbf{A}),\ell} \frac{\mathbf{V}(\partial_t\mathbf{A})}{m_e^2} + C_{\mathbf{V}(\nabla\mathbf{A}),\ell} \frac{\mathbf{V}(\nabla\mathbf{A})}{m_e^2} + C_{\mathbf{V}\mathbf{A}\mathbf{A},\ell} \frac{\mathbf{V}\mathbf{A}\mathbf{A}}{m_e^2} \right] \hat{\mathcal{O}}_{\ell} . \end{aligned} \quad (48)$$

In addition to presenting the C coefficients, we also summarize the results in App. A for the cases where only one of the y_s, y_p, g_v, g_a, d_M coefficients is non-zero. For each table in App. A the first row is the assumed high-energy interaction Lagrangian and the second row is the corresponding NR EFT interaction Lagrangian. The later sections of the tables give the Feynman rules for different vertices that arise in NR EFT, and their derivation is discussed in detail next in Sec. IV.

IV. FEYNMAN RULES IN THE NR EFT

To understand the phenomenology of a given DM interaction, one typically uses Feynman diagrams to compute matrix elements associated with the observable of interest. This procedure is usually straightforward since the Feynman rules for usual relativistic QFTs are well known textbook material [95, 101]. However, here our NR EFT Lagrangian (the theory defined by the NR operators from Eqs. (46) - (48)) is not a familiar relativistic QFT, and therefore the corresponding Feynman rules do not apply in the usual way.

Usually in relativistic QFT, one perturbs around the free theory, which is defined as the limit where interactions and background fields are absent. This means that the building blocks for relativistic Feynman rules (incoming/outgoing states, propagators, and vertex insertions) assume relatively simple forms since the states involved are mainly plane waves. For NR electrons in generic direct detection targets, these simple elements are inappropriate for two key reasons:

- Unlike in the free theory, here the electrons in atoms, molecules or crystals are embedded in background potentials, which alter their wavefunctions and energy levels.
- Electrons occupy filled states within the target, so the background here is no longer the pure vacuum. This situation bears conceptual resemblance to QFT at finite temperature, where the background is thermally populated [102, 103], so the primitive elements that define the Feynman rules differ from their familiar vacuum expressions.

However, despite these differences, there are also many recognizable similarities, as the central quantity of interest is a transition matrix element, which is systematically calculated using intuitive Feynman diagrams.

The purpose of this section is to derive the Feynman rules for an NR EFT Lagrangian with electrons in an arbitrary environment at zero temperature. Similar discussions can also be found in condensed matter textbooks [104], although these usually focus on the NR QED Lagrangian without including the effects of new physics. Furthermore, as emphasized in Sec. III, because the NR EFT prescription only modifies the electron degrees of freedom, no other fields in the high-energy Lagrangian need to change. Thus, in this section we develop new Feynman rules for the NR electrons, without affecting the familiar QFT Feynman rules that apply to dark fields, which can even be relativistic in our treatment.

This section is organized as follows: in Sec. IV A we define the transition matrix element. In Sec IV B we expand the electron field in terms of the “in-medium” eigenstates. In Secs. IV C, IV D we use these definitions to derive the

three and four point Feynman rules, respectively. Next in Sec. IV E we derive the NR electron propagator needed to connect more complicated diagrams. In Sec. IV F we use the three point, four point, and propagator Feynman rules to derive loop diagrams, and provide new Feynman rules for those diagrams as well. The Feynman rules will be introduced along the way in each subsection, and bulleted. In Sec. IV G we summarize all the new Feynman rules introduced.

A. NR Matrix Element Definition

We begin by defining the matrix element \mathcal{M} which we will build using Feynman rules. In familiar relativistic QFT this quantity is defined as

$$\langle \mathcal{F} | i\mathcal{T}_{UV} | \mathcal{I} \rangle = (2\pi)^4 \delta^4(\sum p) (i\mathcal{M}_{UV}) \quad , \quad 1 + i\mathcal{T}_{UV} \equiv T \left\{ \exp \left(i \int d^4x \mathcal{L}_{\text{int}} \right) \right\} , \quad (49)$$

where $|\mathcal{I}\rangle$ and $|\mathcal{F}\rangle$ are the initial and final states, respectively, \mathcal{M}_{UV} is the relativistic matrix element, \mathcal{T}_{UV} is the transfer matrix, $T\{\}$ is a time-ordered product and \mathcal{L}_{int} is interaction Lagrangian. The sum over all external state four-momenta $\sum p$ is shorthand for $\sum p \equiv \sum_{\text{in}} p_{\text{in}} - \sum_{\text{out}} p_{\text{out}}$, which enforces energy-momentum conservation for the $\mathcal{I} \rightarrow \mathcal{F}$ transition.

In the presence of a background field, or if boundary conditions apply to the system, spatial momentum is not necessarily conserved since continuous translation symmetry is absent. Therefore, the factor of $\delta^3(\sum \mathbf{p})$ does not necessarily appear in NR transition amplitudes, so we define

$$\langle \mathcal{F} | i\mathcal{T}_{NR} | \mathcal{I} \rangle \equiv 2\pi \delta(\sum E) (i\mathcal{M}_{NR}) \quad , \quad 1 + i\mathcal{T}_{NR} = T \left\{ \exp \left(i \int d^4x \mathcal{L}_{\text{int}}^{\text{NR}} \right) \right\} , \quad (50)$$

where $\delta(\sum E)$ is an energy conserving delta function, the transfer matrix is written analogously with its UV counterpart, and $\mathcal{L}_{\text{int}}^{\text{NR}}$ is the interaction part of an NR EFT Lagrangian. Note that due to the definition in Eq. (50), \mathcal{M}_{NR} and \mathcal{M}_{UV} have different mass dimensions.

The absence of momentum conservation is important because it changes how four-momentum flows through a diagram. In relativistic QFT four-momentum is conserved at each vertex, so the delta function can be removed when the remaining undetermined four-momenta are integrated over. By contrast, in the NR theory developed here, four-momentum conservation is reduced to just energy conservation, so the corresponding Feynman rules for calculating the NR matrix element \mathcal{M}_{NR} become

- Build $i\mathcal{M}_{NR}$ out of NR vertex factors and propagators (as discussed below)
- Conserve *energy* at each vertex
- For each undetermined energy E , insert factors of $\left(\int \frac{dE}{2\pi} \right)$ and perform the corresponding integral
- Multiply the resulting expression by $-i$

These rules consistently remove all remaining energy conserving delta functions that appear at intermediate steps as we will see below.

B. NR Electron Field Quantization

Before computing the matrix elements in Eq. (50) we must first quantize the electron field, ψ . In relativistic QFT, this is done by expanding the field in a basis of solutions to the Dirac equation and imposing anti-commutation

relation on the raising and lowering operators that create single-particle states [95, 101]. Here in the NR EFT, the situation is different since the NR electron field ψ no longer satisfies the Dirac equation. Using Eq. (27) to extract the equation of motion for ψ , to leading order in $1/m_e$, we recover the Schrödinger equation

$$i\partial_t\psi \approx \left(\frac{\mathbf{k}^2}{2m_e} + e\Phi \right) \psi, \quad (51)$$

where \mathbf{k} is the electron momentum. The electron field can then be expanded in a basis of states as [105]

$$\psi(\mathbf{x}, t) = \sum_J e^{-iE_J t} \psi_J(\mathbf{x}) b_J, \quad (52)$$

where the ψ_J satisfy Eq. (51) and the b_J are single-electron annihilation operators. To satisfy the equal time anti-commutation relations, we impose canonical anti-commutation relations for b_J and require the states form a complete basis

$$\{\psi(\mathbf{x}, 0), \psi^\dagger(\mathbf{y}, 0)\} = \delta^3(\mathbf{x} - \mathbf{y}) \iff \{b_J, b_K^\dagger\} = \delta_{JK} \quad , \quad \sum_J \psi_J(\mathbf{x}) \psi_J^\dagger(\mathbf{y}) = \delta^3(\mathbf{x} - \mathbf{y}), \quad (53)$$

and these conditions demand that the wavefunctions form an orthonormal basis

$$\int d^3\mathbf{x} \psi_J^\dagger(\mathbf{x}) \psi_K(\mathbf{x}) = \delta_{IJ}, \quad (54)$$

where single-electron states are $|J\rangle = b_J^\dagger |0\rangle$ and $|0\rangle$ is the vacuum state of the target.

While these expressions are similar to their relativistic analogues, there are two key differences: 1) the appearance of a state sum over J instead of an integral over spatial momenta in Eq. (52), and 2) the Kronecker delta symbol δ_{JK} instead of a Dirac delta function in Eq. (54). Although free-electron states can be indexed with their spatial momenta, this is not possible for general electronic states in direct detection targets (e.g., bound states with discrete energy levels). In general, NR electronic states must be indexed with some combination of discrete and continuous variables, and our formalism is sufficiently flexible to accommodate both cases: if a given state is continuously indexed (as in the free-electron case), then we replace the discrete elements with their continuous counterparts

$$\delta_{JK} \rightarrow 2\pi \delta(J - K) \quad , \quad \sum_J \rightarrow \int \frac{dJ}{2\pi}. \quad (55)$$

For example, if $J = \mathbf{k}$, then all the indices are continuous and this replacement yields

$$\delta_{\mathbf{k}\mathbf{k}'} \rightarrow (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \quad , \quad \sum_{\mathbf{k}} \rightarrow \int \frac{d^3\mathbf{k}}{(2\pi)^3}, \quad (56)$$

which recovers the familiar relativistic formalism. However, note that with these conventions, free-electron states are normalized according to

$$\langle \mathbf{k} | \mathbf{k} \rangle = \mathcal{V}, \quad (57)$$

which differs from the commonly used $\langle \mathbf{k} | \mathbf{k} \rangle_{\text{rel}} = 2E_{\mathbf{k}} \mathcal{V}$ relativistic convention [101], but has the virtue of simplifying the NR electron Feynman rules that we derive below.

C. Three-Point Vertices

With the NR electron states defined, we can now begin to derive vertex Feynman rules for a general three-point interaction between NR electrons and linearly coupled dark fields.

Warmup

We begin with the simplest three-point vertex example: a Yukawa coupling in the NR Lagrangian

$$\mathcal{L}_{\text{int}}^{\text{NR}} = g \phi \psi^\dagger \psi, \quad (58)$$

where ϕ is a scalar field and g is a small coupling constant. To derive the Feynman rule for \mathcal{M}_{NR} , we use Eq. (50) and expand \mathcal{T}_{NR} to leading order in g , which yields

$$\langle \mathcal{F} | i\mathcal{T}_{\text{NR}} | \mathcal{I} \rangle = 2\pi\delta(\sum E) (i\mathcal{M}_{\text{NR}}) \approx \langle \mathcal{F} | \left(ig \int d^4x \phi \psi^\dagger \psi \right) | \mathcal{I} \rangle, \quad (59)$$

where the initial state consists of a *free* (and possibly relativistic) ϕ particle in an eigenstate of momentum \mathbf{q} , and an electron in some arbitrary NR state $|J\rangle$; the final state consists only of an electron in an NR state $|K\rangle$. We can write the corresponding $|\mathcal{I}\rangle$ and $|\mathcal{F}\rangle$ as the product states

$$|\mathcal{I}\rangle = |\mathbf{q}\rangle \otimes |J\rangle, \quad |\mathcal{F}\rangle = |0\rangle \otimes |K\rangle, \quad (60)$$

so the matrix element from Eq. (59) satisfies

$$2\pi\delta(\sum E) (i\mathcal{M}_{\text{NR}}) = ig \int dt \int d^3\mathbf{x} \langle 0 | \phi(\mathbf{x}) | \mathbf{q} \rangle \langle K | \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) | J \rangle, \quad (61)$$

and ϕ is quantized according to the usual relativistic formalism [95, 101]

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle, \quad \phi(\mathbf{x}, t) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} \right), \quad (62)$$

where $a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$ are respectively the creation and annihilation operators for ϕ states, and the normalization convention satisfies

$$\langle \mathbf{p} | \mathbf{p} \rangle = 2E_{\mathbf{p}} \mathcal{V}, \quad (63)$$

which *differs* from the corresponding convention for the electron field in Eq. (57). Inserting Eq. (62) into Eq. (61) and expanding ψ in a set of basis states from Eq. (52) transforms the time integral into an energy conserving delta function. As noted in Sec. IV, the delta function cancels out of the final expression for \mathcal{M}_{NR} , so we obtain

$$\mathcal{M}_{\text{NR}} = g \int d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \psi_K^\dagger(\mathbf{x}) \psi_J(\mathbf{x}). \quad (64)$$

Thus the Feynman rules for calculating \mathcal{M}_{NR} for a Yukawa interaction are: at each vertex insert a factor of

$$ig \int d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \psi_K^\dagger(\mathbf{x}) \psi_J(\mathbf{x}), \quad (65)$$

and then multiply the final expression by $-i$, as noted in Sec. IV A. This procedure recovers the desired NR matrix element in Eq. (64) and justifies the steps outlined in Sec. IV A.

Note that when ψ_J, ψ_K are free-electron states with $J = \mathbf{p}$ and $K = \mathbf{p}'$, Eq. (65) can be reduced using the integral representation of the delta function

$$(2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q} - \mathbf{p}') \equiv \int d^3\mathbf{x} e^{-i(\mathbf{p} + \mathbf{q} - \mathbf{p}')\cdot\mathbf{x}}, \quad (66)$$

which yields $ig(2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q} - \mathbf{p}')$. As expected from the usual relativistic calculation a momentum conserving delta function appears.

Three-Point Scalar Vertex

We now consider a more general three-point interaction in the NR EFT Lagrangian using the notation introduced in Sec. III

$$\mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell=1}^8 \left[C_{\phi,\ell} \phi + C_{\nabla\phi,\ell} \frac{(\nabla\phi)}{m_e} + \dots \right] \left[\psi^\dagger \hat{\mathcal{O}}_\ell \psi \right], \quad (67)$$

where the terms in the left square bracket contain all field content linear in ϕ (or its derivatives), and all dimensionless coefficients C contain only a single ϕ (including $(\nabla\phi)$ or $(\nabla\nabla\phi)$, etc.) in their left subscript. Following the same steps as in the Yukawa example above, after removing the energy conserving delta function, the NR matrix element becomes

$$i\mathcal{M}_{\text{NR}} = i \sum_{\ell=1}^8 \left[C_{\phi,\ell} + C_{\nabla\phi,\ell} \left(\frac{i\mathbf{q}}{m_e} \right) + \dots \right] \left[\int d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \psi_K^\dagger(\mathbf{x}) \hat{\mathcal{O}}_\ell \psi_J(\mathbf{x}) \right], \quad (68)$$

where $\nabla^i = -\partial^i \rightarrow i\mathbf{q}^i$ in momentum space. This organizational scheme generalizes straightforwardly to an NR EFT interaction containing terms with more derivatives acting on ϕ : replace each ∇ with $i\mathbf{q}$.

Since three-point vertices appear frequently in standard calculations, for future convenience we define the dimensionless combination of NR Lagrangian coefficients C and momenta \mathbf{q} to be

$$f_{\phi,\ell}(\mathbf{q}) \equiv C_{\phi,\ell} + C_{\nabla\phi,\ell} \left(\frac{i\mathbf{q}}{m_e} \right) + \dots, \quad (69)$$

which collect all C terms with only one ϕ subscript (including derivatives acting on ϕ) and all higher order terms represented as (\dots) are normalized with appropriate powers of m_e to maintain consistent dimensionality. Note that this coefficient f contains all the *model-dependent* information about the DM-electron interaction and does not depend on the details of the target. Furthermore we define the basis integral in Eq. (68) as

$$\hat{\mathcal{M}}_{JK,\ell}(\mathbf{q}) \equiv \int d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \psi_K^\dagger(\mathbf{x}) \hat{\mathcal{O}}_\ell \psi_J(\mathbf{x}), \quad (70)$$

which contains all the *target-dependent* information about the DM-electron interaction. Each NR basis operator $\hat{\mathcal{O}}_\ell$ has a corresponding amplitude, $\hat{\mathcal{M}}_{JK,\ell}$, once the external electron states are specified. Thus, the general three-point matrix element is

$$i\mathcal{M}_{\text{NR}} = i \sum_{\ell=1}^8 f_{\phi,\ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK,\ell}(\mathbf{q}), \quad (71)$$

and in terms of f and $\hat{\mathcal{M}}$, the general three-point scalar vertex Feynman rule becomes

- **Three-Point Scalar Vertex:** At each three-point vertex between an incoming particle ϕ with momentum \mathbf{q} and electronic states J, K insert

$$\phi \xrightarrow{\mathbf{q}} \text{---} \bigcirc \begin{matrix} \nearrow K \\ \searrow J \end{matrix} \quad \Longrightarrow \quad i \sum_{\ell=1}^8 f_{\phi,\ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK,\ell}(\mathbf{q}),$$

where the hashed circle represents the sum of all three-point interactions in the NR EFT and arrows represent the flow of momentum through the diagram.

Three-Point Vector Vertex

A vector particle may also appear on the external line of a three-point vertex. Consider the NR interaction Lagrangian

$$\mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{V^0, \ell} V^0 + C_{(\nabla V^0), \ell} \frac{(\nabla V^0)}{m_e} + C_{\mathbf{V}, \ell} \mathbf{V} + C_{(\nabla \mathbf{V}), \ell} \frac{(\nabla \mathbf{V})}{m_e} + \dots \right] \left[\psi^\dagger \hat{\mathcal{O}}_{\ell} \psi \right], \quad (72)$$

which arises in the NR limit from the interaction in Sec. III, where $V^\mu = (V^0, \mathbf{V})$ is a real DM particle quantized according the relativistic QFT prescription

$$|\mathbf{q}, \lambda\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}, \lambda}^\dagger |0\rangle \quad , \quad V^\mu(x) = \sum_{\lambda} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\epsilon_{\lambda}^{\mu}}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} \left(a_{\mathbf{p}, \lambda} e^{-iE_{\mathbf{p}}t} + a_{-\mathbf{p}, \lambda}^\dagger e^{iE_{\mathbf{p}}t} \right), \quad (73)$$

where ϵ_{λ}^{μ} is a polarization four-vector, the sum is over λ polarization states, and the state normalization convention is

$$\langle \mathbf{p}, \lambda | \mathbf{p}, \lambda \rangle = 2E_{\mathbf{p}} \mathcal{V}, \quad (74)$$

which, again, differs from the NR electron state normalization from Eq. (57). Following the same steps as in the general three-point scalar vertex calculation, and adding a polarization index λ to the initial state, the NR matrix element is

$$i\mathcal{M}_{\text{NR}} = i \sum_{\ell} \left\{ \epsilon_{\lambda}^0 \left(C_{V^0, \ell} + [C_{(\nabla V^0), \ell}]^i \left(\frac{i\mathbf{q}^i}{m_e} \right) + \dots \right) + \epsilon_{\lambda}^i \left([C_{\mathbf{V}, \ell}]^i + [C_{(\nabla \mathbf{V}), \ell}]^{ji} \left(\frac{i\mathbf{q}^j}{m_e} \right) + \dots \right) \right\} \hat{\mathcal{M}}_{JK, \ell}, \quad (75)$$

where we have kept the vector field spatial indices explicit.

The key difference here relative to the scalar case concerns the Lorentz indices: in relativistic QFT, a matrix element with an external vector line has its Lorentz index contracted with a polarization vector via $\mathcal{M} \sim \epsilon_{\mu} \mathcal{M}^{\mu}$, so the Cartesian index is lowered relative to Eq. (75). For consistency with familiar relativistic QFT conventions, we define the Feynman rule coefficients of the spatial vector components with a minus sign to yield

$$[f_{V, \ell}]^0 \equiv C_{V^0, \ell} + C_{(\nabla V^0), \ell}^i \left(\frac{i\mathbf{q}^i}{m_e} \right) + \dots \quad , \quad [f_{V, \ell}]^i \equiv - \left[C_{\mathbf{V}, \ell}^i + C_{(\nabla \mathbf{V}), \ell}^{ji} \left(\frac{i\mathbf{q}^j}{m_e} \right) + \dots \right], \quad (76)$$

which respectively collect all C coefficients with one V^0 and one \mathbf{V} (plus their derivatives) in their subscripts, which can be explicitly related to the corresponding expressions in Table II. The expressions in Eq. (76) may also be combined into a four-vector Feynman rule coefficient

$$[f_{V, \ell}]^{\mu} \equiv \left([f_{V, \ell}]^0, [f_{V, \ell}]^i \right), \quad (77)$$

so the three-point NR matrix element with an external vector becomes

$$\mathcal{M}_{\text{NR}} = \epsilon_{\mu}^{\lambda} \mathcal{M}_{\text{NR}}^{\mu} \quad , \quad i\mathcal{M}_{\text{NR}}^{\mu} = i \sum_{\ell=1}^8 [f_{V, \ell}]^{\mu} \hat{\mathcal{M}}_{JK, \ell}, \quad (78)$$

which is the same Feynman rule as in the previous section, but with an added Lorentz index to the Feynman coefficients, which matches that of the incoming vector DM particle

- **Three-Point Vector Vertex:** At each three-point vertex between an incoming vector particle V with momentum \mathbf{q} and electronic states J, K insert

$$V^\mu \xrightarrow{\mathbf{q}} \text{Vertex} \begin{matrix} \nearrow K \\ \searrow J \end{matrix} \quad \Longrightarrow \quad i \sum_{\ell=1}^8 [f_{V,\ell}(\mathbf{q})]^\mu \hat{\mathcal{M}}_{JK,\ell}(\mathbf{q})$$

D. Four-Point Vertices

In general there are many possible four-point interactions (involving two fields in \mathcal{O}_{NR}) that can arise in the NR EFT developed in Sec. III. However, given the interactions in our UV example Lagrangian presented in Sec. III B, all dark fields are linear in \mathcal{O}_{UV} in the UV, so there are only two four-point interactions that derive in the NR from this UV example to order m_e^{-2} : $\psi^\dagger \hat{\mathcal{O}} \psi$ coupled to a photon plus one dark scalar or one dark vector.

Four-Point Scalar-Vector Vertex

As in Sec. IV C, we consider a general four-point interaction in the NR EFT Lagrangian, where we collect all operators that contain one ϕ and one photon A (including both A^0 and \mathbf{A}) plus derivatives acting on these fields. Using the notation of Sec. III, we can write these terms as

$$\begin{aligned} \mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{\phi\mathbf{A},\ell} \frac{\phi\mathbf{A}}{m_e} + C_{(\nabla\phi)\mathbf{A},\ell} \frac{(\nabla\phi)\mathbf{A}}{m_e^2} + C_{\phi(\nabla\mathbf{A}),\ell} \frac{\phi(\nabla\mathbf{A})}{m_e^2} \right. \\ \left. + C_{\phi A^0,\ell} \frac{\phi A^0}{m_e} + C_{(\nabla\phi)A^0,\ell} \frac{(\nabla\phi)A^0}{m_e^2} + C_{\phi(\nabla A^0),\ell} \frac{\phi(\nabla A^0)}{m_e^2} + \dots \right] \left[\psi^\dagger \hat{\mathcal{O}}_\ell \psi \right], \end{aligned} \quad (79)$$

which arise in the NR limit from the interaction in Sec. III. We now repeat the steps from Sec. IV C, taking the incoming ϕ particle to have momentum \mathbf{p}_1 , and outgoing photon (quantized according to Eq. (73)) to have momentum \mathbf{p}_2 and polarization λ . Keeping the field indices explicit, the four-point NR amplitude is

$$\begin{aligned} i\mathcal{M}_{\text{NR}} = i \sum_{\ell} \left\{ \epsilon_{\lambda}^j \left([C_{\phi\mathbf{A},\ell}]^j \frac{1}{m_e} + [C_{(\nabla\phi)\mathbf{A},\ell}]^{ij} \frac{(i\mathbf{p}_1^i)}{m_e^2} + [C_{\phi(\nabla\mathbf{A}),\ell}]^{ij} \frac{(-i\mathbf{p}_2^i)}{m_e^2} + \dots \right) \right. \\ \left. + \epsilon_{\lambda}^0 \left(C_{\phi A^0,\ell} \frac{1}{m_e} + [(C_{(\nabla\phi)A^0,\ell})]^i \frac{(i\mathbf{p}_1^i)}{m_e^2} + [C_{\phi(\nabla A^0),\ell}]^i \frac{(-i\mathbf{p}_2^i)}{m_e^2} + \dots \right) \right\} \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1 - \mathbf{p}_2), \end{aligned} \quad (80)$$

where the terms with \mathbf{p}_1 and \mathbf{p}_2 have different signs because A is an outgoing particle.

As in the previous derivation for the three-point vertex, each term in the general four-point vertex from Eq. (80) factorizes into a model-dependent function (the term inside the brackets) and a target-dependent expression, $\hat{\mathcal{M}}_{JK,\ell}$. Thus, following the notation convention from Sec. IV C, we group the model-dependent pieces into an overall coefficient

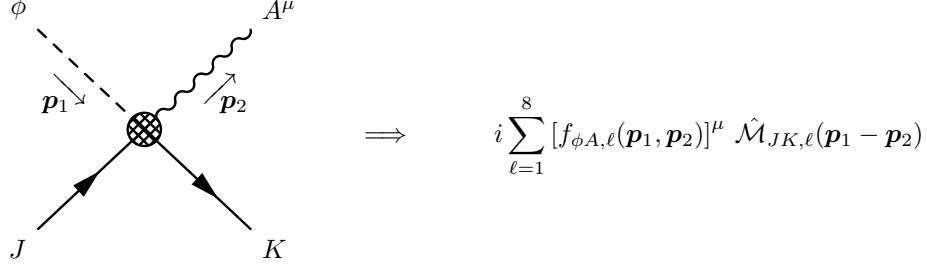
$$\begin{aligned} [f_{\phi A,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^0 &\equiv C_{\phi A^0,\ell} \frac{1}{m_e} + C_{(\nabla\phi)A^0,\ell} \frac{(i\mathbf{p}_1)}{m_e^2} + C_{\phi(\nabla A^0),\ell} \frac{(-i\mathbf{p}_2)}{m_e^2} + \dots \\ [f_{\phi A,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^i &\equiv - \left[[C_{\phi\mathbf{A},\ell}]^i \frac{1}{m_e} + [C_{(\nabla\phi)\mathbf{A},\ell}]^{ji} \frac{(i\mathbf{p}_1^j)}{m_e^2} + [C_{\phi(\nabla\mathbf{A}),\ell}]^{ji} \frac{(-i\mathbf{p}_2^j)}{m_e^2} + \dots \right], \end{aligned} \quad (81)$$

and the four-point NR matrix element can be written

$$\mathcal{M}_{\text{NR}} = \epsilon_\mu^\lambda \mathcal{M}_{\text{NR}}^\mu, \quad i\mathcal{M}_{\text{NR}}^\mu = i \sum_\ell [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2), \quad (82)$$

from which we read off the Feynman rule:

- **Four-Point Scalar-Vector Vertex:** At each four-point vertex between an incoming scalar particle ϕ with momentum \mathbf{p}_1 , an outgoing vector particle A with momentum \mathbf{p}_2 , and electronic states J, K , insert



Four-Point Vector-Vector Vertex

The final four-point vertex we consider is between an incoming vector, $V^\mu = (V^0, \mathbf{V})$, and outgoing photon, $A^\mu = (A^0, \mathbf{A})$. The general interaction Lagrangian which generates a four-point interaction between these fields can be written abstractly using the notation from Sec. III as

$$\begin{aligned} \mathcal{L}_{\text{int}}^{\text{NR}} = \sum_\ell \left[& C_{V^0 A^0, \ell} \frac{V^0 A^0}{m_e} + C_{(\nabla V^0) A^0, \ell} \frac{(\nabla V^0) A^0}{m_e^2} + C_{V^0 (\nabla A^0), \ell} \frac{V^0 (\nabla A^0)}{m_e^2} \right. \\ & + C_{\mathbf{V} A^0, \ell} \frac{\mathbf{V} A^0}{m_e} + C_{(\nabla \mathbf{V}) A^0, \ell} \frac{(\nabla \mathbf{V}) A^0}{m_e^2} + C_{\mathbf{V} (\nabla A^0), \ell} \frac{\mathbf{V} (\nabla A^0)}{m_e^2} \\ & + C_{V^0 \mathbf{A}, \ell} \frac{V^0 \mathbf{A}}{m_e} + C_{(\nabla V^0) \mathbf{A}, \ell} \frac{(\nabla V^0) \mathbf{A}}{m_e^2} + C_{V^0 (\nabla \mathbf{A}), \ell} \frac{V^0 (\nabla \mathbf{A})}{m_e^2} \\ & \left. + C_{\mathbf{V} \mathbf{A}, \ell} \frac{\mathbf{V} \mathbf{A}}{m_e} + C_{(\nabla \mathbf{V}) \mathbf{A}, \ell} \frac{(\nabla \mathbf{V}) \mathbf{A}}{m_e^2} + C_{\mathbf{V} (\nabla \mathbf{A}), \ell} \frac{\mathbf{V} (\nabla \mathbf{A})}{m_e^2} + \dots \right] [\psi^\dagger \hat{\mathcal{O}}_\ell \psi]. \quad (83) \end{aligned}$$

We now repeat the steps from the four-point scalar-photon matrix element calculation. The incoming V particle has momentum \mathbf{p}_1 and polarization λ , and the outgoing photon has momentum \mathbf{p}_2 and polarization λ' . The four-point NR amplitude is then

$$\begin{aligned} i\mathcal{M}_{\text{NR}} = i \sum_\ell \left[& \epsilon_{V, \lambda}^0 \epsilon_{A, \lambda'}^0 \left(C_{V^0 A^0, \ell} \frac{1}{m_e} + [C_{(\nabla V^0) A^0, \ell}]^i \frac{(i\mathbf{p}_1^i)}{m_e^2} + [C_{V^0 (\nabla A^0), \ell}]^i \frac{(-i\mathbf{p}_2^i)}{m_e^2} \right) \right. \\ & + \epsilon_{V, \lambda}^i \epsilon_{A, \lambda'}^0 \left([C_{\mathbf{V} A^0, \ell}]^i \frac{1}{m_e} + [C_{(\nabla \mathbf{V}) A^0, \ell}]^{ji} \frac{(i\mathbf{p}_1^j)}{m_e^2} + [C_{\mathbf{V} (\nabla A^0), \ell}]^{ij} \frac{(-i\mathbf{p}_2^j)}{m_e^2} \right) \\ & + \epsilon_{V, \lambda}^0 \epsilon_{A, \lambda'}^i \left([C_{V^0 \mathbf{A}, \ell}]^i \frac{1}{m_e} + [C_{(\nabla V^0) \mathbf{A}, \ell}]^{ji} \frac{(i\mathbf{p}_1^j)}{m_e^2} + [C_{V^0 (\nabla \mathbf{A}), \ell}]^{ji} \frac{(-i\mathbf{p}_2^j)}{m_e^2} \right) \\ & \left. + \epsilon_{V, \lambda}^i \epsilon_{A, \lambda'}^j \left([C_{\mathbf{V} \mathbf{A}, \ell}]^{ij} \frac{1}{m_e} + [C_{(\nabla \mathbf{V}) \mathbf{A}, \ell}]^{kij} \frac{(i\mathbf{p}_1^k)}{m_e^2} + [C_{\mathbf{V} (\nabla \mathbf{A}), \ell}]^{ikj} \frac{(-i\mathbf{p}_2^k)}{m_e^2} \right) + \dots \right] \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2), \quad (84) \end{aligned}$$

where $\epsilon_{A, \lambda}^\mu, \epsilon_{V, \lambda}^\mu$ are the photon and V polarization vectors, respectively.

As in the previous sections this matrix element is greatly simplified by defining the Feynman rule coefficients $[f_{VA}]^{\mu\nu}$ for which

$$[f_{VA,\ell}]^{00} = \left([C_{V^0 A^0,\ell}] \frac{1}{m_e} + [C_{(\nabla V^0) A^0,\ell}]^i \frac{(i\mathbf{p}_1^i)}{m_e^2} + [C_{V^0(\nabla A^0),\ell}]^i \frac{(-i\mathbf{p}_2^i)}{m_e^2} \right) \quad (85)$$

$$[f_{VA,\ell}]^{i0} = - \left([C_{\mathbf{V} A^0,\ell}]^i \frac{1}{m_e} + [C_{(\nabla \mathbf{V}) A^0,\ell}]^{ji} \frac{(i\mathbf{p}_1^j)}{m_e^2} + [C_{\mathbf{V}(\nabla A^0),\ell}]^{ij} \frac{(-i\mathbf{p}_2^j)}{m_e^2} \right) \quad (86)$$

$$[f_{VA,\ell}]^{0i} = - \left([C_{V^0 \mathbf{A},\ell}]^i \frac{1}{m_e} + [C_{(\nabla V^0) \mathbf{A},\ell}]^{ji} \frac{(i\mathbf{p}_1^j)}{m_e^2} + [C_{V^0(\nabla \mathbf{A}),\ell}]^{ji} \frac{(-i\mathbf{p}_2^j)}{m_e^2} \right) \quad (87)$$

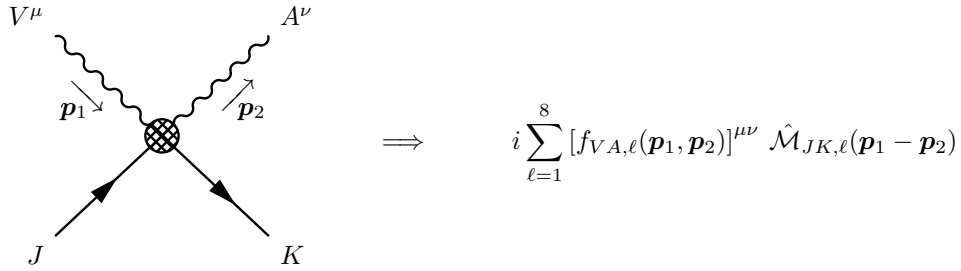
$$[f_{VA,\ell}]^{ij} = \left([C_{\mathbf{V} \mathbf{A},\ell}]^{ij} \frac{1}{m_e} + [C_{(\nabla \mathbf{V}) \mathbf{A},\ell}]^{kij} \frac{(i\mathbf{p}_1^k)}{m_e^2} + [C_{\mathbf{V}(\nabla \mathbf{A}),\ell}]^{ikj} \frac{(-i\mathbf{p}_2^k)}{m_e^2} \right) \quad (88)$$

With this notation the vector-vector four-point matrix element can be written succinctly as,

$$\mathcal{M}_{\text{NR}} = \epsilon_{V,\mu}^\lambda \epsilon_{A,\nu}^{\lambda'} \mathcal{M}_{\text{NR}}^{\mu\nu} \quad , \quad i\mathcal{M}_{\text{NR}}^{\mu\nu} = i \sum_{\ell} [f_{VA,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1 - \mathbf{p}_2), \quad (89)$$

from which we read off the Feynman rule:

- **Four-Point Vector-Vector Vertex:** At each four-point vertex between an incoming vector particle V with momentum \mathbf{p}_1 , an outgoing photon A with momentum \mathbf{p}_2 , and electronic states I, J , insert



E. NR Electron Propagator

Since higher order diagrams involve internal, off-shell electron lines, we need to determine the Dirac propagator in the NR limit. Since the relativistic propagator can be written

$$D_{\text{QFT}}(p) = \int d^4x e^{ip \cdot x} \langle 0 | T \{ \Psi(x) \bar{\Psi}(0) \} | 0 \rangle = \frac{i(\not{p} + m_e)}{p^2 - m_e^2 + i\epsilon}, \quad (90)$$

naively it would seem that the corresponding quantity in the NR EFT would be the Fourier transform of $\langle T \{ \psi(\mathbf{x}, t) \psi^\dagger(\mathbf{y}, t') \} \rangle$. However, defining the NR propagator in this way would *double count* the electronic wavefunctions $\psi_J(\mathbf{x})$, which are already included at the vertices of a given diagram inside $\hat{\mathcal{M}}_{JK,\ell}$ (see, for example, Eq. (65)).

Therefore, instead of taking the NR limit of D_{QFT} in Eq. (90), we first define the *time-dependent* creation/annihilation operators $b_J^\dagger(t)$ and $b_J(t)$ by absorbing the time-dependent phase factors in Eq. (52). The electron field is then

$$\psi(\mathbf{x}, t) = \sum_J \psi_J(\mathbf{x}) b_J(t) \quad , \quad b_J(t) \equiv e^{-iE_J t} b_J, \quad (91)$$

and we can use $b_J(t)$ to define the NR propagator in both time and energy

$$D_{JK}(t) \equiv \langle T \{ b_J(t) b_K^\dagger(0) \} \rangle \quad , \quad D_{JK}(\omega) \equiv \int dt e^{i\omega t} D_{JK}(t), \quad (92)$$

which connects the time evolution between the J^{th} and K^{th} eigenstates without the spatial wavefunctions. The propagator in Eq. (92) appears frequently in condensed matter field theory and should be regarded as a constituent part of the full propagator, $\langle T\{\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{y}, t')\} \rangle$, which can be derived in terms of D_{JK} – for a full discussion of this relationship, see Ch. 2 in Ref. [104].

Another subtle difference between D_{QFT} and D_{JK} is that the ground state of the former is the zero-particle vacuum, whereas the latter ground state contains electrons at finite density. Thus, the expectation value in Eq. (92) must be evaluated with respect to the ground state $|\Omega\rangle$, defined according to

$$|\Omega\rangle \equiv \prod_L b_L^\dagger |0\rangle, \quad (93)$$

where $|0\rangle$ is the zero-particle vacuum and the product of b_L^\dagger operators defines the target ground state.⁵ We define the Fermi surface to be the zero energy point, $E = 0$, and therefore all electronic states in the product in Eq. (93) have negative energy. Furthermore, the existence of this ground state electron population in Eq. (93) affects how creation and annihilation operators act on the ground state

$$b_J^\dagger b_J |\Omega\rangle = \theta(-E_J) |\Omega\rangle, \quad (94)$$

where θ is the Heaviside theta function. Note that b_J only annihilates the vacuum if E_J is positive.

Incorporating these NR subtleties, we can directly evaluate the integral in Eq. (92) to obtain

$$\begin{aligned} D_{JK}(\omega) &= \int dt e^{i(\omega - E_J)t} \left[\theta(t) \langle \Omega | b_J b_K^\dagger | \Omega \rangle - \theta(-t) \langle \Omega | b_K^\dagger b_J | \Omega \rangle \right] \\ &= \delta_{JK} \int dt e^{i(\omega - E_J)t} [\theta(t)\theta(E_J) - \theta(-t)\theta(-E_J)], \end{aligned} \quad (95)$$

and we introduce an $i\epsilon$ prescription to render the integrals finite

$$D_{JK}(\omega) = i\delta_{JK} \left[\frac{\theta(E_J)}{\omega - E_J + i\epsilon} + \frac{\theta(-E_J)}{\omega - E_J - i\epsilon} \right] = \frac{i\delta_{JK}}{\omega - E_J + i\epsilon_J}, \quad \epsilon_J \equiv \epsilon \text{sign}(E_J), \quad (96)$$

so the Feynman rule for propagating internal electron lines can be written

- **NR Electron Propagator:** For every internal electron line labelled by J , insert the propagator from Eq. (95)

$$\begin{array}{c} J \\ \longrightarrow \blacktriangleright \end{array} \quad \Longrightarrow \quad \frac{i}{\omega - E_J + i\epsilon_J}, \quad \epsilon_J = \epsilon \text{sign}(E_J).$$

F. Loop Diagrams

With the vertex functions and electron propagators defined above, we now develop the formalism for calculating loop diagrams. Following the conventions of earlier subsections, we begin with a “warmup” example involving only one NR interaction and then generalize this to an arbitrary set of NR operators.

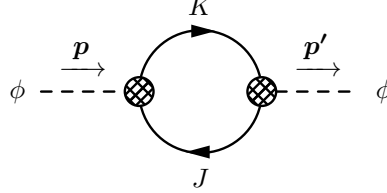
⁵ This distinction is not important for the three and four-point vertex Feynman rules discussed in Secs. IV C, and IV D, respectively. One can show that the transition probability for $|J\rangle \rightarrow |K\rangle$ is the same as taking the initial state be the in-medium vacuum, $|\Omega\rangle$, and the final state to be the in-medium vacuum with J removed, and K added, i.e., $|\Omega\rangle \rightarrow b_K^\dagger b_J |\Omega\rangle$.

Warmup

We now revisit the simple NR interaction Lagrangian from Eq. (58) with the Yukawa coupling

$$\mathcal{L}_{\text{int}}^{\text{NR}} = g \phi \psi^\dagger \psi, \quad (97)$$

and consider an initial state with an incoming ϕ of momentum \mathbf{p} , where $|I\rangle = |\mathbf{p}\rangle$, and a final state with outgoing ϕ of \mathbf{p}' , where $|F\rangle = |\mathbf{p}'\rangle$. The simplest self-energy diagram for this interaction is



which arises from expanding \mathcal{T}_{NR} to second order in the coupling g and inserting the result into Eq. (50). The corresponding matrix element satisfies

$$2\pi\delta(\sum E) i\mathcal{M}_{\text{NR}} = \frac{(ig)^2}{2!} \int d^4x d^4x' \langle \mathbf{p}' | T \{ \phi(x) \psi^\dagger(x) \psi(x) \phi(x') \psi^\dagger(x') \psi(x') \} | \mathbf{p} \rangle, \quad (98)$$

so using Wick's theorem and inserting an overall factor of -1 for a closed electron loop (as in relativistic QFT [95, 101]), the right hand side (RHS) of Eq. (98) becomes

$$\text{RHS} = g^2 \int d^3\mathbf{x} d^3\mathbf{x}' e^{i\mathbf{p}\cdot\mathbf{x} - i\mathbf{p}'\cdot\mathbf{x}'} \sum_{JKLM} \psi_J^\dagger(\mathbf{x}) \psi_K(\mathbf{x}) \psi_L^\dagger(\mathbf{x}') \psi_M(\mathbf{x}') \int dt dt' e^{i(E_{\mathbf{p}}t - E_{\mathbf{p}'}t')} D_{KL}(t - t') D_{MJ}(t' - t), \quad (99)$$

where we have used the relativistic form of ϕ from Eq. (62) and the definition of the propagator $D_{JK}(t)$ from Eq. (95). Using the energy representation for $D_{JK}(\omega)$, this becomes

$$\text{RHS} = -g^2 2\pi\delta(\sum E) \sum_{JK} \left[i\hat{\mathcal{M}}_{JK,1}(\mathbf{p}) \right] \left[i\hat{\mathcal{M}}_{KJ,1}(-\mathbf{p}') \right] \int \frac{dE}{2\pi} \left(\frac{i}{E - E_J + i\epsilon_J} \right) \left(\frac{i}{\omega + E - E_K + i\epsilon_K} \right), \quad (100)$$

where we have used the integral representation of the Dirac delta function to trade the time integrals for an energy conserving delta function and an integration over the undetermined energy E .⁶

Putting it all together, we can now write the loop amplitude from Eq. (100) in terms of the explicit Feynman rules given in Sec. IV A:

$$\mathcal{M}_{\text{NR}} = (-i)(-1) \left(\sum_{JK} \int \frac{dE}{2\pi} \right) \left[ig\hat{\mathcal{M}}_{JK,1}(\mathbf{p}) \right] \left[ig\hat{\mathcal{M}}_{KJ,1}(-\mathbf{p}') \right] \left(\frac{i}{E - E_J + i\epsilon_J} \right) \left(\frac{i}{\omega + E - E_K + i\epsilon_K} \right), \quad (101)$$

where the first factor comes from the rule to multiply by $-i$ (Sec. IV A), the second factor of (-1) is from a closed electron loop, the integral over E is over the undetermined loop energy (Sec. IV A), the fourth and fifth terms are the three-point Feynman rules (Sec. IV C), and the last two factors are from the NR electron propagators (Sec. IV E).

Note that the only new aspect of Eq. (101), which is not determined by the previously stated Feynman rules, is the sum over electron states. Therefore we explicitly add the additional rule:

⁶ Note that to maintain consistency with the overall EFT prescription the energy of the electronic states involved in Eq. (100) must be much less than the electron mass.

- **Loop Feynman Rule:** Sum over all internal electron lines labelled by J, K, \dots by inserting $\sum_{JK\dots}$.

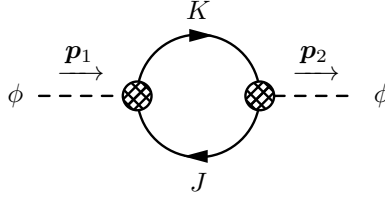
This prescription replaces the three-momentum integrals which are familiar from relativistic loop calculations. This correspondence is restored in the free-electron limit, in which the states are indexed by momentum $J \rightarrow \mathbf{p}$, and the sum over the electronic states becomes an integral over spatial momentum \mathbf{p} with the replacement $\sum_J \rightarrow \int d^3\mathbf{p}/(2\pi)^3$, as discussed in Sec. IV B. Combining this three-dimensional integral with the integral over the undetermined energy E in Eq. (101) yields the replacement

$$\sum_J \int \frac{dE}{2\pi} \rightarrow \int \frac{d^4p}{(2\pi)^4}, \quad (102)$$

thereby recovering the familiar four-dimensional phase space integral over an undetermined loop four-momentum from relativistic QFT.

Scalar Loop Diagram (Type I)

We now generalize the scalar field loop calculation to the NR EFT interaction Lagrangian given in Eq. (67). The leading self-energy Feynman diagram arising from these interactions is



and the corresponding matrix element can be written

$$i\mathcal{M}_{\text{NR}} = - \sum_{JK} \int \frac{dE}{2\pi} \left[i \sum_{\ell} f_{\phi,\ell}(\mathbf{p}_1) \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1) \right] \left[i \sum_m f_{\phi,m}(-\mathbf{p}_2) \hat{\mathcal{M}}_{KJ,m}(-\mathbf{p}_2) \right] \left[\frac{i}{E - E_J + i\epsilon_J} \right] \left[\frac{i}{\omega + E - E_K + i\epsilon_K} \right], \quad (103)$$

which is analogous to Eq. (101), but features sums over the NR basis elements indexed with ℓ, m to include additional operators beyond $\hat{\mathcal{O}}_1$ which defined the NR Yukawa coupling in the warmup example above. After performing the E integral with the identity [57]

$$\int \frac{dE}{2\pi} \left(\frac{1}{E - E_J + i\epsilon_J} \right) \left(\frac{1}{\omega + E - E_K + i\epsilon_K} \right) = i \left[\frac{\theta(-E_J)\theta(E_K) - \theta(E_J)\theta(-E_K)}{\omega - E_K + E_J + i\epsilon \text{sign}(E_K - E_J)} \right], \quad (104)$$

Eq. (103) simplifies to become

$$i\mathcal{M}_{\text{NR}} = -i \sum_{\ell m} f_{\phi,\ell}(\mathbf{p}_1) f_{\phi,m}(-\mathbf{p}_2) \sum_{IF} \left[\frac{\hat{\mathcal{M}}_{IF,\ell}(\mathbf{p}_1) \hat{\mathcal{M}}_{FI,m}(-\mathbf{p}_2)}{\omega - E_F + E_I + i\epsilon} - \frac{\hat{\mathcal{M}}_{IF,m}(-\mathbf{p}_2) \hat{\mathcal{M}}_{FI,\ell}(\mathbf{p}_1)}{\omega + E_F - E_I - i\epsilon} \right], \quad (105)$$

where I and F only sum over the filled ($E_I < 0$), and unfilled ($E_F > 0$) states, respectively.⁷ As with the three-point and four-point vertex Feynman rules derived in Secs. IV C, IV D, respectively, the self-energy factorizes in to a model-dependent contribution (the f 's) and a target-dependent contribution (the $\hat{\mathcal{M}}$'s). To make this separation manifest

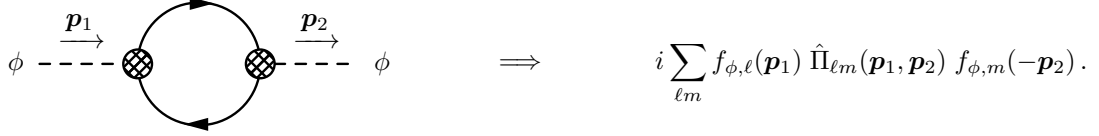
⁷ Note that I indexes the negative energy states of the *entire* target, and \sum_I sums over all filled states in the entire target. For example, consider a target of N hydrogen atoms, labelled by j , each with an electron in the $n = 2, \ell = 1, m = 0$ state. Then $I = \{j, n, \ell, m\}$, and $\sum_I = \sum_j \sum_{n\ell m} \delta_{n2} \delta_{\ell 1} \delta_{m0} = N \sum_{n\ell m} \delta_{n2} \delta_{\ell 1} \delta_{m0}$.

we define the target dependent self-energy as

$$\hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_2) \equiv - \sum_{IF} \left[\frac{\hat{\mathcal{M}}_{IF,\ell}(\mathbf{p}_1) \hat{\mathcal{M}}_{FI,m}(-\mathbf{p}_2)}{\omega - E_F + E_I + i\epsilon} - \frac{\hat{\mathcal{M}}_{IF,m}(-\mathbf{p}_2) \hat{\mathcal{M}}_{FI,\ell}(\mathbf{p}_1)}{\omega + E_F - E_I - i\epsilon} \right], \quad (106)$$

so the Feynman rule and diagram for a loop becomes

- **Scalar Loop Diagram (Type I):** For each single electron loop diagram with ϕ external legs and incoming momentum \mathbf{p}_1 and outgoing momentum \mathbf{p}_2 , insert



$$\phi \xrightarrow{\mathbf{p}_1} \text{---} \bigotimes \text{---} \bigcirc \text{---} \bigotimes \text{---} \xrightarrow{\mathbf{p}_2} \phi \quad \Rightarrow \quad i \sum_{\ell m} f_{\phi, \ell}(\mathbf{p}_1) \hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_2) f_{\phi, m}(-\mathbf{p}_2).$$

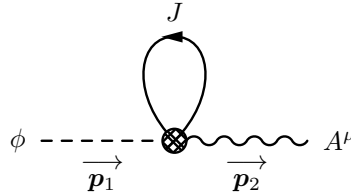
An important subtlety is that in general \mathbf{p}_1 does not have to equal \mathbf{p}_2 , because momentum does not have to be conserved inside the target medium. However, most previous studies do not consider this “off-diagonal” momentum non-conserving contribution to the self-energies, and make the approximation [38, 57]

$$\hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_2) \approx \hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_1) \delta_{\mathbf{p}_1 \mathbf{p}_2}. \quad (107)$$

We will adopt this approximation when computing observables below. See Ref. [38] for additional discussion.

Scalar-Vector Loop Diagram (Type II)

The final loop diagram we compute is for a second loop topology, which we will refer to as a “type II” loop diagram



Since the Feynman rule for the four-point vertex was already derived in Sec. IV D, we apply the previously developed Feynman rules based on Eq. (79) to compute the matrix element

$$\begin{aligned} i\mathcal{M}_{\text{NR}}^\mu &= (-1) \left(\sum_J \int \frac{dE}{2\pi} \right) \left(i \sum_\ell [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\mathcal{M}}_{JJ, \ell}(\mathbf{p}_1 - \mathbf{p}_2) \right) \left(\frac{i}{E - E_J + i\epsilon_J} \right) \\ &= i \sum_\ell [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \sum_J \theta(-E_J) \hat{\mathcal{M}}_{JJ, \ell}(\mathbf{p}_1 - \mathbf{p}_2), \end{aligned} \quad (108)$$

where we have used the Cauchy integral formula to integrate over energy. As in previous sections, the self-energy matrix element cleanly separates into a model-dependent contribution (the f 's) and a target-dependent contribution (the sum over $\hat{\mathcal{M}}$'s). To make this factorization manifest we define

$$\hat{\Pi}_\ell(\mathbf{p}_1 - \mathbf{p}_2) \equiv \sum_J \theta(-E_J) \hat{\mathcal{M}}_{JJ, \ell}(\mathbf{p}_1 - \mathbf{p}_2) = \sum_I \int d^3\mathbf{x} e^{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}} \psi_I^\dagger(\mathbf{x}) \hat{\mathcal{O}}_\ell \psi_I(\mathbf{x}), \quad (109)$$

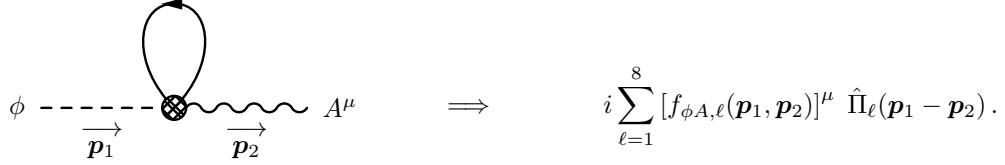
where the sum over I is over filled states in the target ($E_I < 0$).

This NR matrix element expression is most commonly evaluated in the limit where $\mathbf{p}_1 = \mathbf{p}_2$, where it simplifies to

$$i\mathcal{M}_{\text{NR}}^\mu(\mathbf{p}_1, \mathbf{p}_1) \approx i \sum_{\ell} [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_1)]^\mu \hat{\Pi}_\ell(0). \quad (110)$$

Note that $\hat{\Pi}_\ell(0)$ is just the expectation value of the operator $\hat{\mathcal{O}}_\ell$ over the target volume. For example, for $\hat{\mathcal{O}}_1 = 1$, $\hat{\Pi}_1(0) = \sum_I = N_e$, where N_e is the total number of electrons in the target. In terms of $\hat{\Pi}_\ell(\mathbf{p}_1 - \mathbf{p}_2)$ the general type-II loop Feynman rule is

- **Scalar-Vector Loop Diagram (Type II):** For every type II loop connecting an incoming scalar ϕ with momentum \mathbf{p}_1 , and an outgoing vector V with momentum \mathbf{p}_2 insert,



$$\phi \xrightarrow{\mathbf{p}_1} \text{---} \bigcirc \text{---} \xrightarrow{\mathbf{p}_2} A^\mu \quad \Rightarrow \quad i \sum_{\ell=1}^8 [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\Pi}_\ell(\mathbf{p}_1 - \mathbf{p}_2).$$

G. Summary

We now collect all of the Feynman rules derived in Secs. [IV C](#) - [IV D](#), and reference the section in which each one is derived. Wherever possible, we factorize the Feynman rules in to a model-dependent contribution (denoted by expressions involving f vertex factors) which is directly a function of the NR EFT coefficients introduced in [Sec. III](#), and target-dependent factors

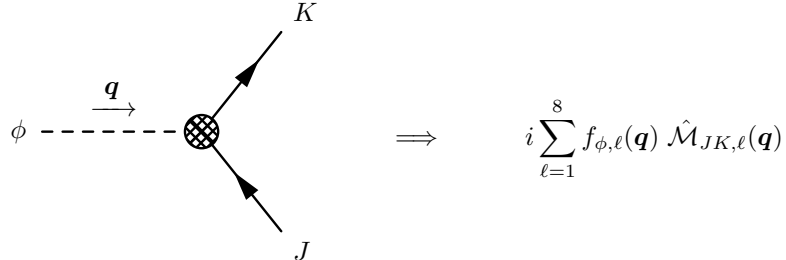
$$\hat{\mathcal{M}}_{IJ, \ell}(\mathbf{q}) \equiv \int d^3 \mathbf{x} e^{i\mathbf{q} \cdot \mathbf{x}} \psi_J^\dagger(\mathbf{x}) \hat{\mathcal{O}}_\ell \psi_I(\mathbf{x}) \quad (111)$$

$$\hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_2) \equiv - \sum_{IF} \left[\frac{\hat{\mathcal{M}}_{IF, \ell}(\mathbf{p}_1) \hat{\mathcal{M}}_{FI, m}(-\mathbf{p}_2)}{\omega - E_F + E_I + i\epsilon} - \frac{\hat{\mathcal{M}}_{IF, m}(-\mathbf{p}_2) \hat{\mathcal{M}}_{FI, \ell}(\mathbf{p}_1)}{\omega + E_F - E_I - i\epsilon} \right] \quad (112)$$

$$\hat{\Pi}_\ell(\mathbf{p}_1 - \mathbf{p}_2) \equiv \sum_I \int d^3 \mathbf{x} e^{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}} \psi_I^\dagger(\mathbf{x}) \hat{\mathcal{O}}_\ell \psi_I(\mathbf{x}), \quad (113)$$

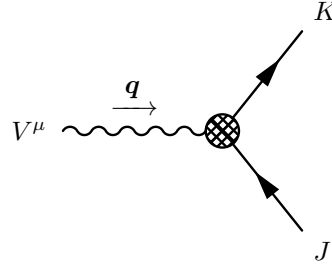
(typically denoted with a “ $\hat{}$ ” symbol) which are directly a function of the target electronic structure. I indexes filled states with $E_I < 0$, and F indexes unfilled states with $E_F > 0$. This list of Feynman rules is not exhaustive; these examples are chosen as a representative set useful in computing the observables discussed in [Secs. V](#), [VI](#), and [VII](#).

- **Three-point Scalar Vertex** ([Sec. IV C](#)): At each three-point vertex between an incoming particle ϕ with momentum \mathbf{q} and electronic states J, K insert,



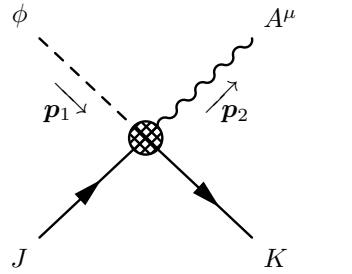
$$\phi \xrightarrow{\mathbf{q}} \text{---} \bigcirc \text{---} \begin{matrix} \nearrow K \\ \searrow J \end{matrix} \quad \Rightarrow \quad i \sum_{\ell=1}^8 f_{\phi, \ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q})$$

- **Three-point Vector Vertex** (Sec. IV C): At each three-point vertex between an incoming vector particle V with momentum \mathbf{q} and electronic states J, K insert



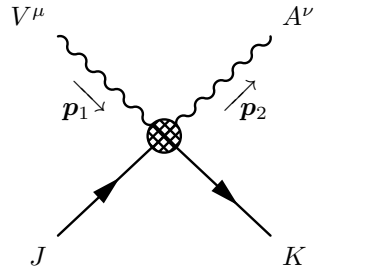
$$\Rightarrow i \sum_{\ell=1}^8 [f_{V,\ell}(\mathbf{q})]^\mu \hat{\mathcal{M}}_{JK,\ell}(\mathbf{q}).$$

- **Four-Point Scalar-Vector Vertex** (Sec. IV D): At each four-point vertex between an incoming scalar particle ϕ with momentum \mathbf{p}_1 , an outgoing vector particle A with momentum \mathbf{p}_2 , and electronic states J, K , insert



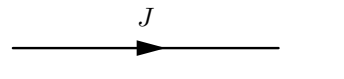
$$\Rightarrow i \sum_{\ell=1}^8 [f_{\phi A,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1 - \mathbf{p}_2),$$

- **Four-Point Vector-Vector Vertex** (Sec. IV D): At each four-point vertex between an incoming vector particle V with momentum \mathbf{p}_1 , an outgoing photon A with momentum \mathbf{p}_2 , and electronic states J, K , insert



$$\Rightarrow i \sum_{\ell=1}^8 [f_{VA,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1 - \mathbf{p}_2)$$

- **NR Electron Propagator** (Sec. IV E): For every internal electron line labelled by J , insert the propagator from Eq. (95)



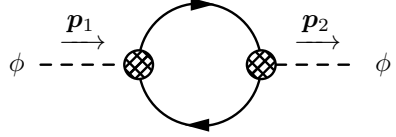
$$\Rightarrow \frac{i}{\omega - E_J + i\epsilon_J}, \quad \epsilon_J = \epsilon \text{sign}(E_J).$$

- Conserve energy at each vertex (Sec. IV A).
- For each undetermined energy E , insert factors of $\left(\int \frac{dE}{2\pi}\right)$ and perform the corresponding integral (Sec. IV A).
- Sum over all internal electron lines labelled by J, K, \dots by inserting $\sum_{JK\dots}$ (Sec. IV F).

- Multiply the resulting expression by $-i$ (Sec. IV A).

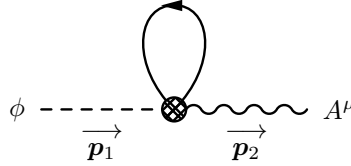
Using these rules one can compute more complicated diagrams. For usability we provide Feynman rules for a few of the composite loop diagrams.

- **Scalar Loop Diagram (Type I)** (Sec. IV F): For each single electron loop diagram between ϕ states with incoming and outgoing momenta \mathbf{p}_1 and \mathbf{p}_2 , respectively, insert



$$\phi \xrightarrow{\mathbf{p}_1} \text{---} \bigcirc \text{---} \bigcirc \text{---} \xrightarrow{\mathbf{p}_2} \phi \quad \Longrightarrow \quad i \sum_{\ell m} f_{\phi, \ell}(\mathbf{p}_1) \hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_2) f_{\phi, m}(-\mathbf{p}_2).$$

- **Scalar-Vector Loop Diagram (Type II)** (Sec. IV F): For every type II loop connecting an incoming scalar ϕ with momentum \mathbf{p}_1 , and an outgoing vector V with momentum \mathbf{p}_2 insert,



$$\phi \xrightarrow{\mathbf{p}_1} \text{---} \bigcirc \text{---} \text{---} \xrightarrow{\mathbf{p}_2} A^\mu \quad \Longrightarrow \quad i \sum_{\ell} [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\Pi}_{\ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

These loop diagrams are typically evaluated in the $\mathbf{p}_1 = \mathbf{p}_2$ limit. To simplify the notation later, when $\mathbf{p}_1 = \mathbf{p}_2$ we write self-energies as, $\Pi(\mathbf{p}_1) \equiv \Pi(\mathbf{p}_1, \mathbf{p}_1)$, $\hat{\Pi}_{\ell m}(\mathbf{p}_1) \equiv \hat{\Pi}_{\ell m}(\mathbf{p}_1, \mathbf{p}_1)$, and $\hat{\Pi}_{\ell}(0) \equiv \hat{\Pi}_{\ell}$.

V. ABSORPTION

Unstable bosonic DM particles can be absorbed in target materials to yield electronic excitations. In this section we calculate general absorption rates for spin-0 and spin-1 particles whose electron interactions have arbitrary Lorentz structure. As in previous sections, we find clean factorization between model-dependent expressions that characterize the DM-electron interaction Lagrangian and target-dependent expressions that characterize properties of the target material. Previous literature on bosonic DM absorption for specific targets and DM models can be found in Refs. [52, 57, 80–82]. Throughout this section, we make frequent use of the resummed in-medium photon propagator:

$$G_{AA}^{\mu\nu} = i \sum_{\lambda} \frac{\epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu}}{\omega^2 - \mathbf{q}^2 - \Pi_{AA}^{\lambda}} \quad , \quad \Pi_{AA}^{\lambda} \equiv -\epsilon_{\mu}^{\lambda} \Pi_{AA}^{\mu\nu} \epsilon_{\nu}^{\lambda} \quad , \quad (114)$$

where ω and \mathbf{q} are the photon energy and momentum, respectively, $\Pi_{AA}^{\mu\nu}$ is the photon self energy tensor,⁸ ϵ_{μ}^{λ} is a polarization vector, and the sum is over photon polarization states λ . A detailed discussion of $G_{AA}^{\mu\nu}$ and its relation to the material dielectric function can be found in Appendix B.

⁸ After this point, $\Pi_{AA}^{\mu\nu}$ will be referred to as $\Pi_{AA}^{\text{UV}, \mu\nu}$ since it is computed with UV matrix elements defined in Eq. (49), as opposed to the NR matrix elements defined in Eq. (50). For further clarification see the discussion before Eq. (117).

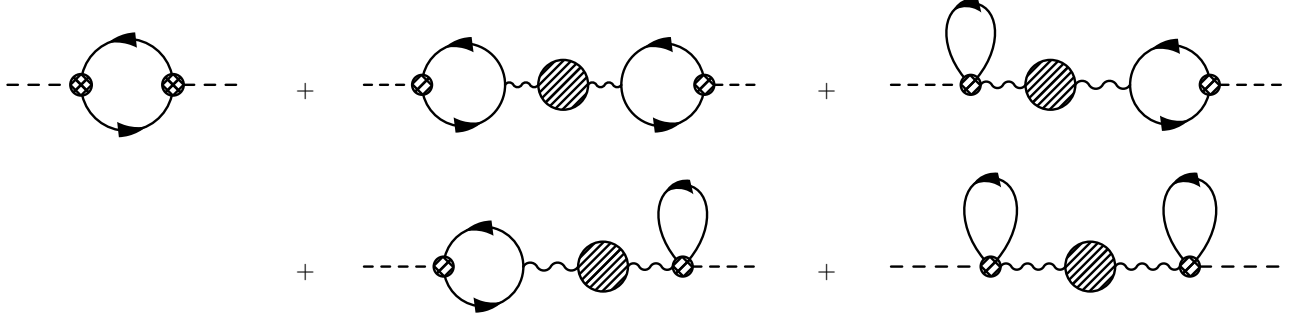


FIG. 1. Example Feynman diagrams for DM (dashed lines) absorption into electronic excitations. The vertex blobs indicate a vertex whose Feynman rules are derived from the DM-electron NR EFT discussed in Secs. III and IV. The Feynman rule coefficients for specific DM models can be found in the tables in App. A. The photon propagator with a shaded internal circle represents a sum over all one particle irreducible (1PI) electron loop diagrams, as discussed in detail in Appendix B. Diagrams involving a photon propagators *screen* the DM-electron interactions.

A. Spin-0 Dark Matter

The absorption rate per incoming spin-0 particle ϕ with momentum \mathbf{p} and energy $\omega^2 = \mathbf{p}^2 + m_\phi^2$ is related to the imaginary part of the particles self-energy via the optical theorem

$$\Gamma_\phi(\mathbf{p}) \approx -\frac{1}{m_\phi} \text{Im} [\Pi_{\phi\phi}^{\text{UV}}(\mathbf{p})], \quad (115)$$

where we have assumed the incoming ϕ is NR, such that $\omega \approx m_\phi$. The total absorption rate per unit of detector exposure (detector mass \times observation time) is then found by multiplying Γ_ϕ by the number of ϕ particles in the detector $N_\phi = \rho_\phi \mathcal{V} / m_\phi$, and dividing by the detector mass, $\rho_T \mathcal{V}$, to yield

$$R_\phi = -\frac{\rho_\phi}{\rho_T m_\phi^2} \text{Im} [\Pi_{\phi\phi}^{\text{UV}}(\mathbf{p})], \quad (116)$$

where ρ_ϕ is the local DM mass density and ρ_T is the target mass density.

The self-energy $\Pi_{\phi\phi}^{\text{UV}}$ is calculated by evaluating the series of diagrams shown in Fig. 1, where the dashed lines represent ϕ particles.⁹ However there is one subtlety: the self-energy in the optical theorem relationship in Eq. (115) is defined in terms of UV matrix elements (Eq. (49)), and therefore has a different mass dimension than the NR self-energy one would compute with the Feynman rules in Sec. IV. The difference is a factor of \mathcal{V} , which can be seen by equating Eqs. (49) and (50) to yield

$$(2\pi)^3 \delta^3(0) \Pi_{\phi\phi}^{\text{UV}}(\mathbf{p}) = \mathcal{V} \Pi_{\phi\phi}^{\text{NR}}(\mathbf{p}) = \Pi_{\phi\phi}^{\text{NR}}(\mathbf{p}), \quad (117)$$

where $\Pi_{\phi\phi}^{\text{NR}}$ is the ϕ self-energy computed using the NR Feynman rules from Sec. IV. For simplicity we will drop the “NR” superscript when the self-energies are defined in terms of NR matrix elements.

⁹ Although we are agnostic about the Lorentz structure of the high-energy ϕ -electron coupling as in Sec. III B, we restrict to interactions linear in ϕ , so there are no ϕ self interactions that contribute to the sum in Fig. 1.

With this subtlety in mind, the first diagram in the upper left of Fig. 1 has been computed in detail in Sec. IV F, and its contribution to the self-energy is

$$-i\Pi_{\phi\phi}(\mathbf{p}) \supset i \sum_{\ell m} f_{\phi,\ell}(\mathbf{p}) \hat{\Pi}_{\ell m}(\mathbf{p}) f_{\phi,m}(-\mathbf{p}), \quad (118)$$

which is just the expression in Eq. (105) with the replacements $\mathbf{p}_1 = \mathbf{p}_2 \equiv \mathbf{p}$. The additional minus sign on the left hand side of Eq. (118) is added because the scalar particle self-energy is defined as the negative of the associated Feynman diagram [95].¹⁰

The other four diagrams in Fig. 1 contain in-medium photon propagators and, as we will see in Sec. V C, these screen the DM interaction. While each can be computed individually it is convenient to split the calculation into pieces. The sum of these screening diagrams factorize as

$$\left(\text{diagram 1} + \text{diagram 2} \right) \times \underbrace{\text{diagram 3}}_{G_{AA}} \times \left(\text{diagram 4} + \text{diagram 5} \right)$$

$\Pi_{\phi A} \qquad \qquad \qquad G_{AA} \qquad \qquad \qquad \Pi_{A\phi}$

where the in-medium photon propagator G_{AA} is computed in detail in App. B. Using the Feynman rules in Sec. IV G, the off-diagonal $\Pi_{\phi A}$ and $\Pi_{A\phi}$ contributions are

$$i\Pi_{\phi A}^{\mu}(\mathbf{p}, \mathbf{p}') = i \sum_{\ell m} f_{\phi,\ell}(\mathbf{p}) \hat{\Pi}_{\ell m}(\mathbf{p}, \mathbf{p}') [f_{A,m}(-\mathbf{p}')]^{\mu} + i \sum_{\ell} [f_{\phi A,\ell}(\mathbf{p}, \mathbf{p}')]^{\mu} \hat{\Pi}_{\ell}(\mathbf{p} - \mathbf{p}') \quad (119)$$

$$i\Pi_{A\phi}^{\mu}(\mathbf{p}', \mathbf{p}) = i \sum_{\ell m} [f_{A,\ell}(\mathbf{p}')]^{\mu} \hat{\Pi}_{\ell m}(\mathbf{p}', \mathbf{p}) f_{\phi,m}(-\mathbf{p}) + i \sum_{\ell} [f_{A\phi,\ell}(\mathbf{p}', \mathbf{p})]^{\mu} \hat{\Pi}_{\ell}(\mathbf{p}' - \mathbf{p}), \quad (120)$$

and \mathbf{p}' is the undetermined momentum flowing through the photon line. The total self-energy can then be written

$$-i\Pi_{\phi\phi}(\mathbf{p}) = i \sum_{\ell m} f_{\phi,\ell}(\mathbf{p}) \hat{\Pi}_{\ell m}(\mathbf{p}) f_{\phi,m}(-\mathbf{p}) + \left(\int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \right) \left[i\Pi_{\phi A}^{\mu}(\mathbf{p}, \mathbf{p}') \right] [G_{AA}(\mathbf{p}')]_{\mu\nu} \left[i\Pi_{A\phi}^{\nu}(\mathbf{p}', \mathbf{p}) \right], \quad (121)$$

where the first term is from the diagonal contribution in Eq. (118) and in the second term we have added the integral over \mathbf{p}' to sum over all undetermined intermediate photon states. With the approximation in Eq. (107), the \mathbf{p}' integral can be removed, which introduces a factor of \mathcal{V} since $\delta_{\mathbf{p}\mathbf{p}'} = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')/\mathcal{V}$. Therefore, the total self-energy simplifies to yield

$$-\Pi_{\phi\phi}(\mathbf{p}) = \sum_{\ell m} f_{\phi,\ell}(\mathbf{p}) \hat{\Pi}_{\ell m}(\mathbf{p}) f_{\phi,m}(-\mathbf{p}) + \frac{i}{\mathcal{V}} \Pi_{\phi A}^{\mu}(\mathbf{p}) [G_{AA}(\mathbf{p})]_{\mu\nu} \Pi_{A\phi}^{\nu}(\mathbf{p}). \quad (122)$$

In typical targets, $\Pi_{\phi\phi}^{\text{UV}}$ is \mathcal{V} independent, such that the absorption rate in Eq. (115) is also \mathcal{V} independent. Therefore, by Eq. (117), $\Pi_{\phi\phi}$ and $\hat{\Pi}_{\ell m}$ must be linear in \mathcal{V} . This \mathcal{V} factor will arise from explicitly evaluating $\hat{\Pi}_{\ell m}$ in Eq. (106) through either $\hat{\mathcal{M}}_{JK,\ell}$ or the state sums.

¹⁰ This is to ensure the scalar propagator resums as

$$G_{\phi\phi} = \frac{i}{q^2 - m_{\phi}^2} + \frac{i}{q^2 - m_{\phi}^2} \left(-i\Pi_{\phi\phi}^{\text{UV}} \right) \left(\frac{i}{q^2 - m_{\phi}^2} \right) + \dots = \frac{i}{q^2 - m_{\phi}^2 - \Pi_{\phi\phi}^{\text{UV}}}.$$

B. Spin-1 Dark Matter

We now consider absorption of a dark spin-1 particle, V , with mass m_V . For a fixed polarization λ , the absorption per incoming particle is

$$\Gamma_V^\lambda(\mathbf{p}) \approx -\frac{1}{m_V \mathcal{V}} \text{Im} [\Pi_{VV}^\lambda(\mathbf{p})] \quad , \quad \Pi_{VV}^\lambda = -\epsilon_\mu^\lambda \Pi_{VV}^{\mu\nu} \epsilon_\nu^\lambda, \quad (123)$$

where we have assumed NR kinematics for the incoming V , written Γ_V^λ in terms of the NR self-energies (related to the usual self-energies used in the optical theorem via Eq. (117)), and projected the self-energy using real polarization vectors satisfying

$$\epsilon_L^\mu = \frac{1}{\sqrt{p^2}}(|\mathbf{p}|, \omega \hat{\mathbf{p}}) \quad , \quad \epsilon_\pm^\mu = (0, \hat{\mathbf{p}}_\pm) \quad , \quad p^\mu \epsilon_\mu^\lambda = 0 \quad , \quad \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{p^2} \quad , \quad \epsilon_\lambda^\mu \epsilon_{\lambda'\mu} = -\delta_{\lambda\lambda'} \quad , \quad (124)$$

where $\hat{\mathbf{p}}_\pm$ are any two vectors chosen to be mutually orthonormal to $\hat{\mathbf{p}}$, $p^\mu = (\omega, \mathbf{p})$ is the incoming four-momentum, and $p^2 = m_V^2$.

The total absorption rate averaged over the polarization components is then¹¹

$$R_V = -\frac{\rho_V}{\rho_T m_V^2 \mathcal{V}} \left(\frac{1}{3} \sum_\lambda \right) \text{Im} [\Pi_{VV}^\lambda(\mathbf{p})] = \frac{\rho_V}{3\rho_T m_V^2 \mathcal{V}} \left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m_V^2} \right) \text{Im} [\Pi_{VV}^{\mu\nu}(\mathbf{p})] \quad , \quad (125)$$

where ρ_V is the mass density of V , and we have replaced the polarization sum with Eq. (124). The self-energy $\Pi_{VV}^{\mu\nu}$ calculation is analogous to the previously considered scalar $\Pi_{\phi\phi}$, except now the self-energies have additional Lorentz indices matching those of the corresponding field V . Explicitly including these Lorentz indices, the relevant off-diagonal self-energies are

$$i\Pi_{VA}^{\mu\nu}(\mathbf{p}) = i \sum_{\ell m} [f_{V,\ell}(\mathbf{p})]^\mu \hat{\Pi}_{\ell m}(\mathbf{p}) [f_{A,m}(-\mathbf{p})]^\nu + i \sum_\ell [f_{VA,\ell}(\mathbf{p})]^{\mu\nu} \hat{\Pi}_\ell \quad (126)$$

$$i\Pi_{AV}^{\mu\nu}(\mathbf{p}) = i \sum_{\ell m} [f_{A,\ell}(\mathbf{p})]^\mu \hat{\Pi}_{\ell m}(\mathbf{p}) [f_{V,m}(-\mathbf{p})]^\nu + i \sum_\ell [f_{AV,\ell}(\mathbf{p})]^{\mu\nu} \hat{\Pi}_\ell \quad (127)$$

and the diagonal self-energy can be written

$$\Pi_{VV}^{\mu\nu}(\mathbf{p}) = \sum_{\ell m} [f_{V,\ell}(\mathbf{p})]^\mu \hat{\Pi}_{\ell m}(\mathbf{p}) [f_{V,\ell}(-\mathbf{p})]^\nu + \frac{i}{\mathcal{V}} \Pi_{VA}^{\mu\rho}(\mathbf{p}) [G_{AA}(\mathbf{p})]_{\rho\sigma} \Pi_{AV}^{\sigma\nu}(\mathbf{p}) \quad , \quad (128)$$

so the final Π_{VV} would be identical in form to the scalar analogue in Eq. (122) if the Lorentz indices were left implicit. As noted in Sec. VA, once $\hat{\Pi}_{\ell m}$ is evaluated explicitly in typical targets both the $\hat{\Pi}$'s and Π 's will be linear in \mathcal{V} , leaving the absorption rate in Eq. (125) \mathcal{V} independent.

C. Screening Effects

Above it was mentioned that the four diagrams in Fig. 1 containing the photon ‘‘screen’’ the DM interaction with electrons and suppress the absorption rate. To justify this, here we investigate DM absorption in two specific DM models – one of these exhibits screening, and the other does not.

¹¹ This is not appropriate when the incoming DM modes have different abundances. For example, the sun produces primarily longitudinal dark photons [78, 106], which must be taken into account when averaging Eq. (125).

An Example With Screening

First, consider the scenario where the V is a kinetically mixed dark photon, whose mass basis UV interaction is

$$\mathcal{L}_{\text{int}} = -\kappa e V_\mu \bar{\Psi} \gamma^\mu \Psi. \quad (129)$$

To simplify the calculation of the absorption rate in Eq. (125) note that $\Pi_{VV}^{\mu\nu}$ satisfies the Ward Identity (WI): $p_\mu \Pi_{VV}^{\mu\nu} = \Pi_{VV}^{\mu\nu} p_\nu = 0$, where $p_\mu = (\omega, \mathbf{p})$ is the V momentum. This eliminates the contribution from contracting $\Pi_{VV}^{\mu\nu}$ with $p_\mu p_\nu / m_V^2$ in Eq. (125). Furthermore, the WI demands that for absorption kinematics, $|\mathbf{p}| \ll \omega$, the temporal components of the V self-energy are negligible since

$$\Pi_{VV}^{00} = \frac{\mathbf{p}^i \Pi_{VV}^{ij} \mathbf{p}^j}{\omega^2}. \quad (130)$$

Thus, in the absorption limit, the absorption rate in Eq. (125) reduces to

$$R_V \approx \frac{\rho_V}{3\rho_T m_V^2 \mathcal{V}} \text{Im} [\Pi_{VV}^{ii}(\mathbf{p})], \quad (131)$$

where we have dropped terms of order $|\mathbf{p}|/m_V$. To compute Π_{VV}^{ii} with Eq. (128) we can use the NR Feynman rule coefficients from Table V and replace $g_v \rightarrow -\kappa e$

$$[f_{V,2}]^{i,j} = i\kappa e \delta^{ij} \quad , \quad [f_{VA,1}]^{ij} = -\frac{\kappa e^2}{m_e} \delta^{ij}. \quad (132)$$

These expressions are identical to the leading order NR Feynman rule coefficients of NR QED (App. A) with the replacement $\kappa \rightarrow 1$, and therefore we can use the relationships in App. B to relate each of the terms in Eq. (128) to the dielectric function $\varepsilon(\omega)$ (assuming an isotropic target for simplicity)

$$\begin{aligned} \text{Im} [f_{V,2}]^i \hat{\Pi}_{22} [f_{V,2}]^j &= \mathcal{V} \kappa^2 \text{Im} [\Pi_{AA}^{\text{UV},ij}] = \mathcal{V} \kappa^2 \omega^2 \text{Im} [\varepsilon(\omega)] \delta^{ij} \\ \Pi_{VA}^{ij} = \Pi_{AV}^{ij} &= -\mathcal{V} \kappa \omega^2 [1 - \varepsilon(\omega)] \delta^{ij} \quad , \quad G_{AA}^{ij} = \frac{i\delta^{ij}}{\omega^2 \varepsilon(\omega)}. \end{aligned} \quad (133)$$

Substituting these expressions with $\omega \approx m_V$ into Eq. (128), and then Eq. (131), leads to the familiar expression for the absorption rate of dark photon DM [56, 57, 76, 78, 79]

$$R_V = \kappa^2 \frac{\rho_V}{\rho_T} \text{Im} \left[\frac{-1}{\varepsilon(m_V)} \right] = \kappa^2 \frac{\rho_V}{\rho_T} \frac{\text{Im} [\varepsilon(m_V)]}{|\varepsilon(m_V)|^2}, \quad (134)$$

which is *screened* due to the appearance of $1/|\varepsilon|^2$, where $|\varepsilon| > 1$ in typical materials.

An Example Without Screening

We now consider a DM model which will not be screened. Consider the interactions of an axion-like particle a with mass m_a

$$\mathcal{L}_{\text{int}} = g_{ae} (\partial_\mu a) \bar{\Psi} \gamma^\mu \gamma^5 \Psi. \quad (135)$$

Since this is a momentum dependent interaction, in the absorption limit, $|\mathbf{p}| \ll \omega$, the zero Lorentz component dominates, and we have

$$\mathcal{L}_{\text{int}} \approx g_{ae} (\partial_t a) \bar{\Psi} \gamma^0 \gamma^5 \Psi, \quad (136)$$

for which the dark operator ($\partial_t a$) has the same electronic interactions as the spin-1 V_0 state coupled to the $\bar{\Psi}\gamma^0\gamma^5\Psi$ current discussed in Sec. III. Therefore the NR EFT interaction Lagrangian coefficients (C) for $\partial_t a$ are identical to those of V^0 in Table II (with $g_a \rightarrow g_{ae}$), and the corresponding Feynman rule coefficient f is

$$[C_{\partial_t a, 5}]^{ij} = -ig_{ae} \delta^{ij} \quad , \quad [f_{a, 5}]^{ij} = -g_{ae} \omega \delta^{ij} . \quad (137)$$

Using the self-energy from Eq. (122) and the relationship between the in-medium propagator and material dielectric in simple targets from Eq. (133), the pseudoscalar self-energy is

$$-\Pi_{aa} = f_{a, 5}(m_a) \hat{\Pi}_{55} f_{a, 5}(-m_a) - \frac{1}{\mathcal{V}} \frac{\Pi_{aA}^i \Pi_{Aa}^i}{m_a^2 \varepsilon(m_a)} . \quad (138)$$

From Eq. (138) we see that there can only be screening if $\Pi_{aA} \neq 0$, i.e., if the pseudoscalar mixes with the photon. Since the axion-like particle only couples to the the fifth NR basis operator, and the photon dominantly couples to the second (App. A), Π_{aA} is only non-zero if $\hat{\Pi}_{25} \neq 0$. Let us focus on the calculation of $\hat{\Pi}_{25}$ for a target where spin is a good quantum number such that $I = \{b, s\}$, and the wavefunctions may be decomposed as

$$\psi_{bs}(\mathbf{x}) = \psi_b(\mathbf{x}) \xi_s \quad , \quad \xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad (139)$$

and the energy levels are spin-degenerate, $E_{b, s} = E_b$. In this limit, for electronic transitions $I = \{b, s\} \rightarrow F = \{b', s'\}$, we have

$$[\hat{\Pi}_{25}]^{ijk} = -\frac{1}{m_e^2} \sum_{bb'} \left[\frac{\left(\int d^3 \mathbf{y} e^{i\mathbf{p} \cdot \mathbf{y}} \psi_{b'}^* \nabla^i \psi_b \right) \left(\int d^3 \mathbf{x} e^{-i\mathbf{p} \cdot \mathbf{x}} \psi_b^* \nabla^k \psi_{b'} \right) \left(\sum_{ss'} \xi_{s'}^\dagger \xi_s \xi_s^\dagger \sigma^j \xi_{s'} \right)}{m_a - E_b + E_{b'} + i\epsilon} - (b, s \leftrightarrow b', s', \epsilon \rightarrow -\epsilon) \right] , \quad (140)$$

where the sum over spins satisfies $\sum_s \xi_s \xi_s^\dagger = 1$, so this expression simplifies to yield

$$[\hat{\Pi}_{25}]^{ijk} \propto \sum_{s'} \xi_{s'}^\dagger \boldsymbol{\sigma} \xi_{s'} = \text{Tr}[\boldsymbol{\sigma}] = 0 , \quad (141)$$

and therefore there is *no screening*, which follows generically from inserting a spin-independent and spin-dependent operator in $\hat{\Pi}_{\ell m}$.

Using similar arguments, we arrive at the general conclusion: at leading order in simple targets, *there is no screening if the dominant NR EFT interaction is spin-dependent* – i.e., if the dark fields couple dominantly to $\hat{\mathcal{O}}_4, \hat{\mathcal{O}}_5, \hat{\mathcal{O}}_6$, or $\hat{\mathcal{O}}_8$ from Eqs. (40) and (41).

VI. SCATTERING

A. General Formalism

In addition to the absorption process discussed in Sec. V, DM may scatter off a target, inducing transitions between the filled and unfilled electronic states. The probability for an interaction to occur given an initial state $|\mathcal{I}\rangle$ and final state $|\mathcal{F}\rangle$, over some time period T due to a transfer matrix \mathcal{T} is [95, 101]

$$\mathcal{P}_{\mathcal{I} \rightarrow \mathcal{F}} = \frac{|\langle \mathcal{F} | i\mathcal{T} | \mathcal{I} \rangle|^2}{\langle \mathcal{I} | \mathcal{I} \rangle \langle \mathcal{F} | \mathcal{F} \rangle} , \quad (142)$$

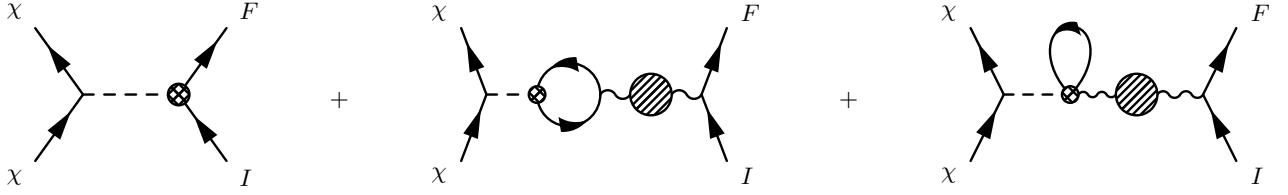


FIG. 2. Feynman diagrams for fermion DM-electron scattering. The first diagram on the left is the tree level process, while the second two *screen* the interactions. As in Fig. 1, the vertex blobs indicate a vertex whose Feynman rules are derived from the DM-electron NR EFT discussed in Secs. III and IV. The Feynman rules for specific models can be found in App. A. The photon propagator includes a resummation of all 1PI diagrams and is discussed in detail in App. B.

and the corresponding interaction rate is

$$\Gamma_{\mathcal{I} \rightarrow \mathcal{F}} \equiv \frac{\mathcal{P}_{\mathcal{I} \rightarrow \mathcal{F}}}{T}. \quad (143)$$

For the scattering process of interest here, the initial state is an incoming DM particle χ with momentum \mathbf{p} , and spin s , and an electron in a filled state $|I\rangle$. The final state is a DM particle with momentum \mathbf{p}' and spin s' , and an electron in an unfilled state $|F\rangle$

$$|\mathcal{I}\rangle = |\mathbf{p}, s\rangle \otimes |I\rangle \quad , \quad |\mathcal{F}\rangle = |\mathbf{p}', s'\rangle \otimes |F\rangle. \quad (144)$$

The total scattering rate per incoming DM particle is the sum over the final state quantum numbers, \mathbf{p}' , F , s' , and initial, filled electron states I

$$\Gamma(\mathbf{p}, s) = \sum_{\mathbf{p}'} \sum_I \sum_F \sum_{s'} \Gamma_{\mathcal{I} \rightarrow \mathcal{F}} = \frac{1}{T} \sum_{\mathbf{p}'} \sum_{IF} \sum_{s'} \frac{|\langle \mathcal{F} | i\mathcal{T} | \mathcal{I} \rangle|^2}{\langle \mathcal{I} | \mathcal{I} \rangle \langle \mathcal{F} | \mathcal{F} \rangle}. \quad (145)$$

By inserting the (relativistic) DM state normalization factors [101]

$$\langle \mathbf{p}, s | \mathbf{p}, s \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^3(0) = 2E_{\mathbf{p}} \mathcal{V} \quad , \quad \langle \mathbf{p}', s' | \mathbf{p}', s' \rangle = 2E_{\mathbf{p}'} (2\pi)^3 \delta^3(0) = 2E_{\mathbf{p}'} \mathcal{V}, \quad (146)$$

and using the fact that, for the continuum index \mathbf{p}' , the sum can be written as an integral

$$\sum_{\mathbf{p}'} = \mathcal{V} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3}, \quad (147)$$

the general formula from Eq. (145) becomes

$$\Gamma(\mathbf{p}, s) = \frac{1}{T} \frac{1}{2E_{\mathbf{p}} \mathcal{V}} \left(\int \frac{d^3 \mathbf{p}'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}'}} \right) \sum_{IF} \sum_{s'} \frac{|\langle \mathcal{F} | i\mathcal{T} | \mathcal{I} \rangle|^2}{\langle I | I \rangle \langle F | F \rangle}, \quad (148)$$

which represents the physical scattering rate per incoming DM particle in a target of volume \mathcal{V} .

Electron State Normalization

The electron state normalization factors in Eq. (148) can be simplified further. Using the electron state normalization factors discussed in Sec. IV B, if I is a discrete index, then

$$\langle I | I \rangle = 1 \quad , \quad \frac{1}{\langle I | I \rangle} \sum_I = \sum_I \quad , \quad (149)$$

and if I is a continuous index (e.g., electron spatial momentum \mathbf{k}), then

$$\langle I|I\rangle = \langle \mathbf{k}|\mathbf{k}\rangle = \mathcal{V} \quad , \quad \frac{1}{\langle I|I\rangle} \sum_I = \frac{1}{\langle \mathbf{k}|\mathbf{k}\rangle} \sum_{\mathbf{k}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \quad , \quad (150)$$

where the electron states are normalized according to Eq. (57). Both cases here are neatly handled by replacing

$$\sum_I \frac{1}{\langle I|I\rangle} \rightarrow \sum_I \quad , \quad (151)$$

for both I and F , and *interpreting the sum on the right-hand side as a sum when I is discrete, and an integral when I is continuous*. This is consistent with the convention introduced in Sec. IV B, where sums over discrete/continuous indices are summed/integrated over. With this convention for the electron state sums, the scattering rate in Eq. (148) can be written

$$\Gamma(\mathbf{p}, s) = \frac{1}{T} \frac{1}{2E_{\mathbf{p}}\mathcal{V}} \left(\int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}'}} \right) \sum_{IF} \sum_{s'} |\langle \mathcal{F} | i\mathcal{T} | \mathcal{I} \rangle|^2 \quad , \quad (152)$$

where the sums over I, F become integrals for the continuous indices.

B. NR Dark Matter Scattering

We now replace \mathcal{T} in Eq. (152) with \mathcal{T}_{NR} from Eq. (50), noting that

$$|\langle \mathcal{F} | i\mathcal{T}_{\text{NR}} | \mathcal{I} \rangle|^2 = [2\pi\delta(\sum E)] [2\pi\delta(0)] |\mathcal{M}_{\text{NR}}|^2 = T [2\pi\delta(\sum E)] |\mathcal{M}_{\text{NR}}|^2 \quad , \quad (153)$$

where we have used

$$2\pi\delta(\omega) = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} dt e^{-i\omega t} \quad , \quad 2\pi\delta(0) \rightarrow T \quad . \quad (154)$$

Substituting Eq. (153) in to Eq. (152), while taking the NR limit of the energy normalization factors $E_{\mathbf{p}} \approx E_{\mathbf{p}'} \approx m_{\chi}$ and trading the $\mathbf{p}' \equiv \mathbf{p} - \mathbf{q}$ integral for an integral over momentum transfer \mathbf{q} , the total scattering rate per incoming DM particle becomes¹²

$$\Gamma(\mathbf{p}, s) = \frac{2\pi}{4m_{\chi}^2\mathcal{V}} \sum_{IF} \sum_{s'} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2 \quad . \quad (155)$$

Since we are interested in NR DM scattering, their momentum and energy are typically approximated in terms of their velocity \mathbf{v} , $\mathbf{p} \approx m_{\chi}\mathbf{v}$, $E_{\mathbf{p}} \approx m_{\chi} + m_{\chi}\mathbf{v}^2/2$. Therefore the argument of the energy conserving delta function becomes

$$\sum E \implies \left(\frac{|\mathbf{p} - \mathbf{q}|^2}{2m_{\chi}} + E_F \right) - \left(\frac{\mathbf{p}^2}{2m_{\chi}} + E_I \right) = -\mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_{\chi}} + E_F - E_I \quad . \quad (156)$$

The average scattering rate per incoming DM particle $\bar{\Gamma}$ is found by averaging over the incoming DM velocity distribution function $f_{\chi}(\mathbf{v})$ and spin s to obtain

$$\bar{\Gamma} = \frac{2\pi}{4m_{\chi}^2\mathcal{V}} \sum_{IF} \left(\frac{1}{2} \sum_{ss'} \right) \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int d^3\mathbf{v} f_{\chi}(\mathbf{v} + \mathbf{v}_e) \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2 \quad , \quad (157)$$

¹² To connect back to the usual QFT free-particle limit, consider the case where the electrons are free particles: $I = \{\mathbf{k}, \sigma\}$, $F = \{\mathbf{k}', \sigma'\}$, and we average over incoming fermion spins, σ, s . In this limit Eq. (155) becomes,

$$\Gamma(\mathbf{p}) = \frac{1}{2m_{\chi}} \int |\overline{\mathcal{M}}|^2 (2\pi)^4 \delta^4(\sum p) \left[\frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2m_e} \right] d\Pi_f \quad , \quad d\Pi_f = \left[\frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{1}{2m_{\chi}} \right] \left[\frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{1}{2m_e} \right] \quad ,$$

where $|\overline{\mathcal{M}}|^2$ is the usual QFT spin-averaged matrix element, we have added factors of $2m_e$ to match the usual QFT state normalization for electrons, and note that $\mathcal{M}_{\text{NR}} = \mathcal{M} (2\pi)^3 \delta^3(\sum \mathbf{p})$. Here the integral over \mathbf{k} arises from the replacement $\sum_I \rightarrow \sum_{\sigma} \int d^3\mathbf{k}/(2\pi)^3$.

where we have boosted to the Earth frame with the velocity of the Earth in the galactic frame, \mathbf{v}_e , and we model f_χ as a truncated Maxwell-Boltzmann distribution

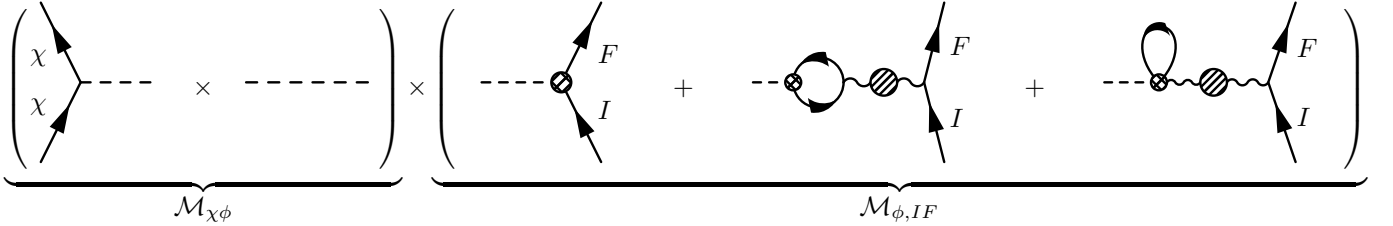
$$f_\chi(\mathbf{v}; v_0, v_{\text{esc}}) = \frac{1}{N_0} e^{-\mathbf{v}^2/v_0^2} \Theta(v_{\text{esc}} - v) \quad , \quad N_0 = \pi^{3/2} v_0^3 \left[\text{erf}\left(\frac{v_{\text{esc}}}{v_0}\right) - \frac{2}{\sqrt{\pi}} \frac{v_{\text{esc}}}{v_0} e^{-v_{\text{esc}}^2/v_0^2} \right]. \quad (158)$$

where v_0 is the velocity dispersion and v_{esc} is the escape velocity [107, 108]. Finally, the total scattering rate R (per unit of detector exposure) is found by multiplying the average scattering rate $\bar{\Gamma}$ by the number of DM particles inside the detector $N_\chi = (\rho_\chi/m_\chi)\mathcal{V}$, and dividing by the mass of the detector $\rho_T\mathcal{V}$ to yield the general scattering rate

$$R = \frac{\rho_\chi}{\rho_T m_\chi} \frac{2\pi}{4m_\chi^2 \mathcal{V}} \sum_{IF} \left(\frac{1}{2} \sum_{ss'} \right) \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int d^3\mathbf{v} f_\chi(\mathbf{v} + \mathbf{v}_e) \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2. \quad (159)$$

From this expression, one simply needs to evaluate the matrix element \mathcal{M}_{NR} using the NR EFT Feynman rules introduced in Sec. IV. The leading order scattering diagrams are given in Fig. 2.

Before computing the NR matrix elements, there is another useful manipulation which separates the DM-mediator interactions from the mediator-electron interactions. Notice that for all the diagrams in Fig. 2 the DM-mediator interaction contribution is identical. We can factorize these sub-diagrams as



where, due to the split, $\mathcal{M}_{\chi\phi}$ and $\mathcal{M}_{\phi,IF}$ have Lorentz indices that match those of the mediator. With this decomposition, the total matrix element is a product of two different matrix elements

$$i\mathcal{M}_{\text{NR}} = [i\mathcal{M}_{\chi\phi}(\mathbf{q}, \mathbf{v})] [i\mathcal{M}_{\phi,IF}(\mathbf{q})] \quad , \quad i\mathcal{M}_{\text{NR}} = [i\mathcal{M}_{\chi V}^\mu(\mathbf{q}, \mathbf{v})] [i[\mathcal{M}_{V,IF}(\mathbf{q})]_\mu], \quad (160)$$

where the NR matrix element on the left applies for scalar mediators, and the one on the right applies for vector mediators. This decomposition is beneficial because \mathbf{v} and s, s' only appear on the DM-mediator side of the calculation. Note that $\mathcal{M}_{\chi\phi}$ and $\mathcal{M}_{\chi V}$ are the familiar QFT matrix elements, which can then be expanded in \mathbf{v} and (analytically) integrated over the DM velocity distribution. The resulting ‘‘DM form factor’’ can then be used *independently* of how the mediator couples to the NR electrons. Defining this form factor for scalar and vector mediators as

$$\begin{aligned} \mathcal{F}_{\chi\phi}(\mathbf{q}, \omega) &\equiv \frac{2\pi}{4m_\chi^2} \frac{1}{2} \sum_{ss'} \int d^3\mathbf{v} f_\chi(\mathbf{v} + \mathbf{v}_e) \delta\left(\omega - \mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_\chi}\right) [\mathcal{M}_{\chi\phi}]_{ss'}^* [\mathcal{M}_{\chi\phi}]_{ss'} \\ \mathcal{F}_{\chi V}^{\mu\nu}(\mathbf{q}, \omega) &\equiv \frac{2\pi}{4m_\chi^2} \frac{1}{2} \sum_{ss'} \int d^3\mathbf{v} f_\chi(\mathbf{v} + \mathbf{v}_e) \delta\left(\omega - \mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_\chi}\right) [\mathcal{M}_{\chi V}]_{ss'}^{\mu,*} [\mathcal{M}_{\chi\phi}]_{ss'}^\nu, \end{aligned} \quad (161)$$

the rate in Eq. (159) simplifies to

$$\begin{aligned} R &= \frac{\rho_\chi}{\rho_T m_\chi \mathcal{V}} \sum_{IF} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{F}_{\chi\phi}(\mathbf{q}, E_F - E_I) \mathcal{M}_{\phi,IF}^* \mathcal{M}_{\phi,IF} \\ R &= \frac{\rho_\chi}{\rho_T m_\chi \mathcal{V}} \sum_{IF} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \mathcal{F}_{\chi V}^{\mu\nu}(\mathbf{q}, E_F - E_I) [\mathcal{M}_{V,IF}]_\mu^* [\mathcal{M}_{V,IF}]_\nu, \end{aligned} \quad (162)$$

for scalar and vector mediators, respectively. Using Eq. (161) the $\mathcal{F}_{\chi\phi}$ and $\mathcal{F}_{\chi V}$ form factors will be explicitly computed for a variety of DM models in Sec. VID.

C. Screening Effects And In-Medium Coupling Coefficients

We now turn to the calculation of the matrix element $\mathcal{M}_{\phi,IF}$ and $\mathcal{M}_{V,IF}$ in Eq. (160). The first term in the diagrammatic expansion is straightforwardly computed with the Feynman rules discussed in Sec. IV. The next two can be written in terms of $\Pi_{\phi A}$ and Π_{VA} , the mediator photon mixing self-energies introduced in Secs. VA VB, and G_{AA} , the in-medium photon propagator discussed in detail in App. B. In total, the matrix elements are

$$\begin{aligned} i\mathcal{M}_{\phi,IF} &= i \sum_{\ell} f_{\phi,\ell} \hat{\mathcal{M}}_{IF,\ell} + \left(\int \frac{d^3 \mathbf{q}'}{(2\pi)^3} \right) \left(i\Pi_{\phi A}^{\mu}(\mathbf{q}, \mathbf{q}') \right) \left([G_{AA}(\mathbf{q}')]_{\mu\nu} \right) \left(i \sum_{\ell} [f_{A,\ell}]^{\nu} \hat{\mathcal{M}}_{IF,\ell} \right) \\ i\mathcal{M}_{V,IF}^{\mu} &= i \sum_{\ell} [f_{V,\ell}]^{\mu} \hat{\mathcal{M}}_{IF,\ell} + \left(\int \frac{d^3 \mathbf{q}'}{(2\pi)^3} \right) \left(i\Pi_{VA}^{\mu\alpha}(\mathbf{q}, \mathbf{q}') \right) \left([G_{AA}(\mathbf{q}')]_{\alpha\beta} \right) \left(i \sum_{\ell} [f_{A,\ell}]^{\beta} \hat{\mathcal{M}}_{IF,\ell} \right), \end{aligned} \quad (163)$$

where the \mathbf{q}' integral arises from summing over the intermediate photon states. These expressions can be further simplified using the $\mathbf{q}' \approx \mathbf{q}$ approximation from Eq. (107)

$$\begin{aligned} i\mathcal{M}_{\phi,IF} &= i \sum_{\ell} \left(f_{\phi,\ell} + \frac{i}{\mathcal{V}} \left[\Pi_{\phi A}^{\mu}(\mathbf{q}) \right] [G_{AA}(\mathbf{q})]_{\mu\nu} [f_{A,\ell}]^{\nu} \right) \hat{\mathcal{M}}_{IF,\ell} \equiv i \sum_{\ell} g_{\phi,\ell} \hat{\mathcal{M}}_{IF,\ell} \\ i\mathcal{M}_{V,IF}^{\mu} &= i \sum_{\ell} \left([f_{V,\ell}]^{\mu} + \frac{i}{\mathcal{V}} \left[\Pi_{VA}^{\mu\alpha}(\mathbf{q}) \right] [G_{AA}(\mathbf{q})]_{\alpha\beta} [f_{A,\ell}]^{\beta} \right) \hat{\mathcal{M}}_{IF,\ell} \equiv i \sum_{\ell} g_{V,\ell}^{\mu} \hat{\mathcal{M}}_{IF,\ell}, \end{aligned} \quad (164)$$

where we have defined *in-medium* coupling coefficients

$$\begin{aligned} g_{\phi,\ell}(\mathbf{q}) &\equiv f_{\phi,\ell}(\mathbf{q}) + \frac{i}{\mathcal{V}} \left[\Pi_{\phi A}^{\mu}(\mathbf{q}) \right] [G_{AA}(\mathbf{q})]_{\mu\nu} [f_{A,\ell}(\mathbf{q})]^{\nu} \\ [g_{V,\ell}(\mathbf{q})]^{\mu} &\equiv [f_{V,\ell}(\mathbf{q})]^{\mu} + \frac{i}{\mathcal{V}} \left[\Pi_{VA}^{\mu\alpha}(\mathbf{q}) \right] [G_{AA}(\mathbf{q})]_{\alpha\beta} [f_{A,\ell}(\mathbf{q})]^{\beta}. \end{aligned} \quad (165)$$

If the mediator mixes with the photon, then $\Pi_{\phi A} \neq 0$ or $\Pi_{VA} \neq 0$ and the scattering rate may be *screened*, thereby reducing $g_{\phi,\ell}$ ($g_{V,\ell}$) from its tree level value, $f_{\phi,\ell}$ ($f_{V,\ell}$).

For example, consider the scenario where the vector mediator V is a kinetically mixed dark photon (see [109] for a review) with electron couplings

$$\mathcal{L}_{\text{int}} \supset -\kappa e V_{\mu} \bar{\Psi} \gamma^{\mu} \Psi, \quad (166)$$

where $\kappa \ll 1$ is a mixing parameter and the Feynman coefficients are all proportional to the photon couplings $f_V = \kappa f_A$. Assuming that scattering is dominated by the zero components of $\Pi_{VA} = \kappa \Pi_{AA}$ and G_{AA} , the in-medium coupling is

$$[g_{V,\ell}]^0 \approx \kappa [f_{A,\ell}]^0 \left[1 + \frac{i}{\mathcal{V}} \Pi_{AA}^{00} G_{00}^{AA} \right] = \kappa [f_{A,\ell}]^0 \left[1 + i \Pi_{AA}^{\text{UV},00} G_{00}^{AA} \right], \quad (167)$$

where we have used $\Pi_{AA}^{00} = \mathcal{V} \Pi_{AA}^{\text{UV},00}$ in Eq. (117) to make the cancellation of the \mathcal{V} factor transparent. Using the results in App. B we can relate $\Pi_{AA}^{\text{UV},00}$ and G_{AA}^{00} to the target dielectric tensor ϵ

$$\Pi_{AA}^{\text{UV},00} = -\mathbf{q}^2 (1 - \hat{\mathbf{q}} \cdot \epsilon \cdot \hat{\mathbf{q}}) \quad , \quad G_{AA}^{00} = \frac{i}{\mathbf{q} \cdot \epsilon \cdot \mathbf{q}}. \quad (168)$$

where $\hat{\mathbf{q}} \equiv \mathbf{q}/|\mathbf{q}|$, and $\mathbf{q} \cdot \epsilon \cdot \mathbf{q} \equiv q^i \epsilon^{ij} q^j$. Thus, for the dark photon mediated model, the effective coupling is

$$[g_{V,\ell}]^0 \approx \frac{\kappa [f_{A,\ell}]^0}{\hat{\mathbf{q}} \cdot \epsilon \cdot \hat{\mathbf{q}}}, \quad (169)$$

so the interaction is *screened* by the dielectric.

In terms of the in-medium coupling coefficients, the DM-electron scattering rate for scalar and vector mediated models are

$$R = \frac{\rho_\chi}{\rho_T m_\chi \mathcal{V}} \sum_{\ell m} \sum_{IF} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \mathcal{F}_{\chi\phi}(\mathbf{q}, E_F - E_I) g_{\phi,\ell}^* g_{\phi,m} \hat{\mathcal{M}}_{IF,\ell}^* \hat{\mathcal{M}}_{IF,m}$$

$$R = \frac{\rho_\chi}{\rho_T m_\chi \mathcal{V}} \sum_{\ell m} \sum_{IF} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \mathcal{F}_{\chi V}^{\mu\nu}(\mathbf{q}, E_F - E_I) [g_{\phi,\ell}^*]_\mu [g_{\phi,m}]_\nu \hat{\mathcal{M}}_{IF,\ell}^* \hat{\mathcal{M}}_{IF,m}. \quad (170)$$

To summarize the meaning of each term in this expression: $\mathcal{F}_{\chi\phi}$ and $\mathcal{F}_{\chi V}$ from Eq. (161) are ‘‘DM form factors’’ and only dependent on the physics of the dark sector. $g_{\phi,\ell}$ and $g_{V,\ell}$ from Eq. (164) are ‘‘in-medium coupling coefficients,’’ which depend on how the mediator couples to electrons in the UV, and may be screened via mixing with the photon. Lastly, $\hat{\mathcal{M}}_{IF,\ell}$ from Eq. (70) are the target-dependent transition matrix elements for each NR basis operator, $\hat{\mathcal{O}}_\ell$. We now compute the DM form factors $\mathcal{F}_{\chi\phi}$, $\mathcal{F}_{\chi V}$ for a variety of fermionic DM models.

D. Dark Matter Form Factors

Scalar Mediated Models

For a fermionic DM particle χ , the most general renormalizable scalar couplings are

$$\mathcal{L} \supset (y_{\chi s} \bar{\chi} \chi + i y_{\chi p} \bar{\chi} \gamma^5 \chi) \phi, \quad (171)$$

where the squared, spin averaged sub-amplitude can be computed using familiar QFT Feynman rules

$$\frac{1}{2} \sum_{ss'} \mathcal{M}_{\chi\phi,ss'}^* \mathcal{M}_{\chi\phi,ss'} \approx \frac{4y_{\chi s}^2 m_\chi^2 + y_{\chi p}^2 \mathbf{q}^2}{(\mathbf{q}^2 + m_\phi^2)^2}, \quad (172)$$

and we have approximated $q^2 \approx -\mathbf{q}^2$ and $p \cdot p' \approx m_\chi^2 + \mathbf{q}^2/2$ in the NR limit. The DM form factor from Eq. (161) can now be written

$$\mathcal{F}_{\chi\phi} = \frac{1}{4m_\chi^2} \left[\frac{4y_{\chi s}^2 m_\chi^2 + y_{\chi p}^2 \mathbf{q}^2}{(\mathbf{q}^2 + m_\phi^2)^2} \right] K_0(\mathbf{q}, \omega, \mathbf{v}_e), \quad (173)$$

where K_0 is the *kinematic function* introduced in Ref. [88, 110, 111]

$$K_0(\mathbf{q}, \omega, \mathbf{v}_e) \equiv 2\pi \int d^3 \mathbf{v} f_\chi(\mathbf{v} + \mathbf{v}_e) \delta \left(\omega - \mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_\chi} \right) = \frac{2\pi^2}{N_0 |\mathbf{q}|} \left(e^{-v_-^2/v_0^2} - e^{-v_*^2/v_0^2} \right), \quad (174)$$

where N_0 is the velocity profile normalization factor defined in Eq. (158) and we have defined the velocities

$$v_- \equiv \min(v_{\text{esc}}, v_*) \quad , \quad v_* \equiv \frac{1}{|\mathbf{q}|} \left(\mathbf{q} \cdot \mathbf{v}_e + \frac{\mathbf{q}^2}{2m_\chi} + \omega \right). \quad (175)$$

Note that the squared sub-amplitude in Eq. (172) only depends on the norm \mathbf{q}^2 , so it can be taken outside of the velocity integral in Eq. (174).

Vector Mediated Models

We now consider DM χ coupled to a vector mediator V_μ with the most general dimension-four interactions

$$\mathcal{L} \supset (g_{\chi v} \bar{\chi} \gamma^\mu \chi + g_{\chi a} \bar{\chi} \gamma^\mu \gamma^5 \chi) V_\mu, \quad (176)$$

and, as in Sec. VID, we first calculate the DM-mediator sub-amplitude $\mathcal{M}_{\chi V}^\mu$ using conventional QFT Feynman rules:

$$i\mathcal{M}_{\chi V}^\mu = \left(\frac{i}{q^2 - m_V^2} \right) \left(-\eta^{\mu\nu} + \frac{q^\mu q^\nu}{m_V^2} \right) \bar{u}_{s'}(\mathbf{p} - \mathbf{q}) [g_{\chi v} \gamma_\nu + i g_{\chi a} \gamma_\nu \gamma^5] u_s(\mathbf{p}). \quad (177)$$

Squaring and spin-averaging yields

$$\begin{aligned} \frac{1}{2} \sum_{ss'} \mathcal{M}_{\chi V}^{*\mu} \mathcal{M}_{\chi V}^\nu &\approx \left(\frac{1}{q^2 + m_V^2} \right)^2 \left\{ g_{\chi v}^2 [-\mathbf{q}^2 \eta^{\mu\nu} + 2(p^\mu p'^\nu + p^\nu p'^\mu)] + 4i g_{\chi v} g_{\chi a} \epsilon^{\mu\nu\rho\lambda} p_\rho p'_\lambda \right. \\ &\quad \left. + g_{\chi a}^2 \left[-(4m_\chi^2 + \mathbf{q}^2) \eta^{\mu\nu} + 2(p^\mu p'^\nu + p'^\mu p^\nu) + \frac{4m_\chi^2(2m_V^2 + \mathbf{q}^2)}{m_V^4} q^\mu q^\nu \right] \right\}, \quad (178) \end{aligned}$$

where, again, p and p' are respectively the incoming and outgoing DM four momenta. Expanding to second order in $|\mathbf{q}|$ and $|\mathbf{p}|$ for each component gives

$$\begin{aligned} \frac{1}{2} \sum_{ss'} \mathcal{M}_{\chi V}^{*\mu} \mathcal{M}_{\chi V}^\nu &\approx \frac{1}{(q^2 + m_V^2)^2} \left\{ g_{\chi v}^2 \begin{pmatrix} 4m_\chi^2 & 2m_\chi(2p^i - q^i) \\ 2m_\chi(2p^i - q^i) & 4p^i p^j - 2(p^i q^j + q^i p^j) + \mathbf{q}^2 \delta^{ij} \end{pmatrix} + 4i g_{\chi v} g_{\chi a} \begin{pmatrix} 0 & -\epsilon^{ijk} p^j q^k \\ \epsilon^{ijk} p^j q^k & m_\chi \epsilon^{ijk} q^k \end{pmatrix} \right. \\ &\quad \left. + g_{\chi a}^2 \left[\begin{pmatrix} 4p^2 - 4\mathbf{q} \cdot \mathbf{p} + \mathbf{q}^2 & 2m_\chi(2p^i - q^i) \\ 2m_\chi(2p^i - q^i) & 4m_\chi^2 \delta^{ij} \end{pmatrix} + \frac{4m_\chi^2(2m_V^2 + \mathbf{q}^2)}{m_V^4} \begin{pmatrix} 0 & 0 \\ 0 & q^i q^j \end{pmatrix} \right] \right\}, \quad (179) \end{aligned}$$

where we have condensed the $\mu\nu$ components to a matrix; the 00 component is in the upper left, the 0*i*, *i*0 components are in the upper right and lower left, respectively, the *ij* components are in the lower right, and we have separated the anomalous contribution to the axial-vector current.

Substituting Eq. (179) in to Eq. (161) we can compute the DM form factor, $\mathcal{F}_{\chi V}^{\mu\nu}$

$$\begin{aligned} \mathcal{F}_{\chi V}^{\mu\nu} &= \frac{1}{(q^2 + m_V^2)^2} \left\{ g_{\chi v}^2 \begin{pmatrix} K_0 & K_1^i - \frac{q^i}{2m_\chi} K_0 \\ K_1^i - \frac{q^i}{2m_\chi} K_0 & K_2^{ij} - \left(K_1^i \frac{q^j}{2m_\chi} + \frac{q^i}{2m_\chi} K_1^j \right) + \frac{\mathbf{q}^2}{4m_\chi^2} \delta^{ij} K_0 \end{pmatrix} \right. \\ &\quad \left. + g_{\chi a}^2 \left[\begin{pmatrix} K_2^{ii} - \frac{q^i}{m_\chi} K_1^i + \mathbf{q}^2 K_0 & K_1^i - \frac{q^i}{2m_\chi} K_0 \\ K_1^i - \frac{q^i}{2m_\chi} K_0 & \delta^{ij} K_0 \end{pmatrix} + \frac{2m_V^2 + \mathbf{q}^2}{m_V^4} \begin{pmatrix} 0 & 0 \\ 0 & q^i q^j K_0 \end{pmatrix} \right] \right. \\ &\quad \left. + 2i g_{\chi v} g_{\chi a} \begin{pmatrix} 0 & -\epsilon^{ijk} K_1^j \frac{q^k}{2m_\chi} \\ \epsilon^{ijk} K_1^j \frac{q^k}{2m_\chi} & \epsilon^{ijk} \frac{q^k}{2m_\chi} K_0 \end{pmatrix} \right\}, \quad (180) \end{aligned}$$

where the velocity integrals have been rewritten in terms the generalized kinematic functions K_1^i and K_2^{ij} , first derived in Ref. [88], where K_1^i satisfies

$$K_1^i(\mathbf{q}, \omega, \mathbf{v}_e) = 2\pi \int d^3\mathbf{v} v^i f_\chi(\mathbf{v} + \mathbf{v}_e) \delta\left(\omega - \mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_\chi}\right) = (v_* \hat{q}^i - v_e^i) K_0(\mathbf{q}, \omega, \mathbf{v}_e), \quad (181)$$

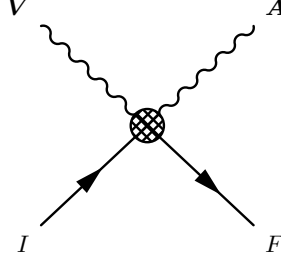


FIG. 3. An example Feynman diagram for the dark Thomson scattering process.

and the K_2^{ij} function can be written

$$\begin{aligned}
K_2^{ij}(\mathbf{q}, \omega, \mathbf{v}_e) &= 2\pi \int d^3\mathbf{v} v^i v^j f_\chi(\mathbf{v} + \mathbf{v}_e) \delta\left(\omega - \mathbf{q} \cdot \mathbf{v} + \frac{\mathbf{q}^2}{2m_\chi}\right) \\
&= v_*^2 \hat{q}^i \hat{q}^j K_0(\mathbf{q}, \omega, \mathbf{v}_e) + (\delta^{ij} - \hat{q}^i \hat{q}^j) \left[\frac{\pi^2 v_0^2}{|\mathbf{q}| N_0} \left(v_0^2 e^{-v_-^2/v_0^2} - (v_0^2 - v_-^2 + v_{\text{esc}}^2) e^{-v_{\text{esc}}^2/v_0^2} \right) \right] \\
&\quad - \left[v_e^i K_1^j(\mathbf{q}, \omega, \mathbf{v}_e) + v_e^j K_1^i(\mathbf{q}, \omega, \mathbf{v}_e) \right] + v_e^i v_e^j K_0(\mathbf{q}, \omega, \mathbf{v}_e). \tag{182}
\end{aligned}$$

VII. DARK THOMSON SCATTERING

The last process we consider is “dark Thomson scattering,” where an incoming DM vector particle V inelastically scatters off an electron and converts into a photon, as depicted Fig. 3. This process is the low-energy limit of dark Compton scattering considered in Ref. [23]. The interaction rate per incoming V particle from Eq. (143) can be written in terms of \mathcal{M}_{NR} as

$$\Gamma_{\mathcal{I} \rightarrow \mathcal{F}} = \frac{2\pi}{\langle \mathcal{I} | \mathcal{I} \rangle \langle \mathcal{F} | \mathcal{F} \rangle} \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2. \tag{183}$$

Using the Feynman rules developed in Sec. IV we can evaluate the NR matrix element for incoming and outgoing states given by,

$$|\mathcal{I}\rangle = |\mathbf{p}, \lambda\rangle \otimes |I\rangle \quad , \quad |\mathcal{F}\rangle = |\mathbf{p} - \mathbf{q}, \lambda'\rangle \otimes |F\rangle, \tag{184}$$

where λ is the polarization of the incoming DM particle and λ' is the polarization of the outgoing photon. The total scattering rate of a polarization λ , per detector exposure, R_λ , is then found by summing over the electron and photon states, multiplying by the total number of DM particles in the detector, $N_V = (\rho_V/m_V)\mathcal{V}$, and dividing by the detector mass, $\rho_T\mathcal{V}$,

$$R_\lambda = \frac{\rho_V}{\rho_T m_V} \frac{2\pi}{4m_V \omega \mathcal{V}} \sum_{IF} \sum_{\lambda'} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2. \tag{185}$$

where $\omega = |\mathbf{p} - \mathbf{q}|$ is the energy of the outgoing photon and, as in Sec. VI, the sums over I, F become integrals when the indices are continuous. Lastly, averaging over the incoming DM polarization gives the expected total scattering rate,

$$R = \frac{\rho_V}{\rho_T m_V} \frac{2\pi}{4m_V \omega \mathcal{V}} \sum_{IF} \left(\frac{1}{3} \sum_{\lambda\lambda'} \right) \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(\sum E) |\mathcal{M}_{\text{NR}}|^2. \tag{186}$$

The NR matrix element for the dark Thomson process can then be computed with the Feynman rules from Sec. IV,

$$i\mathcal{M}_{\text{NR}} = \epsilon_{V,i}^\lambda \epsilon_{A,j}^{\lambda'} \left(i \sum_\ell [f_{VA}]^{ij} \hat{\mathcal{M}}_{IF,\ell} \right), \quad (187)$$

where ϵ_V, ϵ_A are the DM and photon polarization vectors, respectively, and we have ignored the zero component contribution because it is higher order.

A Dark Thomson Scattering Example

To provide an example calculation we consider dark Thomson scattering in a simple model and target. As our model we take the kinetically mixed dark photon ($g_v = -\kappa e$ in the representative UV Lagrangian in Sec. III). In this model the dominant Feynman rule coefficient contributing to dark Thomson scattering is

$$[f_{VA,1}]^{ij} = -\frac{\kappa e^2}{m_e} \delta^{ij}. \quad (188)$$

For the target we assume that the electrons are free ($\Phi = 0$), and the initial state is $N_e = n_e \mathcal{V}$ electrons with momentum much less than m_V . This is equivalent to the approximations made in Ref. [23], and the final scattering rate will match Ref. [23] in the limit of $m_V \ll m_e$. Therefore the electron states are indexed by momentum and spin quantum numbers,

$$|I\rangle = |\mathbf{k}, s\rangle \quad , \quad \sum_I \rightarrow \sum_s \int \frac{d^3\mathbf{k}}{(2\pi)^3} \quad , \quad |F\rangle = |\mathbf{k}', s'\rangle \quad , \quad \sum_F \rightarrow \sum_{s'} \int \frac{d^3\mathbf{k}'}{(2\pi)^3}. \quad (189)$$

The relevant transition matrix element is $\hat{\mathcal{M}}_{\mathbf{k}s\mathbf{k}'s',1}$

$$\hat{\mathcal{M}}_{\mathbf{k}s\mathbf{k}'s',1} = \delta_{ss'} (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{k}' + \mathbf{k}), \quad (190)$$

and upon substitution the rate in Eq. (186) becomes

$$R = \frac{\rho_V}{\rho_T m_V} \frac{\kappa^2 e^4}{m_e^2} \frac{2\pi}{4m_V^2} \left(\frac{1}{3} \sum_{\lambda\lambda'} |\epsilon_{V,\lambda} \cdot \epsilon_{A,\lambda'}|^2 \right) \left(\sum_{ss'} \delta_{ss'} \right) \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \delta(m_V - |\mathbf{k}'|). \quad (191)$$

This can be further simplified using the polarization sum relationships from Eq. (124)

$$\sum_\lambda \epsilon_{V,\lambda}^i \epsilon_{V,\lambda}^j \approx \delta^{ij} \quad \implies \quad \sum_{\lambda\lambda'} |\epsilon_{V,\lambda} \cdot \epsilon_{A,\lambda'}|^2 \approx \sum_{\lambda'} \epsilon_{A,\lambda'} \cdot \epsilon_{A,\lambda'} = 2, \quad (192)$$

which simplifies Eq. (191) to yield

$$R = \frac{n_e}{\rho_T} \frac{\rho_V}{m_V} \frac{e^4 \kappa^2}{6\pi m_e}, \quad (193)$$

noting that the sum over initial electron states is simply their number density $n_e = 2 \int d^3\mathbf{k}/(2\pi)^3$.

VIII. CONCLUSIONS

Understanding DM-electron interactions in a variety of targets is necessary to maximize the discovery potential of current, and future, electron based direct detection experiments. In this work we formulate the NR EFT of DM-electron interactions as a tool to compute any DM-electron observable, in any target. The EFT is developed from a ‘‘top-down’’ perspective, where the starting point is a Lagrangian defining DM-electron interactions at energies above

the electron mass m_e . The high-energy theory is then matched to a low-energy theory describing the interactions of the DM and NR electron field, which satisfies the Schrödinger equation (including corrections to an arbitrary order in $1/m_e$). The mapping between the high and low-energy electron fields, discussed in Sec. II, can be found by “integrating out the positron” field which is done with the Foldy-Wouthuysen method [90–94] and maps QED to NR QED [89]. In Sec. III this mapping is then applied to the interaction operators in the high-energy theory to find the NR EFT Lagrangian of DM-electron interactions.

While the NR EFT dictates the structure of the DM-electron interactions, it is still one step removed from the observables one wants to compute, e.g., absorption or scattering rates. Typically in relativistic QFT this connection is made by using Feynman rules and diagrams to compute a matrix element, which is then used to compute the observable. In Sec. IV we develop analogous Feynman rules for the NR EFT. This allows any process to be computed diagrammatically which has the benefit of composability: more complicated diagrams and observables can be constructed from a few primitive vertices and propagators. These Feynman rules are then used to compute the DM absorption, scattering, and dark Thomson scattering rates for a wide variety of DM models in Secs. V, VI, and VII, respectively.

A major benefit of the formalism developed here is that the interaction rates of a given DM model can be directly applied to any electronic target. This is because the target electronic structure enters only through the evaluation of the Feynman rules. Furthermore we identified a basis of eight NR operators, Eqs. (39) - (41), which can determine the electronic response of any DM model to order m_e^{-2} , assuming the electron is in only a background electric potential. Extending the existing codes for DM-electron interaction rates in crystals (EXCEED-DM [67, 112], QCDark [113], QEDark [65], QEDark-EFT [69]), or atomic targets (DarkART [68]) to compute these eight transition matrix elements in Eq. (70), in any kinematic regime, would allow them to compute all the DM-electron interaction rates discussed here.

There are other interesting DM induced processes which deposit an NR amount of energy and momentum to the electron degrees of freedom, and therefore could be similarly generalized with the results of this paper. For example, relativistic absorption (e.g., absorption of solar axions [114–116]), or even those which involve some nuclear degrees of freedom, e.g., the Migdal effect [117–123] or “fermionic absorption” [124–127]. Given a term in the NR EFT which describes these interactions, similar Feynman rules may be analogously derived for those processes, and therefore generalized to any target. Additionally, an advantage of using an EFT for only the electron degrees of freedom is that the usual tools from relativistic QFT can be used to handle any other degrees of freedom. For example, if DM couples to quarks, and therefore only interacts with electrons via a loop [128], the DM-quark and mixing diagrams can be computed as usual, and one simply needs to replace the electron vertices with the Feynman rules developed here.

ACKNOWLEDGMENTS

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Appendix A: NR Interaction Lagrangian Summary Tables

High-Energy Interaction Lagrangian

$$\mathcal{L}_{\text{int}} = y_s \phi \bar{\Psi} \Psi$$

NR EFT Interaction Lagrangian

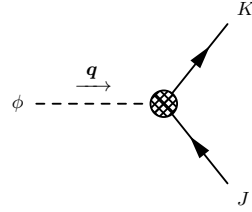
$$\begin{aligned} \mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{\phi, \ell} \phi + C_{(\nabla\phi), \ell} \frac{(\nabla\phi)}{m_e} + C_{(\nabla\nabla\phi), \ell} \frac{(\nabla\nabla\phi)}{m_e^2} \right. \\ \left. + C_{\phi\mathbf{A}, \ell} \frac{\phi\mathbf{A}}{m_e} + C_{(\nabla\phi)\mathbf{A}, \ell} \frac{(\nabla\phi)\mathbf{A}}{m_e^2} + C_{\phi(\nabla\mathbf{A}), \ell} \frac{\phi(\nabla\mathbf{A})}{m_e^2} + C_{\phi\mathbf{A}\mathbf{A}, \ell} \frac{\phi\mathbf{A}\mathbf{A}}{m_e^2} \right] [\psi^\dagger \hat{\mathcal{O}}_\ell \psi] \end{aligned}$$

$$C_{\phi,1} = y_s \quad , \quad [C_{\phi,3}]^{ij} = \frac{y_s}{2} \delta^{ij} \quad , \quad [C_{(\nabla\phi),2}]^{ij} = \frac{y_s}{2} \delta^{ij} \quad , \quad [C_{(\nabla\phi),5}]^{ijk} = \frac{-iy_s}{4} \epsilon^{ijk} \quad , \quad [C_{(\nabla\nabla\phi),1}]^{ij} = \frac{y_s}{8} \delta^{ij}$$

$$[C_{\phi\mathbf{A},2}]^{i,j} = -iey_s \delta^{ij} \quad , \quad [C_{(\nabla\phi)\mathbf{A},1}]^{ij} = \frac{-iey_s}{2} \delta^{ij} \quad , \quad [C_{(\nabla\phi)\mathbf{A},4}]^{ij,k} = \frac{ey_s}{4} \epsilon^{ijk} \quad , \quad [C_{\phi(\nabla\mathbf{A}),1}]^{ij} = \frac{-iey_s}{2} \delta^{ij}$$

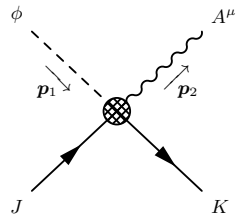
$$[C_{\phi(\nabla\mathbf{A}),4}]^{ij,k} = \frac{ey_s}{2} \epsilon^{ijk} \quad , \quad [C_{\phi\mathbf{A}\mathbf{A}}]^{ij} = -\frac{e^2 y_s}{2} \delta^{ij}$$

NR Feynman Rules



$$\Rightarrow i \sum_{\ell=1}^8 f_{\phi, \ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q}).$$

$$f_{\phi,1} = y_s \left(1 - \frac{\mathbf{q}^2}{8m_e^2} \right) \quad , \quad [f_{\phi,2}]^i = \frac{iy_s}{2} \frac{q^i}{m_e} \quad , \quad [f_{\phi,3}]^{ij} = \frac{y_s}{2} \delta^{ij} \quad , \quad [f_{\phi,5}]^{ij} = \frac{y_s}{4} \epsilon^{ijk} \frac{q^k}{m_e}$$



$$\Rightarrow i \sum_{\ell=1}^8 [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^\mu \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$[f_{\phi A,1}]^i = -\frac{ey_s}{2m_e^2} (p_1^i - p_2^i) \quad , \quad [f_{\phi A,2}]^{i,j} = \frac{iey_s}{m_e} \delta^{ij} \quad , \quad [f_{\phi A,4}]^{i,j} = -\frac{iey_s}{4m_e^2} \epsilon^{ijk} (p_1^k - 2p_2^k)$$

TABLE III. Summary of the NR EFT Interaction Lagrangian and Feynman rules generated by the UV interaction Lagrangian $\mathcal{L}_{\text{int}} = y_s \phi \bar{\Psi} \Psi$. See Sec. IV to see how to find the f 's in terms of the C 's.

High-Energy Interaction Lagrangian

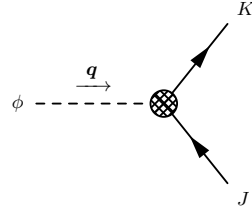
$$\mathcal{L}_{\text{int}} = iy_p \phi \bar{\Psi} \gamma^5 \Psi$$

NR EFT Interaction Lagrangian

$$\mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{\phi, \ell} \phi + C_{(\nabla \phi), \ell} \frac{(\nabla \phi)}{m_e} + C_{\phi(\partial_t \mathbf{A}), \ell} \frac{\phi(\partial_t \mathbf{A})}{m_e^2} + C_{\phi(\nabla A^0), \ell} \frac{\phi(\nabla A^0)}{m_e^2} \right] [\psi^\dagger \hat{\mathcal{O}}_{\ell} \psi]$$

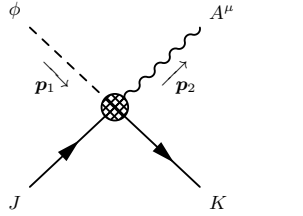
$$[C_{\phi, 8}]^{ij} = -\frac{y_p}{2} \delta^{ij} \quad , \quad [C_{(\nabla \phi), 4}]^{i,j} = -\frac{y_p}{2} \delta^{ij} \quad , \quad [C_{\phi(\partial_t \mathbf{A}), 4}]^{i,j} = -\frac{ey_p}{2} \delta^{ij} \quad , \quad [C_{\phi(\nabla A^0), 4}]^{i,j} = -\frac{ey_p}{2} \delta^{ij}$$

NR Feynman Rules



$$\Rightarrow i \sum_{\ell=1}^8 f_{\phi, \ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q}).$$

$$[f_{\phi, 4}]^i = -\frac{iy_p}{2} \frac{q^i}{m_e} \quad , \quad [f_{\phi, 8}]^{ij} = -\frac{y_p}{2} \delta^{ij}$$



$$\Rightarrow i \sum_{\ell=1}^8 [f_{\phi A, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu} \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$[f_{\phi A, 4}]^{0,i} = \frac{iey_p}{2m_e} \frac{p_2^i}{m_e} \quad , \quad [f_{\phi A, 4}]^{i,j} = \frac{iey_p}{2m_e} \frac{\omega_2}{m_e} \delta^{ij}$$

TABLE IV. Summary of the NR EFT Interaction Lagrangian and Feynman rules generated by the UV interaction Lagrangian $\mathcal{L}_{\text{int}} = iy_p \phi \bar{\Psi} \gamma^5 \Psi$. See Sec. IV to see how to find the f 's in terms of the C 's.

High-Energy Interaction Lagrangian

$$\mathcal{L}_{\text{int}} = g_v V_\mu \bar{\Psi} \gamma^\mu \Psi$$

NR EFT Interaction Lagrangian

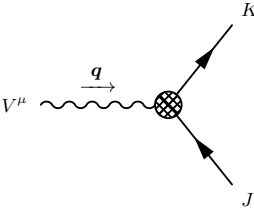
$$\mathcal{L}_{\text{int}}^{\text{NR}} = \sum_\ell \left[C_{V^0, \ell} V^0 + C_{(\nabla V^0), \ell} \frac{(\nabla V^0)}{m_e} + C_{(\nabla \nabla V^0), \ell} \frac{(\nabla \nabla V^0)}{m_e^2} + C_{(\nabla V^0) \mathbf{A}, \ell} \frac{(\nabla V^0) \mathbf{A}}{m_e^2} \right. \\ \left. + C_{\mathbf{V}, \ell} \mathbf{V} + C_{(\nabla \mathbf{V}), \ell} \frac{(\nabla \mathbf{V})}{m_e} + C_{\mathbf{V} \mathbf{A}, \ell} \frac{\mathbf{V} \mathbf{A}}{m_e} + C_{\mathbf{V}(\partial_t \mathbf{A}), \ell} \frac{\mathbf{V}(\partial_t \mathbf{A})}{m_e^2} + C_{\mathbf{V}(\nabla A^0), \ell} \frac{\mathbf{V}(\nabla A^0)}{m_e^2} \right] [\psi^\dagger \hat{\mathcal{O}}_\ell \psi]$$

$$C_{V^0, 1} = g_v \quad , \quad [C_{(\nabla V^0), 5}]^{i, jk} = \frac{ig_v}{4} \epsilon^{ijk} \quad , \quad [C_{(\nabla \nabla V^0), 1}]^{ij} = \frac{g_v}{8} \delta^{ij} \quad , \quad [C_{(\nabla V^0) \mathbf{A}, 4}]^{i, jk} = -\frac{eg_v}{4} \epsilon^{ijk}$$

$$[C_{(\nabla \mathbf{V}), 1}]^{i, j} = \frac{ig_v}{2} \delta^{ij} \quad , \quad [C_{(\nabla \mathbf{V}), 4}]^{i, j, k} = -\frac{g_v}{2} \epsilon^{ijk} \quad , \quad [C_{\mathbf{V}, 2}]^{i, j} = ig_v \delta^{ij} \quad , \quad [C_{\mathbf{V}, 8}]^{i, j, k} = -\frac{g_v}{2} \epsilon^{ijk}$$

$$[C_{\mathbf{V} \mathbf{A}, 1}]^{ij} = eg_v \delta^{ij} \quad , \quad [C_{\mathbf{V}(\partial_t \mathbf{A}), 4}]^{i, j, k} = \frac{eg_v}{2} \epsilon^{ijk} \quad , \quad [C_{\mathbf{V}(\nabla A^0), 4}]^{i, j, k} = \frac{eg_v}{2} \epsilon^{ijk}$$

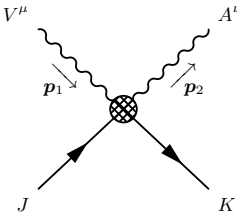
NR Feynman Rules



$$\Rightarrow i \sum_\ell [f_{V, \ell}(\mathbf{q})]^\mu \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q}).$$

$$f_{V, 1}^0 = g_v \left(1 - \frac{\mathbf{q}^2}{8m_e^2} \right) \quad , \quad [f_{V, 5}]^{0, ij} = -\frac{g_v}{4} \epsilon^{ijk} \frac{q^k}{m_e}$$

$$[f_{V, 1}]^i = \frac{g_v}{2m_e} q^i \quad , \quad [f_{V, 2}]^{i, j} = -ig_v \delta^{ij} \quad , \quad [f_{V, 4}]^{i, j} = \frac{ig_v}{2m_e} \epsilon^{ijk} q^k \quad , \quad [f_{V, 8}]^{i, j, k} = \frac{g_v}{2} \epsilon^{ijk}$$



$$\Rightarrow i \sum_\ell [f_{VA, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$[f_{VA, 4}]^{0i, j} = \frac{ieg_v}{4m_e} \epsilon^{ijk} \frac{p_1^k}{m_e} \quad , \quad [f_{VA, 4}]^{i0, j} = \frac{ieg_v}{2m_e} \epsilon^{ijk} \frac{p_2^k}{m_e}$$

$$[f_{VA, 1}]^{ij} = \frac{eg_v}{m_e} \delta^{ij} \quad , \quad [f_{VA, 4}]^{i, j, k} = \frac{ieg_v}{2m_e} \frac{\omega_2}{m_e} \epsilon^{ijk}$$

TABLE V. Summary of the NR EFT Interaction Lagrangian and Feynman rules generated by the UV interaction Lagrangian $\mathcal{L}_{\text{int}} = g_v V_\mu \bar{\Psi} \gamma^\mu \Psi$. See Sec. IV to see how to find the f 's in terms of the C 's.

High-Energy Interaction Lagrangian

$$\mathcal{L}_{\text{int}} = g_a V_\mu \bar{\Psi} \gamma^\mu \gamma^5 \Psi$$

NR EFT Interaction Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{V^0, \ell} V^0 + C_{(\nabla V^0), \ell} \frac{(\nabla V^0)}{m_e} + C_{V^0 \mathbf{A}, \ell} \frac{V^0 \mathbf{A}}{m_e} + C_{\mathbf{V}, \ell} \mathbf{V} + C_{(\nabla \mathbf{V}), \ell} \frac{(\nabla \mathbf{V})}{m_e} + C_{\mathbf{V} \mathbf{A}, \ell} \frac{\mathbf{V} \mathbf{A}}{m_e} \right. \\ \left. + C_{(\nabla \nabla \mathbf{V}), \ell} \frac{(\nabla \nabla \mathbf{V})}{m_e^2} + C_{(\nabla \mathbf{V}) \mathbf{A}, \ell} \frac{(\nabla \mathbf{V}) \mathbf{A}}{m_e^2} + C_{\mathbf{V}(\nabla \mathbf{A}), \ell} \frac{\mathbf{V}(\nabla \mathbf{A})}{m_e^2} + C_{\mathbf{V} \mathbf{A} \mathbf{A}, \ell} \frac{\mathbf{V} \mathbf{A} \mathbf{A}}{m_e^2} \right] [\psi^\dagger \hat{\mathcal{O}}_\ell \psi] \end{aligned}$$

$$[C_{V^0, 5}]^{ij} = -ig_a \delta^{ij} \quad , \quad [C_{(\nabla V^0), 4}]^{i, j} = -\frac{ig_a}{4} \delta^{ij} \quad , \quad [C_{V^0 \mathbf{A}, 4}]^{i, j} = -eg_a \delta^{ij}$$

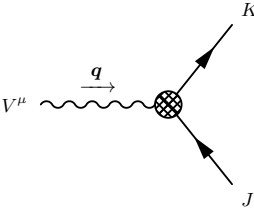
$$[C_{\mathbf{V}, 4}]^{i, j} = -g_a \delta^{ij} \quad , \quad [C_{\mathbf{V}, 6}]^{i, jkl} = \frac{g_a}{2} H_1^{ijkl} \quad , \quad [C_{\nabla \mathbf{V}, 2}]^{ij, k} = \frac{ig_a}{4} \epsilon^{ijk} \quad , \quad [C_{\nabla \mathbf{V}, 5}]^{ij, kl} = \frac{g_a}{4} H_2^{ijkl}$$

$$[C_{\mathbf{V} \mathbf{A}, 5}]^{ij, kl} = -\frac{ieg_a}{2} H_2^{ijkl} \quad , \quad [C_{(\nabla \nabla \mathbf{V}), 4}]^{ijk, l} = -\frac{g_a}{8} \delta^{ij} \delta^{kl} \quad , \quad [C_{(\nabla \mathbf{V}) \mathbf{A}, 1}]^{ijk} = \frac{eg_a}{4} \epsilon^{ijk}$$

$$[C_{(\nabla \mathbf{V}) \mathbf{A}, 4}]^{ijk, l} = -\frac{ieg_a}{4} H_2^{ijkl} \quad , \quad [C_{\mathbf{V}(\nabla \mathbf{A}), 1}]^{ijk} = -\frac{eg_a}{2} \epsilon^{ijk} \quad , \quad [C_{\mathbf{V}(\nabla \mathbf{A}), 4}]^{ijk, l} = -\frac{ieg_a}{4} H_2^{ijkl}$$

$$[C_{\mathbf{V} \mathbf{A} \mathbf{A}, 4}]^{ijk, l} = -\frac{g_a}{2} H_1^{ijkl}$$

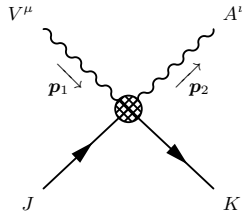
NR Feynman Rules



$$\Rightarrow i \sum_{\ell} [f_{V, \ell}(\mathbf{q})]^\mu \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q}).$$

$$[f_{V, 4}]^{0, i} = \frac{g_a}{4} \frac{q^i}{m_e} \quad , \quad [f_{V, 5}]^{0, ij} = -ig_a \delta^{ij}$$

$$[f_{V, 2}]^{i, j} = \frac{g_a}{4} \epsilon^{ijk} \frac{q^k}{m_e} \quad , \quad [f_{V, 4}]^{i, j} = g_a \delta^{ij} \left(1 - \frac{\mathbf{q}^2}{8m_e^2} \right) \quad , \quad [f_{V, 5}]^{i, jk} = -\frac{ig_a}{4} H_2^{lijk} \frac{q^l}{m_e} \quad , \quad [f_{V, 6}]^{i, jkl} = -\frac{g_a}{2} H_1^{ijkl}$$



$$\Rightarrow i \sum_{\ell} [f_{V \mathbf{A}, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$[f_{V \mathbf{A}, 4}]^{0i, j} = \frac{eg_a}{m_e} \delta^{ij}$$

$$[f_{V \mathbf{A}, 1}]^{ij} = \frac{ieg_a}{4m_e^2} \epsilon^{ijk} (p_1^k - p_2^k) \quad , \quad [f_{V \mathbf{A}, 4}]^{i, jk} = \frac{eg_a}{4m_e^2} (p_1^l - p_2^l) H_1^{iljk} \quad , \quad [f_{V \mathbf{A}, 5}]^{ij, kl} = -\frac{ieg_a}{2m_e} H_1^{ijkl}$$

TABLE VI. Summary of the NR EFT Interaction Lagrangian and Feynman rules generated by the UV interaction Lagrangian $\mathcal{L}_{\text{int}} = g_a V_\mu \bar{\Psi} \gamma^\mu \gamma^5 \Psi$. See Sec. IV to see how to find the f 's in terms of the C 's. Furthermore for readability we have defined, $H_1^{ijkl} \equiv \delta^{ik} \delta^{jl} - \delta^{ij} \delta^{kl}$, $H_2^{ijkl} \equiv \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} - 2\delta^{il} \delta^{jk}$.

NR QED Interaction Lagrangian

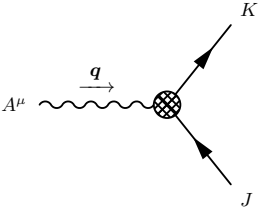
$$\mathcal{L}_{\text{int}}^{\text{NR}} = \sum_{\ell} \left[C_{A_0, \ell} A^0 + C_{\mathbf{A}, \ell} \mathbf{A} + C_{(\nabla A^0), \ell} \frac{(\nabla A^0)}{m_e} + C_{(\partial_t \mathbf{A}), \ell} \frac{(\partial_t \mathbf{A})}{m_e} + C_{(\nabla \mathbf{A}), \ell} \frac{(\nabla \mathbf{A})}{m_e} + C_{AA, \ell} \frac{\mathbf{A} \mathbf{A}}{m_e} \right. \\ \left. + C_{(\partial_t \nabla \mathbf{A}), \ell} \frac{(\partial_t \nabla \mathbf{A})}{m_e^2} + C_{(\nabla \nabla A^0), \ell} \frac{(\nabla \nabla A^0)}{m_e^2} + C_{(\nabla A^0) \mathbf{A}, \ell} \frac{(\nabla A^0) \mathbf{A}}{m_e^2} \right] [\psi^\dagger \hat{O}_\ell \psi]$$

$$C_{A_0, 1} = -e \quad , \quad [C_{\mathbf{A}, 2}]^{i, j} = -ie \delta^{ij} \quad , \quad [C_{\mathbf{A}, 8}]^{i, j, k} = \frac{e}{4} \epsilon^{ijk} \quad , \quad [C_{(\nabla A^0), 5}]^{i, j, k} = -\frac{e}{4} \epsilon^{ijk} \quad , \quad [C_{(\partial_t \mathbf{A}), 5}] = -\frac{e}{4} \epsilon^{ijk}$$

$$[C_{(\nabla \mathbf{A}), 1}]^{ij} = -\frac{ie}{2} \delta^{ij} \quad , \quad [C_{(\nabla \mathbf{A}), 4}]^{ij, k} = \frac{e}{2} \epsilon^{ijk} \quad , \quad [C_{AA, 1}]^{ij} = -\frac{e^2}{2} \delta^{ij} \quad , \quad [C_{(\partial_t \nabla \mathbf{A}), 1}]^{ij} = -\frac{e}{8} \delta^{ij}$$

$$[C_{(\partial_t \nabla \mathbf{A}), 4}]^{ij, k} = -\frac{ie}{8} \epsilon^{ijk} \quad , \quad [C_{(\nabla \nabla A^0), 1}]^{ij} = -\frac{e}{8} \delta^{ij} \quad , \quad [C_{(\nabla A^0) \mathbf{A}, 4}]^{ij, k} = \frac{e^2}{4} \epsilon^{ijk}$$

NR Feynman Rules

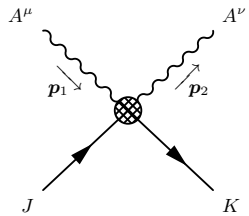


$$\Rightarrow i \sum_{\ell=1}^8 [f_{A, \ell}(\mathbf{q})]^\mu \hat{\mathcal{M}}_{JK, \ell}(\mathbf{q}).$$

$$[f_{A, 1}]^0 = -e \left(1 - \frac{\mathbf{q}^2}{8m_e^2} \right) \quad , \quad [f_{A, 5}]^{0, ij} = -\frac{ie}{4} \epsilon^{ijk} \frac{q^k}{m_e}$$

$$[f_{A, 1}]^i = \frac{e}{2} \frac{q^i}{m_e} \left(1 + \frac{\omega}{4m_e} \right) \quad , \quad [f_{A, 2}]^{i, j} = e \delta^{ij} \quad , \quad [f_{A, 4}]^{i, j} = \frac{ie}{2} \epsilon^{ijk} \frac{q^k}{m_e} \left(1 + \frac{\omega}{4m_e} \right) \quad [f_{A, 5}]^{i, j, k} = -\frac{i\omega}{4m_e} \epsilon^{ijk}$$

$$[f_{A, 8}]^{i, j, k} = -\frac{e}{4} \epsilon^{ijk}$$



$$\Rightarrow i \sum_{\ell=1}^8 [f_{AA, \ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK, \ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$[f_{AA, 1}]^{ij} = -\frac{e^2}{m_e} \delta^{ij} \quad , \quad [f_{AA, 4}]^{0i, j} = -\frac{e^2}{4m_e} \epsilon^{ijk} \frac{q^k}{m_e}$$

TABLE VII. Feynman rules for NR QED to order m_e^{-2} . The NR QED Lagrangian is derived in Sec. II. Note that four-point amplitude Feynman coefficients with identical fields is multiplied by a symmetry factor of two.

Dipole Interaction NR Limit

The high-energy/UV magnetic dipole interaction may be decomposed in to dark electric and magnetic fields (\mathbf{E}' , \mathbf{B}' , respectively) as

$$\mathcal{L}_{\text{int}} = \frac{d_M}{2} V_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi = id_M E'^i \bar{\Psi} \gamma^0 \gamma^i \Psi - \frac{id_M}{2} B'^k \bar{\Psi} [\epsilon^{kij} \gamma^i \gamma^j] \Psi, \quad (\text{A1})$$

where $V_{0i} = -E'_i$, $V_{ij} = \epsilon_{ijk} B'^k$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. The antisymmetric field strength tensor is written in terms of these two vector fields, and we construct the NR Lagrangian in terms of these coefficients

$$\begin{aligned} \mathcal{L}_{\text{int}}^{\text{NR}} = \frac{1}{m_e} \sum_{\ell=1}^8 & \left[C_{\mathbf{E}',\ell} \mathbf{E}' + C_{(\nabla \mathbf{E}'),\ell} \frac{(\nabla \mathbf{E}')}{m_e} + C_{\mathbf{E}'\mathbf{A},\ell} \frac{\mathbf{E}'\mathbf{A}}{m_e} + C_{\mathbf{E}'(\nabla A^0),\ell} \frac{\mathbf{E}'(\nabla A^0)}{m_e^2} + C_{\mathbf{E}'(\partial_t \mathbf{A}),\ell} \frac{\mathbf{E}'(\partial_t \mathbf{A})}{m_e^2} \right. \\ & + C_{\mathbf{B}',\ell} \mathbf{B}' + C_{(\nabla \mathbf{B}'),\ell} \frac{\nabla \mathbf{B}'}{m_e} + C_{(\nabla \nabla \mathbf{B}'),\ell} \frac{(\nabla \nabla \mathbf{B}')}{m_e^2} + C_{\mathbf{B}'\mathbf{A},\ell} \frac{\mathbf{B}'\mathbf{A}}{m_e} + C_{(\nabla \mathbf{B}')\mathbf{A},\ell} \frac{(\nabla \mathbf{B}')\mathbf{A}}{m_e^2} \\ & \left. + C_{\mathbf{B}'(\nabla \mathbf{A})} \frac{\mathbf{B}'(\nabla \mathbf{A})}{m_e^2} + C_{\mathbf{B}'\mathbf{A}\mathbf{A},\ell} \frac{\mathbf{B}'\mathbf{A}\mathbf{A}}{m_e^2} \right] [\psi^\dagger \hat{O}_\ell \psi]. \quad (\text{A2}) \end{aligned}$$

Where the coefficients are given by

$$\begin{aligned} [C_{\mathbf{E}',5}]^{i,jk} &= -id_M m_e \epsilon^{ijk}, \quad [C_{\mathbf{E}',7}]^{i,j} = -\frac{d_M m_e}{2} \delta^{ij}, \quad [C_{\nabla \mathbf{E}',1}]^{ij} = -\frac{1}{2} d_M m_e \delta^{ij}, \quad [C_{\nabla \mathbf{E}',4}]^{ij,k} = -\frac{1}{2} id_M m_e \epsilon^{ijk} \\ [C_{\mathbf{E}'\mathbf{A},4}]^{ij,k} &= ed_M m_e \epsilon^{ijk}, \quad [C_{\mathbf{E}'(\partial_t \mathbf{A}),1}]^{ij} = -\frac{1}{2} ed_M m_e \delta^{ij}, \quad [C_{\mathbf{E}'(\nabla A^0),1}]^{i,j} = -\frac{1}{2} ed_M m_e \delta^{ij} \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} [C_{\mathbf{B}',4}]^i &= -d_M m_e \delta^{ij}, \quad [C_{\mathbf{B}',6}]^{i,jkl} = -\frac{1}{2} d_M m_e \delta^{ik} \delta^{jl}, \quad [C_{\nabla \mathbf{B}',2}]^{ij,k} = -\frac{1}{4} id_M m_e \epsilon^{ijk} \\ [C_{\nabla \mathbf{B}',5}]^{ij,kl} &= -\frac{1}{4} d_M m_e (\delta^{ik} \delta^{jl} + \delta^{ij} \delta^{kl}), \quad [C_{\nabla \nabla \mathbf{B}',4}]^{ijk,l} = -\frac{1}{8} d_M m_e \delta^{ij} \delta^{kl} \\ [C_{\mathbf{B}'\mathbf{A},5}]^{ij,kl} &= \frac{1}{2} ied_M m_e (\delta^{il} \delta^{jk} + \delta^{ij} \delta^{kl}), \quad [C_{(\nabla \mathbf{B}')\mathbf{A},1}]^{ijk} = -\frac{1}{4} ed_M m_e \epsilon^{ijk} \\ [C_{(\nabla \mathbf{B}')\mathbf{A},4}]^{ijk,l} &= \frac{1}{4} ied_M m_e (\delta^{il} \delta^{jk} + \delta^{ij} \delta^{kl}), \quad [C_{\mathbf{B}'(\nabla \mathbf{A}),4}]^{ijk,l} = \frac{1}{4} ied_M m_e (\delta^{ik} \delta^{jl} + \delta^{ij} \delta^{kl}) \\ [C_{\mathbf{B}'\mathbf{A}\mathbf{A},4}]^{ijk,l} &= \frac{1}{2} e^2 d_M m_e \delta^{ik} \delta^{jl} \end{aligned} \quad (\text{A4})$$

Since \mathbf{E}' is related to V^0 and V^i , we can write the coefficients of the expansion of the NR Lagrangian in terms of those in the \mathbf{E}' , \mathbf{B}' expansion. For instance,

$$C_{(\nabla V^0)\dots,\ell} = C_{(\partial_t \mathbf{V})\dots,\ell} = -C_{\mathbf{E}\dots} \quad (\text{A5})$$

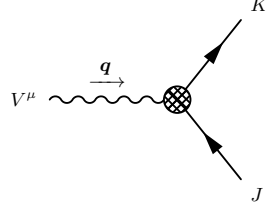
$$C_{(\nabla \mathbf{V})\dots,\ell} = \varepsilon^{ijk} [C_{\mathbf{B}\dots,\ell}]^{k\dots} \quad (\text{A6})$$

Where the ellipses indicate the possibility of additional photon terms. See Table VIII for the Feynman rule coefficients.

High-Energy Interaction Lagrangian

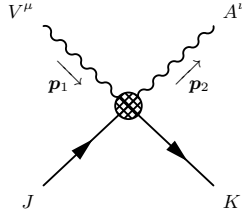
$$\mathcal{L}_{\text{int}} = \frac{d_M}{2} V_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi$$

NR Feynman Rules



$$\Rightarrow i \sum_{\ell=1}^8 f_{V,\ell}(\mathbf{q}) \hat{\mathcal{M}}_{JK,\ell}(\mathbf{q}).$$

$$\begin{aligned} [f_{V,1}]^0 &= -\frac{d_M \mathbf{q}^2}{2m_e} \quad , \quad [f_{V,5}]^{0,ij} = -d_M \epsilon^{ijk} q^k \quad , \quad [f_{V,7}]^{0,i} = -\frac{d_M}{2} q^i \\ [f_{V,1}]^i &= -d_M \frac{\omega q^i}{2m_e} \quad , \quad [f_{V,2}]^i = -\frac{id_M}{4m_e} (q^i q^j - \delta^{ij} q^2) \quad , \quad [f_{V,4}]^{i,j} = \frac{id_M}{4} \epsilon^{ijk} q^k \left(1 - \frac{2\omega}{m_e} - \frac{\mathbf{q}^2}{2m_e^2} \right) \\ [f_{V,5}]^{i,jk} &= -d_M \left(\omega \epsilon^{ijk} + \frac{q^j \epsilon^{ijk} q^k}{4m_e} \right) \quad , \quad [f_{V,6}]^{i,jkl} = \frac{id_M}{4} q^m \left(\delta^{jl} \epsilon^{ikm} + \delta^{jk} \epsilon^{ilm} \right) \quad , \quad [f_{V,7}]^{i,j} = -\frac{id_M \omega}{2} \delta^{ij} \end{aligned}$$



$$\Rightarrow i \sum_{\ell=1}^8 [f_{VA,\ell}(\mathbf{p}_1, \mathbf{p}_2)]^{\mu\nu} \hat{\mathcal{M}}_{JK,\ell}(\mathbf{p}_1 - \mathbf{p}_2).$$

$$\begin{aligned} [f_{VA,1}]^{00} &= \frac{ed_M}{2m_e^2} (\mathbf{p}_1 \cdot \mathbf{p}_2) \quad , \quad [f_{VA,1}]^{0i} = \frac{ed_M}{2m_e^2} \omega_2 p_1^i \quad , \quad [f_{VA,4}]^{0i,j} = \frac{ied_M}{m_e} \epsilon^{ijk} p_2^k \quad , \quad [f_{VA,1}]^{i0} = \frac{ed_M}{2m_e^2} \omega_1 p_2^i \\ [f_{VA,1}]^{ij} &= \frac{ed_M}{4m_e^2} \left(p_1^i p_1^j - \delta^{ij} (\mathbf{p}_1^2 - 2\omega_1 \omega_2) \right) \quad , \quad [f_{VA,4}]^{ij,k} = \frac{ied_M}{4m_e^2} \left(\epsilon^{ijm} (p_2^k p_1^m - p_2^m p_1^k) + \delta^{ij} \epsilon^{iml} p_2^m p_1^l + 4\omega_1 \epsilon^{ijk} m_e \right) \\ [f_{VA,5}]^{ij,kl} &= -\frac{ed_M}{2m_e} \left(\delta^{kl} \epsilon^{ijm} + \delta^{jk} \epsilon^{ilm} \right) p_1^m \end{aligned}$$

TABLE VIII. Summary of the Feynman rules generated by the UV interaction Lagrangian $\mathcal{L}_{\text{int}} = d_M V_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi / 2$. See App. A for the NR EFT interaction Lagrangian.

Appendix B: In-Medium Photon Propagator

In the absence of interactions, the photon propagator is

$$G_{\mu\nu}^0 = -\frac{i \eta_{\mu\nu}}{q^2}, \quad (\text{B1})$$

where $q^\mu = (\omega, \mathbf{q})$ is a momentum four-vector, and we have neglected gauge-dependent contributions which vanish from physical processes due the Ward identity (WI). In the presence of interactions the photon propagator is modified. The modification to the propagator due to virtual insertions of 1PI diagrams, can be written as the infinite sum



where the left-hand side is the resummed propagator, and the right-hand side is contribution from many 1PI diagrams (represented by dotted blobs).

Computing the resummed propagator is aided with a different representation of the photon propagator

$$G_{\mu\nu}^0 = i \sum_{\lambda} \frac{\epsilon_{\mu}^{\lambda} \epsilon_{\nu}^{\lambda}}{q^2}, \quad (\text{B2})$$

where ϵ_{λ}^{μ} are three polarization vectors satisfying

$$\epsilon_{\pm}^{\mu} = (0, \hat{\mathbf{q}}_{\pm}) \quad , \quad \epsilon_L^{\mu} = \frac{1}{\sqrt{q^2}}(|\mathbf{q}|, \omega \hat{\mathbf{q}}) \quad , \quad q_{\mu} \epsilon_{\lambda}^{\mu} = 0 \quad , \quad \epsilon_{\lambda}^{\mu} \epsilon_{\mu}^{\lambda'} = -\delta_{\lambda\lambda'} \quad , \quad \sum_{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu} = -\eta^{\mu\nu} + \frac{q^{\mu} q^{\nu}}{q^2}, \quad (\text{B3})$$

where $\hat{\mathbf{q}}_{\pm}$ are any two vectors mutually orthonormal to $\hat{\mathbf{q}}$. The propagator in Eq. (B2) is physically equivalent to that in Eq. (B1), since the terms proportional to q^{μ} vanish due to the WI. The 1PI diagram is the photon self-energy, Π_{AA} , and can be approximately decomposed in to components along each polarization vector¹³

$$\Pi_{AA}^{\mu\nu} = - \sum_{\lambda} \Pi_{AA}^{\lambda} \epsilon_{\lambda}^{\mu} \epsilon_{\lambda}^{\nu} \quad , \quad \Pi_{AA}^{\lambda} = -\epsilon_{\mu}^{\lambda} \Pi_{AA}^{\mu\nu} \epsilon_{\nu}^{\lambda}. \quad (\text{B4})$$

Note that this Π_{AA} is referred to as Π_{AA}^{UV} in the main text. The resummed propagator is then computed as

$$G_{\mu\nu} = G_{\mu\nu}^0 + G_{\mu\alpha}^0 \left(i \Pi_{AA}^{\alpha\beta} \right) G_{\beta\nu}^0 + \dots = \left(i \sum_{\lambda} \frac{\epsilon_{\mu}^{\lambda} \epsilon_{\nu}^{\lambda}}{q^2} \right) + \left(i \sum_{\lambda} \frac{\epsilon_{\mu}^{\lambda} \epsilon_{\alpha}^{\lambda}}{q^2} \right) \left(i \Pi_{AA}^{\alpha\beta} \right) \left(i \sum_{\lambda} \frac{\epsilon_{\beta}^{\lambda} \epsilon_{\nu}^{\lambda}}{q^2} \right) + \dots = i \sum_{\lambda} \frac{\epsilon_{\mu}^{\lambda} \epsilon_{\nu}^{\lambda}}{q^2 - \Pi_{AA}^{\lambda}}. \quad (\text{B5})$$

As discussed in Sec. IV A in a more general context momentum conservation need not apply, and therefore the resummed propagator may be a function of both the incoming and outgoing momentum, \mathbf{q}, \mathbf{q}' , respectively. However the non-interacting propagator should be unchanged and we can write it as a function of the incoming and outgoing momentum as

$$G_{\mu\nu}^0(\mathbf{q}, \mathbf{q}') = G_{\mu\nu}^0(\mathbf{q}) \delta_{\mathbf{q}\mathbf{q}'} = G_{\mu\nu}^0(\mathbf{q}) \frac{(2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}')}{\mathcal{V}} \quad (\text{B6})$$

¹³ In complicated materials the polarization vectors in Eq. (B3) may not allow for $\Pi_{AA}^{\mu\nu}$ to be decomposed directly as in Eq. (B4); there may be terms mixing the transverse and longitudinal polarization components. However, there is always a basis of polarization vectors (a linear combination of those in Eq. (B3)) which allow for a polarization decomposition as in Eq. (B4). The rest of the derivation in more complicated materials follows analogously once the diagonalizing basis of polarization vectors is used.

Following the Feynman rules discussed in Sec. IV we compute the NR resummed propagator as

$$\begin{aligned} G_{\mu\nu}^{\text{NR}}(\mathbf{q}, \mathbf{q}') &= G_{\mu\nu}^{\text{NR},0}(\mathbf{q}, \mathbf{q}') + \int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \frac{d^3\mathbf{q}_2}{(2\pi)^3} (G_{\mu\alpha}^{\text{NR},0}(\mathbf{q}, \mathbf{q}_1)) \left(i\Pi_{AA}^{\text{NR},\alpha\beta}(\mathbf{q}_1, \mathbf{q}_2) \right) \left(G_{\beta\nu}^{\text{NR},0}(\mathbf{q}_2, \mathbf{q}') \right) + \dots \\ &= (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') G_{\mu\nu}^0(\mathbf{q}) + G_{\mu\alpha}^0(\mathbf{q}) \left(i\Pi_{AA}^{\text{NR},\alpha\beta}(\mathbf{q}, \mathbf{q}') \right) G_{\beta\nu}^0(\mathbf{q}') + \dots, \end{aligned} \quad (\text{B7})$$

where $G_{\mu\nu}^{\text{NR},0}(\mathbf{q}, \mathbf{q}') = (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') G_{\mu\nu}^0(\mathbf{q})$ connects the NR propagator diagram to the usual relativistic quantity. If the 1PI diagram can be approximated as

$$\Pi_{AA}^{\text{NR},\mu\nu}(\mathbf{q}, \mathbf{q}') \approx \Pi_{AA}^{\text{NR},\mu\nu}(\mathbf{q}) \delta_{\mathbf{q}\mathbf{q}'} = \Pi_{AA}^{\text{NR},\mu\nu}(\mathbf{q}) \frac{(2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}')}{\mathcal{V}}, \quad (\text{B8})$$

as discussed in Sec. IV F, then the propagator in Eq. (B7) can be resummed as it was previously, using $\Pi_{AA}^{\text{NR},\mu\nu}(\mathbf{q}) = \mathcal{V}\Pi_{AA}^{\mu\nu}(\mathbf{q})$,

$$\begin{aligned} G_{\mu\nu}^{\text{NR}}(\mathbf{q}, \mathbf{q}') &\approx (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') \left(G_{\mu\nu}^0(\mathbf{q}) + G_{\mu\alpha}^0(\mathbf{q}) \left(i\Pi_{AA}^{\alpha\beta}(\mathbf{q}) \right) G_{\beta\nu}^0(\mathbf{q}) + \dots \right) \\ &= (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{q}') \left(i \sum_{\lambda} \frac{\epsilon_{\mu}^{\lambda} \epsilon_{\nu}^{\lambda}}{q^2 - \Pi_{AA}^{\lambda}} \right), \end{aligned} \quad (\text{B9})$$

where the momentum conserving delta function is usually cancelled if the Feynman rule was defined in the usual way, i.e., if the left-hand side of Eq. (B9) had a momentum-conserving delta function. Note that in this approximation one can either explicitly insert $G_{\mu\nu}^{\text{NR}}(\mathbf{q}, \mathbf{q}')$ in to a diagram and integrate over the extra undetermined momentum, or require momentum conservation and insert only the final term in the brackets in Eq. (B9) (which is $G_{\mu\nu}$ in Eq. (B5)), ignoring the momentum conserving delta function.

The *in-medium* photon propagator is the resummed photon propagator in Eq. (B9) with the 1PI diagrams evaluated in a medium (as opposed to a vacuum in the usual relativistic QFT scenario). The in-medium photon propagator is usually written in terms of the dielectric function, $\epsilon(\mathbf{q}, \omega)$. The relationship between the projected photon self-energies, Π_{AA}^{λ} and ϵ is usually made using the constitutive relationships of electrodynamics and equations of motion for A^{μ} , $\partial_{\nu} F^{\mu\nu} = -\Pi_{AA}^{\mu\nu} A_{\nu}$, from which we can identify $-\Pi_{AA}^{\mu\nu} A_{\nu}$ as an effective four-current, $J^{\mu} = (J^0, \mathbf{J})$

$$\mathbf{J} = \boldsymbol{\sigma} \mathbf{E} \quad \iff \quad J^{\mu} = -\Pi_{AA}^{\mu\nu} A_{\nu} \quad (\text{B10})$$

where $\boldsymbol{\sigma}$ is the conductivity, related to the dielectric by $\boldsymbol{\sigma} = i\omega(1 - \epsilon)$. Since $E^i \supset -\partial_t A^i = i\omega A^i$ we can identify

$$\Pi_{AA}^{ij} = -\omega^2(1 - \epsilon^{ij}). \quad (\text{B11})$$

Therefore the transverse projected self-energies are

$$\Pi_{AA}^{\pm} = -\epsilon_{\mu}^{\pm} \Pi_{AA}^{\mu\nu} \epsilon_{\nu}^{\pm} = \omega^2(1 - \hat{\mathbf{q}}_{\pm} \cdot \boldsymbol{\epsilon} \cdot \hat{\mathbf{q}}_{\pm}). \quad (\text{B12})$$

The longitudinal projected self-energies can be written in terms of Π_{AA}^{00}

$$\Pi_{AA}^L = -\epsilon_{\mu}^L \Pi_{AA}^{\mu\nu} \epsilon_{\nu}^L = -\frac{q^2}{|\mathbf{q}|^2} \Pi_{AA}^{00}, \quad (\text{B13})$$

which, using the WI, $q_{\mu} \Pi_{AA}^{\mu\nu} = \Pi_{AA}^{\nu\mu} q_{\mu} = 0$, can be written in terms of Π_{AA}^{ij}

$$\Pi_{AA}^{00} = \frac{\mathbf{q}^i \Pi_{AA}^{ij} \mathbf{q}^j}{\omega^2}. \quad (\text{B14})$$

Therefore,

$$\Pi_{AA}^L = q^2(1 - \hat{\mathbf{q}} \cdot \boldsymbol{\epsilon} \cdot \hat{\mathbf{q}}). \quad (\text{B15})$$

The in-medium photon propagator is then written in terms of the dielectric as

$$G^{\mu\nu} = i \left(\frac{\epsilon_L^\mu \epsilon_L^\nu}{q^2 (\hat{\mathbf{q}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}})} + \frac{\epsilon_+^\mu \epsilon_+^\nu}{\omega^2 (\hat{\mathbf{q}}_+ \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}_+) - q^2} + \frac{\epsilon_-^\mu \epsilon_-^\nu}{\omega^2 (\hat{\mathbf{q}}_- \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}_-) - q^2} \right). \quad (\text{B16})$$

This can be further simplified in the limit of scattering or absorption kinematics, as considered in Secs. VI, V respectively,

$$G^{\mu\nu} = \begin{cases} \frac{i}{\omega^2} \left(\frac{\epsilon_{\hat{\mathbf{q}}}^\mu \epsilon_{\hat{\mathbf{q}}}^\nu}{\hat{\mathbf{q}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}} + \frac{\epsilon_+^\mu \epsilon_+^\nu}{\hat{\mathbf{q}}_+ \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}_+} + \frac{\epsilon_-^\mu \epsilon_-^\nu}{\hat{\mathbf{q}}_- \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}_-} \right) & \omega \gg |\mathbf{q}| \text{ (absorption)} \\ \frac{i}{q^2} \left(\frac{\epsilon_0^\mu \epsilon_0^\nu}{\hat{\mathbf{q}} \cdot \boldsymbol{\varepsilon} \cdot \hat{\mathbf{q}}} - \epsilon_+^\mu \epsilon_+^\nu - \epsilon_-^\mu \epsilon_-^\nu \right) & \omega \ll |\mathbf{q}| \text{ (scatter)} \end{cases}, \quad (\text{B17})$$

where $\epsilon_{\hat{\mathbf{q}}}^\mu \equiv (0, \hat{\mathbf{q}})$ and $\epsilon_0^\mu \equiv (1, 0, 0, 0)$ are limits of ϵ_L^μ . In the limit of an isotropic target ($\varepsilon^{ij} = \delta^{ij} \varepsilon$) these can be simplified even further to

$$G^{\mu\nu} = \begin{cases} \frac{i}{\varepsilon \omega^2} P^{\mu\nu} & \omega \gg |\mathbf{q}| \text{ (absorption)} \\ \frac{i}{q^2} \left(\frac{\epsilon_0^\mu \epsilon_0^\nu}{\varepsilon} - \epsilon_+^\mu \epsilon_+^\nu - \epsilon_-^\mu \epsilon_-^\nu \right) & \omega \ll |\mathbf{q}| \text{ (scatter)}, \end{cases} \quad (\text{B18})$$

where $P^{\mu\nu} = \text{diag}(0, 1, 1, 1)$.

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