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A model-independent description of $B \to D\pi\ell\nu$ decays

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We introduce a new parameterization of $B \to D\pi\ell\nu$ form factors using a partial-wave expansion and derive bounds on the series coefficients using analyticity and unitarity. This is the first generalization of the model-independent formalism developed by Boyd, Grinstein, and Lebed for $B \to D\ell\nu$ to semileptonic decays with multi-hadron final states, and enables data-driven form-factor determinations with robust, systematically-improvable uncertainties. Using this formalism, we extract the form-factor parameters for $B \to D_2^*(\to D\pi)\ell\nu$ decays in a model-independent way from fits of data from the Belle Experiment, and, for the first time, study the two-pole structure in the $D\pi$ S-wave in semileptonic decays employing lineshapes from unitarized chiral perturbation theory.

Motivation — Experimental measurements of tree-level semileptonic B-meson decays enable theoretically clean determinations of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $|V_{ub}|$ and $|V_{cb}|$, allowing for sensitive tests of the Standard Model by overconstraining the CKM unitarity triangle [1–3]. Further, $|V_{ub}|$ and $|V_{cb}|$ are parametric inputs to predictions for loop-level flavor-changing processes that are sensitive to new high-scale physics beyond the reach directly detectable by the LHC [4, 5].

A major challenge for both inclusive and exclusive determinations of $|V_{ub}|$ is suppressing the CKM-favored $B \to X_c\ell\nu$ background, which exhibits a similar experimental signature and is $\mathcal{O}(100)$ times more abundant than $B \to X_u\ell\nu$ decays. The background subtraction process is further complicated by the orbitally excited states, collectively referred to as D^{**}, whose kinematic distributions remain poorly understood and branching fractions exhibit uncertainties of approximately 20% [6]. In measurements performed by the Belle and Belle II Collaborations, the remaining "gap" between the sum of all considered exclusive modes and the inclusive $B \to X\ell\nu$ branching fraction, comprising unmeasured non-resonant $B \to X_c\ell\nu$ decays, is generally treated in simulation by assuming a composition of equal parts of $B \to D^{(*)}\eta\ell\nu$ decays, as prescribed in Ref. [7]. Because neither experimental evidence nor theoretical predictions exist for $B \to D^{(*)}\eta\ell\nu$ decays, a 100% uncertainty is assumed for the corresponding branching fractions. For these reasons, the $X_c\ell\nu$ modeling uncertainty is hard to quantify and becomes dominant for studies of inclusive $B \to X_{c/u}\ell\nu$ decays [7–11].

Exclusive measurements relying on tagged methods, in which machine learning algorithms are employed to fully reconstruct the companion B meson through exclusive decay modes [12, 13], do not rely as directly on $X_c\ell\nu$ modeling as inclusive analyses. However, significant differences in these reconstruction algorithms' performances between data and simulation is accounted for by performing a calibration using a decay with a well known branching fraction: inclusive $B \to X\ell\nu$ [14]. This calibration, in turn, becomes a leading source of systematic error for tagged analyses [15–17]. In addition, the limited knowledge of $B \to D^{**}\ell/\tau\nu$ branching fractions and form factors are large systematic uncertainties in studies of rare processes such as $B \to K\nu\nu$ at Belle II or $R(D^*)$ at the LHCb experiment [18, 19].

The most commonly used description of $B \to D^{**}\ell\nu$ decays is the Leibovich-Ligeti-Stewart-Wise (LLSW) parameterization [20, 21] with central values from the fit given in Refs. [22, 23]. This parameterization includes a single D_0^* resonance. Studies in the context of unitarized chiral perturbation theory (UChPT), however, have shown that the scalar member of the D^{**} family, the $D_0^*(2300)$, is an overlap of two states with poles near $(2.1 - i0.1)$ and $(2.45 - i0.13)$ GeV [24–26]. Consequently, the S-wave lineshape is not described by a simple Breit-Wigner distribution, but has a more complex structure. This conclusion is supported by lattice quantum chromodynamics (LQCD) results from a coupled-channel analysis performed by the Hadron Spectrum Collaboration [27] and a reinterpretation [28] of the partial-wave analysis of $B^+ \to D^-\pi^+\pi^+$ decays by LHCb [29]. Further, in Ref. [30], Le Yaouanc, Leroy, and Roudeau point out that in the fits to the LLSW parametrization tail effects from the D^* resonance are omitted and consequently overestimates the D_0^* contribution.

To address these and other limitations of existing parameterizations, in this letter we develop the first model-independent description of resonant and non-resonant $B \to D\pi\ell\nu$ decays based on analyticity and unitarity. We then apply our formalism, which accommodates arbitrary lineshapes, to fit experimental spectrum measurements and draw conclusions about the pole structure of the S-wave channel.

Form-factor parameterization — Semileptonic $B \to D\pi\ell\nu$ decays are characterized by five kinematic variables: the momentum transfer square q^2,[1] the helicity angle of the charged lepton $\cos\theta_l$, the helicity angle of the D meson $\cos\theta$, the azimuthal angle between the $\ell\nu$ and $D\pi$ planes χ, and the invariant mass of the hadronic system $M_{D\pi}$.

Form-factor decompositions for charged-current semileptonic decays involving two final state hadrons have been performed for $B\to\pi\pi\ell\nu$ decays [31] and involve a partial-wave decomposition in $\cos\theta$ to disentangle contributions from different hadronic resonances. Following a similar strategy, we express the $B\to D\pi\ell\nu$ hadronic matrix elements as

$$\begin{aligned}\langle D(p_D)\pi(p_\pi)|V_\mu|B(p_B)\rangle &= \frac{2i}{M_B+M_{D\pi}}\epsilon_{\mu\nu\rho\sigma}p^\rho_{D\pi}p^\sigma_B\sum_{l>0}L^{(l),\nu}V^{(l)}(q^2,M^2_{D\pi})\,,\\ \langle D(p_D)\pi(p_\pi)|A_\mu|B(p_B)\rangle &= M_{D\pi}\frac{q^\mu}{q^2}\sum_{l\geq 0}L^{(l),\nu}q_\nu A_0^{(l)}(q^2,M^2_{D\pi})+\sum_{l>0}\left(L^{(l),\mu}-\frac{L^{(l),\nu}q_\nu}{q^2}q^\mu\right)A_1^{(l)}(q^2,M^2_{D\pi})\\ &\quad+\left[(p_B+p_{D\pi})^\mu-\frac{M_B^2-M^2_{D\pi}}{q^2}q^\mu\right]\sum_{l\geq 0}L^{(l),\nu}q_\nu A_2^{(l)}(q^2,M^2_{D\pi})\,.\end{aligned}\tag{1}$$

The vector $L^{(l)}$ is related to the angular momentum of the final-state hadron system in the B-meson rest frame and fulfills

$$L^{(l)}_\mu p^\mu_{D\pi}=0,\;\; L^{(0)}_\mu q^\mu=1,\;\; L^{(l)}_\mu q^\mu=M_BW^lP_l(\cos\theta),\tag{2}$$

where $W=|\vec{q}||\vec{p_D}|/(M_BM_{D\pi})$ and P_l are the Legendre polynomials.

For $l=0$ only two form factors — f_0 and f_+ — contribute ($V^{(l)}$ and $A_1^{(l)}$ do not contribute to S-wave decays since they directly couple the weak current to the angular momentum of the hadronic system), and we define them as Boyd, Grinstein, and Lebed (BGL) do for $B\to D\ell\nu$ decays [32–34]:

$$A_0^{(0)}=\frac{M_B^2-M^2_{D\pi}}{M_{D\pi}}f_0\,,\quad A_2^{(0)}=f_+\,.\tag{3}$$

For $l>0$, we re-write the form factors in Eq. (1) analogously:

$$\begin{aligned}V^{(l)}&=\frac{M_B+M_{D\pi}}{2}g_l\,,\quad A_0^{(l)}=\mathcal{F}_{2,l}\,,\quad A_1^{(l)}=\frac{1}{2}f_l\,,\\ A_2^{(l)}&=\frac{1}{\lambda_B}\left[M_{D\pi}(M_B^2-M^2_{D\pi})\mathcal{F}_{1,l}-(p_{D\pi}\cdot q)f_l\right]\,,\end{aligned}\tag{4}$$

where the threshold factor $\lambda_B=M_B^4+M_{D\pi}^4+q^4-2(M_B^2M^2_{D\pi}+M^2_{D\pi}q^2+q^2M_B^2)$. With this choice (and a suitably defined polarization vector $L^{(l)}_\mu$), the $l=1$ and $l=2$ terms match the standard expressions for $B\to D^*\ell\nu$ and $B\to D_2^*\ell\nu$ decays, respectively [35, 36].

Using Eqs. (1), (3), and (4), it is straightforward to derive the $B\to D\pi\ell\nu$ differential decay rate. After integrating over all angles, the double differential decay rate for massless leptons can be written on a single line:

$$\frac{d^2\Gamma}{dM^2_{D\pi}dq^2}=\frac{G_F^2|V_{cb}|^2}{(4\pi)^5}M_B\left(W\frac{\lambda_B}{M_B^2}\frac{4|f_+|^2}{3}+M^2_{D\pi}\sum_{l>0}W^{2l+1}\left[\frac{4(M_B^2-M^2_{D\pi})^2}{3(2l+1)}\frac{|\mathcal{F}_{1,l}|^2}{\lambda_B}+\frac{l(l+1)}{(2l+1)}q^2\left(|g_l|^2+\frac{|f_l|^2}{\lambda_B}\right)\right]\right)\,.\tag{5}$$

The fully general five-fold differential decay rate allowing for interference effects between different partial waves is provided in the supplemental material [37].

Unitarity bounds — Model-independent constraints on the $B\to D\pi\ell\nu$ form factors arise from analyticity and unitarity. We begin with the two-point functions

$$\Pi^{L/T}_{(J)}(q)\equiv i\int d^4x\;e^{iq\cdot x}\;\langle 0|\,J^{L/T}(x)\,J^{L/T}(0)\,|0\rangle\,,\tag{6}$$

where J^μ denotes a $b\to c$ flavor-changing vector or axial-vector current and L/T denotes the component longitudinal or transverse to q^μ. Susceptibilities $\chi^{L/T}_{(J)}$ are defined from derivatives of $\Pi^{L/T}_{(J)}(q)$ as

$$\begin{aligned}\chi^L_{(J)}(Q^2)&\equiv\frac{\partial\Pi^L_{(J)}}{\partial q^2}\Big|_{q^2=Q^2}=\frac{1}{\pi}\int_0^\infty dq^2\frac{\mathrm{Im}\,\Pi^L_{(J)}(q^2)}{(q^2-Q^2)^2}\,,\\ \chi^T_{(J)}(Q^2)&\equiv\frac{1}{2}\frac{\partial^2\Pi^T_{(J)}}{\partial(q^2)^2}\Big|_{q^2=Q^2}=\frac{1}{\pi}\int_0^\infty dq^2\frac{\mathrm{Im}\,\Pi^T_{(J)}(q^2)}{(q^2-Q^2)^3}\,,\end{aligned}\tag{7}$$

and are related to integrals over the imaginary part of

[1] It is sometimes useful to instead consider the dependence on the recoil parameter $w=(M_B^2+M^2_{D\pi}-q^2)/(2M_BM_{D\pi})$.

$\Pi^{L/T}_{(J)}(q)$ via dispersion relations. The susceptibilities $\chi^{L/T}_{(V/A)}$ are perturbatively calculable for spacelike q^2 and have been computed to $\mathcal{O}(\alpha_S^2)$ at $Q^2=0$ in Ref. [38].

Separately, the optical theorem relates $\mathrm{Im}\,\Pi^T_{(J)}(q^2)$ to a sum of squared amplitudes for all intermediate states that can appear between the currents in Eq. (6). This sum includes terms with the matrix element $\langle \overline{B} D\pi | J^{L/T} | 0\rangle$, which is related to $|\langle D\pi | J^{L/T} | B\rangle|^2$ by crossing symmetry and can therefore be parameterized by the form-factor decomposition in Eq. (1). The contribution of the $B\to D\pi\ell\nu$ channel to the dispersion relations in Eq. (7) is then given by evaluating the phase space integrals arising in the sum over states,

$$\begin{aligned}
\mathrm{Im}\,\Pi^L_A\Big|_{D\pi} &= \frac{1}{32\pi^3}\frac{M_B^4}{q^4}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\left(M^2_{D\pi}\sum_{l>0}\frac{W^{2l+1}}{2l+1}|\mathcal{F}_2^{(l)}|^2+\frac{(M_B^2-M^2_{D\pi})^2}{M_B^2}|f_0|^2\right)\,,\\
\mathrm{Im}\,\Pi^T_A\Big|_{D\pi} &= \frac{1}{96\pi^3}\frac{M_B^4}{q^2\lambda_B}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\left(M^2_{D\pi}\sum_{l>0}\frac{W^{2l+1}}{2l+1}\left(\frac{|\mathcal{F}_1^{(l)}|^2}{q^2}+\frac{l+1}{l}|f^{(l)}|^2\right)+\frac{|f_+|^2}{q^2M_B^2}\right)\,,\\
\mathrm{Im}\,\Pi^T_V\Big|_{D\pi} &= \frac{1}{96\pi^3}\frac{M_B^4}{q^2}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\,M^2_{D\pi}\sum_{l>0}W^{2l+1}\frac{l+1}{l(2l+1)}|g^{(l)}|^2\,.
\end{aligned} \tag{8}$$

Positivity of the squared amplitudes appearing in the sum over states implies $\mathrm{Im}\,\Pi^{L/T}_J\Big|_{D\pi}\le \mathrm{Im}\,\Pi^{L/T}_J$. Inequalities for the $B\to D\pi\ell\nu$ form factors can then be derived from this inequality by inserting Eq. (8) and the perturbative expression for $\mathrm{Im}\,\Pi^{L/T}_J$. These so-called unitarity bounds provide q^2-dependent constraints that should be incorporated in determinations of the $B\to D\pi\ell\nu$ form factors. Conveniently, the unitarity bounds apply only to either the single form factor $\mathcal{F}_2^{(l)}$, the pair of form factors $\mathcal{F}_1^{(l)}/f^{(l)}$, or the single form factor $g^{(l)}$ rather than more general linear combinations. A parameterization of the q^2-dependence of the form factors is required to concretely specify how the bounds are imposed; we turn to this next.

Coupled-channel S-wave and z-expansion — The model-independent parameterization and bounds presented in previous sections make no assumptions about the number, energies, or lineshapes of possible resonances. To render fitting the measured $B\to D\pi\ell\nu$ decay spectra to our parameterization more tractable, it is helpful to include additional theoretical information and make some plausible assumptions.

The semileptonic B-decay form factors can be factorized into a part describing the short-distance weak decay and a part encoding the long-ranged final-state interactions between the hadrons [39, 40]:

$$f^{(l)}(q^2,M^2_{D\pi})=\hat{f}^{(l)}(q^2,M^2_{D\pi})g^{(l)}(M^2_{D\pi})\,. \tag{9}$$

The weak-interaction contribution to the form factors of QCD resonances is approximately independent of $M_{D\pi}$ [41]:

$$\hat{f}^{(l)}(q^2,M^2_{D\pi})\approx\tilde{f}^{(l)}(q^2)+\mathcal{O}((M_R^2-M^2_{D\pi})/M_B^2)\,, \tag{10}$$

Indeed, studies of $B\to\pi\pi(K)$ in the context of light cone sum rules (LCSR) [42] and recent LQCD studies of the $B\to\rho$ form factors [43] point towards the smallness of the neglected contributions.

The interaction potentials relevant for coupled-channel S-wave $D\pi$, $D\eta$ and D_sK scattering have been determined to next-to-leading order in UChPT [44] and the S-matrix has been obtained from those results in Ref. [24]. To connect the semileptonic-decay form factors to the known S-matrix, we make use of the dispersion relation [45]

$$\begin{aligned}
&\mathrm{Im}\,\vec{f}(q^2,M^2_{D\pi}+i\epsilon)\\
&=T^*(M^2_{D\pi}+i\epsilon)\Sigma(M^2_{D\pi})\vec{f}(q^2,M^2_{D\pi}+i\epsilon)\,,
\end{aligned} \tag{11}$$

where the form-factor f is a vector in channel space and stands for either f_+ or f_0. Here, T is the T-matrix, and Σ contains the relevant phase-space factors and is defined in Ref. [45]. The solution of Eq. (11) is well known and given by the Muskhelishvili-Omnès (MO) matrix Ω [45, 46]

$$\begin{aligned}
&\vec{f}(q^2,M^2_{D\pi})=\Omega(M^2_{D\pi})\vec{P}(q^2,M^2_{D\pi})\,,\\
&\mathrm{Im}\,\Omega(s+i\epsilon)=\frac{1}{\pi}\int_{s_{\mathrm{thr}}}^{\infty}\frac{T^*(s')\Sigma(s')\Omega(s')}{s'-s-i\epsilon}\mathrm{d}s'\,,
\end{aligned} \tag{12}$$

where, $\vec{P}$ are boundary functions and we neglect higher order terms in $M_{D\pi}$ following the same arguments as for Eq. (10). A numerical algorithm to solve the above integral equation is outlined in Ref. [47].

The q^2-dependent remainder of a given form factor f defined in Eq. (10), as well as the boundary functions $\vec{P}$ in Eq. (12) can be expanded as [32–34]

$$\tilde{f}_l(q^2)=\frac{1}{\phi_l^{(f)}(q^2)B_f(q^2)}\sum_{i=0}^{\infty}a_{li}^{(f)}z^i \tag{13}$$

where B_f is a Blaschke product taking into account the poles of all subthreshold B_c resonances for a given channel and $\phi_l^{(f)}$ is the relevant outer function. Additional details on the derivation and numerical calculation of the outer functions can be found in the supplemental material [37] and Ref. [48]. The variable z is given by

$$z(q^2, q_0^2) = \frac{q_0^2 - q^2}{(\sqrt{q_+^2 - q^2} - \sqrt{q_+^2 - q_0^2})^2} \tag{14}$$

where $q_+^2 = (M_B + M_D + m_\pi)$ and ranges from 0 to 0.06 for $q_0^2 = 0$ GeV^2 . In terms of the coefficients $a_{li}^{(f)}$ the unitarity bounds in Eq. (8) take the form

$$\sum_{i,l} |a_{li}^{(f)}|^2 < 1 \ , \tag{15}$$

allowing for an easy integration of the form factors in a fit, including priors on the coefficients. Recently discussed problems associated to lower-lying branch cut at $q^2 = (M_B + M_D)^2$ can be incorporated as outlined in Refs. [49–52].

Experimental fits — To test our new $B \to D\pi\ell\nu$ form-factor description and extract the coefficients of the z expansion from data, we proceed in two steps. First, we fit the measured w- and $\cos\theta$-dependence of the $B \to D_2^*\ell\nu$ differential decay width [53] and $B^0 \to D_2^{*-}\pi^+$ branching fraction [6] constraining the $a_{li}^{(f)}$ with Gaussian priors centered at zero with unit width. We employ the least-squares fitting package `lsqfit` and use the augmented $\chi^2_{\rm aug}$ defined in Refs. [54, 55] to assess the goodness of fit. Additional numerical inputs are taken from Refs. [56–59] as discussed in the supplemental material [37]. The loose constraints help the fit converge more quickly but have little impact on the final results since the magnitudes of the resulting z-coefficients are all of order a tenth or smaller.

As shown in Fig. 1, we find a harder, *i.e.*, enhanced for high values of q^2, D_2^* w-spectrum than Refs. [22, 23] and also better describe the data. The most likely reason for this is the greater flexibility in the model independent approach employed here in comparison to the heavy-quark effective theory (HQET) based approach of these works.

Next, we fit the $B \to D\pi\ell\nu$ $M_{D\pi}$-spectrum measured recently by Belle [60] using the z-expansion coefficients from the first fit as priors to constrain the shape the D_2^* form factors. Following Refs. [30, 61], we parameterize the D^* and D_2^* lineshapes by a Breit-Wigner distribution and with Blatt-Weisskopf damping factors [62, 63]. In contrast to Ref. [30] we allow the Blatt-Weisskopf radius to be determined in the fit. As shown in Fig. 2, our form-factor parameterization provides a good description of the data over the entire invariant-mass range.

Our fit to the $B \to D\pi\ell\nu$ invariant-mass spectrum can be used to make predictions for related quantities. Figure 3 shows the predicted partial-wave contributions to the q^2-spectrum. After integrating over the momentum transfer, we obtain for the D-wave channel ${\rm Br}(B \to D_2^*(\to D\pi^\pm)\ell\nu) = (1.90 \pm 0.11) \times 10^{-3}$, which is larger than Belle's determination in Ref. [60]. This is because the smooth falling function employed by Belle to describe the seemingly nonresonant contributions overlaps with the D_2^* resonance, whereas in our description, the S-wave and D^* components are negligible near the resonance. For the S-wave contribution, we obtain ${\rm Br}(B \to (D\pi)_S\ell\nu) = (1.03\pm0.27)\times10^{-3}$, which agrees with the arguments made in Ref. [30] but is smaller than the branching fraction usually assigned in experimental analyses.

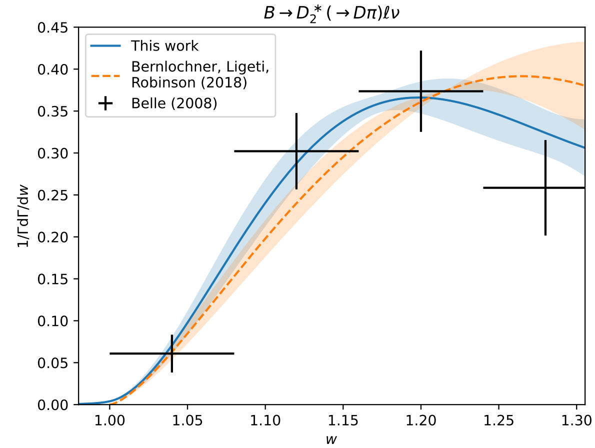

FIG. 1. Normalized $B \to D_2^*\ell\nu$ w-spectrum. The black data points are from Ref. [53]. The blue solid curve with error band is our fit result, while the orange dashed curve and band are from Refs. [22, 23]. The $\chi^2_{\rm aug}/{\rm dof} = 6.4/12$ and $Q = 0.9$.

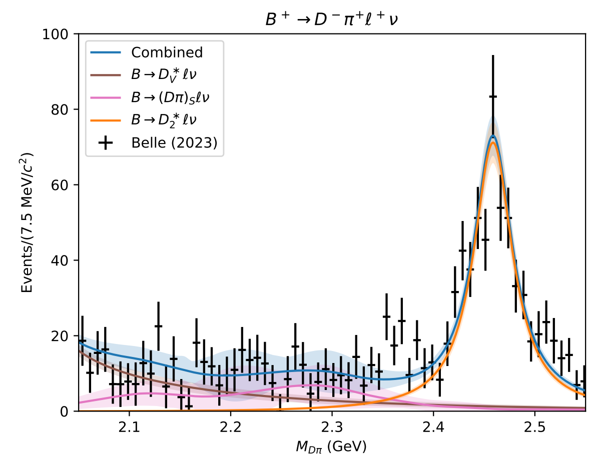

FIG. 2. Fit of the measured $M_{D\pi}$-spectrum [60] using the z-expansion to parameterize D_2^* and S-wave form factors. The $\chi^2_{\rm aug}/{\rm dof} = 124.4/133$ and $Q = 0.69$. Only data for the more precise B^+ mode is shown.

Implications and outlook — We present the first model-independent description of $B \to D\pi\ell\nu$ decays based on unitarity and analyticity of the relevant form fac-

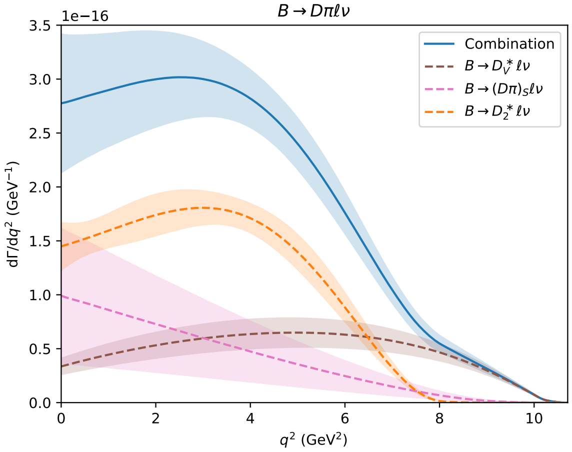

FIG. 3. Predicted partial-wave decomposition of the $B \to D\pi\ell\nu$ q^2-spectrum (dashed and dotted curves with error bands) and their total (solid curve with error band) from the fit in Fig. 2.

tors and the factorization of final-state interactions. This constitutes the first generalization of the BGL parameterization to multi-hadron final states and provides the first step towards a model-independent study of semileptonic B-meson decays into higher resonances and non-resonant final states. Our framework does not include any assumptions about lineshapes of resonances and is extendable to other decay processes with charmed mesons in the final state such as $B \to D^*\pi\ell\nu$ or $B_s \to DK\ell\nu$. Further, it is also valid for final states with more than two hadrons, and can be combined with other known $b \to c$ form factors in a global fit to obtain constraints on less well-known form factors (see e.g Ref. [64]). By replacing the D meson by a pion, the unitarity bounds can be applied to $B \to \pi\pi\ell\nu$ decays, including the phenomenologically interesting $B \to \rho\ell\nu$ channel, which is the target of first LQCD calculations beyond the narrow-width limit [43], as well as non-resonant backgrounds, which constitute the dominant systematic uncertainty [65].

Taking into account recent theoretical considerations and measurements of $B \to D\pi\ell\nu$ decays by Belle, we provide precise predictions for semileptonic decays into the broad two-pole structure in the S-wave and determine the form-factor parameters for $B \to D_2^*\ell\nu$ decays from data. This marks the first time in which a three-component hypothesis consisting of S-wave contributions, D^* virtual contributions and D_2^* contributions, is compared to the measured $M_{D\pi}$-spectrum. Previous works either do not include all three components simultaneously [22, 53, 66] or do not compare to the measured $M_{D\pi}$-spectra [30]. We demonstrate, in contrast to existing literature, that our treatment of the S-wave is compatible with the $M_{D\pi}$-spectrum measured by Belle and thus is favored over previous models that assume a single, broad S-wave state, the $D_0^*(2300)$. While more careful studies need to be conducted, the change in the shape for $B \to D_2^*\ell\nu$ decays, as well as the inclusion of the virtual D^* contribution lead to an overall harder q^2-spectrum, potentially resolving some of the discrepancies seen in inclusive analyses at high q^2 [7, 10].

The coupled channel nature of the S-wave contribution enables us to obtain predictions for $B \to D\eta\ell\nu$ and $B \to D_sK\ell\nu$ decays purely based on measurements of $B \to D\pi\ell\nu$ decays. For the $D\eta$ S-wave contribution we obtain $\mathrm{Br}(B \to (D\eta)_S\ell\nu) = (1.9 \pm 1.7) \times 10^{-5}$, two orders of magnitude too small to constitute a sizeable portion of the semileptonic gap. Consequently, both approaches utilized by the Belle and Belle II Collaborations in recent measurements to fill the semileptonic gap in terms of $B \to D^{(*)}\eta\ell\nu$ decays, either via a broad S-wave resonance or equidistributed in phase space, are ruled out.

Additional theoretical work, such as LCSR computations of the S-wave form factors [67], would greatly improve the results presented in this letter.

Future experimental measurements of the q^2- and $\cos\theta_l$-spectra of $B \to D_2^*\ell\nu$ decays by Belle II with the already available data set, as well as updated angular analyses of $B^0 \to D^0\pi^-\pi^+$ and $B^0 \to D^0\pi^-K^+$ decays by LHCb, would improve the precision of the form factors presented in this letter. In the long term, a full partial-wave analysis of $B \to D\pi\ell\nu$ decays is required to ultimately determine the exact composition of the $D\pi$ spectrum in semileptonic decays. Additionally, the final state interactions between D mesons and light hadrons can be tested by measuring femtoscopic correlation functions at the ALICE experiment [68]. This result could provide a direct, orthogonal test of the S-wave two-pole structure in heavy ion collisions [69].

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SUPPLEMENTAL MATERIAL

Fivefold differential decay rate

The fivefold differential decay rate for $B \to D\pi\ell\nu$ decays is given by

$$\frac{\mathrm{d}\Gamma_{B\to D\pi\ell\nu}}{\mathrm{d}M^2_{D\pi}\mathrm{d}q^2\mathrm{d}\cos\theta\mathrm{d}\cos\theta_l\mathrm{d}\chi} = \frac{G_F^2|V_{cb}|^2}{M_B}\frac{1}{(4\pi)^6}\left(1-\frac{m_l^2}{q^2}\right)W\sum_{a,b}|\mathcal{M}_{ab}|^2 \ , \tag{16}$$

where a, b label the partial waves and G_F is Fermi's constant and the matrix.

The matrix element squared is given by

$$|\mathcal{M}_{ab}|^2 = \langle D(p_D)\pi(p_\pi)|V_\mu - A_\mu|B(p_B)\rangle \langle D(p_D)\pi(p_\pi)|V_\nu - A_\nu|B(p_B)\rangle L^{\mu\nu}(p_l,p_\nu) \tag{17}$$

where

$$L^{\mu\nu}(p_l,p_\nu) = \mathrm{Tr}\left(\gamma^\mu P_L(\not{p}_l - m_l)\gamma^\nu P_L \not{p}_\nu\right) \tag{18}$$

is the leptonic tensor and $P_L = (\mathbb{1}-\gamma_5)/2$.

Inserting Eq. (1), we obtain

$$\begin{aligned}
|\mathcal{M}_{ab}|^2 = M_B^2 M_{D\pi}^2(q^2-m_l^2)W^{a+b}\Bigg\{ &\left[\frac{\tilde{\mathcal{F}}_{1,a}\mathcal{F}^*_{1,b}}{\lambda(M_B^2,M_{D\pi}^2,q^2)q^2} + \frac{m_l^2}{q^4}\mathcal{F}_{2,a}\mathcal{F}^*_{2,b} - \frac{f_a f_b^*}{\lambda(M_B^2,M_{D\pi}^2,q^2)} - g_a g_b^*\right]P_a^0 P_b^0 \\
&- \left(1-\frac{m_l^2}{q^2}\right)\frac{\tilde{\mathcal{F}}_{1,a}\mathcal{F}^*_{1,b}}{\lambda(M_B^2,M_{D\pi}^2,q^2)q^2}\cos^2\theta_l P_a^0 P_b^0 \\
&+ \left(\frac{f_a f_b^*}{\lambda(M_B^2,M_{D\pi}^2,q^2)} + g_a g_b^*\right)\left[P^0_{a-1}P^0_{b-1} + \frac{P^1_{a-1}P^1_{b-1}}{ab}\right] - \left(1-\frac{m_l^2}{q^2}\right)g_a g_b^*\frac{P_a^1 P_b^1}{ab}(1-\cos^2\theta_l) \\
&- \left(1-\frac{m_l^2}{q^2}\right)\left(\frac{f_a f_b^*}{\lambda(M_B^2,M_{D\pi}^2,q^2)} - g_a g_b^*\right)\frac{P_a^1 P_b^1}{ab}\cos^2\chi(1-\cos^2\theta_l) \\
&- \frac{f_a g_b^* + g_a f_b^*}{\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}}\left[P^0_{a-1}P^0_{b-1} + \frac{P^1_{a-1}P^1_{b-1}}{ab}\right]\cos\theta_l + \left(\frac{f_a g_b^* + g_a f_b^*}{\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}} + \frac{m_l^2}{q^4}\frac{\tilde{\mathcal{F}}_{1,a}\mathcal{F}^*_{2,b} + \mathcal{F}_{2,a}\mathcal{F}^*_{1,b}}{\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}}\right)\cos\theta_l P_a^0 P_b^0 \\
&+ i\left(1-\frac{m_l^2}{q^2}\right)\frac{g_a f_b^* - f_a g_b^*}{\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}}\frac{P_a^1 P_b^1}{ab}(1-\cos^2\theta_l)\cos\chi\sin\chi \\
&+ \left(1-\frac{m_l^2}{q^2}\right)\left[\frac{\tilde{\mathcal{F}}_{1,a}f_b^*}{\sqrt{q^2}\lambda(M_B^2,M_{D\pi}^2,q^2)}\frac{P_a^0 P_b^1}{b} + \frac{f_a\mathcal{F}^*_{1,b}}{\sqrt{q^2}\lambda(M_B^2,M_{D\pi}^2,q^2)}\frac{P_a^1 P_b^0}{a}\right]\cos\theta_l\sin\theta_l\cos\chi \\
&+ i\left(1-\frac{m_l^2}{q^2}\right)\left[\frac{\tilde{\mathcal{F}}_{1,a}g_b^*}{\sqrt{q^2}\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}}\frac{P_a^0 P_b^1}{b} - \frac{g_a\mathcal{F}^*_{1,b}}{\sqrt{q^2}\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}}\frac{P_a^1 P_b^0}{a}\right]\cos\theta_l\sin\theta_l\sin\chi \\
&- \left[\left(\tilde{\mathcal{F}}_{1,a}g_b^* + \frac{m_l^2}{q^2}\mathcal{F}_{2,a}f_b^*\right)\frac{P_a^0 P_b^1}{b} + \left(g_a\mathcal{F}^*_{1,b} + \frac{m_l^2}{q^2}f_a\mathcal{F}^*_{2,b}\right)\frac{P_a^1 P_b^0}{a}\right]\frac{\cos\chi\sin\theta_l}{\sqrt{q^2}\sqrt{\lambda(M_B^2,M_{D\pi}^2,q^2)}} \\
&- i\left[\left(\tilde{\mathcal{F}}_{1,a}f_b^* + \frac{m_l^2}{2q^2}\mathcal{F}_{2,a}g_b^*\right)\frac{P_a^0 P_b^1}{b} - \left(f_a\mathcal{F}^*_{1,b} + \frac{m_l^2}{2q^2}g_a\mathcal{F}^*_{2,b}\right)\frac{P_a^1 P_b^0}{a}\right]\frac{\sin\chi\sin\theta_l}{\sqrt{q^2}\lambda(M_B^2,M_{D\pi}^2,q^2)}\Bigg\} \ ,
\end{aligned} \tag{19}$$

where we have introduced $\tilde{\mathcal{F}}_{1,i} = (M_B^2 - M_{D\pi}^2)\mathcal{F}_{1,i}$ for brevity. The P_a^i are the associated Legendre polynomials with arguments of $\cos\theta$. Note, that this expression is also valid for massive leptons and incorporates all interference terms between different partial waves.

Unitarity bounds and outer functions

The derivation of outer functions from the contributions to the unitarity bounds in Eq.(8) is complicated by the appearance of integrals of the form

$$\mathcal{I}^{(l,i)}(q^2) = \int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M_{D\pi}^2\, M_{D\pi}^2 W^{2l+1}\lambda^i(M_B^2, M_{D\pi}^2, q^2) g^{(l)}(M_{D\pi}^2)\ . \tag{20}$$

In the narrow-width limit these integrals can be evaluated analytically and for $l = 1$ lead to the standard expressions of Ref. [34].

For more complicated $g^{(l)}$ no simple analytic solution exists. Consequently the contribution to the outer functions from the integrals needs to be computed numerically through the equation [48]

$$\tilde{\phi}_l^{(f)}(q^2) = e^{i\varphi}\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}t\frac{e^{it}+z}{e^{it}-z}\log|\mathcal{I}^{(l,i)}(e^{it})|\right)\ . \tag{21}$$

Here φ is an arbitrary phase and the full outer function $\phi_l^{(f)}$ is the product of the appropriate $\tilde{\phi}_l^{(f)}(q^2)$, as well as constants and simple functions of z that can be derived following Ref. [34].

Fit of experimental data

To fit to the available data from the Belle experiment [53, 60] we need to parametrize the lineshapes of the D^* and D_2^*. To this end, we write the $M_{D\pi}$-dependent part of form factors such as $g^{(l)}$ in the decomposition of Eq. (9) as

$$g^{(l)}(M_{D\pi}) = \frac{g_l}{(M_{D\pi}^2 - M_{R,l}^2) + iM_{R,l}\Gamma_R(M_{D\pi}^2)} X^{(l)}(|\vec{p}_D|r_{\mathrm{BW}}, |\vec{p}_{D,0}|r_{\mathrm{BW}})\ . \tag{22}$$

Here, $M_{R,l}$ are the respective nominal resonance masses, g_l is the coupling of the resonances to a D-meson and a pion, Γ_R the energy-dependent width and the $X^{(l)}$ are Blatt-Weisskopf damping factors. We take the masses, widths and branching ratios for the D^* and D_2^* from Ref. [6] and determine g_l from them. The damping factors are given by

$$X^{(1)}(z, z_0) = \sqrt{\frac{1+z_0^2}{1+z^2}}\ , \qquad X^{(2)}(z, z_0) = \sqrt{\frac{9+3z_0^2+z_0^4}{9+3z^2+z^4}}\ . \tag{23}$$

The D-meson three-momentum is evaluated in the resonance center-of-mass frame, $|\vec{p}_{D,0}|$ denotes the D-meson three-momentum for $M_{D\pi} = M_R$. The Blatt-Weisskopf radius r_{BW} is treated as a parameter in the fit with a prior of $4 \pm 1\ \mathrm{GeV}^{-1}$, where the central value corresponds to the value chosen by the LHCb experiment for their amplitude analysis of $B^- \to D^+\pi^-\pi^-$ decays [29].

Uncertainties in the scattering phase shifts from Ref. [24] are obtained using resampling techniques. In particular, we compute our results using an ensemble of 150 samples provided by the authors of Ref. [24] that describe the combined 1σ confidence interval of the parameters fitted in that work [56], compute statistical uncertainties from the variation of our results across these resampled ensembles, and combine these uncertainties in quadrature with the other uncertainties appearing in our analysis.

To obtain the P-Wave form factors, we integrate the five-fold differential decay rate over $M_{D\pi}$ and match the four-fold differential decay rate to the one obtained from the LQCD determination of the $B \to D^*\ell\nu$ form factors by the Fermilab/MILC collaboration [57]:

$$\int \mathrm{d}M_{D\pi}^2 \frac{\mathrm{d}^5\Gamma}{\mathrm{d}M_{D\pi}^2\mathrm{d}q^2\mathrm{d}\cos\theta\mathrm{d}\cos\theta_l\mathrm{d}\chi} = \mathrm{Br}(D^* \to D\pi)\frac{\mathrm{d}^4\Gamma}{\mathrm{d}q^2\mathrm{d}\cos\theta\mathrm{d}\cos\theta_l\mathrm{d}\chi}\Bigg|_{\mathrm{FNAL/MILC}}\ . \tag{24}$$

In a first step, we obtain the D-Wave form factor parameters by performing a simultaneous fit to the w- and $|\cos\theta|$-spectra from Ref. [53], as well as the world averages for the nonleptonic rates for $B^0 \to D_2^{*-}\pi^+$ and $B^0 \to D_2^{*-}K^+$ [6]. We require regularity at $q^2 = 0$

$$\begin{aligned} A_1^{(l)}(0, M_{D\pi}^2) &= M_{D\pi}A_0^{(l)}(0, M_{D\pi}^2) \\ &\quad - (M_B^2 - M_{D\pi}^2)A_2^{(l)}(0, M_{D\pi}^2)\ , \end{aligned} \tag{25}$$

connecting the longitudinal form factor A_2, which appears in the nonleptonic decays under consideration, to the other three form factors. We subtract the contribution of B_c meson states for which the leptonic decay constants are known from the susceptibilities to strengthen the unitarity bounds following Refs. [58, 59] and truncate the z-expansion of the form factors at linear order in z.

We then perform a fit to the $M_{D\pi}$-spectra of Ref. [60] with the S-, P- and D-Wave form factor parameters entering the fit with priors set to the previously extracted values and covariances, masses and widths constrained by their world averages and r_{BW} treated as described above. Both charge modes are fitted simultaneously, unlike in experimental fits to the mass spectrum, with the sole difference being the exact values of the involved masses. Since $D^{*0} \to D^+\pi^-$ decays are not allowed on-shell, this leads to slightly different shapes in the $M_{D\pi}$ spectrum at low masses. As a cross-check, we also considered the $M_{D\pi}$-spectra of Ref. [53], finding good agreement and compatible results with the main fit.