Muon g - 2 with overlap valence fermion

(\chi QCD Collaboration)

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We present a lattice calculation of the leading order (LO) hadronic vacuum polarization (HVP) contribution to the muon anomalous magnetic moment for the connected light and strange quarks, \( a_{\mu \text{con,l,s}} \), in the widely used window \( l_0 = 0.4 \) fm, \( t_1 = 1.0 \) fm, \( \Delta = 0.15 \) fm, and also of \( a_{\mu \text{con,l/s}} \) in the short distance region. We use the overlap fermion on 4 physical-point ensembles. Two 2+1 flavors ensembles use the domain wall fermion (DWF) and Iwasaki gauge actions at \( a = 0.084 \) and 0.114 fm, and two 2+1+1 flavors ensembles use the highly-improved-staggered-quark (HISQ) and Symanzik gauge actions at \( a = 0.088 \) and 0.121 fm. They have incorporated infinite volume corrections from 3 additional DWF ensembles at \( L = 4.8, 6.4 \) and 9.6 fm and physical pion mass. For the \( a_{\mu \text{con,l/s}} \), we find that our results on the two smaller lattice spacings are consistent with those using the unitary setup, but those at the two coarser lattice spacings have obviously small differences compared to the unitary ones.

I. INTRODUCTION

The anomalous magnetic moment of the muon \((a_\mu \equiv (g_\mu - 2)/2)\) is one of the crucial benchmarks to verify the correctness of the standard model. The current analysis of the Fermilab experiment [1, 2] is consistent with the previous BNL E821 [3] result with comparable precision, and the Fermilab experiment is planned to reduce the uncertainty by a factor of 4. Those results are higher than the current standard model predictions [4] using phenomenological estimates by around 4\( \sigma \), and so have attracted much theoretical interest about possible new physics.

But such a deviation is very sensitive to the theoretical prediction of the strong interaction contribution to \( a_\mu \), especially the leading order hadronic vacuum polarization (LO-HVP) contribution, \( a_{\mu \text{LO-HVP}} \). The most recent determinations from the hadronic \( R \)-ratio, a dispersion integral over hadronic cross section ratio \( \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-) \), are \( \bar{a} \equiv a_{\mu \text{LO-HVP}} \times 10^{10} = 693.9(4.0) \) [5] and 692.8(2.4) [6], while that required for no new physics is 718(3) [1-4]. Comparing to \( a_{\mu \text{LO-HVP}} \), the electron mass suppresses \( a_{e \mu \text{LO-HVP}} \), the corresponding quantity for the electron, by a factor of \((m_e/m_\mu)^2 \sim 10^{-4}\), and so the theoretical uncertainty of HVP will not affect the agreement between the current theoretical and experimental determinations of \( a_\mu \).

\( a_{\mu \text{LO-HVP}} \) can be obtained using first-principles Lattice QCD calculations, which avoid the possible phenomenological uncertainty from the \( R \)-ratio determination. There are a few recent Lattice QCD results [7-12], and the most precise one [12] obtains \( \bar{a} = 707.5(5.5) \) and agrees with the “no new physics” requirement within 1.5\( \sigma \). A recent study [13] suggests that the so-called “window” value [8] of \( a_\mu \), which picks the contribution around the p meson pole, is sensitive to the discretized fermion action used by the \( a_\mu \) calculation. In this work, we will calculate the window value of \( a_\mu \) using the overlap valence fermion action on ensembles with either the Domain Wall fermion (DWF) sea or HISQ sea, to study the fermion action dependence. We will also examine the lattice spacing dependence in these cases.

The numerical setups of this work are collected in Sec. II and Sec. III presents our results with their statistical and systematic uncertainties. The summary and extended discussions are given in Sec. IV.

II. NUMERICAL SETUP

One can calculate \( a_{\mu \text{LO-HVP}} \) with the following expression,
\[ a_{\mu}^{\text{LO-HVP}} = 4\alpha^2 \int_0^\infty \frac{d^2 q^2}{m_\mu^2} f\left(\frac{q^2}{m_\mu^2}\right)((\Pi(q^2) - \Pi(0)), \] 
\[ f(r) = \frac{Z^2(r)(1 - \sqrt{Z(r)})}{\sqrt{r}(1 + Z^2(r))}, \quad Z(r) = \sqrt{\frac{r + 4 - \sqrt{r}}{2}}, \]
\[ \alpha = \frac{e^2}{4\pi} \sim 1/137 \text{ is the fine structure constant, } m_\mu \text{ is the muon mass, the HVP } \Pi(q^2) \text{ can be obtained from the Fourier transform of the vector current two-point function in Euclidean space-time,} \]
\[ \Pi^{\mu\nu}(q) = \int d^4x e^{iqx} (j_\mu(x)j_\nu(0)) = \Pi(q^2)(q^2\delta_{\mu\nu} - q_\mu q_\nu), \]
and the electromagnetic current
\[ j_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f \]
is summed over all the quark flavors \( f = u, d, s, c, \ldots \) with their electric charge \( Q_f \) in units of the electron charge \( e \). The factor \( Z \) has the properties \( Z(0) = 1 \) and \( Z(r) \asymp 1/\sqrt{r} \) at \( r \to \infty \), which ensures that the major contribution in the \( q^2 \) integration comes from the small \( q^2 \) region.

One can select the momentum \( q \) to be along the temporal direction to simplify the expression of \( \Pi(q^2) \) to
\[ \Pi(q^2) = \int dt \frac{\cos(tq) - 1}{q^2} C(t), \]
where \( C(t) \equiv \frac{1}{3} \sum_i \langle \int d^3x j_i(x,t)j_i(0,0) \rangle \). This definition includes the \( 1/q^2 \) divergence, but then the subtracted \( \Pi(q^2) \) \[14\] is
\[ \Pi(q^2) - \Pi(0) = \int dt \left[ \frac{\cos(tq) - 1}{q^2} + \frac{1}{2} t^2 \right] C(t). \]

One can show that \( \Pi(q^2) \propto 1/M^2 \) if \( C(t) \propto e^{-Mt} \) and then \( a_{\mu}^{\text{HVP}} \propto M^2/M^2 \) if \( C(t) \) is dominated by a single state with mass \( M \gg m_\mu \), and then the heavy quark contribution to \( a_{\mu}^{\text{HVP}} \) is also suppressed by \( 1/m_Q^2 \).

Eventually one can rewrite \( a_{\mu}^{\text{LO-HVP}} \) in terms of \( C(t) \) and the weight function \( \omega(t) \),
\[ a_{\mu}^{\text{LO-HVP}} = \int dt \omega(t)C(t), \]
\[ \omega(t) = 4\alpha^2 \int_0^\infty \frac{d^2 q^2}{m_\mu^2} f\left(\frac{q^2}{m_\mu^2}\right) \left[ \frac{\cos(tq) - 1}{q^2} + \frac{1}{2} t^2 \right]. \]

On the lattice, one would use the original \( \omega(t) \) or replace \( \omega(t) \) by its lattice version \[8\],
\[ \tilde{\omega}(t) = 4\alpha^2 \int_0^\infty \frac{d^2 q^2}{m_\mu^2} f\left(\frac{q^2}{m_\mu^2}\right) \left[ \frac{\cos(tq) - 1}{q^2} + \frac{1}{2} t^2 \right], \]
and sum \( t \) over all the discretized time slices \( (-\frac{T a}{2}, \frac{T a}{2}) \):
\[ a_{\mu}^{\text{LO-HVP,}\omega} = \left[ \sum \omega(t)C^{\text{lat}}(t, a) \right]_{a \to 0, T \to \infty}, \]
\[ a_{\mu}^{\text{LO-HVP,}\tilde{\omega}} = \left[ \sum \tilde{\omega}(t)C^{\text{lat}}(t, a) \right]_{a \to 0, T \to \infty}, \]
to obtain the final prediction of \( a_{\mu}^{\text{LO-HVP}} \), where \( a \) is the lattice spacing, \( T \) is the dimensionless number of lattice sites along the temporal direction, and \( C^{\text{lat}}(t, a) = a \{ C(t) + O(a^2) \} \) is the correlation function on the lattice. Based on numerical calculation, changing the upper limit of the integral in Eq. 6 to a finite constant, such as \((\pi/a)^2\), changes the value of the integral by less than 0.005\%, which is much smaller than our other uncertainties. Also, Ref. \[7\] suggests that the correction is about 0.02\%, by introducing an integral cut-off \( Q^2_{\text{max}} = 3 \text{ GeV}^2 \) in Eq. 6, which is also much smaller than all our total uncertainties.

In the practical calculation on the ensemble with 2 degenerate light flavors, \( C(t) \) can be decomposed into several pieces \[8\],
\[ C(t) = \frac{5}{9} C^{\text{con}}(t; m_t) + \frac{1}{9} C^{\text{con}}(t; m_s) + \frac{4}{9} C^{\text{con}}(t; m_c) + C^{\text{dis}}(t) + \alpha C^{\text{QED}}(t) + \Delta m C^{\text{SIB}}(t) + O(\alpha^2, \alpha \Delta m, \Delta m^2), \]
where \( m_t = (m_u + m_d)/2 \) is the iso-symmetric light quark mass, \( m_s, c \) are the strange and charm quark masses respectively, \( C^{\text{con}}(t; m_f) = C^{\text{con}}(t; S(x = \bar{x}, t = 0; m_f)) \),
\[ C^{\text{con}}(t, S(x_0, t_0; m_f)) = \frac{1}{3} \sum_i \langle \int d^3x \text{Tr}[\gamma_i 5 \gamma_i 5 \gamma_i \gamma_i S^4(x, t + t_0; \bar{x}, t_0; m_f) \gamma_5 \gamma_i S(x, t + t_0; \bar{x}, t_0; m_f)] \rangle \]
is the connected correlation function with the quark propagator \( S(m) \equiv 1/(D + m) \), \( C^{\text{dis}} \) is the contribution from the disconnected quark diagram, \( C^{\text{QED}} \) is the leading order QED correction which can be accessed from the 4-point correlation function with an infinite-volume photon \[15\] (the next order contribution is negligible for the precision required by \( a_{\mu} \)), and \( C^{\text{SIB}} \) is the strong isospin breaking (SIB) effect which is proportional to \((m_d - m_s)/2\).

In this work, we use the overlap fermion on several gauge ensembles with 1-step HYP smearing to calculate \( a_{\mu}^{\text{HVP}} \). The ensembles include the 2+1 flavor DWF ensembles with the Iwasaki gauge action from the RBC/UQCD collaboration \[16\-18\] and the 2+1+1 flavor HISQ ensemble with the Symanzik gauge action from the MILC collaboration \[19\].

The overlap fermion action uses a matrix sign function \( \epsilon(H_w) = \frac{H_w}{\sqrt{m_z^2}} \) of the hermitian Wilson Dirac operator \( H_w(-M_0) = \gamma_5 D_w(-M_0) \) to construct
\[ D_{\infty} = M_0\left(1 + \gamma_5 \epsilon(H_w(-M_0))\right), \]
which was proposed in Ref. \[20, 21\] as a discretized fermion operator satisfying the Ginsburg-Wilson relation \( D_{\infty} \gamma_5 + \gamma_5 D_{\infty} = \frac{2}{m_\mu} D_{\infty} \gamma_5 D_{\infty} \) \[22\], where \( D_{\infty} \) is the Wilson Dirac operator with a negative mass such as
\[ C_{\text{con.LMS}}(t) = \frac{1}{N_{\text{src}}N_{g}^{3}} \sum_{i} \{ C_{\text{con}}(t, S_{L}^{\text{grid}}(\vec{x}_{i}, t_{i})) - C_{\text{con}}(t, S_{L}^{\text{grid}}(\vec{x}_{0}, t_{0})) \} + \frac{1}{L^{3}T} \sum_{\vec{x}_{0}, \vec{t}_{0}} C_{\text{con}}(t, S_{L}(\vec{x}_{0}, t_{0})), \]

where \( S_{L}^{\text{grid}}(\vec{x}_{i}, t_{i}) = \sum_{y \in \text{grid}} S(\vec{x}_{i} + \vec{y}, t_{i} + t_{0}; \vec{y}, t_{0}) \) is the quark propagator using a grid source with \( N_{g} \) points at each spacial dimension, \( N_{\text{src}} \) is the number of \( S_{L}^{\text{grid}} \) located at different origins \((\vec{x}_{i}, t_{i})\), \( S_{L}(m) \equiv \sum_{\lambda < \lambda_{c}} \frac{1}{\lambda_{c} - \lambda} v_{\lambda} v_{\lambda}^{\dagger} \), \( v_{\lambda} \) satisfies \( D_{c}v_{\lambda} = \lambda v_{\lambda} \) and \( \lambda_{c} \sim 200 \) MeV is the upper bound of the eigenvalue \( \lambda \). The source point \( \vec{x}_{0}, t_{0} \) loops over the whole lattice volume to have full statistics for the low-mode parts. The information of the gauge ensembles, grid source parameters and \( \lambda_{c} \) are listed in Table I.

TABLE I. Information of the ensembles, grid sources and \( \lambda_{c} \) used in this calculation, including the DWF+Iwasaki ensembles 48I and 64I [17, 18] and also the HISQ+Symanzik ensembles a12m130 and a09m130 [19] with the lattice spacings from Ref. [13]. Three DWF+Iwasaki+DSDR ensembles 24D/32D/48D [16, 18] are used to estimate the finite-volume effect, and four HISQ+Symanzik ensembles a09m310/a06m310/a09m310/a12m310 [19] with the lattice spacings from Ref. [26] are used to study the continuum extrapolation. The pion mass \( m_{\pi} \) and upper bound \( \lambda_{c} \) of the eigenvalues are in unit of MeV.

\[
\begin{array}{cccccccc}
\text{Symbol} & L^{3} \times T & a \ (\text{fm}) & m_{\pi} & N_{\text{cf}} & N_{\text{src}} & N_{g} & \lambda_{c} \\
48I & 48^{3} \times 96 & 0.11406(26) & 139 & 100 & 12 & 4 & 234 \\
64I & 64^{3} \times 128 & 0.08365(25) & 139 & 92 & 8 & 4 & 187 \\
a12m130 & 48^{3} \times 64 & 0.12121(64) & 131 & 23 & 8 & 4 & 180 \\
a09m130 & 64^{3} \times 96 & 0.08786(47) & 128 & 22 & 8 & 4 & 200 \\
a12m310 & 24^{3} \times 64 & 0.12129(98) & 305 & 54 & 16 & 1 & 224 \\
a09m310 & 32^{3} \times 96 & 0.08821(71) & 313 & 39 & 16 & 1 & 195 \\
a06m310 & 48^{3} \times 144 & 0.05740(50) & 319 & 32 & 8 & 1 & 243 \\
a04m310 & 64^{3} \times 192 & 0.04250(40) & 310 & 54 & 2 & 1 & 167 \\
24D & 24^{3} \times 64 & 0.1940(19) & 141 & 232 & 8 & 2 & 263 \\
32D & 32^{3} \times 64 & 0.1940(19) & 141 & 134 & 8 & 4 & 230 \\
48D & 48^{3} \times 64 & 0.1940(19) & 141 & 47 & 8 & 6 & 116 \\
\end{array}
\]

III. RESULTS AND SYSTEMATICS

The window method proposed in Ref. [8, 14] allows a more precise prediction to combine the “window” value of \( a_{\mu}^{L-O-HYP} \) using the \( C(t) \) from lattice QCD

\[
a_{\mu}^{W} = \int dt \ w^{W}(t) \omega(t) C(t),
\]

\[
w^{W}(t) \equiv \theta(t, t_{0}, \Delta) - \theta(t, t_{1}, \Delta),
\]

\[
\theta(t, t', \Delta) \equiv \frac{1}{2} (1 + \tanh[(t - t')/\Delta]),
\]

with the remaining parts

\[
a_{\mu}^{s} = a_{\mu} + a_{\mu}^{L},
\]

\[
a_{\mu}^{W} = \int dt \ \omega^{W}(t) C(t) w^{W}(t), \quad w^{W}(t) \equiv 1 - \theta(t, t_{0}, \Delta),
\]

\[
a_{\mu}^{L} = \int dt \ \omega^{L}(t) C(t) w^{L}(t), \quad w^{L}(t) \equiv \theta(t, t_{1}, \Delta).
\]

with the \( C(t) \) from the \( R \)-ratio. The extra weight functions \( w^{S,W,L} \) pick the short, medium and long distance contributions of \( C(t) \), respectively, and separate \( a_{\mu} \) into three pieces. With typical parameters \( t_{0} = 0.4 \) fm, \( t_{1} = 1.0 \) fm, \( \Delta = 0.15 \) fm, \( a_{\mu}^{s} \) suppresses the contribution of \( C(t) \) from \( t < t_{0} \) and \( t > t_{1} \), and can have smaller uncertainty using \( C(t) \) from lattice QCD compared to that from the \( R \)-ratio. \( a_{\mu}^{W} \) also provides a good reference to compare the independent lattice QCD results with good precision, in order to check the systematic uncertainties due to different lattice actions and their respective discretization errors.

We shall define the re-scaled connected light and strange quark contributions as

\[
\tilde{a}_{\text{con},f}^{X} = a_{\text{con}}^{X}(m_{t}), \quad \tilde{a}_{\text{con,s}}^{X} = \frac{1}{5} a_{\text{con}}^{X}(m_{s}),
\]

\[
\tilde{a}_{\text{con}}^{X}(m_{q}) \equiv \frac{5}{9} \int dt \ w^{X}(t) \omega(t) C_{f}^{\text{con}}(t; m_{q}) \times 10^{10},
\]

where \( X \in \{ S, W, L \} \) and \( m_{t} \) and \( m_{s} \) are the physical light and strange quark masses, respectively. We use the local vector current in the calculation and apply the
axial-vector normalization constant $Z_A$ obtained from PCAC [27] since the local vector current normalization constant obeys $Z_V = Z_A$ for the overlap fermion. As shown in Fig. 1, $Z_A$ agrees with $Z_V$ very well at the massless limit. ($Z_V$ is determined from the forward matrix element $Z_V \equiv 2E/(\pi(p)|V_4|\pi(p))$ with $\tilde{p} = 0$ in Ref. [28].) The uncertainty of $Z_A$ is at the 0.01% level and can be ignored based on the precision target. Note that the last systematic uncertainty of the $Z_A$ in Ref. [27] is not necessary here as we don’t need to extrapolate the strange quark mass in the sea to the chiral limit.

Since the long distance contribution has been investigated in Ref. [8] and suppressing the statistical uncertainty using the bounding method can be non-trivial, we will concentrate on the medium-range contribution, $\tilde{a}_\text{con}^W$, and the short-range one, $a_\text{con}^W$, in this work. In Fig. 2, we show the result of $\tilde{a}_\text{con}^W(m_q)$ on the four physical-point ensembles as a function of the quark mass renormalized to the MS scheme at 2 GeV through the RI/MOM method [27]. Compared to the values in the upper panel which use the original $\omega$, those in the lower panel using the discretized $\tilde{\omega}$ have smaller differences between ensembles except for the light quark mass region of the OV/HISQ case. It suggests that the modified definition might suppress the discretization error. But the $m_q$ dependence in the small $m_q$ region is non-linear and then the difference between the OV/HISQ result from the a12m130 and a09m130 ensembles is larger with $\tilde{\omega}$ compared to that using $\omega$.

By interpolating the partially quenched valence quark mass to the corresponding pion mass 135 MeV, we obtain the light quark contribution $a_\text{con}^W$ as shown in Fig. 3. We note that the modification of $\omega$ to $\tilde{\omega}$ suppresses the discretization error in the OV/DWF setup (cyan and purple) but has the reversed effect on the OV/HISQ setup (green and gray). For comparison, we also show the results for unitary DWF [8] (open black box) and unitary HISQ [13] (open triangles). One can see that while those at larger lattice spacings have obvious differences, our results at $a \sim 0.08$–0.09 fm are consistent with the unitary results within uncertainty. Such a difference would be a discretization effect since it decreases with smaller lattice spacing. Thus, one expects that there will be no valence fermion action dependence at this and smaller lattice spacings. However, the OV/DWF and OV/HISQ results are conspicuously different at $a \sim 0.08$–0.09 fm. This is an indication that there is still sea dependence at this lattice spacing. After the linear $a^2$ continuum extrapolation, the OV/DWF result 206.7(1.5) and the OV/HISQ result 207.7(3.1), both using $\tilde{\omega}$, are consistent with each other, and each is independent of using $\omega$ or $\tilde{\omega}$. These results are consistent with the Budapest-Marseille-Wuppertal collaboration (BMWc) value 207.3(1.4) [12], but are a few $\sigma$ larger than the RBC value 202.9(1.5) [8]. Since all the cases use linear $a^2$ continuum extrapolation,
it suggests that the $O(a^4)$ effect is not negligible in a least one of these cases. We will elaborate on this point in the Summary and Discussion section.

One possible source of the OV/DWF-OV/HISQ discrepancy at $a \sim 0.08 - 0.09$ fm could be related to the gauge actions used in the DWF and HISQ ensembles, as different improvements make the bare gauge coupling in the RBC ensembles (2.13–2.25) and MILC ensembles (3.60–3.78) differ by a factor of $\sim 1.7$. Such a possibility can be checked with HISQ+Iwasaki and DWF+Iwasaki simulations at $a = 0.08$ fm on a smaller lattice.

Since the volumes of the 4 ensembles are close to each other ($1/L \in [0.172,0.184]$ fm$^{-1}$), we use the Mobius+Iwasaki+DSDR ensembles [16, 18] from the RBC/UKQCD collaboration to estimate the finite volume effect. As shown in Fig. 4, the finite volume effect with an empirical form $a + b \exp(-m_\pi L)$ for the case with $1/L \sim 0.18$ fm$^{-1}$ is $-0.36(56)$ for the light quarks (red boxes, left y-axis) and $0.01(18)$ for the strange quark (blue triangles, right y-axis). They cannot explain the difference between our linear $a^2$ continuum extrapolated $\bar{a}^W_{con,l}$ and the corresponding RBC value.

Combining the physical light quark mass $m_{\pi}^\text{SM}(2 \text{ GeV}) = 3.74(9)$ MeV [29] via the RI/MOM scheme [27], and the ratio $m_s/m_l = 27.42(12)$ from the FLAG review [30], we can estimate the physical strange quark mass to be $m_{\pi}^\text{SM}(2 \text{ GeV}) = 103(3)$ MeV. With this estimate, we show the strange quark contribution $\bar{a}^W_{con,s}$ in Fig. 5. Similar to the light quark case, the linear $a^2$ extrapolated OV/DWF result of 26.7(3) and OV/HISQ result of 27.5(6) are consistent within the systematic uncertainty due to the strange quark mass (which is about 0.4). They are also consistent with the RBC value of 27.0(2) (open black box in the figure).

Next, we turn to the short distance contribution $\bar{a}^S_{con,l/s}$, with the results shown in Fig. 6. We can see that the linear $O(a^2)$ lines are almost the same on both the RBC and MILC ensembles for both the light and strange quark mass cases, and the linear $a^2$ continuum extrapolated values of the OV/DWF and OV/HISQ setups are consistent within the uncertainty (except for the case of $\bar{a}^S_{con,s}$ using $\omega$ where the extrapolated values from the two setups differ by 0.07(2)). But it is interesting that the discretization error of $\bar{a}^S_{con}$ using $\omega$ is much smaller compared to that using $\hat{\omega}$, and the extrapolated values using the two definitions are separated by more than $5\sigma$ ($\sim 5\%$ difference) for both the light and strange quark cases.

This motivates us to repeat the calculation on the HISQ ensembles at $a \sim 310$ MeV pion mass with a larger lattice spacing range $a \in [0.04,0.12]$ fm to check the lattice spacing dependence. Fig. 7 shows that the linear $a^2$ extrapolation still works well for $\bar{a}^W_{con,l}$ (blue data points with the right y-axis), and using $\hat{\omega}$ can suppress the discretization error (similar to the OV/DWF results at the physical point). On the other hand, we can also see that $\bar{a}^S_{con,l}$ (red data points with the left y-axis) is less sensitive to the lattice spacing when we use the original $\omega$ instead of $\hat{\omega}$, and the tension between the linear $a^2$ extrapolated
values using either \(\omega\) or \(\tilde{\omega}\) still exists. It suggests that using \(\tilde{\omega}\) introduces an extra discretization error in the small \(t\) region. Adding \(a^2\) terms in the continuum extrapolation of \(\tilde{a}_{con,l}^S\) using \(\tilde{\omega}\) can suppress the inconsistency.

\(\tilde{a}_{con,l}^S\) corresponds to the integral of \(\omega(t)C(t)\) in the small \(t\) region, and thus its discretization effect can be illustrated through the values of \(\omega(t)C(t)\) (Fig. 8, upper panel) and those of \(\tilde{\omega}(t)C(t)\) (lower panel). The definition of \(\tilde{\omega}\) forces \(\tilde{\omega}(a) \propto a^2\), and, as a consequence, the integral of \(\tilde{\omega}(t)C(t)\) has a sizable discretization error around \(t \sim 1a\) [8]. This is illustrated in the lower panel of Fig. 8. Since \(\tilde{\omega}(t)C(t)\) is not linear in \(t\) in the range of \(t \in [0.0, 0.2] \text{ fm}\), the extra \(O(a^4)\) effect is not avoidable and cannot be mocked up with a linear \(a^2\) continuum extrapolation.

Fig. 8 also shows the values of \(\omega(t)C(t)\) (pink band), with \(C(t) = 1/(12\pi^2) \int_0^\infty d(\sqrt{s}) R(s) e^{-\sqrt{s}t}\) from [14], using the most recent analysis of the \(R\)-ratio data [6]. The pink band is about 40% higher than the connected light quark contribution at \(t \sim 0.2 \text{ fm}\), it should be primarily due to the connected strange and charm contributions and is worth further investigation in the future.

IV. SUMMARY AND DISCUSSION

In this work, we calculated the light and strange contribution of \(a_\mu\) from the connected vector correlators in the medium window (\(t_0 = 0.4 \text{ fm}, t_1 = 1.0 \text{ fm}, \Delta = 0.15 \text{ fm}\) using the overlap fermion, on the physical point ensembles using either DWF+Iwasaki (at \(a = 0.084/0.114 \text{ fm}\)) or HISQ+Symanzik (at \(a = 0.088/0.121 \text{ fm}\)) configurations. Our linear \(a^2\) extrapolated \(\tilde{a}_{con,s}^W\) result is 26.7(3) using the OV/DWF setup; it is consistent with the value 27.5(6) using the OV/HISQ setup and also with the unitary DWF value from RBC [8].

For \(\tilde{a}_{con,l}^W\), the mixed action results on the ensembles with \(a < 0.1 \text{ fm}\) are consistent with the unitary DWF or HISQ calculations, but those at \(a > 0.1 \text{ fm}\) are different from their respective unitary results by many sigmas. After linear \(a^2\) continuum extrapolations, the OV/DWF result 206.7(1.5) and OV/HISQ result 207.7(3.1) are consistent with each other and also with the BMWc value 207.3(1.4) [12]. But these are a few \(\sigma\) larger than the unitary DWF value 202.9(1.5) [8]. It suggests that the \(O(a^4)\) effect should be important in certain lattice setups.

Note that if the upcoming RBC result on a \(96^3\) lattice at \(a \sim 0.07 \text{ fm}\) [31] turns out to lie on the black line of Fig. 3, it would be a strong indication that the \(O(a^4)\) contribution is small for unitary DWF and that the mixed OV/DWF and OV/HISQ results could each have a sizable \(O(a^4)\) correction. Such \(O(a^4)\) behaviors in OV/DWF and OV/HISQ have been observed in \(\Delta_{mix}\), a low-energy constant used to measure the mixed action effect in the pion mass in LO chiral perturbation theory [29, 32].

We also calculated the short range contribution and predict \(\tilde{a}_{con,l+s}^S = 57.8(0.1)(1.5)\) with the systematic error estimated from half of the difference between the predictions of the results from \(\omega\) and \(\tilde{\omega}\). Even though using \(\tilde{\omega}\) introduces an extra discretization error, it is not a big issue after linear \(a^2\) continuum extrapolation. It would be valuable to verify our observation on other lattice setups.

Since the tension among the \(\tilde{a}_{con}^W\) and \(\tilde{a}_{con}^S\) values using different kinds of lattice setups is much smaller than that between \(a_{\mu}^{HV}\) from the \(R\)-ratio and that inferred from the Fermilab experiment, we expect that the \(O(a^4)\) effect
is more important in the long-distance contribution $a_L$, and should be examined carefully. We also suggest a unitary HISQ+Iwasaki simulation at around $a = 0.09$ fm to test the gauge action dependence.

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