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Recent work on tessellations of hyperbolic geometries

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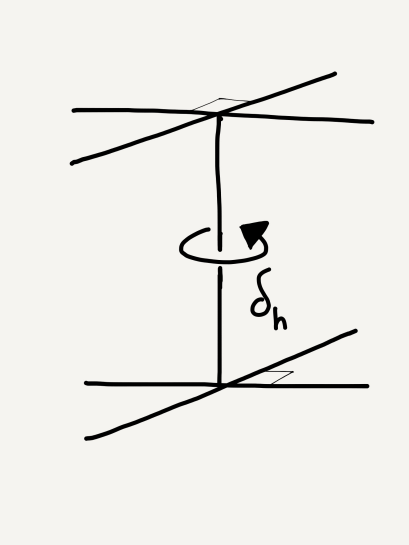
Introduction

- Flat, positive, and negative curvature spaces.

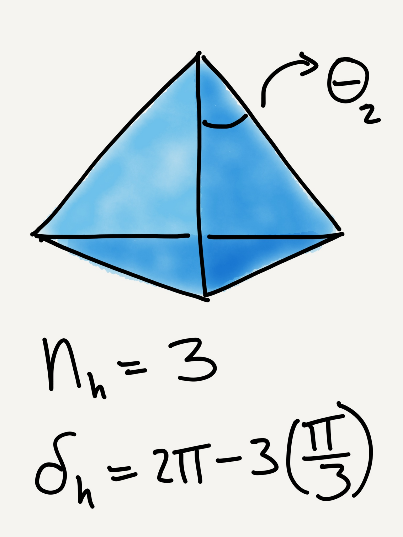
  deficit angle

  $$\delta_h = 2\pi - n_h \times \theta_D$$

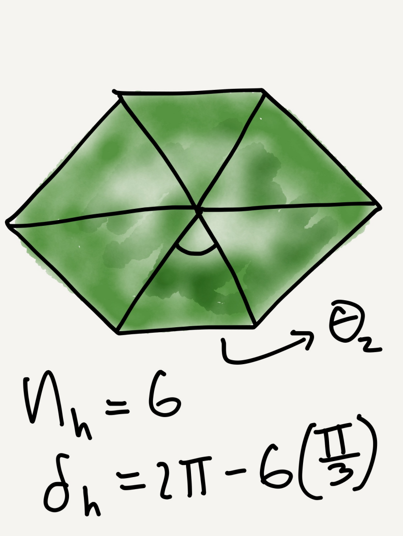
- δ_h measure the deviation from **flat**.
- h a "hinge"
- h is $D - 2$ object
- n_h number of D-dimensional gons around h.
- θ_D is angle between $D - 1$ faces around h.



- Positively curved, closed surface
- Equilateral triangles
- $\theta_2 = \pi/3$
- Hinge is a vertex (site)
- **Three** triangles around vertex



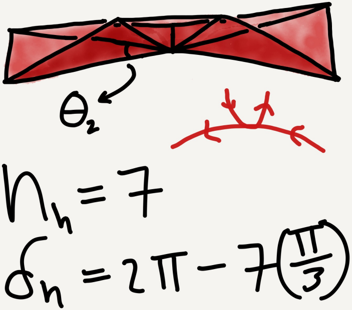
- Flat, open surface
- Equilateral triangles
- $\theta_2 = \pi/3$
- **Six** triangles around vertex



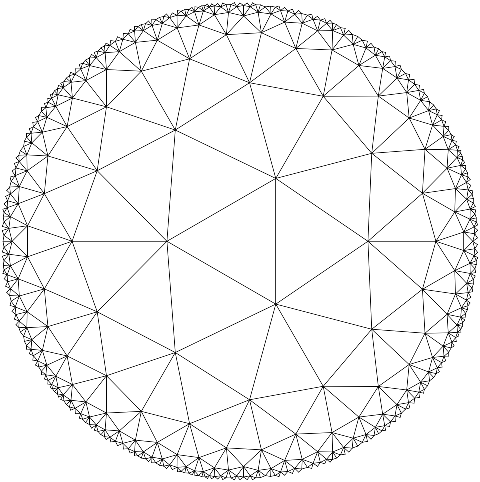
Introduction

Hyperbolic lattices

- Negatively curved, open surface
- Equilateral triangles
- $\theta_2 = \pi/3$
- **Seven** triangles around vertex

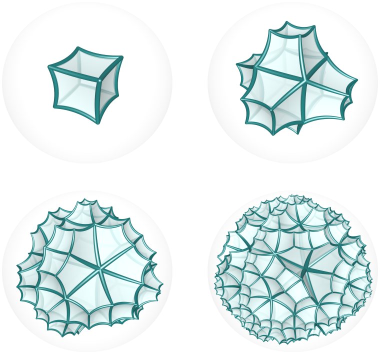


- **Poincaré disk**
- Equilateral triangles
- Seven triangles around vertex
- Schläfli notation $\{p, q, r, \ldots\}$
  - $\rightarrow$ p sided convex polygon, q of them around a vertex, r of these around each edge etc...
- $\{p, q\} \rightarrow \{3, 7\}$ lattice



Excess polygons $\implies$ strange boundary behavior

- Boundary grows **exponentially**
- Boundary **constant fraction** of total
- Boundary is never "negligible"



Including fields

Free scalar field:

$$S_{\text{cont}} = \int d^2x\sqrt{g}\frac{1}{2}(\partial_\mu\phi\partial_\mu\phi + m_0^2\phi^2) \to S_{\text{lat}} = \sum_{x,y}\phi_x L_{xy}\phi_y$$

$\Longrightarrow$ 2-pt. propagator:

$$C(|x-y|) = L_{xy}^{-1}$$

Interacting field theory:

$$S_{\text{lat}} = -\beta\sum_{\langle xy\rangle}\sigma_x\sigma_y - h\sum_x\sigma_x$$

$\Longrightarrow$ 2-pt. correlator:

$$C(|x-y|) = \langle\sigma_x\sigma_y\rangle - \langle\sigma_x\rangle\langle\sigma_y\rangle$$

Including fields

Bulk thermodynamics & propagators

- Critical temperatures: [Wu, 1996, Jiang et al., 2019]
  - Existence of transition
  - Multiple transitions
- Critical exponents: [Ueda et al., 2007, Krcmar et al., 2008, Baek et al., 2011, Benedetti, 2015, Breuckmann et al., 2020]
  - Mean-field exponents for Ising?
  - Two and three dimensions
- General correlators and continuum limit [Brower et al., 2021]
  - ϕ^4 theory
  - Thorough investigation of 2-pt. and 4-pt correlators.
  - Large-small mass limits
  - Refinement $\to$ continuum
  - Talks **Theoretical developments**:
    - July 27$^{\text{th}}$, 21:15 (Richard Brower)
    - July 27$^{\text{th}}$, 21:30 (Evan Owen)
    - July 29$^{\text{th}}$, 22:15 (Cameron Cogburn)

Including fields

Boundary propagators

Boundary correlators and holography [Asaduzzaman et al., 2020]

- Two, and three dimensions
- Multiple tessellations of $\mathbb{H}_2$
- $d = 1$ and $d = 2$ boundary power-law correlators
- Klebanov-Witten behavior [Klebanov and Witten, 1999]
- Talk **Theoretical developments**:
  - July 27$^{\text{th}}$, 21:45 (Muhammad Asaduzzaman)

Expectations

Free field, continuum, **boundary** 2-pt propagator,

$$C(r) \propto r^{-2\Delta_\pm}$$

Power-law behavior known explicitly:

$$2\Delta_\pm = d \pm \sqrt{d^2 + 4L^2 m_0^2}$$

- d is the boundary dimension
- $\Delta_\pm$ correspond to different boundary conditions
- L is radius of curvature

Expectations

Power-law boundary correlations

- High-temperature expansion of the Ising model

$$Z \propto \sum_{\{\Gamma\}} \tanh^{\Gamma}(\beta)$$

  Γ: closed, intersecting loops

- At leading order, the two-point correlator between boundary points

$$C(R) \propto \tanh^{R}(\beta) = e^{-\log(\coth\beta)R}$$

  R is **bulk** geodesic distance.

- The **boundary** distance, r: $R \sim \log r$,

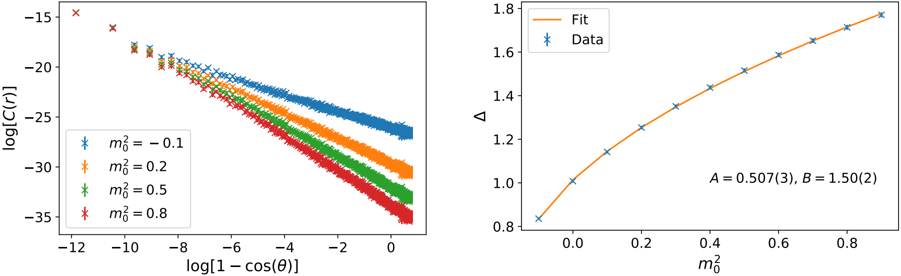
$$C(r) \propto r^{-\log(\coth\beta)}$$

Results

Results

Two dimensions

Free, massive scalar field theory:



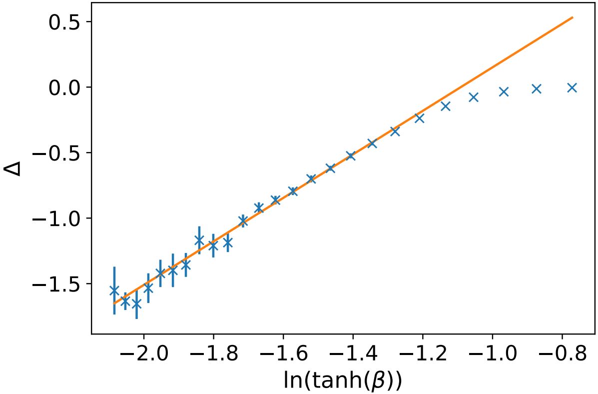
$\{3, 7\}$ lattice.

$$\Delta = A + \sqrt{A^2 + B m_0^2}$$

A, and B match analytic formula well.

Ising model, boundary correlator

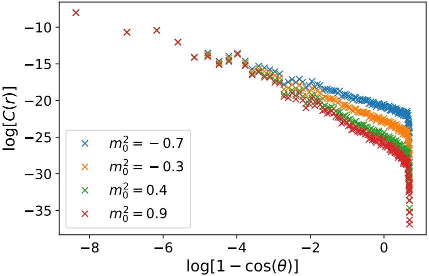
$$C(r) \sim r^{\log(\tanh \beta)}$$

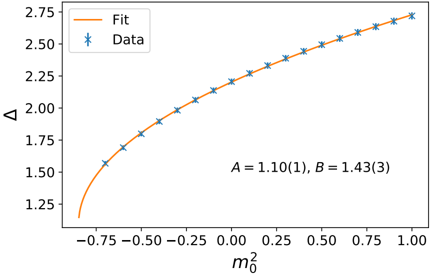


Results

Three dimensions

Free, massive scalar field theory:





- $\{4, 3, 5\}$ lattice $\rightarrow$ **five** cubes around an edge.
- A is close to 1.

Conclusions & Future work

Remarks:

- CFTs appear supported on the boundary.
- The "magic" of holography can be traced back to
  - exponential growth of boundary relative to bulk.
  - "$R \sim \log r$"
- Free scalar field recovers Klebanov-Witten
- Ising model may have mean-field exponents in a variety of cases.

Future Work:

- More studies in *two & three dimensions*: $d = 1$ CFT, $\mathbb{H}_3$, AdS_3.
- Gauge theories and fermions
- Effective boundary action

Thank you!

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