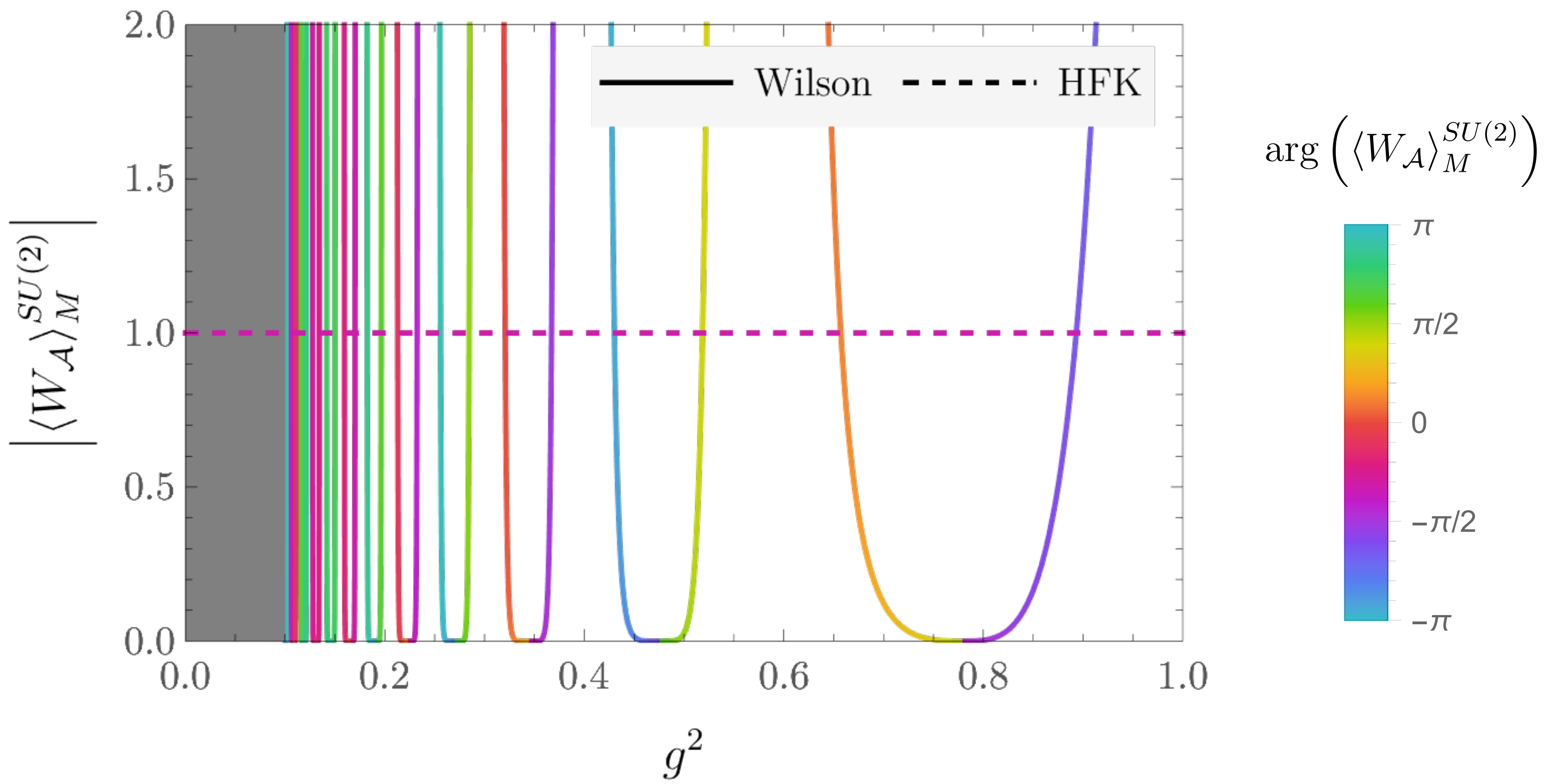
Real time lattice gauge theory actions

Michael Wagman



Kanwar, MW, arXiv:2103.02602

Fermilab

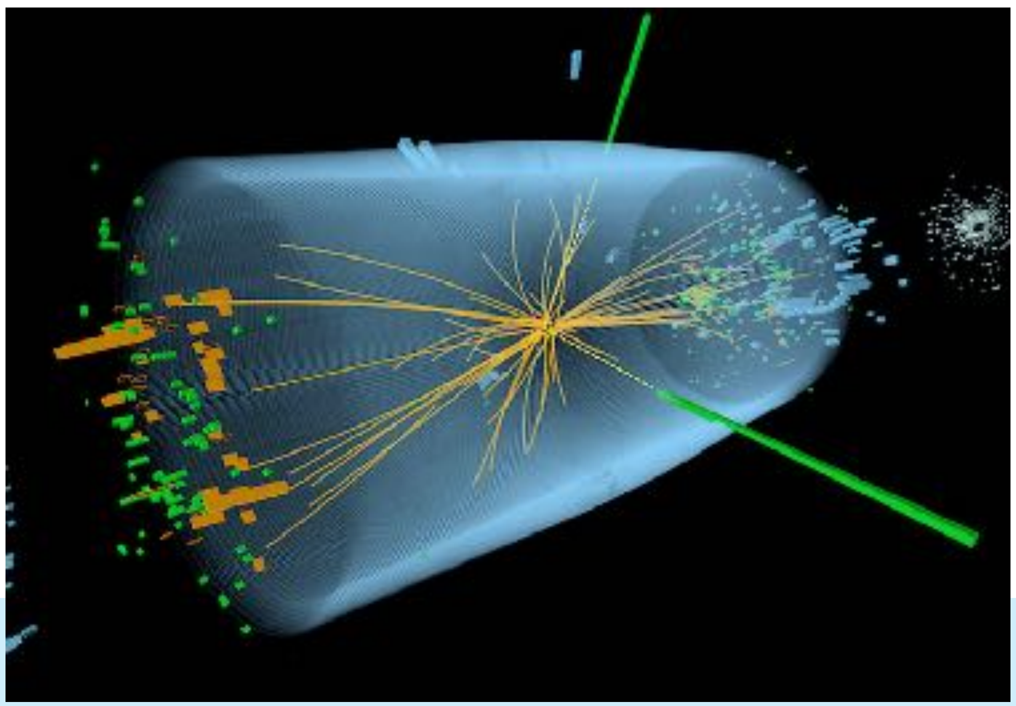
MIT faculty lunch

March 4, 2021

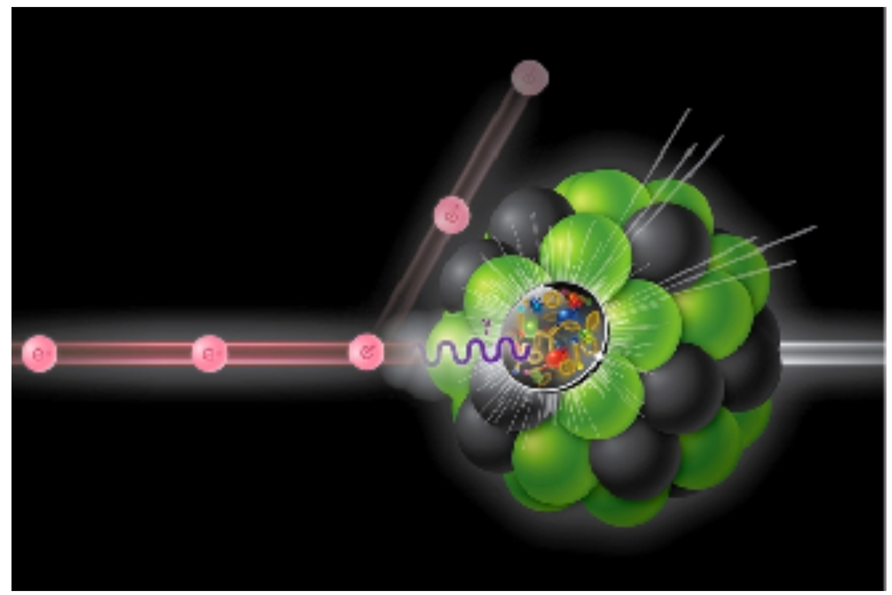
Real-time scattering experiments

Interesting hadron-hadron and hadron-lepton colliders abound

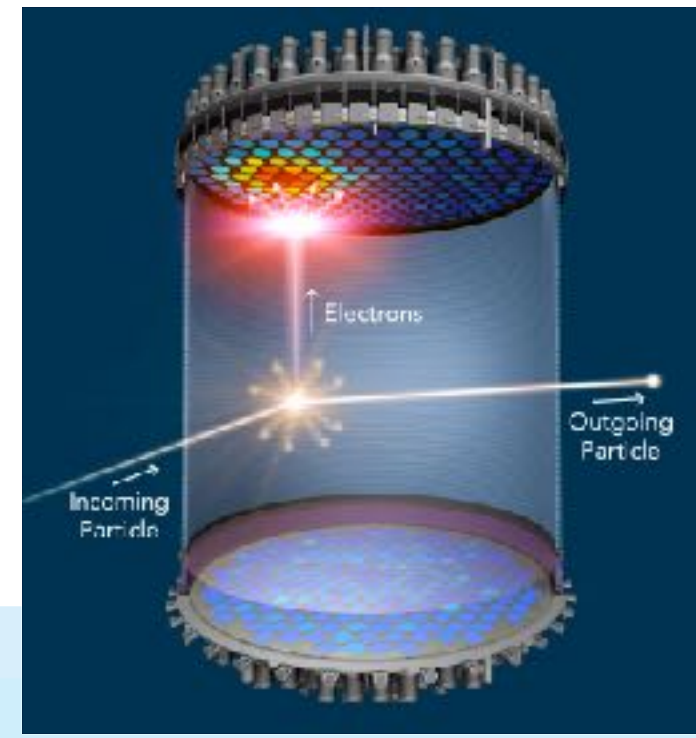
LHC



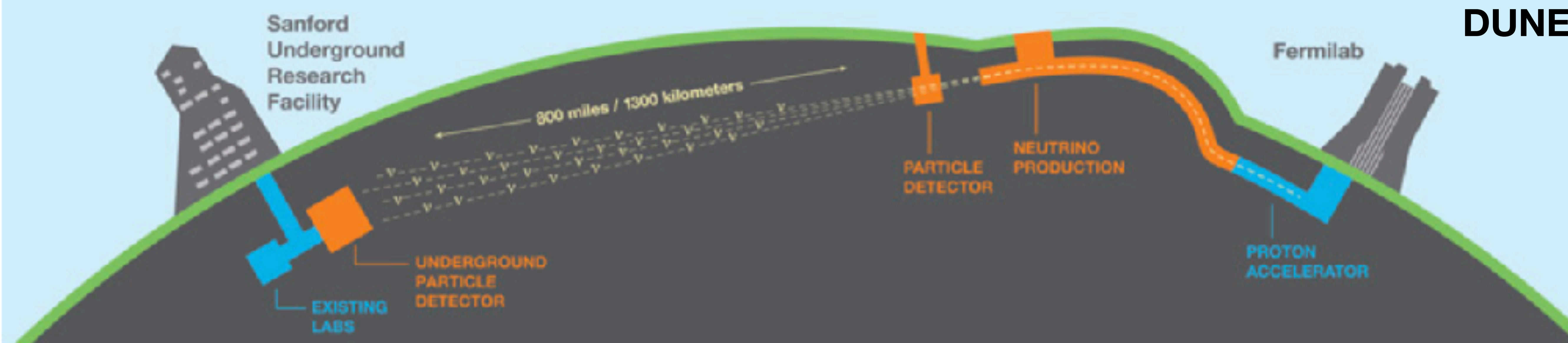
EIC



LUX

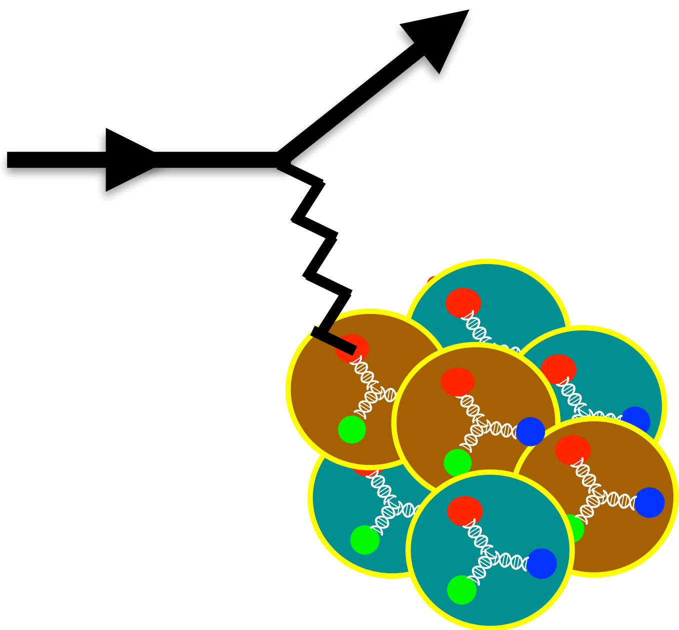


DUNE



Real-time scattering theory

Lepton-hadron cross-sections can be predicted using QCD + electroweak perturbation theory



Event rate / flux

Perturbative

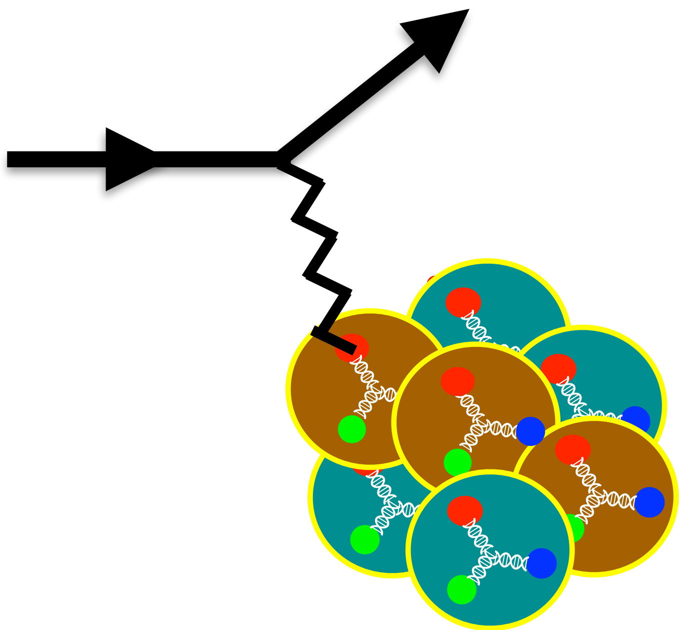
$$\frac{d^2\sigma}{dE'd\Omega} \propto L_{\mu\nu}W^{\mu\nu}$$

Kinematics of final state lepton in detector

$$W^{\mu\nu} = \langle f|J^\mu(x^0,\vec{x})J^\nu(y^0,\vec{y})|i\rangle$$

All the QCD stuff

It's hard to imagine



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$$W^{\mu\nu} = \langle f|J^\mu(x^0,\vec{x})J^\nu(y^0,\vec{y})|i\rangle$$

$$= \sum_n e^{-iE_n(x^0-y^0)}\rho^{\mu\nu}(E_n) \neq \sum_n e^{-E_n(x^0-y^0)}\rho^{\mu\nu}(E_n)$$

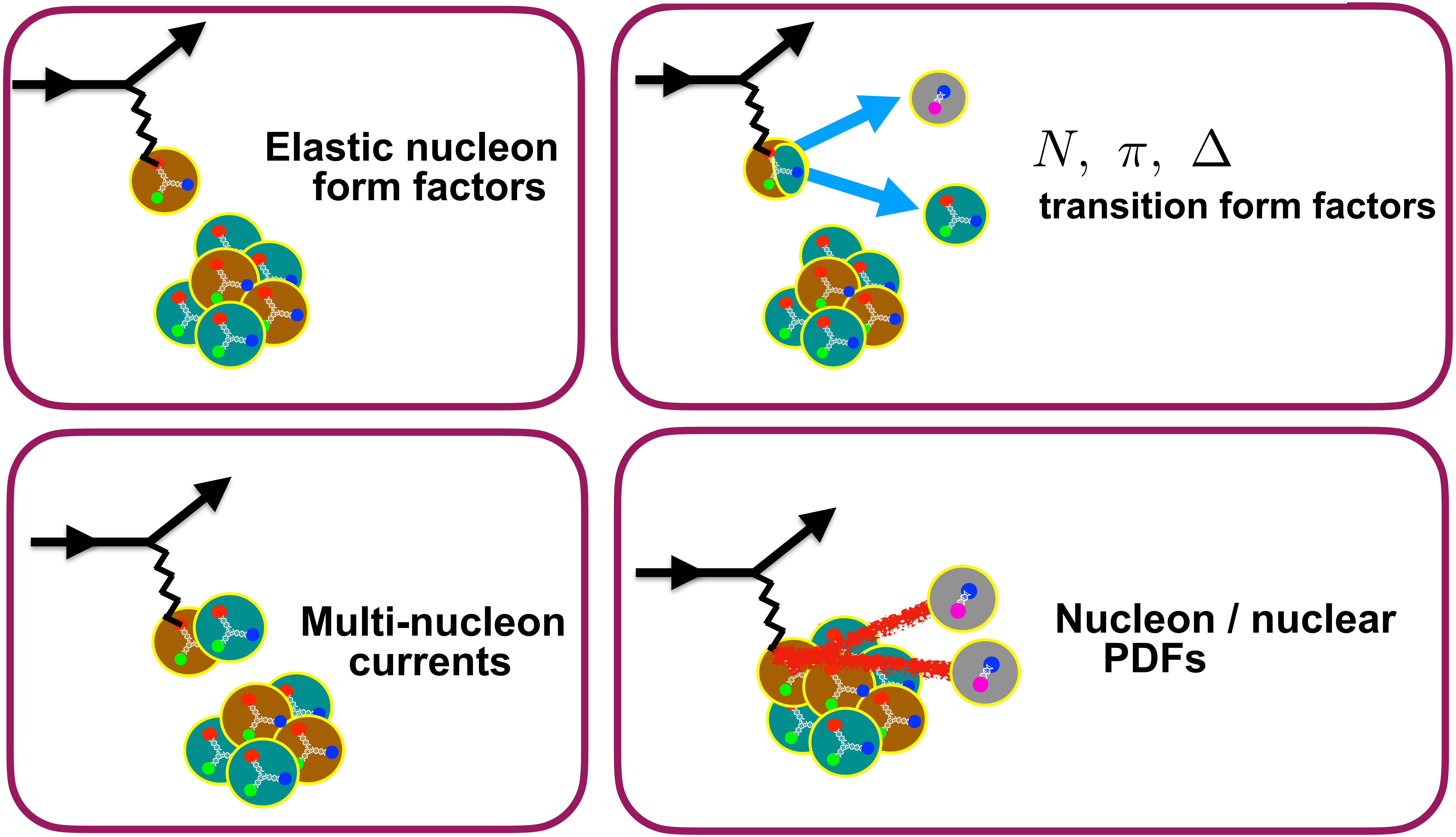
Real-time hadron tensor not simply related to imaginary time version

For strategies and practical challenges see e.g. Hansen, Meyer, Robaina PRD 96 (2017)
Gambino and Hashimoto PRL 125 (2020)

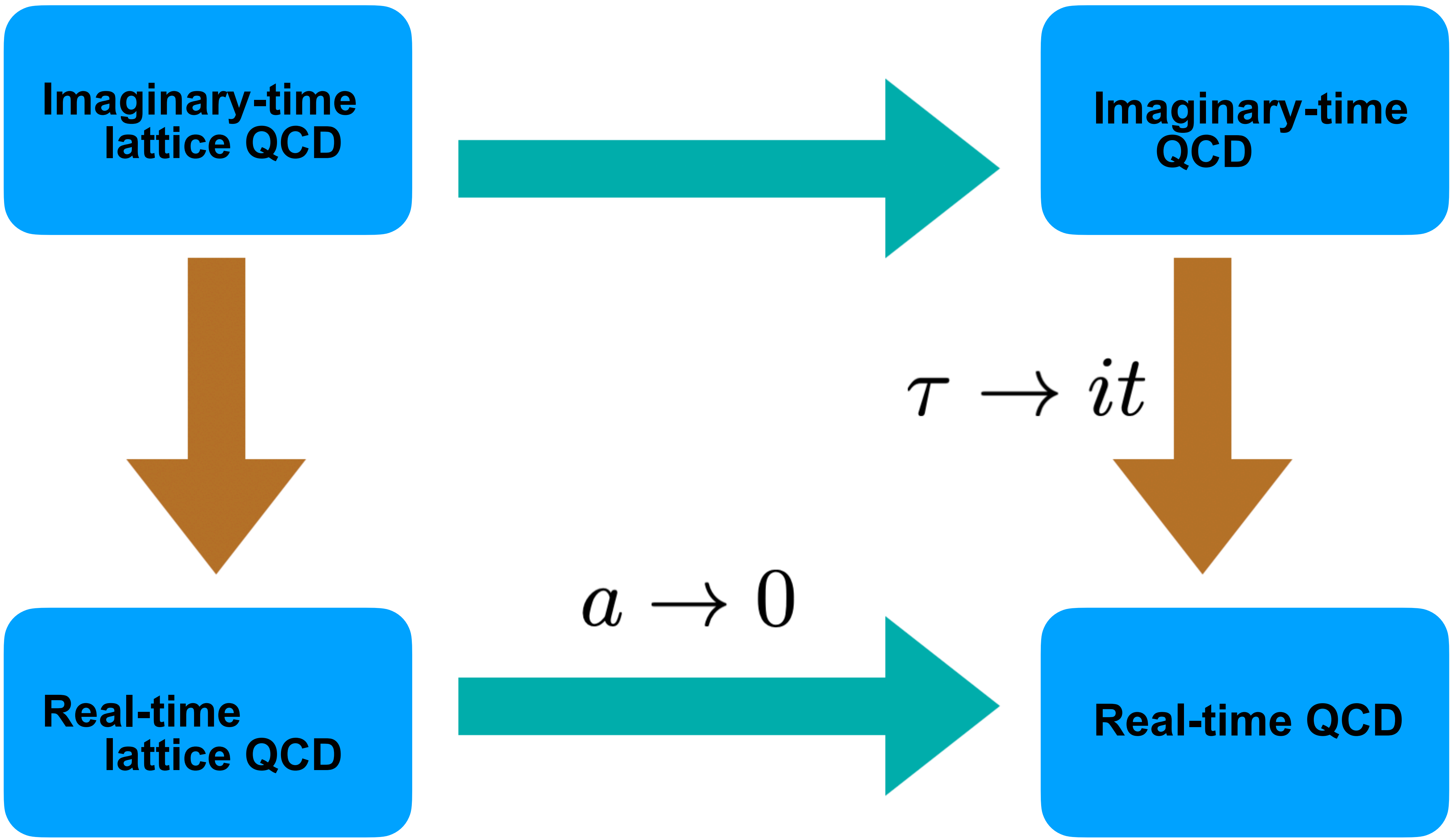
Lattice QCD and νA

LQCD can provide accurate constraints on νA cross sections at a wide range of energies with complementary strengths and weaknesses to experiment

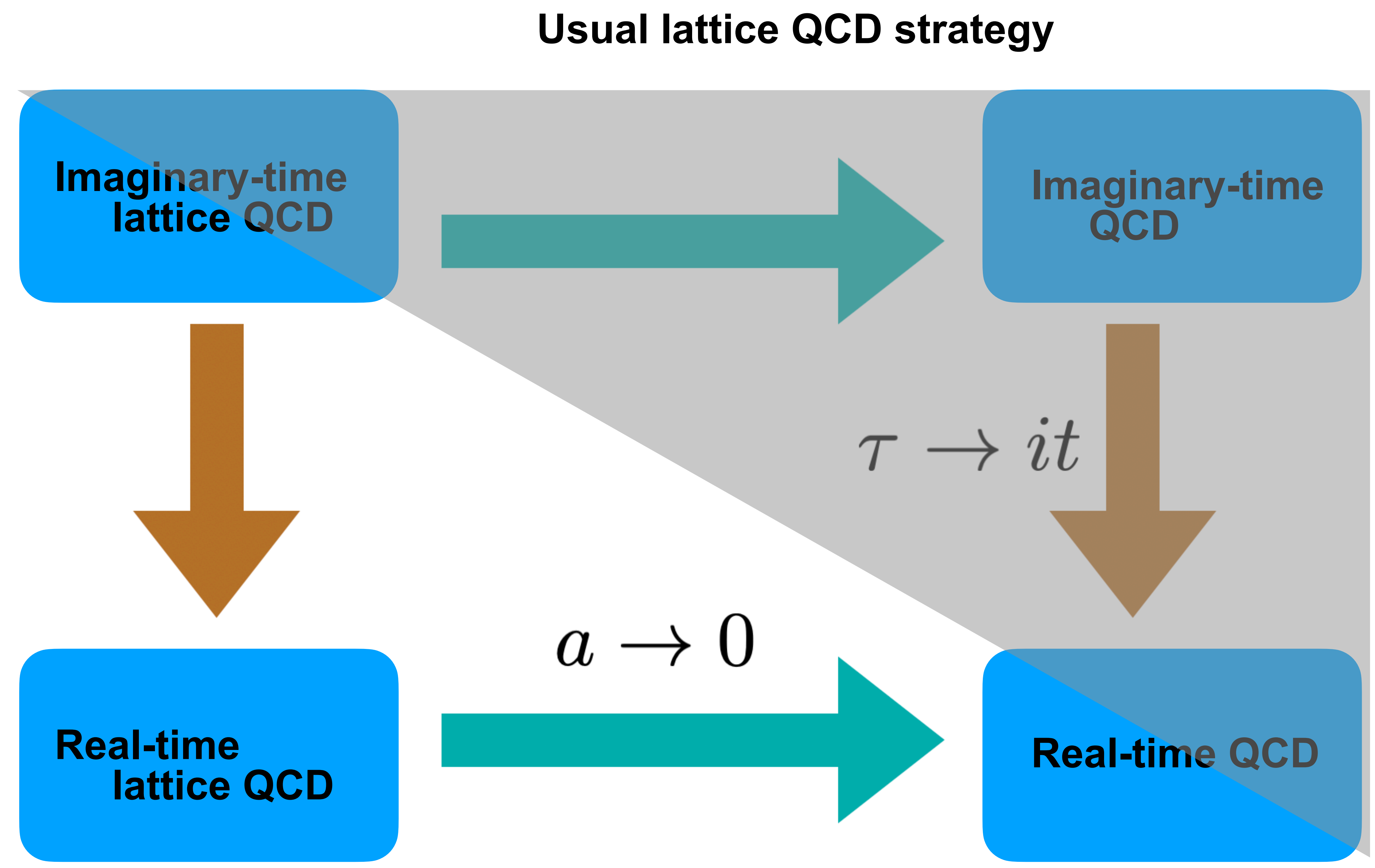
See USQCD νA white paper: Kronfeld et al Eur. Phys. J. A 55 (2019)



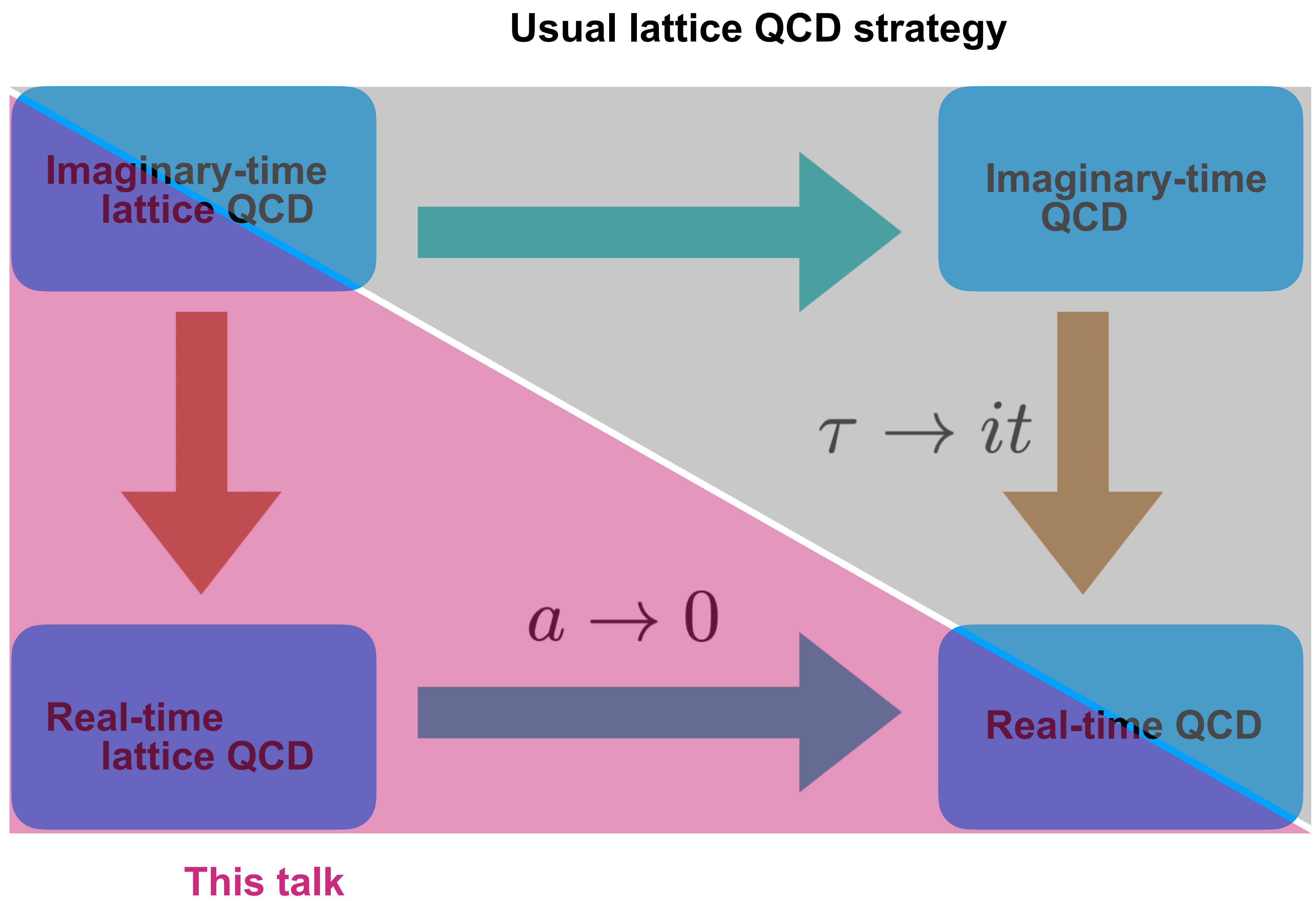
Orders of limits



Orders of limits



Orders of limits



Signs of trouble first pointed out in Hoshina, Fujii, Kikukawa, PoS LATTICE2019, 190 (2020)

The Simple Harmonic Oscillator

Continuum SHO action:

$$S_M[x(t)] = \int dt \frac{1}{2}(\partial_t x(t))^2 - \frac{\omega^2}{2} x(t)^2$$

Real-time

$$S_E[x(t)] = \int dt \frac{1}{2}(\partial_t x(t))^2 + \frac{\omega^2}{2} x(t)^2$$

Imaginary-time

Path integral definition:

$$< x'|e^{-i\hat{H}L_T}|x >= \int_{x_0=x}^{x_{L_T}=x'} \mathcal{D}x \; e^{iS_M}$$

$$< x'|e^{-\hat{H}L_T}|x >= \int_{x_0=x}^{x_{L_T}=x'} \mathcal{D}x \; e^{-S_E}$$

The lattice SHO

Lattice SHO action

$$S_M(x_t) = \sum_{n=0}^{L_T/a-1} \frac{1}{2a}(x_{na+a} - x_{na})^2 - \frac{\omega^2}{2} x_{na}^2$$

$$S_E(x_t) = \sum_{n=0}^{L_T/a-1} \frac{1}{2a}(x_{na+a} - x_{na})^2 + \frac{\omega^2}{2} x_{na}^2$$

Lattice SHO path intgerals

$$\int_{x_0=x}^{x_{L_T}=x'} \mathcal{D}x \; e^{iS_M} =< x'| \prod_{n=0}^{L_T/a-1} \hat{T}_M |x >$$

$$\int_{x_0=x}^{x_{L_T}=x'} \mathcal{D}x \; e^{-S_E} =< x'| \prod_{n=0}^{L_T/a-1} \hat{T}_E |x >$$

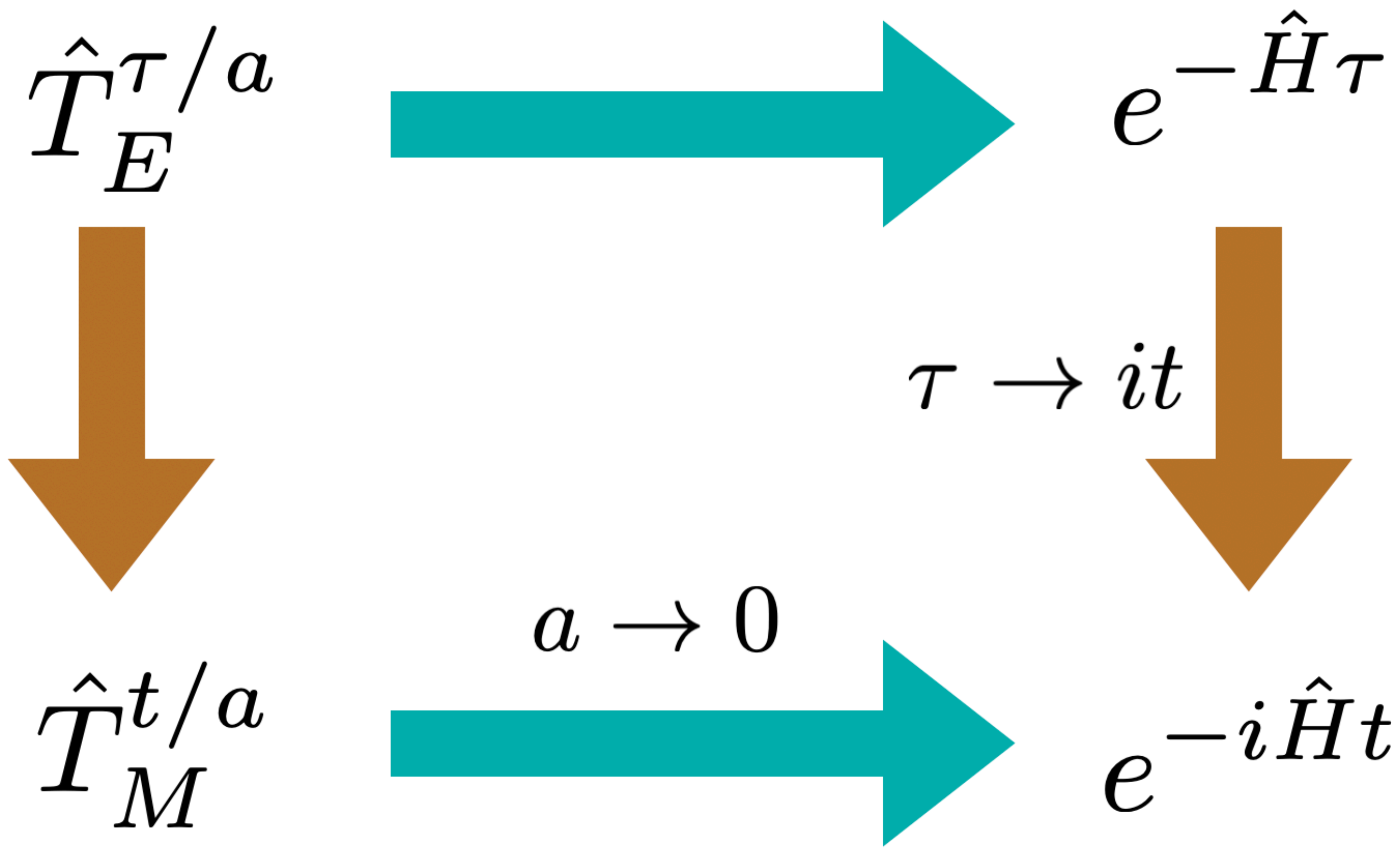
Transfer matrix:

$$\left\langle x'_{(n+1)a} | \hat{T}_M | x_{na} \right\rangle = e^{\frac{i}{2a}(x_{na+a} - x_{na})^2 - \frac{i\omega^2}{2} x_{na}^2}$$

$$\left\langle x'_{(n+1)a} | \hat{T}_E | x_{na} \right\rangle = e^{-\frac{1}{2a}(x_{na+a} - x_{na})^2 - \frac{\omega^2}{2} x_{na}^2}$$

Orders of limits

For the SHO, the continuum limit commutes with analytic continuation between real and imaginary time



Real-time transfer matrix is unitary

$$\hat{T}_M = e^{-ia\hat{V}/2} e^{-ia\hat{K}} e^{-ia\hat{V}/2}$$

Imaginary-time transfer matrix is positive

$$\hat{T}_E = e^{-a\hat{V}/2} e^{-a\hat{K}} e^{-a\hat{V}/2}$$

The quantum rotator

Free particle constrained to move on a circle

$$x(t) \in [0, 2\pi]$$

Same continuum action can be used as free SHO

Naive discretization breaks periodicity, usual prescription in Euclidean is to use different action with same small-a behavior

$$S_M(x_t) = \frac{1}{a} \sum_{n=0}^{L_T/a-1} 1 - \cos(x_{na+a} - x_{na})$$

$$S_E(x_t) = \frac{1}{a} \sum_{n=0}^{L_T/a-1} 1 - \cos(x_{na+a} - x_{na})$$

Real-time transfer matrix:

$$\hat{T}(x_{t+a}, x_t) = e^{\frac{i}{a} - \frac{i}{a} \cos(x_{t+a} - x_t)}$$

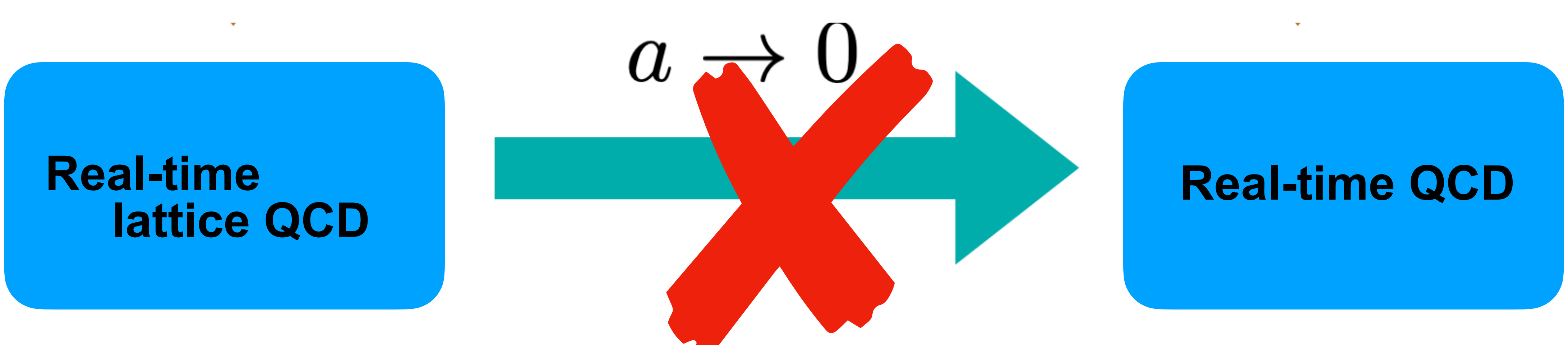
Non-unitarity

Explicit calculations shows the quantum rotator real-time transfer matrix is non-unitary:

$$\hat{T}_M(x,y)\hat{T}_M^\dagger(y,x') \neq \delta(x-x')$$

Unitarity requires eigenvalue ratios to have magnitude 1 as $a \to 0$

But $a \to 0$ limits of these eigenvalues ratios do not exist for quantum rotators



Lattice gauge theory

Gauge transformations act on matter fields as

$$\psi_x^a \to \Omega_x^{ab} \psi_x^b$$

$$\Omega_x \in SU(N),\ U(1)$$

Gauge field acts as parallel transporter in color space

$$U_{x,\mu} \in SU(N),\ U(1)$$

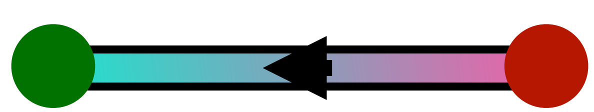
$$U_{x,\mu} = e^{iaA_\mu(x)}$$

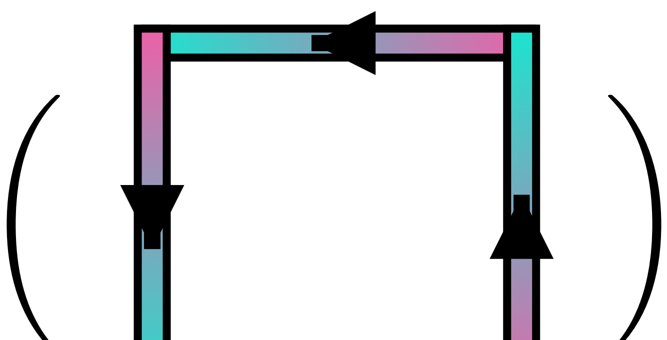
$$U_{x,\mu} \to \Omega_x U_{x,\mu} \Omega_{x+\hat{\mu}}^\dagger$$

$$D_\mu^{ab} \psi_x^b = U_{x,\mu}^{ab} \psi_{x+\hat{\mu}}^b - \psi_x^a$$

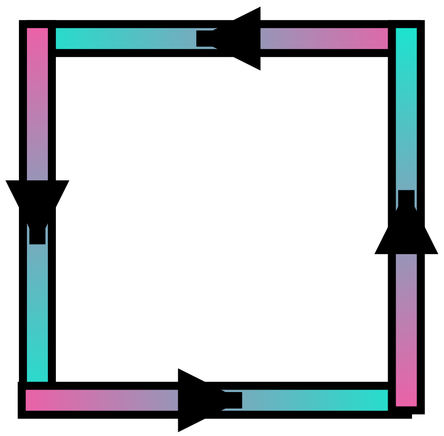
$$D_\mu \psi_x^a \to \Omega_x^{ab} D_\mu \psi_x^b$$

Gauge invariant building blocks:



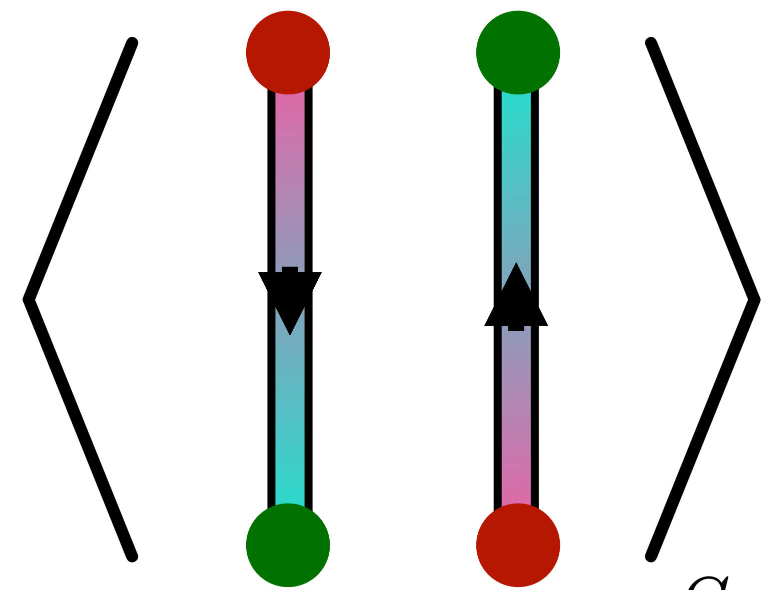


Wilson loops



$$W_{\mathcal{A}} = \prod_{x,\mu \in \partial\mathcal{A}} U_{x,\mu}$$

$=$



$S_{\psi,\mathrm{static}}$

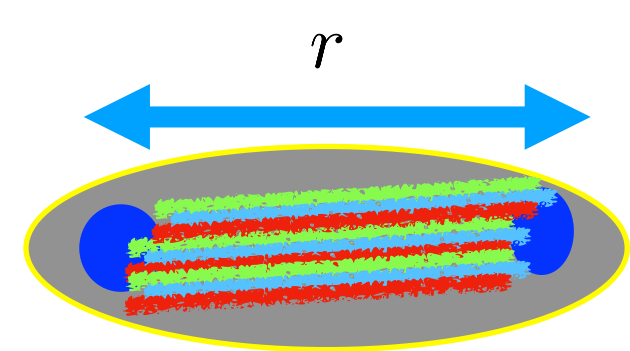
Wilson loops are equivalent to static quark propagators

$$S_{\psi,\mathrm{static}} = \sum_x \overline{\psi}_x D_4 \psi_x$$

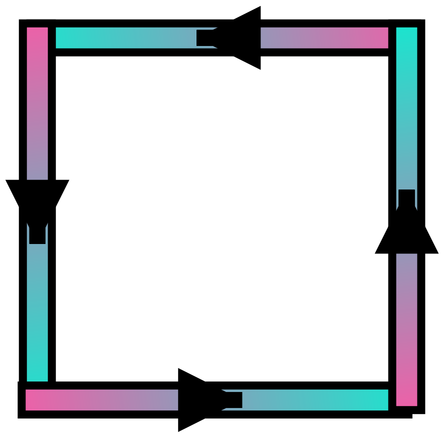
Since by equations of motion $\psi_{(\vec{x},\tau)} = \prod_{\tau'=0}^{\tau} U^{-1}_{(\vec{x},\tau'),4} \psi_{(\vec{x},0)}$

Static quark potential accessible from Wilson loops

$$\langle W_{r\times\tau} \rangle = \sum_n Z_n \, e^{-E_n(r)\tau} = e^{-V(r)\tau} + \ldots$$



The Wilson action



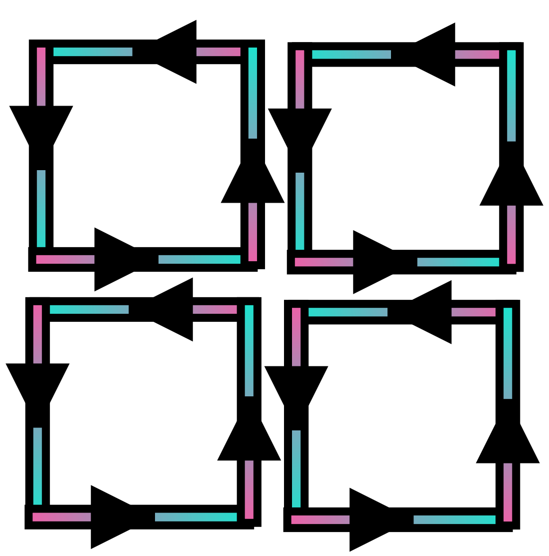
“Plaquettes” are 1x1 Wilson loops

$$P_{x,\mu\nu} = U_{x,\mu} U_{x+\hat{\mu},\nu} U^{-1}_{x+\hat{\mu}+\hat{n}u,\mu} U^{-1}_{x+\hat{\nu},\nu}$$

Wilson action provides simple, gauge-invariant action with correct naive continuum limit

$$S_W(U) = \frac{1}{g^2} \sum_x \sum_{\mu<\nu} \mathrm{Tr}\left[2 - P_{x,\mu\nu} - P^{-1}_{x,\mu\nu}\right]$$

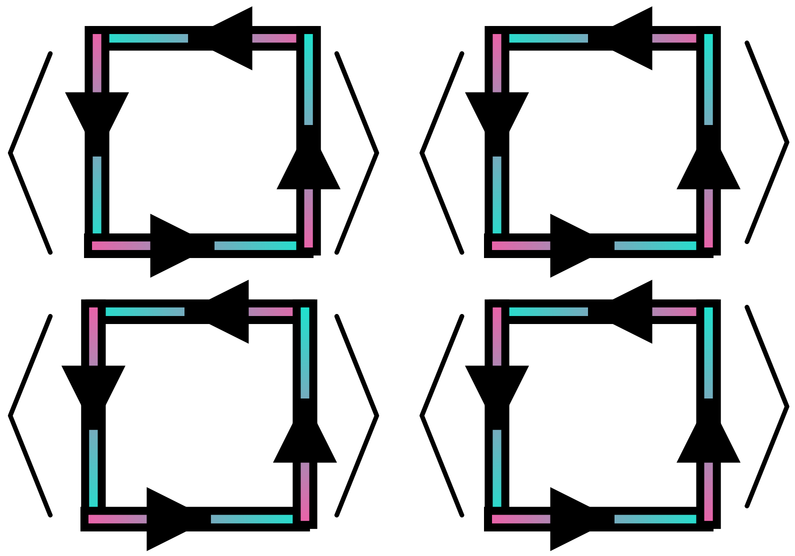
Wilson loops can be expressed using plaquettes, most simply taking open boundary conditions and gauge-fixing $U_{x,\nu} = 1$



$$W_{\mathcal{A}} = \prod_{x,\mu \in \partial\mathcal{A}} U_{x,\mu} = \prod_{x \in \mathcal{A}} P_{x,\mu\nu}$$

2D Confinement

In 2D, Wilson loop expectation values further factorize into products of single-plaquette expectation values



$$\langle \mathrm{Tr}(W_{\mathcal{A}}) \rangle = \prod_{x \in \mathcal{A}} \langle \mathrm{Tr}(P_x) \rangle = \langle \mathrm{Tr}(P) \rangle^A$$

Implies confinement

$$\frac{1}{N} \langle \mathrm{Tr}(W_{\mathcal{A}}) \rangle = e^{-\sigma A}$$

Static quark potential $V(r) = \sigma r$

Confining potential arises for any gauge group in 2D from factorization

$$\sigma_{U(1)} = \ln \left(\frac{I_0(1/e^2)}{I_1(1/e^2)} \right)$$

$$\sigma_{SU(2)} = \ln \left(\frac{I_1(4/g^2)}{I_2(4/g^2)} \right)$$

Gross and Witten, PRD 21 (1980)

Wadia, arXiv:1212.2906 (1979)

The real-time Wilson action

Wilson action splits into kinetic (timelike plaquettes) and potential (spacelike plaquttes) terms

Making usual sign flips, a real-time Wilson action is obtained

$$S_{M,W}(U) = \frac{1}{g^2} \sum_x \sum_k \mathrm{Tr}\left[2 - P_{x,0k} - P^{-1}_{x,0k}\right]$$

$$-\frac{1}{g^2} \sum_x \sum_{i<j} \mathrm{Tr}\left[2 - P_{x,ij} - P^{-1}_{x,ij}\right]$$

In (1+1)D only the kinetic term appears, and path integrals are simply related between real- and imaginary-time

$$e^{iS_{M,W}(U,g^2)} = e^{-S_{E,W}(U,ig^2)}$$

The Wilson action is non-unitary

Wick rotate action

$$g^2 \to ig^2$$

$$\left\langle \frac{1}{2}\mathrm{Tr}(W_{\mathcal{A}}) \right\rangle_{M,W,SU(2)} = \left[\frac{I_1(4i/g^2)}{I_2(4i/g^2)}\right]^{-Lt}$$

Non-unitary time evolution

$$\left\langle \frac{1}{2}\mathrm{Tr}(W_{\mathcal{A}}) \right\rangle_{E,W,SU(2)} = \left[\frac{I_1(4/g^2)}{I_2(4/g^2)}\right]^{-L\tau}$$

Analytically continue time

$$\tau \to it$$

$$\left\langle \frac{1}{2}\mathrm{Tr}(W_{\mathcal{A}}) \right\rangle_{M,HFK,SU(2)} = \left[\frac{I_1(4/g^2)}{I_2(4/g^2)}\right]^{-iLt}$$

The HFK action

The non-unitarity of the Wilson real-time transfer matrix was first pointed out by Hoshina, Fujii, and Kikukawa (HFK)

Hoshina, Fujii, Kikukawa, PoS LATTICE2019, 190 (2020)

Starting from the character expansion of the Wilson action

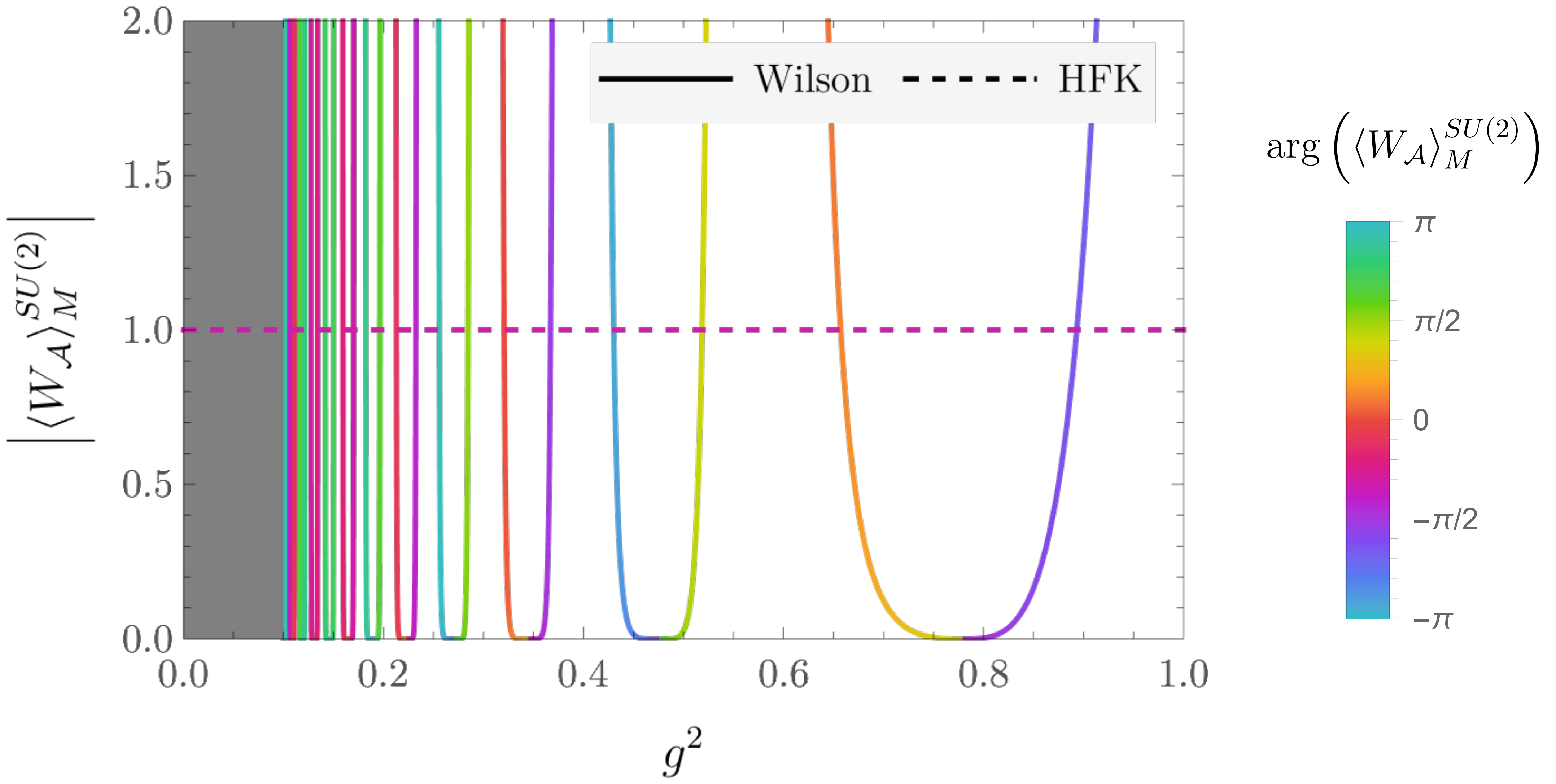
$$e^{-S_{E,W}(U)} = e^{-\frac{i}{a}V_W(U)} \prod_{x,k} \left[\sum_r c_r^W(e^2)\chi_r(P_{x,0k}) \right]$$

The real-time HFK action is defined by replacing the eigenvalues with pure phases to give a unitary transfer matrix by construction

$$e^{iS_{M,HKF}(U)} = e^{-\frac{i}{a}V_W(U)} \prod_{x,k} \left[\sum_r [c_r^W(e^2)]^i \chi_r(P_{x,0k}) \right]$$

Wilson and HFK in (1+1)D

Analytic calculations using HFK action in (1+1)D recover exact results for analytic continuation of Euclidean Wilson to real time



Analytic results for real-time Wilson action show non-unitary continuum limit or have singularities obstructing limit (depends on N)

Divergences

HFK action well-defined for analytic calculations, but character expansion defining HFK action is a divergent function of gauge field

Pure phase

$$e^{iS_{M,HKF}(U)} = e^{-\frac{i}{a}V_W(U)} \prod_{x,k} \left[\sum_r [c_r^W(e^2)]^i \chi_r(P_{x,0k}) \right]$$

Non-zero for all r for some or all field configurations

Rapid phase fluctuations lead to convergence of HFK path integrals, but without absolute convergence impossible to perform sum over representations using Monte Carlo methods

Changing paths

Consider a path integral with a sign problem

$$\langle \mathcal{O} \rangle_M = \frac{1}{Z_M} \int_{\mathcal{M}} \mathcal{D}U \; e^{iS_M(U)} \; \mathcal{O}(U)$$

Deform the integration contour

$$= \frac{1}{Z_M} \int_{\widetilde{\mathcal{M}}} \mathcal{D}U \; e^{iS_M(\widetilde{U})} \; \mathcal{O}(\widetilde{U})$$

$$= \frac{1}{Z_M} \int_{\mathcal{M}} \mathcal{D}U \; J(U) \; e^{iS_M(\widetilde{U}(U))} \; \mathcal{O}(\widetilde{U}(U))$$

Deformed integrand can have less severe sign problem

$$= \frac{1}{Z_M} \int_{\mathcal{M}} \mathcal{D}U \; |J(U)| \; e^{-\mathrm{Im}[S_M(\widetilde{U}(U))]} \; \mathcal{O}(\widetilde{U}(U))$$

$$\times e^{i\mathrm{Re}[S_M(\widetilde{U}(U))] + i\arg[J(U)]}$$

Many previous works:

Witten, AMS/IP Stud.Adv.Math. 50 (2011)

Cristoforetti, Di Renzo, Scorzato, PRD 86 (2012)

…

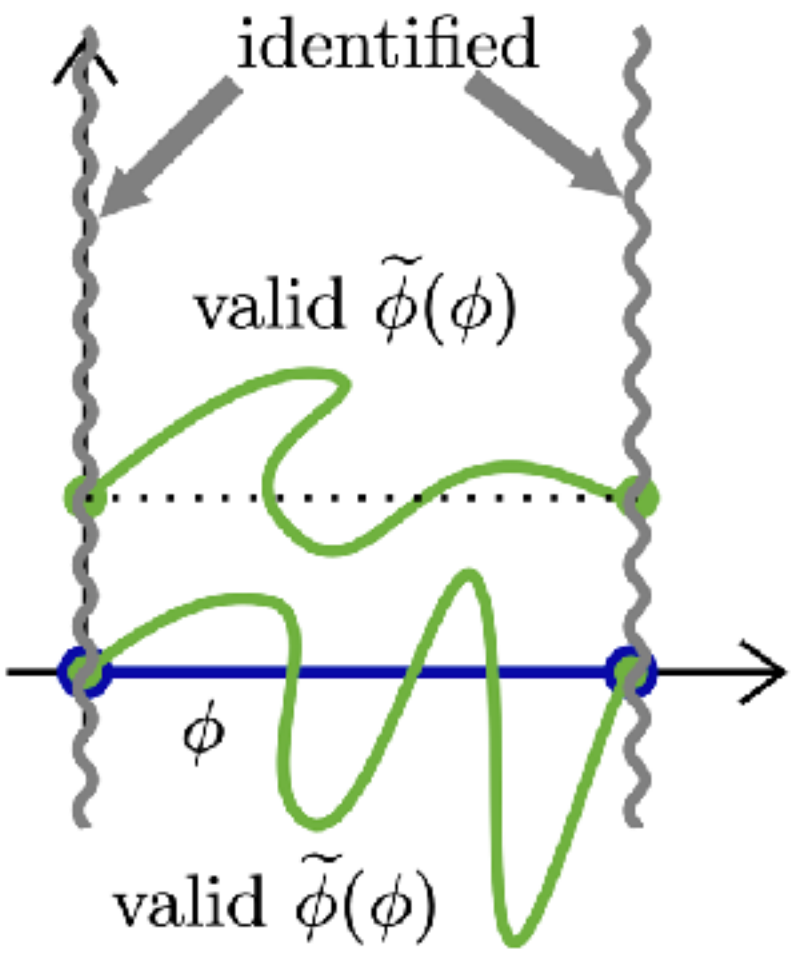
Recent review:

Alexandru, Basar, Bedaque, Warrington, arXiv:2007.05436

A toy sign problem

$$\left\langle e^{i\phi} \right\rangle_\beta = \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, e^{i\phi}\, e^{\beta \cos(\phi)} = \frac{I_1(\beta)}{I_0(\beta)}$$

Stokes' theorem + holomorphic integrand → integral result unaffected by contour deormation



Constant vertical deformation:

$$\left\langle e^{i\phi} \right\rangle_\beta = \frac{1}{Z} \int_{-\pi+if}^{\pi+if} \frac{d\phi}{2\pi}\, e^{i\phi}\, e^{\beta \cos(\phi)}$$

$$= \frac{1}{Z} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, e^{i\phi - f}\, e^{\beta \cos(\phi + if)}$$

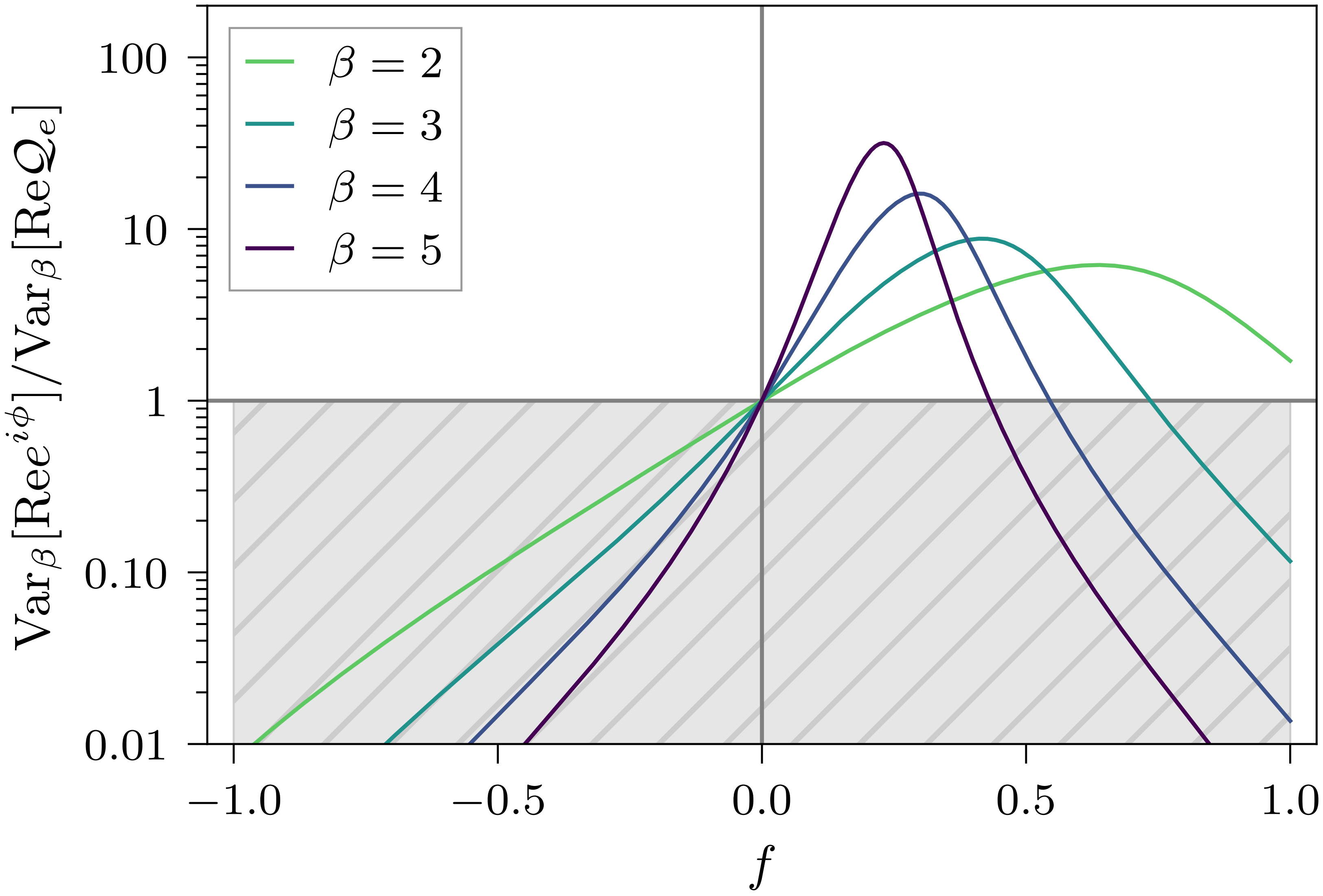
$$= \left\langle e^{i\phi - f} e^{\beta \cos(\phi + if) - \beta \cos(\phi)} \right\rangle_\beta \equiv \left\langle \mathcal{Q}_e \right\rangle_\beta$$

Detmold, Kanwar, Lamm, MW, Warrington, arXiv:2101.12668

Variance reduction

The variance involves non-holomorphic integrands

$$\mathrm{Var}_\beta[\mathrm{Re}\ \mathcal{Q}_e] = \left\langle (\mathrm{Re}\ \mathcal{Q}_e)^2 \right\rangle_\beta - \left\langle e^{i\phi} \right\rangle_\beta^2 \neq \mathrm{Var}_\beta[\mathrm{Re}\ e^{i\phi}]$$



$$\mathcal{Q}_e = e^{-f} e^{i\phi} e^{\Delta S}$$

Detmold, Kanwar, Lamm, MW, Warrington, arXiv:2101.12668

Deformed observables

Deformed observables method: contour deformations without modifying Monte Carlo sampling

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{D}U \; e^{-S(U)} \; \mathcal{O}(U)$$

$$= \frac{1}{Z} \int_{\widetilde{\mathcal{M}}} \mathcal{D}\widetilde{U} \; e^{-S(\widetilde{U})} \; \mathcal{O}(\widetilde{U})$$

$$= \frac{1}{Z} \int_{\mathcal{M}} \mathcal{D}U \; e^{-S(U)} \; \det\left(\frac{\partial \widetilde{U}}{\partial U}\right) \; e^{-S(\widetilde{U}(U))+S(U)} \; \mathcal{O}(\widetilde{U}(U))$$

$$\equiv \frac{1}{Z} \int_{\mathcal{M}} \mathcal{D}U \; e^{-S(U)} \; \mathcal{Q}(U)$$

$$\langle \mathcal{O} \rangle = \langle \mathcal{Q} \rangle$$

$$\mathrm{Var}[\mathcal{O}] \neq \mathrm{Var}[\mathcal{Q}]$$

Detmold, Kanwar, MW, Warrington, PRD 102 (2020)

2D U(1) contour deformations

Using the parameterization

$$P = e^{i\phi} \in U(1)$$

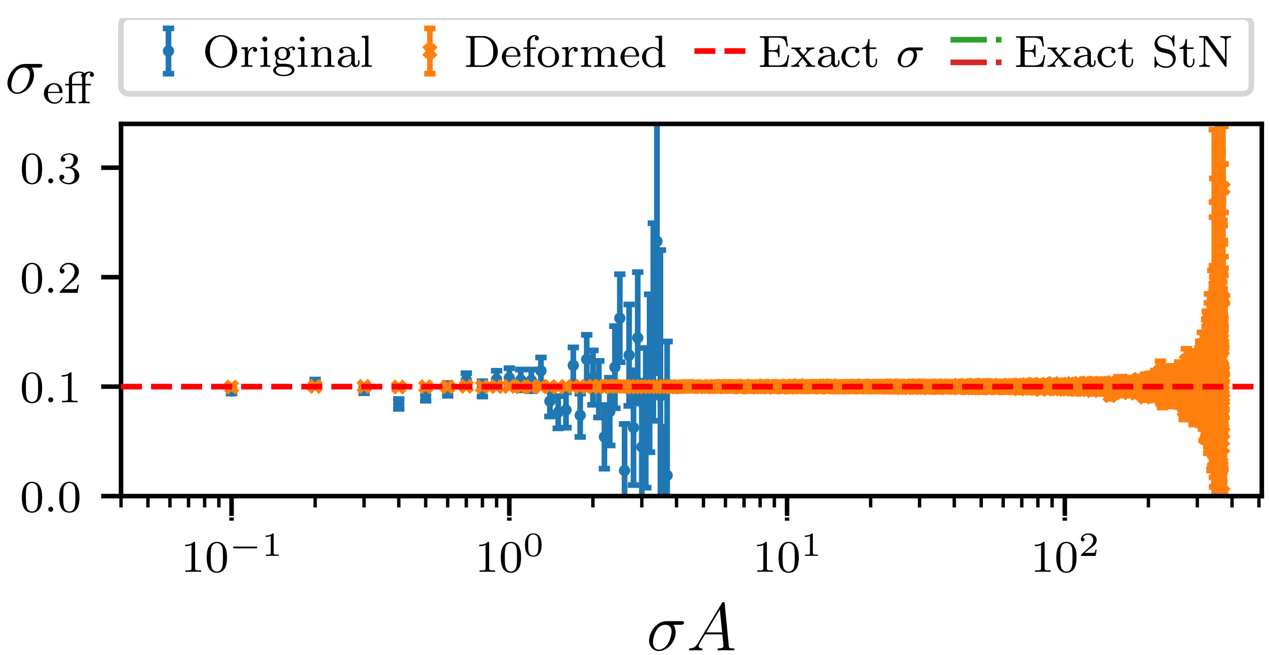
$$\langle W_{\mathcal{A}} \rangle = \left(\int \frac{dP}{2\pi I_0(1/e^2)} \; P \; e^{\frac{1}{2e^2}(P+P^{-1})} \right)^A$$

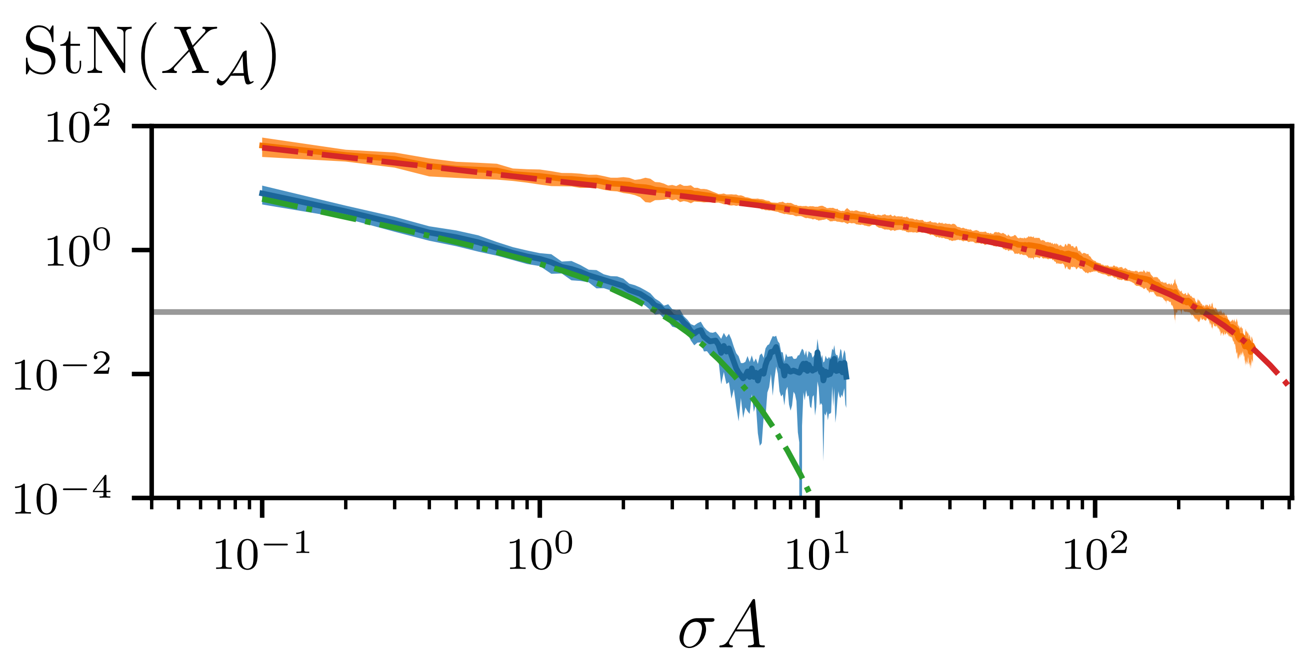
U(1) Wilson loops are products of toy sign problem integrals

$$= \left(\int_{-\pi}^{\pi} \frac{d\phi}{2\pi I_0(1/e^2)} \; e^{i\phi} \; e^{\frac{1}{e^2}\cos(\phi)} \right)^A$$

Contour deformation analogous to toy problem for U(1) Wilson loops

$$e^{i\phi} \rightarrow e^{i(\phi + if)}$$



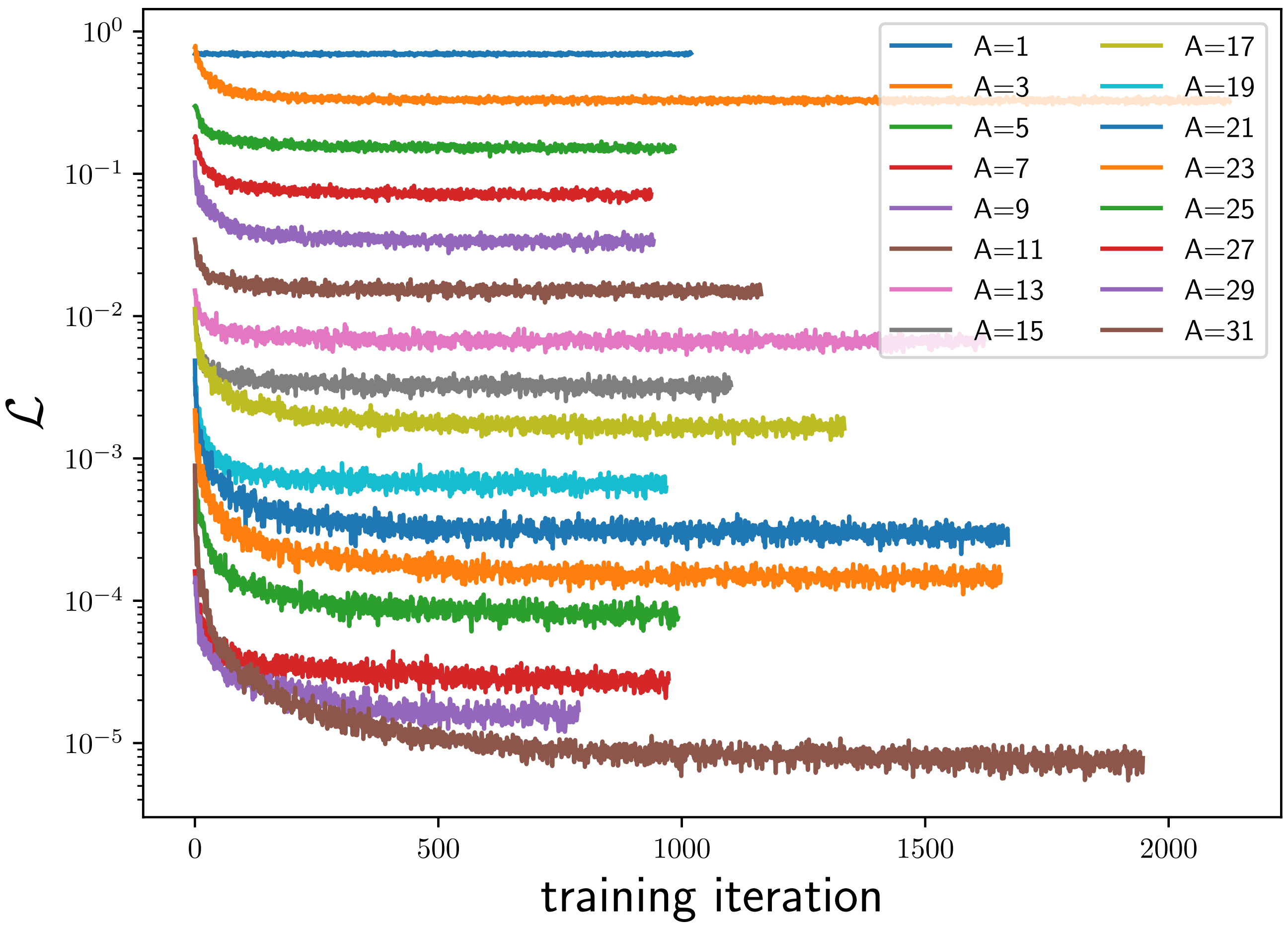


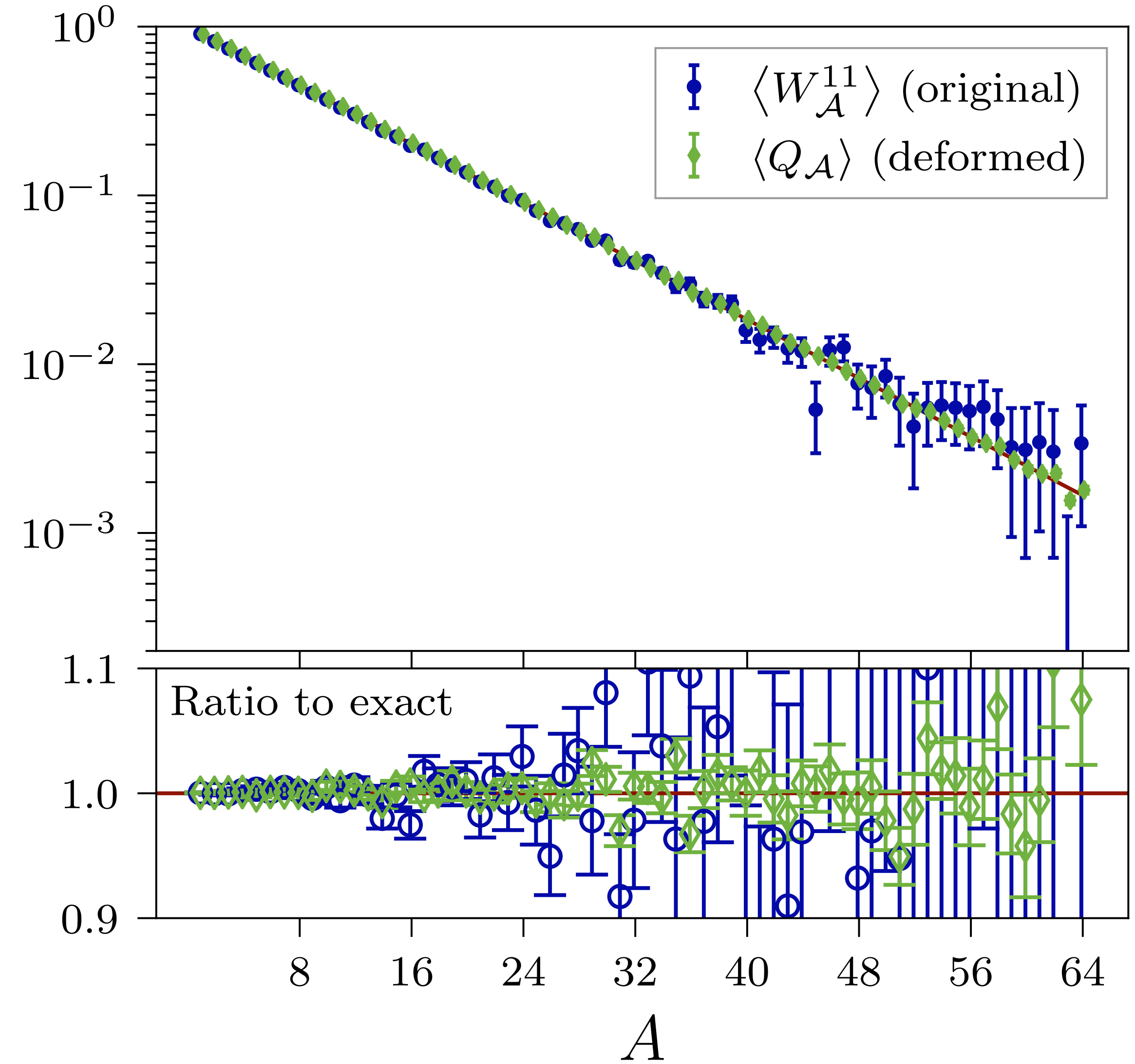
Detmold, Kanwar, MW, Warrington, PRD 102 (2020)

2D SU(3) contour deformations

Variance minimization of parameterized deformations is a well-posed optimization problem suitable for machine learning techniques

Parameterization and optimization strategies recently explored for SU(N)





Detmold, Kanwar, Lamm, MW, Warrington, arXiv:2101.12668

Deformations and convergence

Contour deformation methods can also improve convergence of real-time unitary actions

$$\sum_{\{n\}} \int \mathcal{D}U e^{iS_M(U,n)} = \sum_{\{n\}} \int \mathcal{D}U \; J(U) \; e^{iS_M(\widetilde{U}(U,n),n)}$$

Convergent, but not absolutely

(can't Monte Carlo)

Possibly absolutely convergent if cutoff provided by

$$e^{-\mathrm{Im}\left[S_M(\widetilde{U}(U,n),n)\right]}$$

If (and only if) absolutely convergent path integral representation exists, can use Monte Carlo to perform joint sum-integral

Kanwar, MW, arXiv:2103.02602

Convergent U(1) HFK ?

A simple contour deformation appears to provide convergence

$$\phi_{x,0k} \to \widetilde{\phi}_{x,0k} = \phi_{x,0k} + i \ \mathrm{sign}(r_{x,k})$$

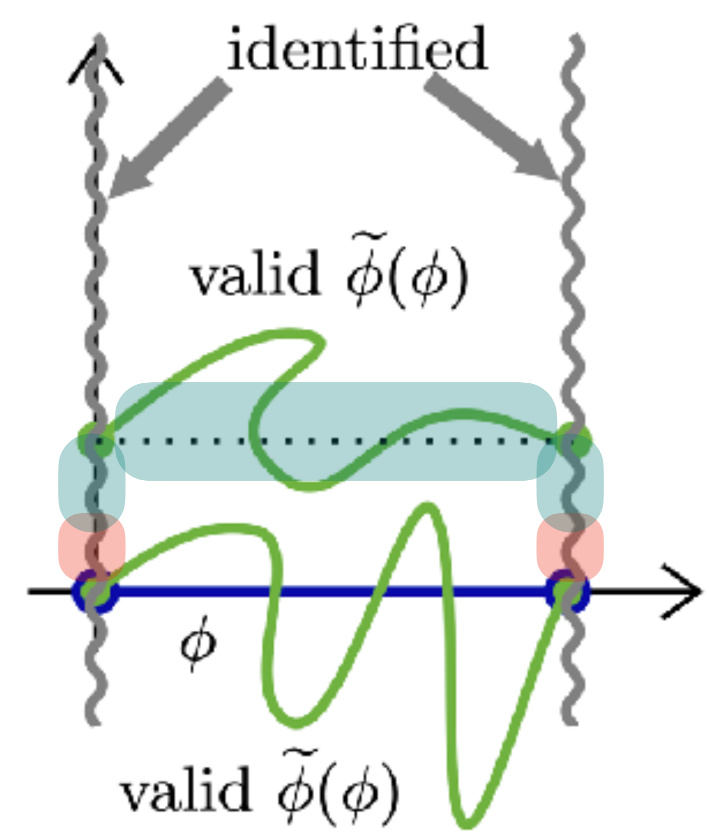
$$e^{iS_{M,HKF}(\widetilde{U},r)} = e^{-\frac{i}{a}V_W(\widetilde{U})} \prod_{x,k} \left[[c_r^W(e^2)]^i e^{ir_{x,k}\phi_{x,0k}} e^{-|r_{x,k}|} \right]$$

Exponential damping leads to absolute convergence everyone on deformed contour

Except on the parts that we implicitly canceled in order to shift continuously…

$$\widetilde{\phi}_{x,0k} = \phi_{x,0k} + i\alpha_{x,k}$$

$$\alpha \in [0, \mathrm{sign}(r_{x,k})]$$



Wick rotation regularization

Minkowski action regularized by introducing “Wick rotation” angle

$\theta \in [0, \pi/2]$

Euclidean: $\theta = 0$

Minkowski: $\theta = \pi/2$

Small for large $|r|$

$$e^{iS_{M,HKF}(U)} \to e^{-\frac{i}{a}V_W(U)} \prod_{x,k} \left[\sum_{r=-\infty}^{\infty} [c_r^W(e^2)]^{e^{i\theta}} e^{ir\phi_{x,0k}} \right]$$

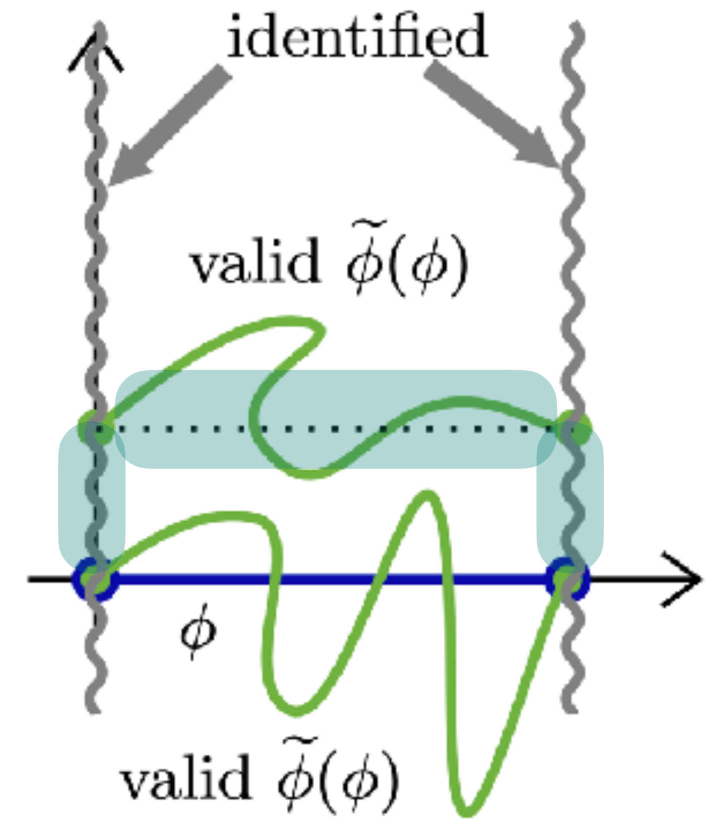
Sum absolutely convergent for $\theta < \pi/2$

Recipe for real-time path integrals

1) Regularize with Wick rotation angle

2) Perform contour deformation, enforcing cancellations arising from shift symmetry

3) Take Minkowski limit on deformed contour

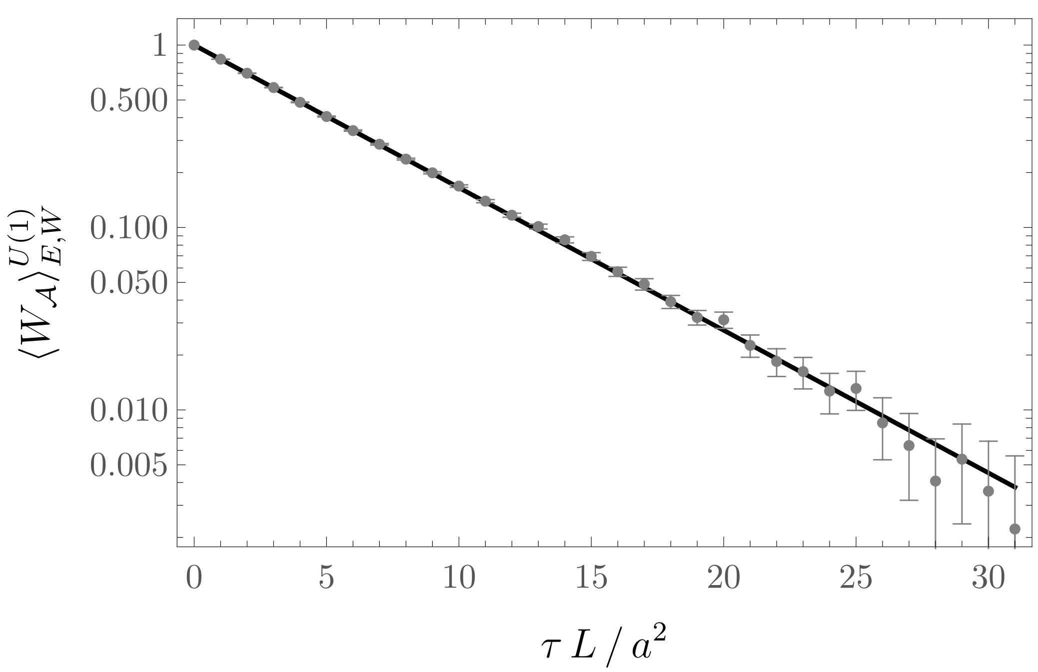


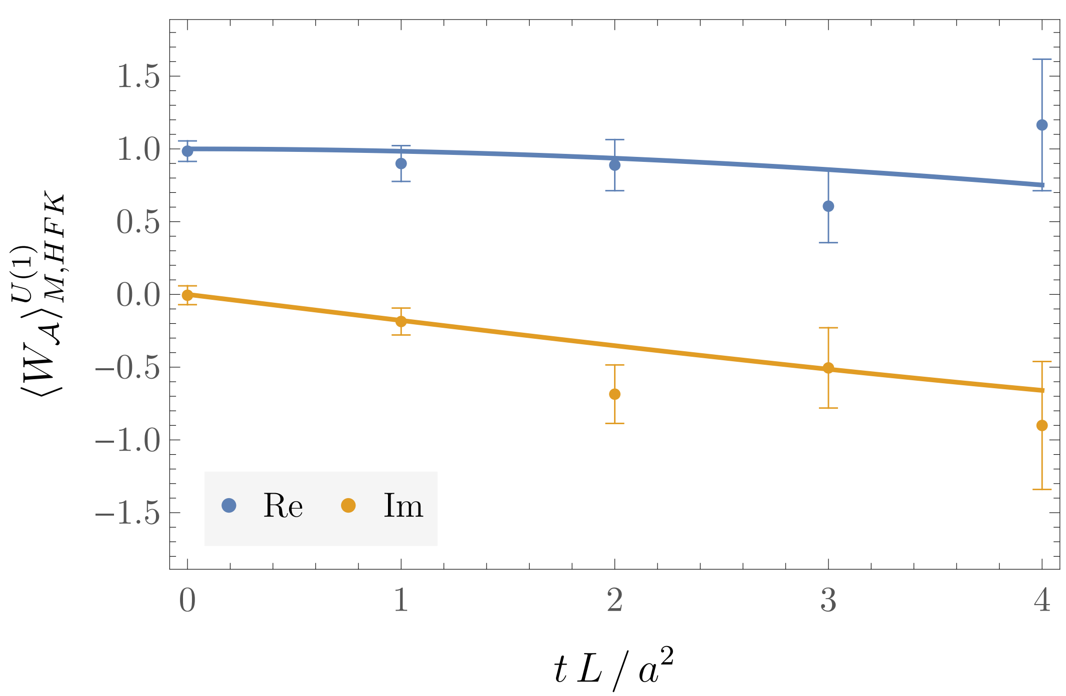
Real-time U(1) HFK results

Infinite sum in contour deformed real-time HFK action can be performed stochastically with integer-valued auxiliary field

$$\int \mathcal{D}U \; J(U)\; e^{-S(\widetilde{U})} = \left(\int \mathcal{D}U \sum_{\{r\}} \right) e^{-S(\widetilde{U},r)}$$

Results consistent with exact (1+1)D analytic continuation





Real-time noisier, contour deform improves but doesn't completely remove sign problem

What about SU(N) ?

Wick rotation of kinetic term still provides regularization

Sum more complicated, involves functions whose magnitudes can't be reduced using vertical deformations

$$e^{ir\phi} \to \frac{\sin((r+1)\phi)}{\sin(\phi)}$$

Analogous definition of convergent HFK path integrals for lattice QCD possible, but we haven't found it

The heat-kernel equation

Alternative starting point — Kogut-Susskind Hamiltonian

$$\hat{H} = -\frac{g^2}{2a}\sum_{x,k}\hat{\Delta}_{x,k} + V_W(\hat{U})$$

Generalization of (minus) Laplacian to gauge group manifold

Wilson action is in eigenbasis of potential

Eigenbasis of kinetic operator - solutions to “heat-kernel” equation

$$\partial_\tau \mathcal{K}_E(U,\tau) = \Delta \mathcal{K}_E(U,\tau)$$

Solution for U(1): $$\mathcal{K}_{E,U(1)}(e^{i\phi}, -e^2) = \sum_{n=-\infty}^{\infty} \exp\left[-\frac{1}{2e^2}(\phi + 2\pi n)^2\right]$$

Generalization to SU(N) starting point for heat-kernel action

The heat-kernel action

Eigenvalue phases

Solution for SU(N):

$$\mathcal{K}_{E,SU(N)}\left(U, \frac{g^2}{2}\right) = \sum_{n_A=-\infty}^{\infty} \mathcal{J}(\{\phi\}, \{n\}) \exp\left[-\frac{1}{g^2}(\phi^A + 2\pi n^A)^2\right]$$

Ugly but known,
non-singular
function

SU(N) constraint: $$\sum_{A=1}^{N} \phi^A = \sum_{A=1}^{N} n^A = 0$$

Isotropic Euclidean action with right naive continuum limit:

$$e^{-S_{E,HK}(U)} = \prod_{x,\mu<\nu} \mathcal{K}_E\left(P_{x,\mu\nu}, \frac{g^2}{2}\right)$$

The Schrödinger equation

Analytic continuation of heat-kernel equation gives Schrödinger equation on gauge group

$$i\partial_t \mathcal{K}_M(U,t) = -\Delta \mathcal{K}_M(U,t)$$

Euclidean solution can be analytically continued straightforwardly

$$\mathcal{K}_{M,U(1)}(e^{i\phi}, e^2) = \sum_{n=-\infty}^{\infty} \exp\left[\frac{i}{2e^2}(\phi + 2\pi n)^2\right]$$

$$\mathcal{K}_{M,SU(N)}\left(U, \frac{g^2}{2}\right) = \sum_{n_A=-\infty}^{\infty} \mathcal{J}(\{\phi\},\{n\}) \exp\left[\frac{i}{g^2}(\phi^A + 2\pi n^A)^2\right]$$

More divergences

Minkowski analog of heat-kernel action

$$e^{iS_{M,HK}(U)} = \prod_{x,k} \mathcal{K}_M\left(P_{x,0k}, \frac{g^2}{2}\right) \prod_{x,i<j} \mathcal{K}_M\left(P_{x,\mu\nu}, -\frac{g^2}{2}\right)$$

Includes different but analogously divergent series

$$\mathcal{K}_{M,U(1)}(e^{i\phi}, e^2) = \sum_{n=-\infty}^{\infty} \exp\left[\frac{i}{2e^2}(\phi + 2\pi n)^2\right]$$

Non-vanishising for large n

$$\mathcal{K}_{M,SU(N)}\left(U, \frac{g^2}{2}\right) = \sum_{n_A=-\infty}^{\infty} \mathcal{J}(\{\phi\}, \{n\}) \exp\left[\frac{i}{g^2}(\phi^A + 2\pi n^A)^2\right]$$

Field configurations with infinitely many winding numbers all contribute to path integrals, suppressed by rapid phase fluctuations

The $\overline{\text{HK}}$ action

No symmetries lost by changing potential term

$$e^{iS_{M,\overline{\text{HK}}}(U)} = e^{-iaV_W(U)} \prod_{x,k} \mathcal{K}_M \left(P_{x,0k}, \frac{g^2}{2} \right)$$

Divergence now only arises in kinetic term and takes the form of sum over Gaussian phases (times ugly but known function)

$$\mathcal{K}_{M,SU(N)} \left(U, \frac{g^2}{2} \right) = \sum_{n_A=-\infty}^{\infty} \mathcal{J}(\{\phi\},\{n\}) \exp \left[\frac{i}{g^2} (\phi^A + 2\pi n^A)^2 \right]$$

Amenable to same strategy as U(1) HFK:

1) regularize kinetic term
2) deform integration contour to provide convergence
3) remove regulator

Convenient variables

In order to perform contour deformations on eigenvalue phases, we need a few changes of variables

Temporal boundary conditions or Euclidean segments can be used to solve equations of motion for links in terms of plaquettes

$$\{U_{x,\mu},\ n^A_{x,k}\} \leftrightarrow \{P_{x,0k},\ U_{x,0},\ n^A_{x,k}\}$$

Eigenvector matrices $V_{x,0k}$ can be "integrated in" freely

$$\{U_{x,\mu},\ n^A_{x,k}\} \leftrightarrow \{\phi^A_{x,0k},\ V_{x,0k},\ U_{x,0},\ n^A_{x,k}\}$$
$$A = 1, \ldots, N$$

Correlations from SU(N) constraint can be diagonalized

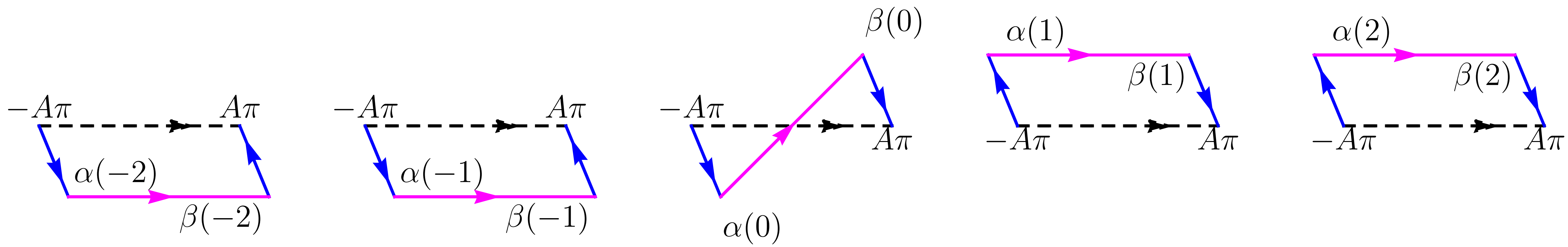
$$\{U_{x,\mu},\ n^A_{x,k}\} \leftrightarrow \{\psi^A_{x,0k},\ V_{x,0k},\ U_{x,0},\ m^A_{x,k}\}$$
$$A = 1, \ldots, N-1$$

Convergent SU(N) $\overline{\text{HK}}$

Wick rotated heat-kernel kinetic term in nice variables

$$\mathcal{G} = \mathcal{J}(\{\phi\},\{n\}) \prod_{A=1}^{N-1} e^{\frac{i}{g^2}\rho^A(\psi^A + 2\pi m^A)^2}$$

n-dependent contour deformation:



Provides exponential convergence everywhere except in neighborhood of endpoints

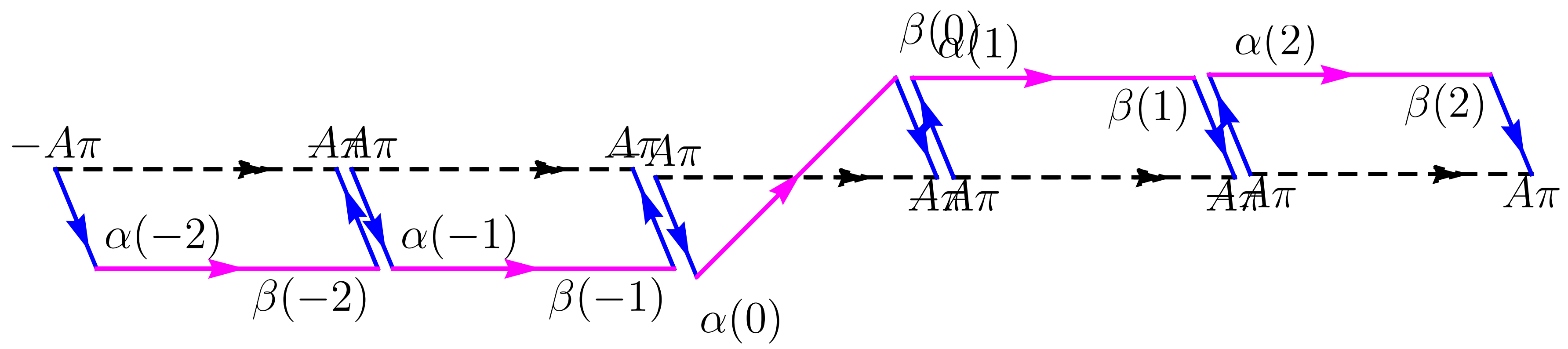
$$\mathcal{G} \sim e^{-\mathcal{C}|n|}$$

Convergent SU(N) $\overline{\text{HK}}$

Wick rotation provides absolutely convergence everywhere on deformed contour

$$\mathcal{G} \to \mathcal{J}(\{\phi\},\{n\}) \prod_{A=1}^{N-1} e^{-\frac{1}{g^2} e^{-i\theta} \rho^A (\psi^A + 2\pi m^A)^2}$$

Blue contours cancel by shift symmetry for all Wick rotation angle



After enforcing cancellation of blue segments, sum-integral on pink contour is absolutely convergent for all gauge field values

Absolutely convergent SU(N) path integrals defined by taking Minkowski limit after cancelling blue contours

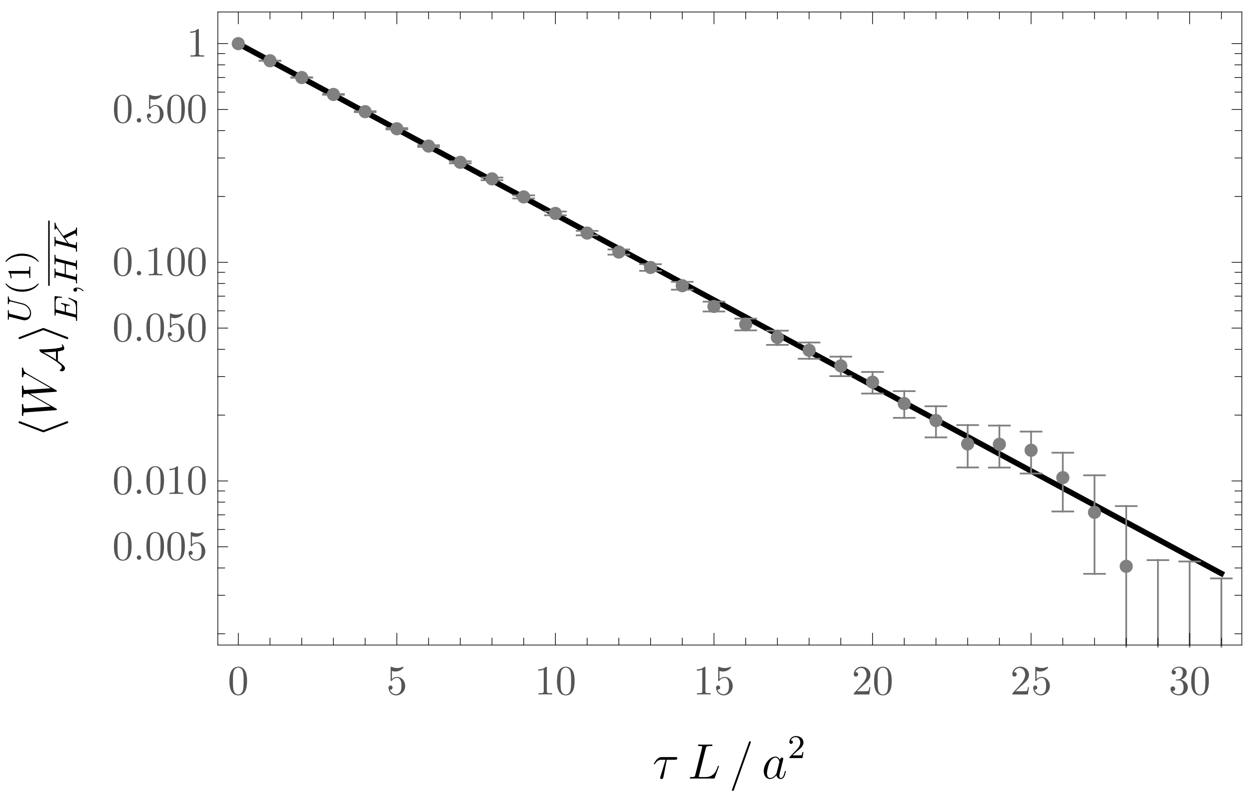
Real-time U(1) $\overline{\text{HK}}$ results

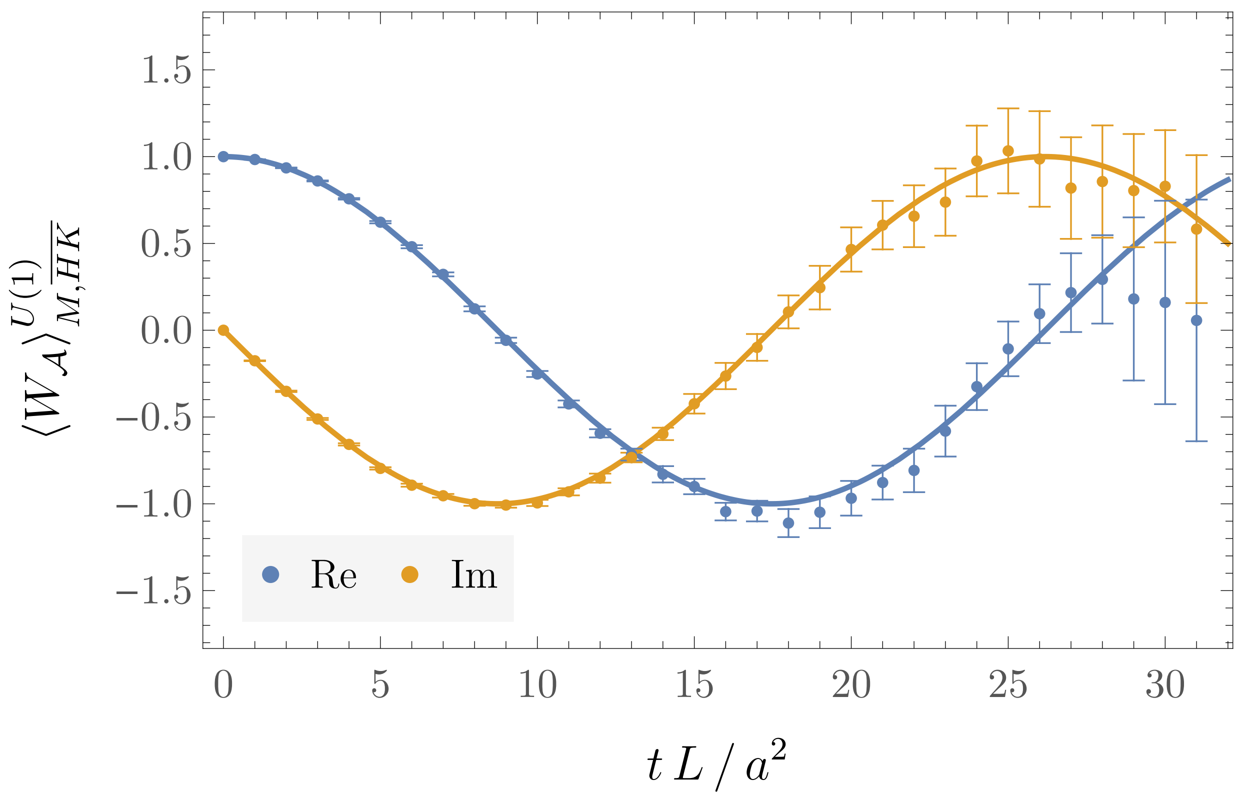
Similar stochastic sampling of auxiliary integer variables works for heat-kernel action

For n=0 terms (dominant in classical approximation), this contour deform completely removes sign problem

$$e^{\frac{i}{2e^2}\phi^2} \rightarrow e^{-\frac{1}{2e^2}\phi^2}$$

Correspondingly no signal-to-noise degradation of $\left\langle e^{i\mathrm{Re}[S_M]} \right\rangle = 1$



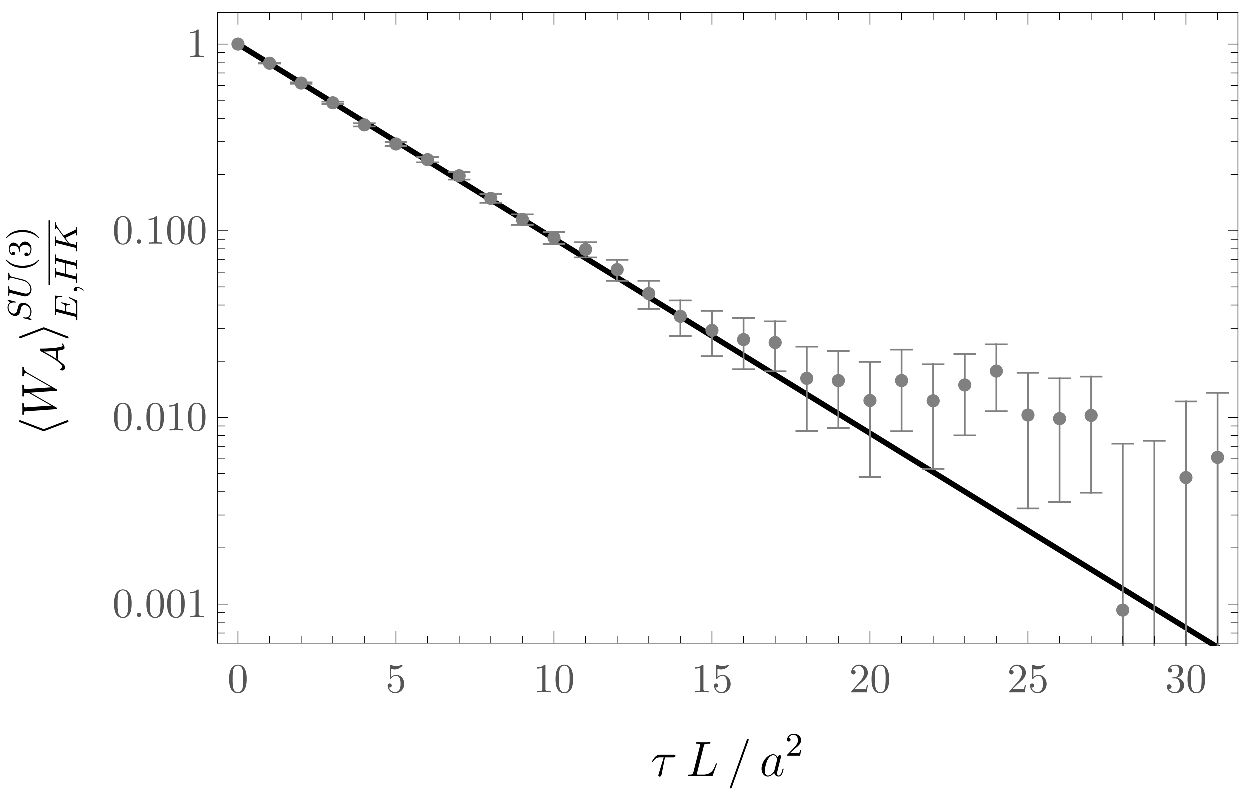


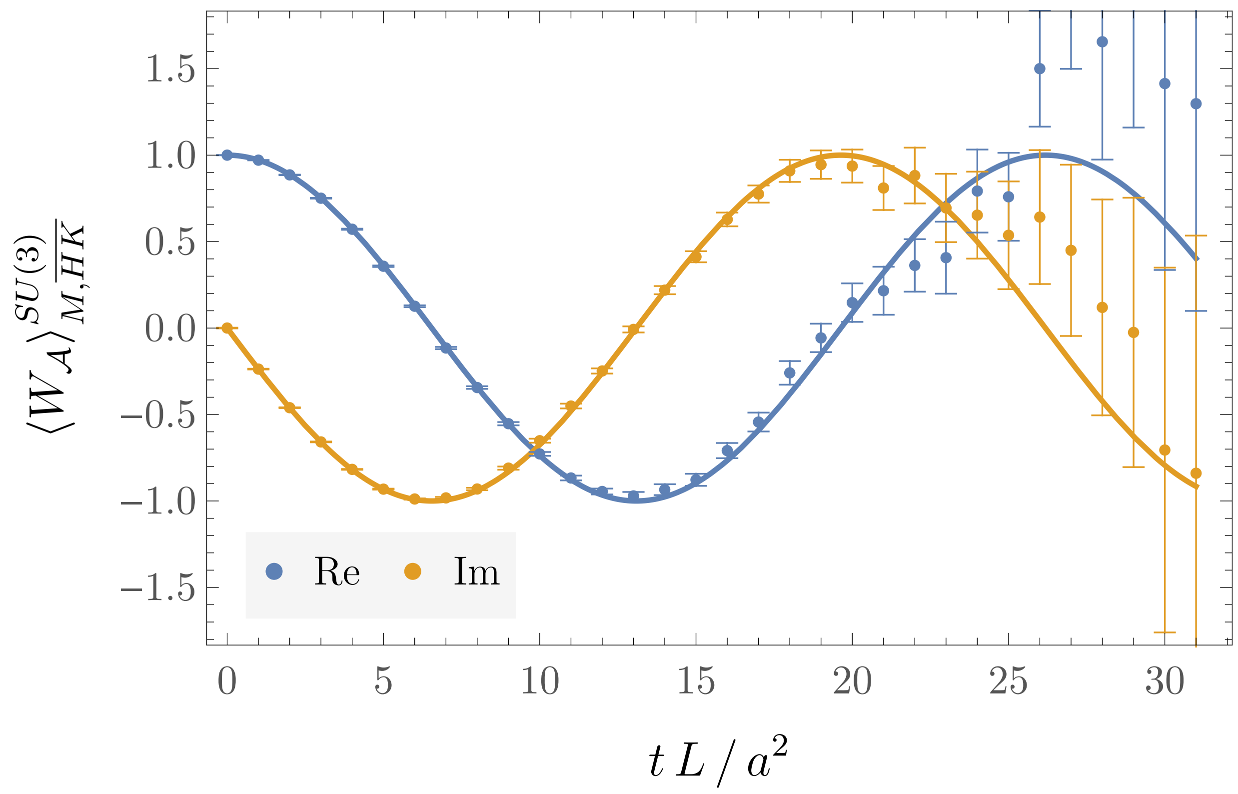
Real-time SU(3) $\overline{\text{HK}}$ results

Similar sampling strategies work for SU(3)

For n=0 terms, contour deform similarly removes Gaussian phase fluctuations

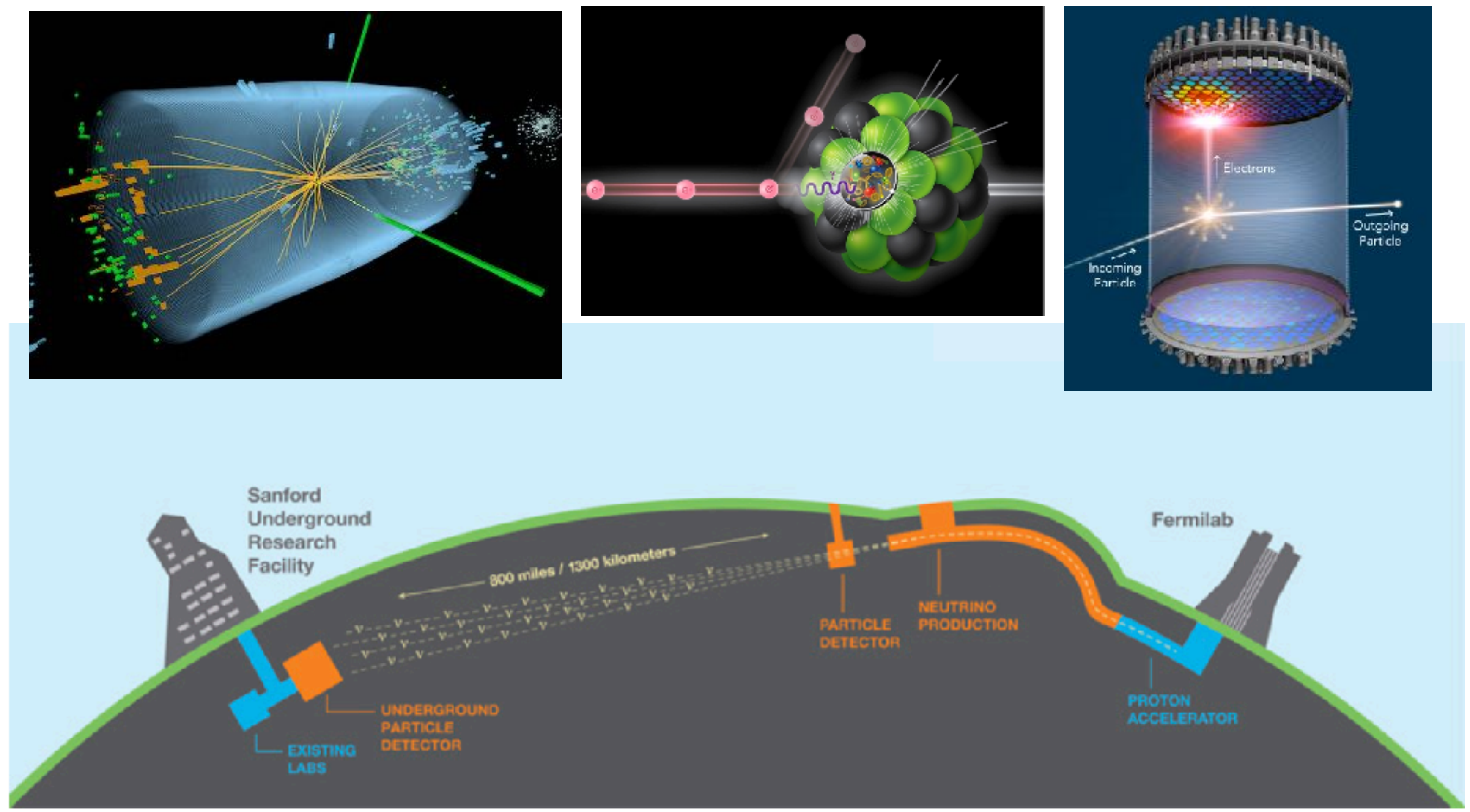
Remaining phase fluctuations from Jacobian and heat-kernel prefactor, partition function sign problem observed to be mild

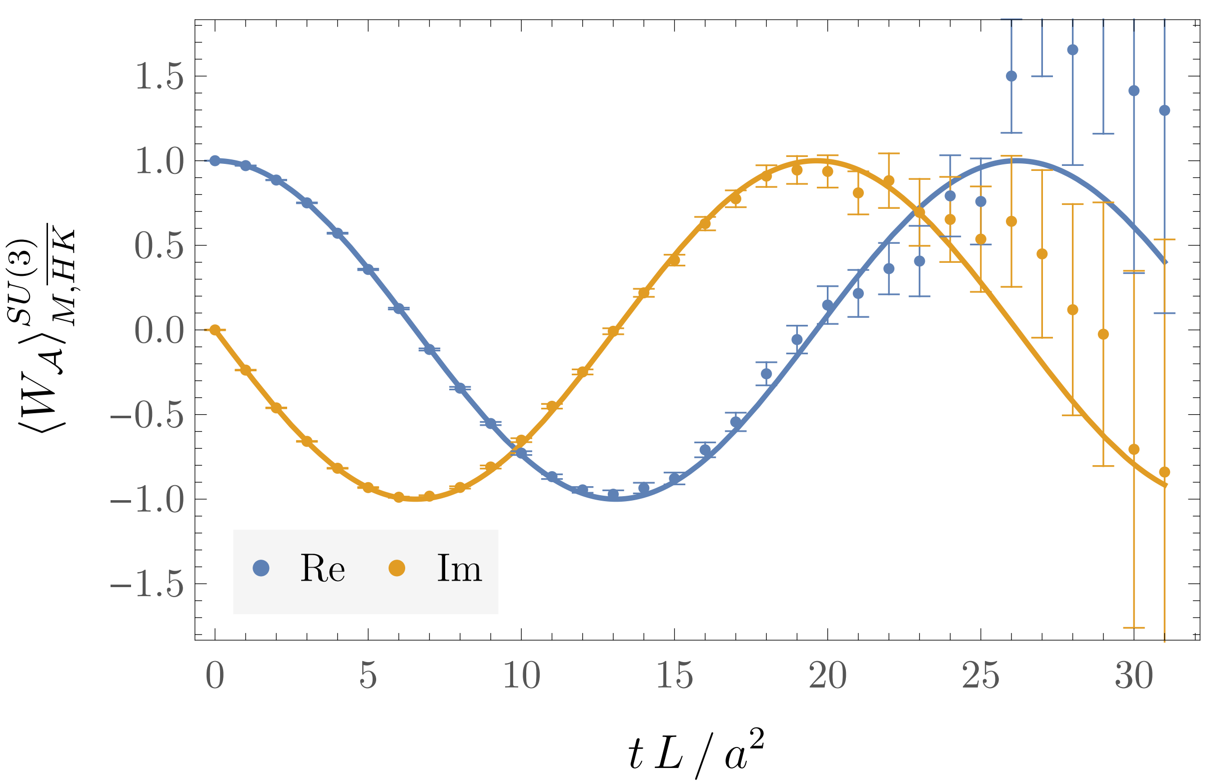




Conclusions

Many interesting questions about gauge theory involve challenges from sign problems





A convergent, unitary action can be constructed for real-time lattice gauge theory

Path integral contour deforms can improve the sign problem, remaining challenge for (3+1)D