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Introduction to Special Relativitiy

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Abstract

The goal of this lecture is to introduce the student to the theory of Special Relativity.

Not to overload the content with mathematics, the author will stick to the simplest cases; in particular only reference frames using Cartesian coordinates and translating along the common x-axis as in Fig. 1 will be used.

The general expressions will be quoted or may be found in the cited literature.

Keywords
special relativity, CAS, accelerator school

1 Introduction

In the second half of the XIX century Maxwell had summarized all known electromagnetic phenomena in four partial differential equations for electric and magnetic fields. These equations contain a numerical constant, c, which has the dimension of a velocity and the value of the speed of light in vacuum. Far from the sources, the Maxwell equations contains also the wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

where the constant c plays the role of the velocity of propagation of the wave. This led to the conclusion that the light was an EM wave which propagates with velocity c with respect to a supporting medium and that Maxwell equations were valid in a frame connected to that medium. Moreover as pointed out by Poincaré and Lorentz, Maxwell equations are not invariant in form (*covariant*) under Galilean transformations which at that time were believed to connect inertial observers. This would mean that the Galilean principle of relativity that Physics laws are the same for all inertial observers would hold good only for Mechanics laws.

In his paper [1] Einstein proposed a different solution which proved to be the correct one.

2 Galilean transformations and Classical Mechanics

The quantitative description of physical phenomena needs a reference frame where the coordinates of the observed objects are specified, a ruler for measuring the distances and a clock for describing the coordinates variation with time. Geometry says how coordinates in two different reference frames are related. If we assume for sake of simplicity two reference frames simply shifted along one of the axis[a], for instance by x_0 along $\hat{x}$, the relationships are (see Fig. 1)

$$x' = x - x_0 \qquad y' = y \qquad z' = z$$

If S' is moving along the common x-axis with speed $\vec{V}$=$\hat{x}V$ with respect to S, assuming the origins coincide at t=0 it is

[a] All other cases can be obtained by introducing a rotation of the axis and a shift of the origin.

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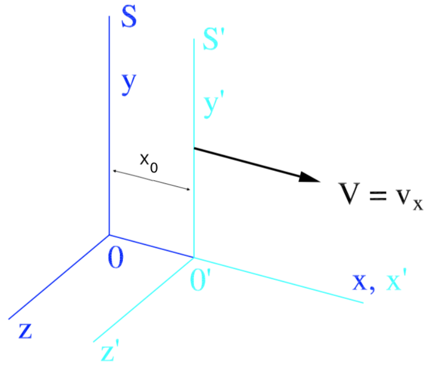

Fig. 1. The accented frame S' is shifted by x_0 with respect to S.

$$x' = x - x_0 = x - Vt \qquad y' = y \qquad z' = z \tag{1}$$

Eqs.(1) are the Galilean coordinate transformations. By differentiating with respect to time it is

$$\dot{x}' = \dot{x} - V \qquad \dot{y}' = \dot{y} \qquad \dot{z}' = \dot{z} \tag{2}$$

where we have implicitly assumed that t'=t and that the lengths are the same. From Eqs.(2) we see that velocities add. If the light from a source on a train propagates in the x-direction with velocity $\hat{x}c$, for an observer at rest on the railway platform it would propagate with velocity $\hat{x}(c+V)$ (see Fig. 2).

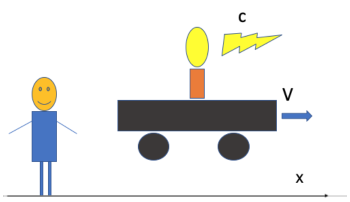

Fig. 2. Light source on a train moving along the x-direction with uniform speed $\hat{x}V$ wrt the railway platform.

By differentiating Eqs.(2) wrt time we get

$$\ddot{x}' = \ddot{x} \qquad \ddot{y}' = \ddot{y} \qquad \ddot{z}' = \ddot{z} \tag{3}$$

that is the acceleration of a body is the same for all observers related by Galilean transformations.
The basic laws of classical dynamics are

1. A *free* body perseveres in its state of rest, or of uniform motion (principle of inertia). Reference frame where the principle of inertia holds good are said inertial.

2. In an inertial reference frame it is $\vec{F} = m\vec{a}$, that is the acceleration, $\vec{a}$, is proportional to the applied force, $\vec{F}$, through a constant, m ("inertial mass"). In other words, if in an inertial frame a body appears to be accelerated it means that there must be something acting on it. Implicitly it is assumed that m is a characteristic of the body which doesn't depend upon its status of motion.

3. Whenever two bodies interact they apply equal and opposite forces to each other.

The second and third laws combined give the total momentum conservation for an isolated system. The three laws of dynamics hold good in inertial frames. If an inertial frame exists, all reference frames in uniform motion with respect to it are inertial. As they are all equivalent it is reasonable to assume that all mechanics laws are the same for inertial observers (principle of relativity). More precisely, the principle states that the laws must have the *same form* (covariance). If we chose a non inertial frame for describing the motion of an object, the numerical results would be the same if the motion of the reference frame itself is accounted for correctly. However the equation of motion for the observed object would take a different form.

Are mechanics laws invariant under Galilean transformations? Suppose that Alex is studying the motion of a ball let to fall under the earth gravitation force. Alex measures that the object is subject to a constant acceleration of $a \approx 9.8\ \mathrm{ms}^{-2}$. By using different balls he finds that the acceleration is always the same, g. He concludes that there must be a force acting on the balls which is directed towards the center of the earth and having magnitude mg. Betty is on a train moving uniformly with velocity $\vec{V}$=$\hat{x}V$ with respect to Alex (see Fig. 3).

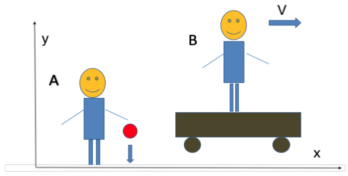

Fig. 3. Alex, at rest on the railway platform, studies the motion of objects under the gravitation force. Betty is on a train moving along the x-direction with uniform speed $\hat{x}V$ wrt the railway platform.

From Eqs.(3)

$$\ddot{x}' = \ddot{x} = 0 \qquad \ddot{y}' = \ddot{y}$$

and as the mass, m is a constant, she will agree with Alex on magnitude and direction of the force. Classic mechanics laws are covariant under Galilean transformations.

The relativity principle allows us to chose the most convenient frame for describing an event.

We want to show in a more formal way that Newton law is invariant under Galilean transformations by using an example. Let us suppose we have a system of particles and that the forces between them depend upon the reciprocal distances, r_{ij}. In the S inertial reference frame it is

$$\vec{F}_i = -\nabla_{r_i} \Sigma_j U(r_{ij}) = m_i \vec{a}_i$$

In the moving frame S' the Newton law must take the same form with the potential U having the same functional dependence upon the new variables as in the old ones. From the Galilean transformations Eqs.(1) and (3) it is $r'_{ij} = r_{ij}$, $\vec{a}'_i = \vec{a}_i$ and $\nabla_{r'_i} = \nabla_{r_i}$ and therefore, as the mass is a scalar invariant, it is indeed

$$\vec{F}'_i = -\nabla_{r'_i} \Sigma_j U(r'_{ij}) = m_i \vec{a}'_i$$

3 Galilean relativity and EM wave equation

Using the cyclic rule[b] the wave equation[c]

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi = 0$$

[b] $\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j}$

[c] For simplicity we have chosen the x-axis along the direction of propagation.

becomes under Galilean transformation

$$\left[\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{V^2}{c^2}\frac{\partial^2}{\partial x'^2} - 2\frac{V}{c^2}\frac{\partial^2}{\partial x'\partial t'}\right]\Phi = 0$$

and it is clearly not covariant. As anticipated, Maxwell equations would describe EM laws in a particular reference frame, and as such, a *privileged* one. It was conjectured the existence of a medium, the *luminiferous aether*, supporting the propagation of EM waves, as the air supports sound waves. This medium had to be extremely rarefied to be undetectable directly and it would permeate the whole space. The speed of light would be c with respect to the medium and, accordingly to Eqs.(2), would be different for an observer moving with respect to the medium.

Experiments for demonstrating the existence of the aether, by measuring the speed of light under different conditions were attempted, the most famous of them being those performed by Michelson and Morley using an interferometer.

The arrangement is schematically shown in Fig.4. The light is split into two orthogonal patterns of equal length by the partially silvered glass M, reflected back by mirrors M1 and M2 and recombined on a screen. If the earth is at rest in the aether, the recombined waves are in phase but if the earth is moving the time needed by the two waves for reaching the screen would be different and an interference pattern should be observed on the screen S.

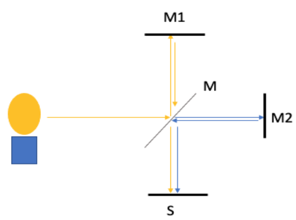

Fig. 4. A schematic view of Michelson-Morley interferometer experiment.

For avoiding errors due to incorrect mirrors angle or to the distances between the two mirrors and the partially silvered glass being not identical, the apparatus can be rotated so that the possible interference fringes would move.

The result of the first experiment in 1887 was negative. It was repeated with higher accuracy apparatuses during the following 50 years, however the result was always negative. Theories proposed to justify the negative result were contradicted by other experiments. A detailed quantitative description of these experiments may be found in [2]

Attempts of modifying the still relatively new EM laws in such a way that they would be invariant under Galilean transformations led to predictions of new phenomena which could not be proved experimentally.

4 Relativistic Kinematics

In 1905 Einstein [1] proposed a solution to the dilemma based on two postulates:

1. Physics laws are the same in all inertial frames, there is no preferred reference frame.

2. The speed of light in the empty space has the same finite value c in all inertial frames.

At that time the existence of the aether was still widely accepted and not yet ruled out by experiments. It is worth noting that Lorentz had found the coordinates transformation which leave Maxwell's equations invariant in 1904, before the publication of Einstein's paper, accompanied however by an erroneous interpretation. It is in Einstein paper that such transformations are *physically* justified and therefore extendable to the whole Physics. In particular, the concept of time was critically addressed and the fact that the time is not universal comes as a consequence of the light having a finite velocity.

Let us summarize Einstein reasoning. In order to describe the motion of an object we need to equip each point of our reference frame with identical clocks and rulers. Is it possible to synchronize the clocks by sending light rays. For instance we can imagine of sending a light ray from a point A to B and B reflecting it back to A (see Fig.5). The two observers sitting in A and B may agree in setting the clock in B at the arrival of the signal to a given value t_B while A will set its own clock to $2t_B$ when receiving back the signal. However if we want the speed of light to be c=3×10^8 m s^{-1} we shall measure the distance, L, between A and B and set t_B=L/c.

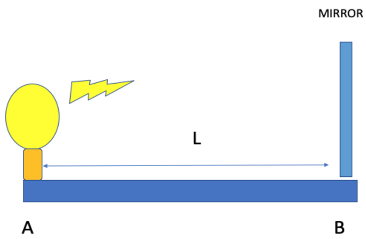

Fig. 5. Synchronization procedure of the clocks in A and B.

Assuming the clocks are identical, they will stay synchronized. For this procedure we use the light because, we have assumed that it propagates in vacuum with constant velocity so that we can be assured that the velocity is the same in both directions.

Once all clocks within one frame are synchronized we can establish the chronological sequence between events happenings in different places within the same frame of reference.

The observer S' moving with respect to S may synchronize its own clocks with the very same procedure. However this synchronization procedure observed by the resting observer is not correct. Suppose A and B lying on the common x-axis with B on the right of A ($x_B > x_A$) as shown in Fig.6: while the light moves to B, B moves further away and once reflected back to A, A moves toward the light. Therefore observed by S the time needed to reach B is obtained by setting

$$ct_B = L + Vt_B$$

($L \equiv x_B - x_A$) which gives

$$t_B = L/(c - V)$$

while the time needed to reach A is obtained from

$$ct_A = L - Vt_A$$

that is

$$t_A = L/(c + V)$$

and

$$t_B - t_A = L\Big(\frac{1}{c-V} - \frac{1}{c+V}\Big) = \frac{2VL}{c^2[1-(V/c)^2]} \neq 0$$

Therefore for S, S' clocks are not synchronized. If the clocks in the moving frame would be synchronous with the stationary ones they wouldn't be synchronous in their own frame. The "stationary" frame would dictate the timing. However stationarity is relative, the inertial frames are all equivalent: if there exist no privileged frame, we must abandon the idea of universal time. Relativity of time is a consequence of the speed of light being finite.

As a consequence events which may be simultaneous for S are in general not simultaneous for S' and the other way round.

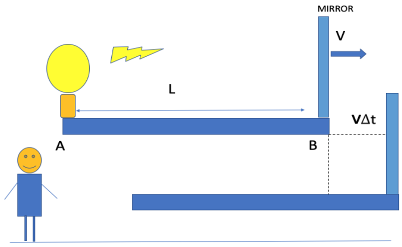

Fig. 6. Synchronization procedure of S' clocks as seen by the “resting" observer.

5 Lorentz transformations

By assuming the speed of light constant in all reference frames, the Galilean transformations, implying the addition of velocity rule, must be modified. The new transformations must reduce to the Galilean ones when the relative motion is slow ($V \ll c$). According to the first Einstein postulate, the empty space is isotrope (all direction are equivalent) and homogeneous (all points are equivalent); it would make no sense to postulate that the laws are invariant in a space which is not homogeneous and isotrope. As time is not universal, it must be included in the coordinate transformation.

Resorting to arguments of space homogeneity and isotropy, and to the Einstein postulates it is relatively simple to work out the correct coordinates transformation.

Homogeneity implies the relationship between the coordinates must be linear:

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t \end{aligned}$$

where the coefficients a_{ij} may depend upon the relative speed V.

The points on the x-axis where y=z=0 must transform to y'=z'=0 at all times which means that a_{21}=a_{31}=a_{24}=a_{34}=0. The points with y=0 (the x-z plan) must transform into y'=0 and therefore it is also a_{23}=0. The points with z=0 (the x-y plan) must transform into z'=0 and therefore it is also a_{32}=0. Because of isotropyy, time must be invariant for a sign inversion of the coordinates y and z which means a_{42}=a_{43}=0. So we are left with 8 unknown coefficients:

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' &= a_{22}y \\ z' &= a_{33}z \\ t' &= a_{41}x + a_{44}t \end{aligned}$$

For a point on the y-axis (x=z=0) it is

$$x' = a_{12}y + a_{14}t$$

and therefore x' value would depends on the sign of y which again contradicts the hypothesis of isotropy. Therefore it must be a_{12}=0. The same argument can be used to set a_{13}=0. We are left with

$$\begin{aligned} x' &= a_{11}x + a_{14}t \\ y' &= a_{22}y \\ z' &= a_{33}z \\ t' &= a_{41}x + a_{44}t \end{aligned}$$

The value of a_{22} is found by observing that

$$y' = a_{22}(V)y = a_{22}(V)a_{22}(-V)y'$$

that is $a_{22}(V)a_{22}(-V)$=1. Because a_{22}=1 for $V \to 0$, the correct choice is a_{22}=1. In the same way it is found a_{33}=1.
The origin of the S' frame is described in S as $x = Vt$ and has by definition x'=0 at any time. Therefore

$$0 = x'_0 = a_{11}x_0 + a_{14}t = a_{11}Vt + a_{14}t$$

that is a_{14} and a_{11} are related by

$$a_{14}/a_{11} = -V$$

and the equation for x' becomes

$$x' = a_{11}(x + a_{14}t/a_{11}) = a_{11}(x - Vt)$$

For finding the values of the remaining coefficients a_{11}, a_{41} and a_{44} we resort to the fact that the speed of light is the same in S and S' and that the wave equation is invariant in form. Suppose an EM spherical wave leaves the origin of the frame S at t = 0. The propagation is described in S by the equation of a sphere which radius increases with time as

$$R(t) = x^2 + y^2 + z^2 = c^2t^2 \tag{4}$$

In S' the wave propagates with the same speed c and therefore

$$R'^2(t') = x'^2 + y'^2 + z'^2 = c^2t'^2$$

which writing the coordinates x', y', z' and t' in terms of x, y, z and t becomes

$$a_{11}^2x^2 + a_{11}^2V^2t^2 - 2a_{11}xVt + y^2 + z^2 = c^2a_{41}^2x^2 + c^2a_{44}^2t^2 + 2a_{41}a_{44}xt$$

Rearranging the terms it is

$$(a_{11}^2 - c^2a_{41}^2)x^2 - 2(a_{11}^2V + c^2a_{41}a_{44})xt + y^2 + z^2 = (c^2a_{44}^2 - a_{11}^2V^2)t^2$$

Comparing this equation with Eq.(4), we get a system of 3 equations in the 3 unknown a_{11}, a_{41} and a_{44}

$$\begin{aligned} a_{11}^2 - c^2a_{41}^2 &= 1 \\ a_{11}^2V + c^2a_{41}a_{44} &= 0 \\ c^2a_{44}^2 - a_{11}^2V^2 &= c^2 \end{aligned}$$

which is solved by

$$a_{11} = a_{44} = \frac{1}{\sqrt{1-(V/c)^2}}$$

$$a_{41} = -\frac{V/c^2}{\sqrt{1-(V/c)^2}}$$

The final coordinates transformation for a uniform motion along the x-axis with relative speed V is therefore

$$x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z \qquad ct' = \gamma(ct - \beta x) \tag{5}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \text{and} \qquad \beta \equiv V/c$$

The inverse transformation from S' to S is obtained by replacing β with $-\beta$. The transformation can be written also in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \mathcal{L} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Successive collinear Lorentz transformations are obtained by matrix multiplication.

The matrix elements may be also written as

$$\mathcal{L}_{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}$$

with x^0=ct, x^1=x, x^2=y and x^3=z.

It is worth noting that for $V \ll c$, that is $\beta \to 0$ and $\gamma \to 1$, Lorentz transformations coincide with the Galilean ones, while if $V > c$, γ becomes imaginary and the transformations are meaningless. The fact that c is the limit velocity is not a Einstein postulates, it is a consequence of the Lorentz transformation. Fig. 7 shows γ as function of β.

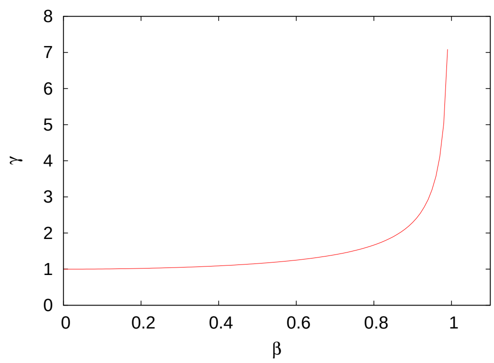

Fig. 7. γ as function of $\beta \equiv v/c$.

The general expression of the Lorentz transformation of parallel translation with arbitrary direction of the relative velocity reads [4]

$$ct' = \gamma\left(ct - \vec{\beta} \cdot \vec{r}\right) \qquad \vec{r}\,' = \vec{r} + \frac{\gamma - 1}{\beta^2}\vec{\beta} \cdot \vec{r}\vec{\beta} - \gamma\vec{\beta}ct \tag{6}$$

with $\vec{\beta} \equiv \vec{V}/c$.

Time is one of the 4 coordinates describing an event and as the spatial coordinates is subject to a (Lorentz) transformation between moving frames.

For spatial coordinates it is always possible if for instance $x_2 > x_1$ to find a new coordinates frame such that $x'_2 < x'_1$.

Is it possible to find a Lorentz transformation which inverts the temporal order of events?

Assume an event happening at the time t_1 at the location x_1 in S and a second event happens at t_2 in x_2 with $t_2 > t_1$. Is it possible to find a Lorentz transformation such that $t'_2 < t'_1$? In S' it is

$$ct'_1 = \gamma(ct_1 - \beta x_1)$$

$$ct'_2 = \gamma(ct_2 - \beta x_2)$$

and therefore

$$c(t'_2 - t'_1) = \gamma[c(t_2 - t_1) - \beta(x_2 - x_1)]$$

Therefore it is $t'_2 < t'_1$ if $\beta(x_2 - x_1) > c(t_2 - t_1)$, that is if $V(x_2 - x_1)/(t_2 - t_1) > c^2$. This may be possible depending on the values of $x_2 - x_1$ and $t_2 - t_1$. However if the first event in S *drives* the second one, x_2 and t_2 are not arbitrary.

If w is the speed of the signal triggering the second event from the first one it is

$$x_2 - x_1 = w(t_2 - t_1)$$
$$c(t'_2 - t'_1) = \gamma[c(t_2 - t_1) - \beta w(t_2 - t_1)] = \gamma c(t_2 - t_1)\left(1 - \frac{Vw}{c^2}\right)$$

which is always positive as $w \leq c$. Causality is not violated.

6 Some consequences of Lorentz transformations: length contraction and time dilation

As a consequence of Lorentz transformations, lengths are not invariant. Consider for instance a rod along the x-axis and at rest in the moving frame S'. The length of the rod in S' is L'. The length in S is determined by the positions of the rod ends at the *same* time and therefore from Eq.(5) with t_1=t_2

$$L' = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma L$$

The moving rod is shorter than in the frame where it is at rest (*length contraction*). However the length of a rod aligned with one of the two axis perpendicular to the direction of motion is invariant. For this reason angles are in general not invariant.

Suppose a clock at rest in S measuring a time interval $t_2 - t_1$ between two events happening at that same location in S. From Eq.(5) with x_1=x_2 the time interval in S' between the two events is

$$t'_2 - t'_1 = \gamma(t_2 - t_1)$$

which is larger than measured in S (*dilation of time*). Moreover events happening at the same time but in *different places* in S, will be no more simultaneous in the moving frame S'. In fact using Eq.(5) with t_1=t_2 it is

$$c(t'_2 - t'_1) = \gamma\beta(x_1 - x_2)$$

which is non vanishing if $x_1 \neq x_2$.

In general it is named as *proper* the interval (in space or time) measured in a inertial frame where the observed object is at rest.

Let us suppose that we have two synchronized clocks, $C1$ and $C2$, at the origin O of S and that at t=0 we set $C2$ in uniform motion along the x-axis with velocity V. After a time t_{C1}=T, when $C2$ accordingly to time dilation strikes T/γ, the clock $C2$ inverts its direction. When $C2$ arrives back in O, $C1$ strikes $2T$ and $C2$ instead $2T/\gamma$. This may look as a paradox because the notion of motion is relative: with respect to $C2$, it was $C1$ moving and therefore $C2$ should strike $2T$ and $C1$ instead $2T/\gamma$. However when they are both in O we can compare their time and only one outcome is possible. The mistake is considering the two situations equivalent, while they are not. $C2$ has been set in motion by the action of some kind of force and some kind of force also is responsible for changing its direction, while $C1$ has experienced no force. Indeed direct experiments involving clocks have shown that time dilation is real [3].

7 Lorentz transformations for velocity and acceleration

The relativistic transformation for the velocity follow from the Lorentz transformations for the coordinates

$$\begin{aligned} v'_x &\equiv \frac{dx'}{dt'} = \frac{dx - V dt}{dt - V dx/c^2} = \frac{v_x - V}{1 - v_x\beta/c} \\ v'_y &\equiv \frac{dy'}{dt'} = \frac{dy}{\gamma(dt - V dx/c^2)} = \frac{v_y}{\gamma(1 - v_x\beta/c)} \\ v'_z &\equiv \frac{dz'}{dt'} = \frac{dz}{\gamma(dt - V dx/c^2)} = \frac{v_z}{\gamma(1 - v_x\beta/c)} \end{aligned} \tag{7}$$

with $\beta \equiv V/c$, $v_x \equiv dx/dt$, $v_y \equiv dy/dt$ and $v_z \equiv dz/dt$. The inverse transformation is obtained by replacing V with $-V$. Unlike the classical case, also the components of the velocity perpendicular to the motion, when non vanishing, are affected by the motion. This is a consequence of the fact that the time is not invariant and therefore, although the lengths perpendicular to the motion direction are unchanged, the time needed to cover them is changed.

As an exercise, let us use these expressions for a light ray. For v_x=c and v_y=v_z=0 it is

$$v'_x = \frac{c - V}{1 - V/c} = c\,\frac{c - V}{c - V} = c \quad \text{and} \quad v'_y = v'_z = 0$$

For v_y=c and v_x=v_z=0 it is v'_x=$-V$, v'_y=c/γ, v'_z=0 and

$$v'^2_x + v'^2_y + v'^2_z = V^2 + c^2[1 - (V/c)^2] = c^2$$

As expected, the speed of light is invariant.

In a similar way as for the velocity it is possible to find the transformation for the acceleration [2]

$$\begin{aligned} a'_x &= \frac{a_x}{\gamma^3(1 - v_x\beta/c)^3} \\ a'_y &= \frac{a_y}{\gamma^2(1 - v_x\beta/c)^2} + \frac{a_x v_y \beta/c}{\gamma^2(1 - v_x\beta/c)^3} \\ a'_z &= \frac{a_z}{\gamma^2(1 - v_x\beta/c)^2} + \frac{a_x v_z \beta/c}{\gamma^2(1 - v_x\beta/c)^3} \end{aligned} \tag{8}$$

Acceleration is not invariant under Lorentz transformations unless both v and $V \to 0$.

8 Experimental evidence of relativistic kinematics

In his papers Einstein suggested possible experiments for confirming the validity of his theory. Here we give some examples: light aberration, (transverse) Doppler effect and lifetime of unstable particles.

Light aberration

Light aberration is the apparent motion of a light source due to the movement of the observer. It was first discovered in astronomy. Consider a source emitting photons at an angle θ with respect to the x-axis in the S frame where v_y=$c\sin\theta$ and v_x=$c\cos\theta$ (see Fig.8). In S' it is v'_y=$c'\sin\theta'$ and v_x=$c'\cos\theta'$. Using Galilean transformations for the velocity components v_x and v_y

$$v'_y = v_y \text{ and } v'_x = v_x - V$$

$$\tan\theta' = v'_y/v'_x = v_y/(v_x - V)$$

$$\tan\theta' = \frac{\sin\theta}{(\cos\theta - \beta)}$$

Using instead Lorentz transformations

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

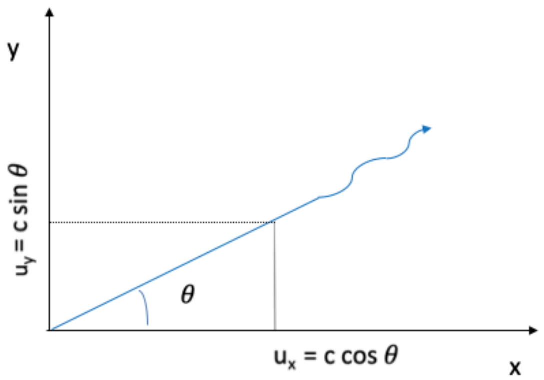

Fig. 8. Source emitting a light ray at an angle θ with respect to the x-axis in S.

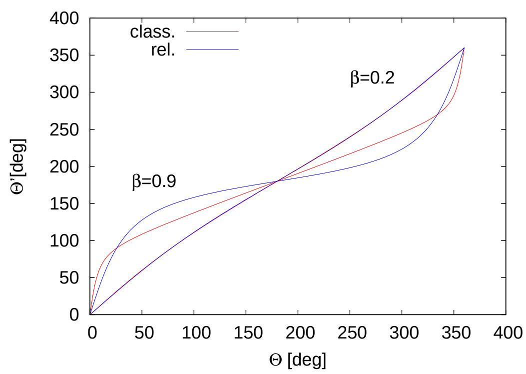

Fig. 9. Angles observed in the moving frame for β=0.2 and β=0.9.

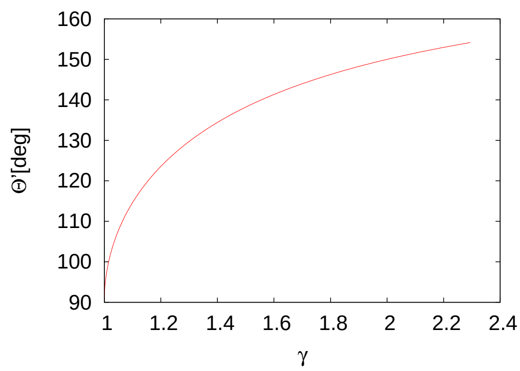

Fig. 10. θ' vs. γ for θ=90^0.

High energy experiments involving emission of photons confirm the relativistic expression. Fig.9 shows the classical and relativistic relationships between the emission angles for β=0.2 and β=0.9. We see that θ'=θ for 0 and 180 degrees for both the classical as well as the relativistic expression. In all other cases it must be paid attention whether the angles are specified in the moving or in the rest frame. Fig.10 shows how the angle θ=90^0 transforms as function of γ.

Doppler effect

In the following we give an alternative computation of light aberration by using the undulatory description of light which allows also to treat the Doppler effect.

Consider a plane light wave propagating in the direction $\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta$

$$A(x, y; t) = \cos\left[k(x\cos\theta + y\sin\theta) - \omega t\right] \tag{9}$$

where k=ω/c is the wave number. The wave must have the same form when observed in S'

$$A(x', y'; t') = \cos\left[k'(x'\cos\theta' + y'\sin\theta') - \omega' t'\right]$$

Expressing the coordinates in S in terms of the coordinates in S', Eq.(9) gives

$$B(x', y'; t') = \cos\left\{[k\gamma(x' + \beta ct)\cos\theta + y'\sin\theta)] - \omega\gamma(t' + \beta x'/c)\right\}$$

Comparing with the expression for $A(x', y'; t')$ we get

$$k'\cos\theta' = k\gamma\cos\theta - \omega\gamma\beta/c = k\gamma(\cos\theta - \beta) \tag{10}$$

$$k'\sin\theta' = k\sin\theta \tag{11}$$

$$\omega' = -k\gamma\beta c\cos\theta + \gamma\omega = \gamma\omega(1 - \beta\cos\theta) \tag{12}$$

From Eqs. (10) and (11) it is

$$\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}$$

which is the result found previously. In addition Eq. (12) gives the wavelength measured by two observers in relative motion. Suppose that the source is at rest in S so that ω is the proper frequency, ω_0. Thus it is

$$\omega' = \omega_0\gamma(1 - \beta\cos\theta)$$

where θ is the propagation angle in the source reference frame. For θ=0 it is

$$\omega' = \omega_0\sqrt{\frac{1-\beta}{1+\beta}}$$

and therefore $\omega' < \omega_0$ for $\beta > 0$ (receiver moving away from source), while $\omega' > \omega_0$ for $\beta < 0$ (receiver moving towards the source).

For θ=90^0 (in the frame where the source is at rest) it is

$$\omega' = \omega_0\gamma$$

Unlike the classical case, relativistically it is expected the existence of a transverse Doppler effect which is a consequence of time being not invariant. This was predicted by Einstein who also suggested in 1907 an experiment using hydrogen ions for measuring it. The experiment realized for the first time by Ives and Stilwell in 1938 proved the correctness of Einstein prediction.

Lifetime of unstable particles

Beside e^-, p and n, in nature there are particles which are produced by scattering process and unlike e^+, $\bar{p}$ and $\bar{n}$, are "short-living". Their number decays in time as

$$N(t) = N_0 e^{-t/\tau}$$

Pions are produced by bombarding a proper target by high energy protons and leave the target with $v \approx 2.97{\times}10^8$ m/s that is β=0.99 and $\gamma \approx 7$. The lifetime of charged pions at rest is τ_0=26$\times$10^{-9} s. The time, $\bar{t}$, needed for the pions at rest to decay by half is

$$N(\bar{t}) = N_0 e^{-\bar{t}/\tau} = \frac{N_0}{2} \quad \rightarrow \quad \bar{t} = 18 \text{ ns}$$

It is observed that they are reduced to the half after 37 m from the target. If their lifetime would be as when at rest they should become the half already after about 5 m. The experimental observation is explained if the pion lifetime in the laboratory frame is

$$\tau = \gamma \tau_0$$

as predicted by time dilation. The decaying pions produce muons which are unstable too. Their lifetime at rest is 2.2 μs, small however larger than pion one. Time dilation may allow us realizing future colliders smashing muons if we are able to accelerate them to high energy very quickly!

9 Relativistic Dynamics

Assuming $\vec{F}$ invariant and m constant, Newton law, $\vec{F} = m\vec{a}$, is not invariant under Lorentz transformations because, as we have seen, $\vec{a}$ is not invariant. In addition the mass cannot be a constant because by applying a constant force to an object its speed would increase indefinitely becoming larger than c. Classical mechanics must be modified to achieve invariance under Lorentz transformations and the new expressions must reduce to the classical ones when the speed of the objects is much smaller than c.

The relativistic mass

In the 1905 paper, Einstein used the Lorentz force and the electro-magnetic field transformations to achieve the generalization of the definition of momentum and energy. We follow here instead the approach by Lewis and Tolman [5].

In 1909 MIT professors of chemistry Lewis and Tolman suggested a more straightforward reasoning with respect to Einstein original one and which involved purely mechanical arguments.

Let us assume there are two observers, Alex and Betty, moving towards each other with the same velocity as seen by a third observer, Charlie (see Fig. 11). Betty sits in S and Alex in S'. Alex and Betty have identical elastic balls. Betty (see Fig. 12) releases the red ball with v_x^B=0 and $v_z^B \neq 0$, while Alex (Fig. 13) releases the green ball with speed $v_x'^A$=0 and $\bar{v}_z'^A$ numerically equal and opposite to the red ball velocity, that is

$$v_z'^A = -v_z^B$$

The experiment is set so up that the two balls collide and rebound as shown in Fig. 14.
Now let us consider Betty point of view. For Betty it is

$$\Delta p_x^B = 0 \quad \Delta p_z^B = 2 m_B v_z^B$$

$$\Delta p_x^A = 0 \quad \Delta p_z^A = -2 m_A v_z^A$$

We recall Lorentz transformation of velocity

$$v_z' = \frac{v_z}{\gamma(1 - v_x \beta / c)}$$

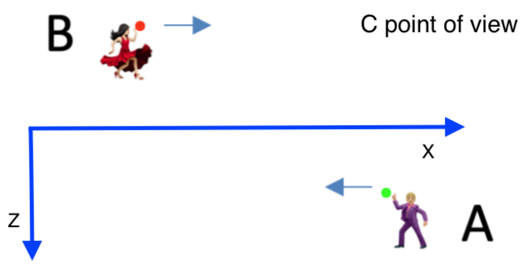

Fig. 11. Betty and Alex frames as seen by the third observer Charlie. They move towards each other with equal and opposite velocity along the x-axis.

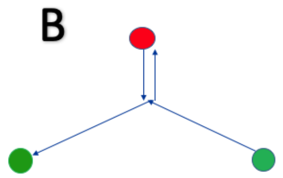

Fig. 12. The elastic collision as observed by Betty.

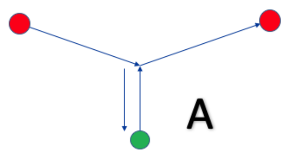

Fig. 13. The elastic collision as observed by Alex.

As we know the values of the velocity components in the moving frame we need here the inverse of the velocity transformation Eq. (7), that is

$$v_x = \frac{v_x' + V}{1 + v_x'\beta/c}$$

$$v_z = \frac{v_z'}{\gamma(1 + v_x'\beta/c)}$$

In our case v_x'=$v_x'^A$=0 and v_z'=$v_z'^A$=$-v_z^B$ and therefore $v_x^A = V$ before and after the collision, and

$$v_z^A = v_z'^A/\gamma = -v_z^B/\gamma \tag{13}$$

with $\gamma = 1/\sqrt{1-(v_x^A/c)^2}$, before the collision and

$$v_z^A = -v_z'^A/\gamma = v_z^B/\gamma \tag{14}$$

after.

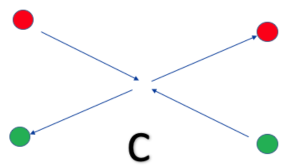

Fig. 14. The elastic collision as observed by Charlie. The two balls are scattered under the same angle of incidence.

Momentum, classically defined as $\vec{p}$=$m\vec{v}$, is conserved if $\Delta(\vec{p}^A + \vec{p}^B)$=0. It is Δp_x^A=Δp_x^B=0. In addition it must be

$$\Delta p_z^B = -\Delta p_z^A$$

which using Eqs. (13) and (14) and the definition of momentum gives

$$m_B v_z^B = -m_A v_z^A = \frac{1}{\gamma} m_A v_z^B \quad \rightarrow \quad m_A = \gamma m_B$$

We may assume that v_z^B is small so that m_B is the *mass at rest*, m_0, and m_A=$m(v)$.
So we have found that

$$m(v) = \gamma m_0$$

where m_0 is the mass in the reference frame where the object is at rest. We can keep the momentum definition, $\vec{p}$=$m\vec{v}$, from classical dynamics by giving up the invariance of mass. It is worth noting that in modern physics language m is used for denoting the rest mass. An approach similar to Lewis and Tolman one is used in [2] where the elastic scattering of two identical particles is observed in the center of mass and in the frame of one of the two particles. The assumption done here (and in [2]) is that the scattering angle is equal to the incidence one which is a possible realization of an elastic scattering.

The relativistic energy

As the mass depends upon v, let us write the Newton law $\vec{F}$=$md\vec{v}/dt$ as

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m_0 \vec{v}) = m_0 \vec{v}\frac{d\gamma}{dt} + m_0 \gamma \frac{d\vec{v}}{dt}$$

By scalar multiplication of the r.hs. and l.h.s. by $\vec{v}$, it is

$$\vec{F} \cdot \vec{v} = \vec{v} \cdot \frac{d\vec{p}}{dt}$$

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = m_0 \gamma \vec{v} \cdot \frac{d\vec{v}}{dt} + m_0 \frac{v^2}{c^2} \gamma^3 v \frac{dv}{dt} = m_0 \gamma v (1 + \frac{v^2 \gamma^2}{c^2}) = m_0 \gamma^3 v \frac{dv}{dt}$$

that is

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} = m_0 \gamma^3 v \frac{dv}{dt}$$

It is easy to verify that this equation is satisfied if we define the energy as

$$E = mc^2 = \gamma m_0 c^2$$

For v=0, it is E_0=$m_0 c^2$ which has the meaning of the *energy at rest*. Relativistically the energy of a free particle at rest is non-vanishing. The (relativistic) kinetic energy is obtained by subtracting the rest energy from the total energy

$$T = mc^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \neq \frac{1}{2} \gamma m_0 v^2$$

which gives the classical kinetic energy $T \simeq m_0 v^2/2$ for $v \ll c$.

Experiments confirmed the validity of the relativistic relationship between $\vec{p}$ and $\vec{v}$.

In particular Bertozzi experiment [6] measured directly the velocity of an e^- beam accelerated in a linear accelerator. The experimental arrangement is shown in Fig. 15. The e^- speed was measured through the time of flight. The kinetic energy was computed from the accelerating field and from the measurement of the heat deposited at the aluminum target. The results in Fig. 16 confirm Einstein prediction and also show clearly the presence of a limit speed, c.

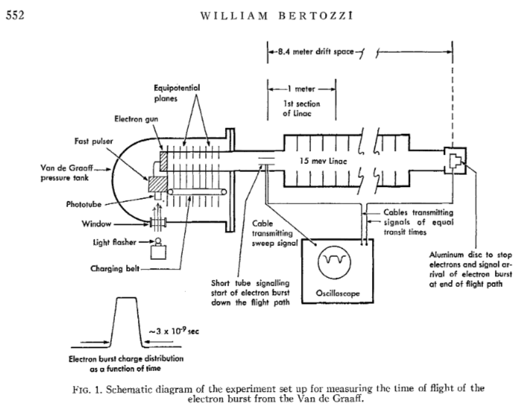

Fig. 15. Bertozzi apparatus (from [6]).

Relativity is of fundamental importance for accelerators where the particles may be accelerated to speed near to c. In a ring accelerator dipole magnets keep the particles on the design orbit and longitudinal

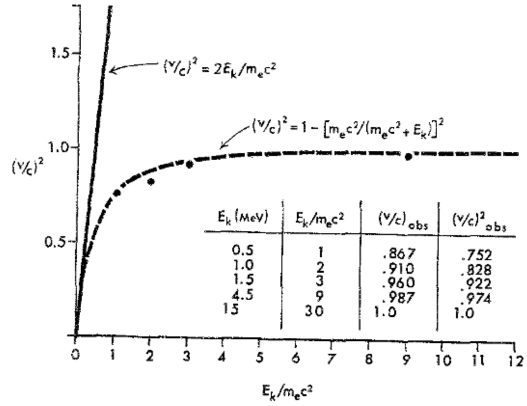

Fig. 16. Bertozzi results (from [6]). The solid line is the calculation using classical formulas, while the dashed line is the relativistic prediction. The dots are the experimental results.

radio-frequency electric fields boost their energy. The relationship between momentum and speed dictates how the frequency of the accelerating electric field and the dipole field must be varied with energy. The dipole field must be ramped up according to momentum for keeping the particles on the design orbit (ρ=p/eB). The electric field frequency, which is a multiple of the revolution frequency, is

$$f_{rf} = hf_{rev} = h\frac{\beta c}{L} = h\frac{c}{\gamma L}\sqrt{\gamma^2 - 1}$$

which for large γ becomes

$$f_{rf} \approx h\frac{c}{L}(1 - \frac{1}{2\gamma^2})$$

At large γ the revolution frequency is almost constant. This is particularly true for $e^{\pm}$ which have 1836 larger γ than protons for the same energy. Fig. 17 shows the CERN PS Booster case where the protons kinetic energy is ramped from 160 MeV to 2 GeV.

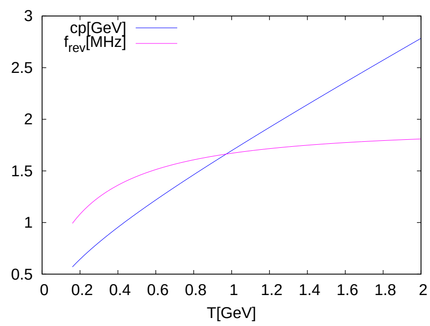

Fig. 17. Momentum and revolution frequency in the CERN PS Booster as a function of the kinetic energy. The ring is about 157 m long.

While in classical mechanics the mass is an invariant scalar conserved in physics processes, relativistically the rest mass alone is not conserved.

To show that the rest mass is not conserved we consider an inelastic scattering (kinetic energy is not conserved) between two identical particles, A and B, with rest mass m_0. In the center of mass, S', it is $\vec{v}'^A = -\vec{v}'^B$. We may assume $\vec{v}'^A = -\vec{v}'^B = \hat{x}v'$ (see Fig. 18). After colliding the two particles glue together in a new particle, C, at rest in S' (see Fig.19) so that momentum is conserved. In the reference frame, S, where A is at rest, the particle B moves before the collision with speed $v_x^B = 2v'/(1+v'^2/c^2)$ while after the collision C moves with speed $v_x^C = v'$ (see Figs. 20, 21). The mass of B in S is

$$m_B = \frac{m_0}{\sqrt{1-(v_x^B/c)^2}} = \frac{m_0[1+(v'/c)^2]}{1-(v'/c)^2}$$

Momentum conservation in S requires

$$\underbrace{p_x^A + p_x^B}_{\text{before}} = \underbrace{p_x^C}_{\text{after}} \quad \rightarrow \quad \frac{m_0 v_x^B}{\sqrt{1-(v_B/c)^2}} = \frac{m_0^C v'}{\sqrt{1-(v'/c)^2}}$$

Using the value found for v_x^B and solving for m_0^C we get

$$m_0^C = \frac{2m_0}{\sqrt{1-(v'/c)^2}}$$

$$m_0^C - 2m_0 = 2m_0\left(\frac{1}{\sqrt{1-(v'/c)^2}} - 1\right)$$

The rest mass of the product particle C is *larger* than the sum of the starting particle rest masses and the difference, multiplied by c^2, is just the initial total kinetic energy in S', $T'_A + T'_B$. The kinetic energy in S' has been completely converted into mass. Although the kinetic energy is not conserved the total energy, kinetic plus energy at rest, is conserved. The fact that the sum of the rest masses is not conserved is a fact well known to every high energy particle physicist. An example is the annihilation of a e^+e^- pair into 2 photons.

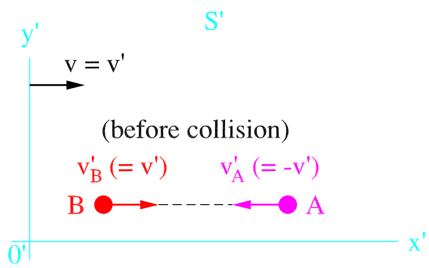

Fig. 18. Identical particle colliding head-on observed in the center of mass frame , S'.

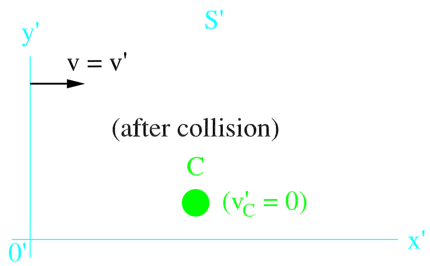

Fig. 19. After inelastic collision the two particles glue together in the particle C at rest in S'.

10 4-vectors

While relativistically lengths and time depend upon the motion of the observer, the interval defined as

$$(ds)^2 \equiv [d(ct)]^2 - (dx)^2 - (dy)^2 - (dz)^2$$

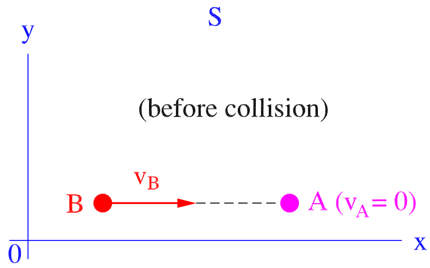

Fig. 20. Particles observed before collision in the frame S where A is at rest.

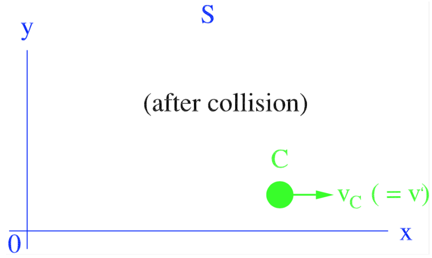

Fig. 21. Particle C after collision as seen in S.

is invariant under Lorentz transformations. Indeed

$$\begin{aligned} ds'^2 &= c^2dt'^2 - \left(dx'^2 + dy'^2 + dz'^2\right) \\ &= \gamma^2\left(c^2dt^2 + \beta^2dx^2 - 2\beta c\,dt\,dx - \beta^2c^2dt^2 - dx^2 + 2\beta c\,dt\,dx\right) - dy^2 - dz^2 \\ &= \gamma^2\left[\left(1-\beta^2\right)\left(c^2dt^2 - dx^2\right)\right] - dy^2 - dz^2 \\ &= c^2dt^2 - dx^2 - dy^2 - dz^2 = ds^2 \end{aligned}$$

If (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2) are the coordinates of two events in S we ask whether it is possible to find an inertial frame S' where the two events happen in the *same place*. As the interval is invariant this means that

$$(\Delta s')^2 = (\Delta s)^2$$

and therefore

$$(c\Delta t')^2 = (c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

where we have set $\Delta t \equiv t_2 - t_1$, $\Delta x \equiv x_2 - x_1$ and so on. The l.h.s. of this equation is always positive. Therefore the answer is affirmative if $(\Delta s)^2 > 0$. Such intervals are called *time-like* intervals. The time in S' between the two events is

$$\Delta t' = \frac{1}{c}\sqrt{c^2\Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)} = \frac{\Delta s}{c}$$

By using the Lorentz transformations we find that the speed of the frame S' with respect to S is $V = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}/\Delta t$ which is smaller than c because $(\Delta s)^2 > 0$.

Now we ask if it is possible to find an inertial frame where the two events happen at the *same time*.

In this case $(\Delta s')^2 = (\Delta s)^2$ implies that

$$(c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) = -(\Delta x'^2 + \Delta y'^2 + \Delta z'^2) < 0$$

that is $(\Delta s)^2$ must be negative. The distance between the two events in S' is

$$\sqrt{\Delta x'^2 + \Delta y'^2 + \Delta z'^2} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2 - (c\Delta t)^2}$$

which is a *real* number as the argument of the square root on the l.h.s is positive.

By using the Lorentz transformations we find

$$0 = c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$$

that is the speed V of the frame S' with respect to S is $V = c^2\Delta t/\Delta x$. The constraint $v < c$ imposes $\Delta x/\Delta t > c$. This means that between the two events there may exist no causality connection. These intervals are called *space-like* intervals. Finally the case $\Delta s'$=Δs=0 corresponds to events connected by a light ray.

Let us consider a particle moving with velocity $\vec{v}(t)$, non necessarily uniform, in S. The time interval $d\tau$ evaluated in a inertial frame S' where the particle is instantaneously at rest is called proper time. It is related to the time measured in S by

$$d\tau = \sqrt{1 - v^2/c^2} dt \equiv \frac{dt}{\gamma}$$

and for a finite time interval

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1 - v^2/c^2}}$$

The proper time is *by definition* an invariant. This results also from the fact that $c^2 d\tau^2$ is the invariant ds^2 evaluated in the frame where the particle is instantaneously at rest. This definition of proper time contains the definition given in Section VI as a particular case when the particle is not accelerated.

An object which 4 components transform as (ct, x, y, z) is a 4-vector. In the same way as done for intervals, one can prove that for any 4-vector the quantities

$$A^\nu B_\nu \equiv A_0 B_0 - \left(A_x B_x + A_y B_y + A_z B_z\right)$$

and in particular

$$A^\nu A_\nu = A_0^2 - \left(A_x^2 + A_y^2 + A_z^2\right)$$

are invariant.

Classically the scalar products, $\vec{A} \cdot \vec{B}$, and in particular the length of vectors, $\vec{A} \cdot \vec{A}$, are invariant.

11 Minkowski space-time

In 1907 the mathematician Hermann Minkowski, who was Einstein professor at Zürich Polytechnic, showed that the special theory of relativity can be formulated by using a 4-dimensional space with metric tensor[d], g, given by

$$g = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Lorentz frames are those where the metric tensor takes this special form; they are connected by Lorentz transformations. The points of Minkowski space-time are the *events* and the vectors in this space have 4 components (*4-vectors*) which transforms according to Lorentz transformations.

The transformations for momentum and energy may be found directly from the definitions and the Lorentz transformations for the velocity, without introducing the notion of 4-vectors. The result is

$$p'_x = \gamma_V (p_x - E\, V/c^2) \qquad p'_y = p_y \qquad p'_z = p_z$$

$$E' = \gamma_V (E - V\, p_x)$$

[d]Here it is a 4×4 matrix defining the scalar product.

with $\gamma_V \equiv 1/\sqrt{1 - V^2/c^2}$. *A posteriori* we notice that the transformations have the same form as the coordinates transformations with $\vec{r} \to \vec{p}$ and $t \to E/c^2$ and therefore $(E/c, \vec{p})$ is a 4-vector.

A more elegant way of reaching the same result is by noticing that $(E/c, \vec{p})$ *must* transform according to Lorentz transformations, and therefore it is a 4-vector. Indeed owing to the fact that the proper time interval $d\tau = dt/\gamma$ is an invariant and that (cdt, dx, dy, dz) transforms obviously as (ct, x, y, z), the quantity (*4-velocity*) defined as

$$\left(\frac{c\,dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \left(\gamma\frac{c\,dt}{dt}, \gamma\frac{dx}{dt}, \gamma\frac{dy}{dt}, \gamma\frac{dz}{dt}\right) \equiv (\gamma c, \gamma\vec{v}) \tag{15}$$

transforms according to Lorentz transformations. Multiplying the 4-velocity by the rest mass we get

$$m_0(\gamma c, \gamma\vec{v}) = (E/c, \vec{p})$$

which is also a 4-vector (energy-momentum or 4-momentum vector) and therefore transforms according to Lorentz transformation.

Relativistically energy and momentum are closely connected. If in one inertial reference frame energy and momentum are conserved ($\Delta\vec{p}$=0 and ΔE=0), for example in a collision between particles, they are conserved in every other inertial frame because a 4-vector having all components vanishing in a reference frame will have vanishing components in any other one too.

Similarly if momentum is conserved for two inertial observers ($\Delta\vec{p}$=$\Delta\vec{p}\,'$=0), the energy too must be conserved.

12 Newton and Minkowski force and their relativistic transformation

We may write the relativistic Newton law $\vec{F}$=$d\vec{p}/dt$ in terms of 4-vectors. In the particle proper frame

$$\frac{dp^\nu}{d\tau} = f^\nu \tag{16}$$

with (p^0, p^1, p^2, p^3)=$(E/c, p_x, p_y, p_z)$ and (f^0, f^1, f^2, f^3)=(f^0, F_x, F_y, F_z). The l.h.s. is a 4-vector and therefore also $\vec{f}$, the Minkowski force, on the r.h.s. must be a 4-vector related to the Newton force $\vec{F}$.

The space part of the equation of motion is

$$\frac{d\vec{p}}{d\tau} = \vec{f} \qquad \to \qquad \gamma\frac{d\vec{p}}{dt} = \vec{f} \qquad \to \qquad \vec{f} = \gamma\vec{F}$$

The time part of Eq. (16) is

$$f^0 = \frac{dp^0}{d\tau} = \frac{1}{2p^0}\frac{d(p^0)^2}{d\tau} = \frac{1}{2p^0}\frac{d(E/c)^2}{d\tau} \tag{17}$$

The invariance of $(E/c)^2 - \vec{p}\cdot\vec{p} = (m_0c)^2$ implies that

$$\frac{d}{d\tau}\left[\left(\frac{E}{c}\right)^2 - \vec{p}\cdot\vec{p}\right] = 0$$

$$\frac{d}{d\tau}\left(\frac{E}{c}\right)^2 = 2\vec{p}\cdot\frac{d\vec{p}}{d\tau}$$

which inserted in Eq. (17) gives

$$f^0 = \frac{1}{2p^0}\frac{d(E/c)^2}{d\tau} = \frac{1}{2p^0}2\vec{p}\cdot\frac{d\vec{p}}{d\tau} = \frac{m_0\gamma\vec{v}}{E/c}\cdot\left(\gamma\vec{F}\right) = \gamma\vec{\beta}\cdot\vec{F} \tag{18}$$

The Minkowski force is therefore

$$(f^0, f^1, f^2, f^3) = (\gamma\vec{\beta}\cdot\vec{F}, \gamma\vec{F})$$

We notice that

$$\frac{dE}{dt} = \frac{1}{\gamma}\vec{v}\cdot\vec{f} = \frac{1}{\gamma}\frac{d\vec{\ell}}{dt}\cdot\vec{f} \qquad \rightarrow \quad dE = \frac{1}{\gamma}d\vec{\ell}\cdot\vec{f}$$

which is the expression of the work done by a force $\vec{F} = \vec{f}/\gamma$.
In absence of external forces ($\vec{F}$=0) it is $\vec{f}$=0 and momentum and energy are conserved.
Being a 4-vector, Minkowski force transforms following Lorentz transformations. It must be paid attention to distinguish between the particle velocity, $\vec{v}$, in the S frame and the frames relative speed that we will denote by $\vec{V}$. Using the general expression of Lorentz transformations Eq. (6) we have

$$\begin{aligned} f'^0 &= \gamma_V\left(f^0 - \vec{\beta}_V\cdot\vec{f}\right) \\ \vec{f}\,' &= \vec{f} + \frac{\gamma_V - 1}{\beta_V^2}(\vec{\beta}_V\cdot\vec{f}\,)\vec{\beta}_V - \gamma_V f^0\vec{\beta}_V \end{aligned}$$

The Newton force transformation writes

$$\gamma'\vec{F}' = \gamma\vec{F} + \frac{\gamma_V - 1}{\beta_V^2}\left[\vec{\beta}_V\cdot\left(\gamma\vec{F}\right)\right]\vec{\beta}_V - \gamma_V\vec{\beta}_V\left(\gamma\vec{\beta}\cdot\vec{F}\right) \tag{19}$$

The inverse transformation is obtained by replacing $\vec{\beta}_V$ with $-\vec{\beta}_V$

$$\gamma\vec{F} = \gamma'\vec{F}' + \frac{\gamma_V - 1}{\beta_V^2}\left[\vec{\beta}_V\cdot\left(\gamma'\vec{F}'\right)\right]\vec{\beta}_V + \gamma_V\vec{\beta}_V\left(\gamma'\vec{\beta}'\cdot\vec{F}'\right) \tag{20}$$

13 Relativistic transformation of EM fields

The force acting on a charged particle moving in a EM field is the Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

The corresponding Minkowski force is

$$f^\nu = \left(\gamma\vec{\beta}\cdot\vec{F}, \gamma\vec{F}\right) = q\left[\gamma\vec{\beta}\cdot\left(\vec{E} + \vec{v}\times\vec{B}\right), \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right)\right]$$

with $\gamma = 1/\sqrt{1-(V/c)^2}$. This equation can be written in matrix form as

$$\begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix} = \frac{q}{c}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix}\begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

Minkowski force and the 4-velocity $(\gamma c, \gamma\vec{v})$ (Eq. 15) are 4-vectors. Using the Lorentz transformation $\mathcal{L}$ from S to S' and $\mathcal{L}^{-1}$ from S' to S we get

$$\begin{pmatrix} f'^0 \\ f'^1 \\ f'^2 \\ f'^3 \end{pmatrix} = \mathcal{L} \begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix} = \frac{q}{c}\mathcal{L} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \mathcal{L}^{-1} \begin{pmatrix} \gamma' c \\ \gamma' v'_x \\ \gamma' v'_y \\ \gamma' v'_z \end{pmatrix}$$

Requiring that the Minkowski force in S' has the same form as in S, it must be

$$\begin{pmatrix} 0 & E'_x & E'_y & E'_z \\ E'_x & 0 & cB'_z & -cB'_y \\ E'_y & -cB'_z & 0 & cB'_x \\ E'_z & cB'_y & -cB'_x & 0 \end{pmatrix} = \mathcal{L} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & cB_z & -cB_y \\ E_y & -cB_z & 0 & cB_x \\ E_z & cB_y & -cB_x & 0 \end{pmatrix} \mathcal{L}^{-1}$$

which yelds the field components in S'

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma_V(E_y - VB_z) & B'_y &= \gamma_V\Big(B_y + \frac{V}{c^2}E_z\Big) \\ E'_z &= \gamma_V(E_z + VB_y) & B'_z &= \gamma_V\Big(B_z - \frac{V}{c^2}E_y\Big) \end{aligned}$$

In alternative to the previous formal approach, we give here a way for finding directly the field transformation from physical considerations.

The force acting on a charged particle moving in a EM field is the Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

The corresponding Minkowski force is

$$f^\nu = \left(\gamma\vec{\beta} \cdot \vec{F}, \gamma\vec{F}\right) = q\left[\gamma\vec{\beta} \cdot \vec{E}, \gamma\left(\vec{E} + \vec{v} \times \vec{B}\right)\right]$$

In a second reference frame, S', the force must have the same form

$$f'^\nu = \left(\gamma'\vec{\beta}' \cdot \vec{F}', \gamma'\vec{F}'\right) = q\left[\gamma'\vec{\beta}' \cdot \vec{E}', \gamma'\left(\vec{E}' + \vec{v}' \times \vec{B}'\right)\right]$$

where we assumed $q' = q$ which is a fact experimentally proven with high precision.

Knowing how the Minkowski force transforms it is possible to get the expressions for the field transformations.

Let us consider the case of a particle at rest in S subject to the fields $\vec{E}$ and $\vec{B}$. In S it is

$$\vec{F} = q\vec{E}$$

In the frame S' moving with translational motion along the common $\hat{x}$-axis with velocity V with respect to S it is v'_x=$-V$ and v'_y=v'_z=0. The force components in S' are

$$\begin{aligned} F'_x &= q(E'_x + v'_y B'_z - v'_z B'_y) = qE'_x \\ F'_y &= q(E'_y - v'_x B'_z + v'_z B'_x) = qE'_y + qVB'_z \\ F'_z &= q(E'_z + v'_x B'_y - v'_y B'_x) = qE'_z - qVB'_y \end{aligned}$$

From Eq. (19) the force components transform as

$$\gamma_V F'_x = F_x + \frac{\gamma_V - 1}{\beta_V^2}\beta_V^2 F_x = \gamma_V F_x$$
$$\gamma_V F'_y = F_y$$
$$\gamma_V F'_z = F_z$$

with $\gamma_V \equiv 1/\sqrt{1-(V/c)^2}$. Writing explicitly the force in terms of the fields we get

$$E_x = E'_x \qquad E_y = \gamma_V(E'_y + VB'_z) \qquad E_z = \gamma_V(E'_z - VB'_y)$$

The inverse transformation are obtained replacing V with $-V$

$$E'_x = E_x \qquad E'_y = \gamma_V(E_y - VB_z) \qquad E'_z = \gamma_V(E_z + VB_y)$$

Finding out the transformation for the magnetic field is a more complicated because the electric force cannot be made vanishing by a convenient choice of the reference frame. We consider again the two frames S and S', with S' moving with velocity V along the common $\hat{x}$-axis. For a charged particle moving in S' along the $\hat{y}'$-axis it is v_x=V, v_y=v'_y/γ_V and v_z=v'_z=0. The force in S' is

$$F'_x = q(E'_x + v'_y B'_z) \qquad F'_y = qE'_y \qquad F'_z = q(E'_z - v'_y B'_x)$$

Using the force transformation we get

$$\gamma' F'_x = \gamma' q(E'_x + v'_y B'_z) = \gamma F_x + \frac{\gamma_V - 1}{\beta_V^2}\beta_V^2 \gamma F_x - \gamma_V \gamma \beta_V(\frac{v_x}{c}F_x + \frac{v_y}{c}F_y)$$
$$= \frac{\gamma}{\gamma_V}F_x - \gamma_V\gamma\frac{v_y V}{c^2}F_y = \frac{\gamma}{\gamma_V}q(E_x + v_y B_z) - \gamma_V\gamma\frac{v_y V}{c^2}q(E_y - v_x B_z)$$
$$\gamma' F'_y = \gamma' q E'_y = \gamma F_y = \gamma q(E_y - v_x B_z)$$
$$\gamma' F'_z = \gamma' q(E'_z - v'_y B'_x) = \gamma F_z = \gamma q(E_z + v_x B_y - v_y B_x)$$

Using the transformations already found for the electric field and the fact that in this case it is $\gamma'\gamma_V = \gamma$, we notice that the equation for F'_y is an identity while the other two equations give

$$\gamma v_y B'_z = \frac{\gamma}{\gamma_V}v_y B_z - \gamma\gamma_V\frac{v_y V}{c^2}(E_y - VB_z)$$

$$\gamma'\gamma_V(E_z + VB_y) - \gamma_V v_y B'_x = \gamma(E_z + VB_y - v_y B_x)$$

The magnetic field component transformations are therefore

$$B'_z = \gamma_V(B_z - \frac{V}{c^2}E_y)$$
$$B'_x = B_x$$

The transformation for B_y is obtained considering a particle moving along the $\hat{z}'$-axis and writes

$$B'_y = \gamma_V(B_y + \frac{V}{c^2}E_z)$$

The expressions found are valid for a translational motion along the x-axis. In the general case when $\vec{V}$ has an arbitrary direction the field transformations write [4]

$$\begin{aligned}\vec{E}' &= \gamma_V\left(\vec{E}+\vec{V}\times\vec{B}\right)-\frac{\gamma_V^2}{\gamma_V+1}(\vec{\beta}_V\cdot\vec{E})\vec{\beta}_V\\ \vec{B}' &= \gamma_V\left(\vec{B}-\frac{\vec{V}}{c^2}\times\vec{E}\right)-\frac{\gamma_V^2}{\gamma_V+1}(\vec{\beta}_V\cdot\vec{B})\vec{\beta}_V\end{aligned} \tag{21}$$

Decomposing the fields in their components parallel and perpendicular to the relative velocity $\vec{V}$, these relations may be written also as

$$\begin{aligned}\vec{E}' &= \vec{E}_{||}+\gamma_V\left(\vec{E}_{\perp}+\vec{V}\times\vec{B}\right)\\ \vec{B}' &= \vec{B}_{||}+\gamma_V\left(\vec{B}_{\perp}-\frac{\vec{V}}{c^2}\times\vec{E}\right)\end{aligned}$$

where we made use of the identity

$$\gamma_V-\frac{\gamma_V^2\beta_V^2}{\gamma_V+1}=1$$

Transformation of a charge distribution

Let us consider a distribution of charges at rest in S'. The charge density is given by

$$\rho'(x',y',z',t')=\frac{qN}{dx'dy'dz'}$$

In S, moving with velocity $-V$ with respect to S' (see Fig. 22), the volume element is

$$dx\,dy\,dz=\frac{dx'}{\gamma}dy'dz'$$

where we have taken into account the length contraction in the x direction.

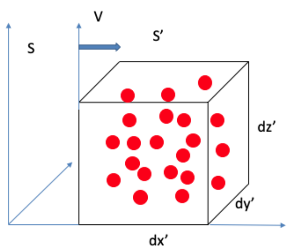

Fig. 22. Charge distribution at rest in S' .

The charge density in S is therefore

$$\rho=\frac{qN}{dx\,dy\,dz}=\gamma\rho'$$

As the charge distribution moves in S with velocity $+\hat{x}V$, in S there is a current moving in the x direction with density

$$j_x=\rho V=\gamma\rho' V$$

and in general

$$\vec{j} = \rho\vec{V} = \gamma\rho'\vec{V}$$

There is an analogy with the energy-momentum 4-vector $(E/c, \vec{p})$ or $(mc, \vec{p})$ with $\rho \ \rightarrow \ m$ and $\vec{j} \ \rightarrow \ \vec{p}$. In fact renaming by ρ_0 the charge density in the rest frame, ρ', we may write

$$\rho = \rho_0 \frac{m}{m_0}$$

$$\vec{j} = \rho_0 \frac{\vec{p}}{m_0}$$

as m/m_0=γ. Therefore $(\rho c, \vec{j}\ )$ is a 4-vector. Indeed the transformations we have found are the (inverse) Lorentz transformations for the particular case $\vec{j}'$=0.

In the Lorentz gauge

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \Phi}{\partial t}$$

the equations for the scalar and vector potential take the form

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{\rho}{\epsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{\vec{j}}{\epsilon_0 c^2}$$

Using the d'Alembert operator

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

these equations can be combined in a single one

$$\Box A^\alpha = \frac{1}{\epsilon_0 c^2}\mu_0 J^\alpha \qquad (22)$$

with A^0=Φ/c, A^1=A_x, A^2=A_y, A^3=A_z and J^0=$c\rho$, J^1=j_x, J^2=j_y, J^3=j_z. We know now that $(c\rho, \vec{j}\ )$ is a 4-vector and it is easy to verify that the d'Alembert operator is invariant under Lorentz transformations. Therefore by requiring covariance of the Eq. (22) also $(\Phi/c, \vec{A})$ must be a 4-vector.

Direct proof of invariance of Maxwell Equations

Knowing how fields and sources transform one can prove that Maxwell equations are invariant under Lorentz transformation.

For example let us prove that

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \Rightarrow \qquad \nabla' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0}$$

The partial derivatives in S' and in S are related by the cyclic rule

$$\frac{\partial}{\partial ct'} = \frac{\partial ct}{\partial ct'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial ct'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial ct'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial ct'}\frac{\partial}{\partial z} = \gamma\left(\frac{\partial}{\partial ct} + \beta\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial x'} = \frac{\partial ct}{\partial x'}\frac{\partial}{\partial ct} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial x'}\frac{\partial}{\partial z} = \gamma\left(\beta\frac{\partial}{\partial ct} + \frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \qquad\qquad \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

By using the cyclic rule, the EM field transformations and the fact that Maxwell equation hold good in S, we find

$$\begin{aligned}
\nabla' \cdot \vec{E}' &= \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} \\
&= \gamma \frac{\partial E'_x}{\partial x} + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} + \gamma\beta \frac{\partial E'_x}{\partial ct} \\
&= \gamma \frac{\partial E_x}{\partial x} + \gamma \frac{\partial E_y}{\partial y} + \gamma \frac{\partial E_z}{\partial z} - \gamma V \frac{\partial B_z}{\partial y} + \gamma V \frac{\partial B_y}{\partial z} + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \nabla \cdot \vec{E} - \gamma V \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \gamma\beta \frac{\partial E_x}{\partial ct} \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \left(\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right)_x \\
&= \gamma \frac{\rho}{\epsilon_0} - \gamma V \frac{j_x}{\epsilon_0 c^2} \\
&= \frac{\gamma}{\epsilon_0 c} (\rho c - \beta j_x) \\
&= \frac{\rho'}{\epsilon_0}
\end{aligned}$$

which prove the invariance of the first Maxwell law under Lorentz transformations.

14 Some geometrical aspects of special relativity

Let us consider our observer O at the origin of the inertial frame S. We can represent the x and $w \equiv ct$ coordinates[e] measured by O on two orthogonal axis (see Fig. 23). This graphical illustration was introduced by Minkowski. Any event is represented by a point in the Minkowski diagram and the trajectory of a particle will be a sequence of points called "world line". The angle between the tangent to a material particle world line and the w-axis is always smaller than 45^0, as the particle speed is always smaller than c. The world line of a light ray is a straight line at 45^0.

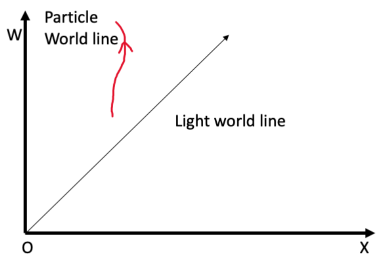

Fig. 23. (x, w) diagram relative to the inertial reference frame S.

Let us consider the (x, w) diagram relative to an inertial reference frame S. The world lines of light waves delimit the grey area in Fig. 24 and define the so called *light cone*. For any event point inside the grey area, P, it is $w^2 - x^2 > 0$. That is the interval Δs between those points and O are time-like and, as seen in the previous section, this means that it is always possible to find a Lorentz transformation where the event happens in the same place and therefore it can be established their chronological sequence. The

[e] For simplicity only the space coordinate x is considered.

events in the upper part of the grey region for which t >0 happen after the event O. This region is called *future* with respect to O). The events represented by points in the lower part of the grey region for which t <0 happen before O (*past*). As the interval Δs^2 is invariant the fact that the P is a future event with respect to O does not depend upon the reference frame.

All points like Q outside the grey area correspond to space-like intervals because $\Delta s^2 = (ct)^2 - x^2 < 0$. As previously shown these are space-like intervals for which it is not possible to find a reference frame where the events happen in the same space point. Therefore it is not possible to establish a chronological sequence between them. This region is called *elsewhere*.

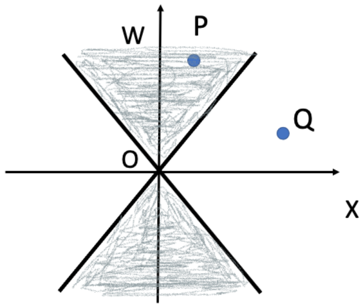

Fig. 24. The light cone relative to the observer O. P is an event in the future, while Q is an "elsewhere" event.

15 SOME APPLICATIONS OF THE EM FIELD TRANSFORMATIONS

The fact that physics laws are the same in any reference frame allows us to solve problems in the most convenient reference frame. Here we show two typical examples which are relevant in accelerator physics.

The field of a moving charge

The EM fields generated by a charge at rest in the origin of the S' frame is

$$\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}'}{r'^3}$$
$$\vec{B}' = 0$$

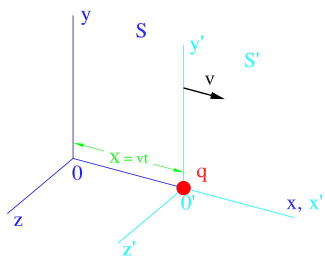

Fig. 25. Particle moving along the x-axis of reference S.

We can use the field transformations found in Section XIII for computing the fields in the frame where the particle is uniformly moving. We chose the frame so that particle moves along the x-axis (see Fig. 25).

The electric field in S is

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{r'^3} \qquad E_y = \gamma E'_y = \gamma \frac{q}{4\pi\epsilon_0} \frac{y'}{r'^3} \qquad E_z = \gamma E'_z = \gamma \frac{q}{4\pi\epsilon_0} \frac{z'}{r'^3}$$

Expressing the accented particle coordinates in terms of the coordinates in S, that is $x' = \gamma(x - vt)$, y'=y and z'=z, the electric field components are

$$E_x = \frac{q}{4\pi\epsilon_0}\frac{\gamma(x-vt)}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$
$$E_y = \frac{q}{4\pi\epsilon_0}\frac{\gamma y}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$
$$E_z = \frac{q}{4\pi\epsilon_0}\frac{\gamma z}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}$$

As the particle is moving, in S there is also a magnetic field. Using the magnetic field transformations it is

$$\vec{B}' = 0 = \vec{B}_{\|} + \gamma_v\left(\vec{B}_{\perp} - \frac{\vec{V}}{c^2}\times\vec{E}\right)$$

which means

$$\vec{B}_{\|} = 0$$
$$\vec{B_{\perp}} = \frac{1}{c^2}\vec{v}\times\vec{E}$$

We may evaluate the electric field at the time $t = 0$ [f]

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{r}}{[\gamma^2x^2+y^2+z^2]^{3/2}}$$

Denoting with θ the angle between the $\hat{x}$-axis and $\vec{r}$ and using the relationship

$$\gamma^2x^2+y^2+z^2 = \gamma^2r^2(1-\beta^2\sin^2\theta)$$

we get

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1-\beta^2}{r^2(1-\beta^2\sin^2\theta)^{3/2}}\frac{\vec{r}}{r}$$

The electric field is still radial and follows the $1/r^2$ law, but has no more a spherical symmetry. The magnetic field is perpendicular to the plane defined by $\vec{r}$ and $\vec{v}$. In accelerators, particles are often "ultra-relativistic" that is their speed in the laboratory frame is almost c. For $\beta \to 1$ it is $\vec{E} \to 0$, unless θ=90^0 or 270^0 where the field is enhanced by a factor γ.

Forces between moving charges

Let us consider an uniform cylindrical beam of radius R of equally charged particles moving with velocity v along $\hat{x}$ (see Fig. 26). Each of them experiences a repulsive electric force and an attractive magnetic force.

In the reference frame S' where the particles are at rest there is no magnetic field. Inside the beam ($r' \leq R$) it is

$$F'_{r'} = qE'_{r'} = \frac{1}{2\pi\epsilon_0 R^2}q^2\lambda' r' \qquad \lambda' = N/L'$$

[f] At a different time $\bar{t}$ the fields take at (x, y, z) the same values as at $(x - v\bar{t}, y, z)$ for t=0.

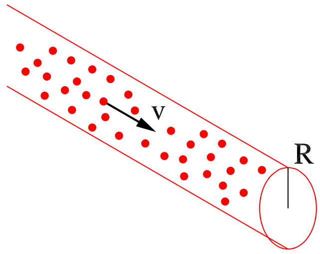

Fig. 26. Uniform cylindrical charge distribution.

that is the force acting on each charge is purely radial.
By using Eq. (20) for the Newton force transformation with γ'=1, β'=0, γ_V=γ, β_V=β and $\vec{\beta}_V \cdot \vec{F}'$=0, and we get

$$F_{\parallel} = 0 \qquad F_r = \frac{1}{\gamma} F'_{r'} = \frac{1}{2\pi\epsilon_0 R^2} q^2 \lambda r \frac{1}{\gamma^2}$$

where the line density in S, λ, is related to the line density in S', λ', by $\lambda = \gamma N/L'$. In the reference frame S the force is still radial and repulsive, but it is reduced by a factor $1/\gamma^2$.
Beam in accelerators may be approximated by a uniform cylindrical charge distribution. We see that the repulsive force between the equally charged particles becomes smaller at high energy.

16 THE CM ENERGY

The center of momentum, usually referred as center of mass, for an isolated ensemble of particles is defined as the inertial frame where it holds

$$\sum_i \vec{p}_i = \sum_i \frac{m_{0,i}\vec{v}_i}{\sqrt{1 - V^2/c^2}} = 0$$

where V is the frame speed with respect to the laboratory.
We have seen that $(E/c)^2 - |\vec{p}|^2$ is an invariant with value $m_0^2 c^2$. For the total energy and momentum of the ensemble

$$E = \sum_i E_i \qquad \text{and} \qquad \vec{P} = \sum_i \vec{p}_i$$

the invariant evaluated in the CM frame is

$$\left(\sum_i E_i/c\right)^2 - \sum_i \vec{p}_i \cdot \sum_i \vec{p}_i = \left(\sum_i E'_i/c\right)^2$$

where E'_i is the energy of the $i - th$ particle in the CM frame. Let us consider two simple cases:

a) two ultra-relativistic particles colliding “head-on”;
b) one ultra-relativistic particle hitting a particle at rest.

For the system of two particles it is

$$\begin{aligned}\frac{(E'_1 + E'_2)^2}{c^2} &= \frac{(E_1 + E_2)^2}{c^2} - (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2) \\ &= \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2\end{aligned}$$

Moreover for ultra-relativistic particles it is

$$p = mv \simeq mc = \frac{E}{c}$$

Case a): $\vec{p}_1/p_1 = -\vec{p}_2/p_2$ (see Fig. 27).

Fig. 27. Two particles colliding head-on.

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1E_2}{c^2} - \frac{E_1^2}{c^2} - \frac{E_2^2}{c^2} + 2\frac{E_1E_2}{c^2} = 4\frac{E_1E_2}{c^2}$$

and thus

$$E_1' + E_2' = 2\sqrt{E_1E_2}$$

For instance, for the LHC pp collider it is E_1=E_2=6.5 TeV and the energy in the center of mass is $E_1' + E_2'$=2×6.5=13 TeV. For the $p/e^{\pm}$ HERA collider, which was in operation until 2007, with E_1=920 GeV and E_2=27.5 GeV it is $E_1' + E_2'$=318 GeV. Case b): $\vec{p}_2 = 0$ and $E_2 = m_{0,2}c^2$ (see Fig. 28).

Fig. 28. Particle hitting a second particle at rest in the laboratory frame S.

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{(E_1 + E_2)^2}{c^2} - p_1^2 - p_2^2 - 2\vec{p}_1 \cdot \vec{p}_2$$

$$\frac{(E_1' + E_2')^2}{c^2} = \frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2\frac{E_1E_2}{c^2} - \frac{E_1^2}{c^2} = \frac{E_2^2}{c^2} + 2\frac{E_1E_2}{c^2}$$

$$(E_1' + E_2') = \sqrt{E_2(E_2 + 2E_1)} = \sqrt{E_2(m_{0,2}c^2 + 2E_1)} \simeq \sqrt{2E_1E_2}$$

For example, with $E_2 = 0.938$ GeV (proton rest mass) to get in the CM an energy of 318 GeV must be E_1 =54 TeV.

From this example we see the advantage of collider experiments with respect to fixed target ones in terms of available energy.

17 THE RELATIVISTIC HAMILTONIAN OF A PARTICLE IN A EM FIELD

Let us consider a physical system in the presence of generalized forces which can be derived from a function $U = U(q_i, \dot{q}_i)$ (*generalized potential*).

It is possible to associate to such system a lagrangian function $\mathcal{L} = T - U$, T being the kinetic energy. The dynamics of the system is described by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$

The coordinates $(q_i, \dot{q}_i)$ may be just the components of $(\vec{r}_j, \dot{\vec{r}}_j)$, but it can be convenient or even necessary to use other variables.

The generalized forces are related to the function U by [7]

$$F_\alpha = \sum_i \frac{\partial q_i}{\partial r_\alpha}\left[-\frac{\partial U}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_i}\right]$$

which if the coordinates $(\vec{r}, \dot{\vec{r}})$ are used ($\partial q_i/\partial r_\alpha$=$\delta_{i\alpha}$) gives

$$F_\alpha = -\left(\nabla U\right)_\alpha + \frac{d}{dt}\frac{\partial U}{\partial v_\alpha} \qquad \to \qquad \vec{F} = -\nabla U + \frac{d}{dt}\nabla_v U$$

Very often physics problems are described by using the Lagrange or the Hamilton formalism. It is therefore useful to derive the relativistic lagrangian function for a particle in an EM field.

The Hamilton principle says that between all patterns connecting the point $(q_{i,1}, \dot{q}_{i,1}; t_1)$ to the point $(q_{i,2}, \dot{q}_{i,2}; t_2)$ the system will actually follow that one for which the integral (action)

$$S = \int_{t_1}^{t_2} dt \mathcal{L}(q_i, \dot{q}_i; t)$$

has a minimum or a maximum. This principle specifies the dynamics as well as the Lagrange equations do. First we find the lagrangian function for a *free* particle, $\mathcal{L}_{free}$, by asking the action to be a Lorentz invariant.

We rewrite the action by using the proper time $d\tau = dt/\gamma$

$$S = \int_{t_1}^{t_2} dt \mathcal{L}_{free} = \int_{\tau_1}^{\tau_2} d\tau \gamma \mathcal{L}_{free}$$

In order to be $\gamma\mathcal{L}_{free}$ an invariant $\mathcal{L}_{free}$ must be proportional to $1/\gamma$ so that the dependence on γ disappears from the integral.

Let us write than $\mathcal{L}_{free} = \alpha/\gamma$. For a free particle the Lagrange equation becomes

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{free}}{\partial v} = 0$$

Inserting our expression for $\mathcal{L}_{free}$ we have

$$\frac{d}{dt}\frac{\partial}{\partial v}\frac{\alpha}{\gamma} = -\frac{1}{c^2}\frac{d}{dt}\alpha\gamma v = 0$$

which reduces to the Newton law $d(m_0\gamma v)/dt = 0$ if we set $\alpha = -m_0 c^2$. Therefore the relativistic lagrangian function of the free particle is

$$\mathcal{L}_{free} = -\frac{m_0 c^2}{\gamma}$$

Now let us compute the lagrangian function related to the EM fields. The Lorentz force is

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

The EM fields in terms of scalar and vector potentials are (MKSA units)

$$\vec{B} = \nabla \times \vec{A} \qquad\qquad \vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

Thus the Lorentz force can be written as

$$\vec{F} = q\big(-\nabla\Phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \nabla \times \vec{A}\big)$$

We use the identity

$$\nabla(\vec{a}\cdot\vec{b}) = \big(\vec{a}\cdot\nabla\big)\,\vec{b} + \big(\vec{b}\cdot\nabla\big)\,\vec{a} + \vec{a}\times\big(\nabla\times\vec{b}\big) + \vec{b}\times\big(\nabla\times\vec{a}\big)$$

for transforming the term $\vec{v}\times\nabla\times\vec{A}$

$$\vec{v}\times\nabla\times\vec{A} = \nabla\left(\vec{A}\cdot\vec{v}\right) - \big(\vec{v}\cdot\nabla\big)\,\vec{A}$$

Thus the Lorentz force is

$$\vec{F} = q\nabla\big(-\Phi + \vec{A}\cdot\vec{v}\big) - q\frac{\partial\vec{A}}{\partial t} - q\big(\vec{v}\cdot\nabla\big)\vec{A} = q\nabla\big(-\Phi + \vec{A}\cdot\vec{v}\big) - q\frac{d\vec{A}}{dt}$$

We recognize that the generalized potential is $U = q\Phi - q\vec{A}\cdot\vec{v}$. Indeed

$$\frac{d}{dt}\nabla_v U = \frac{d}{dt}\nabla_v\big(q\Phi - q\vec{A}\cdot\vec{v}\big) = -q\frac{d\vec{A}}{dt}$$

because the EM potentials do not depend upon the particle velocity.
In conclusion, the Lorentz force for a particle in an EM field may be written in terms of a generalized potential U as

$$\vec{F} = -\nabla U + \frac{d}{dt}\nabla_v U$$

with

$$U = q\left(\Phi - \vec{A}\cdot\vec{v}\right)$$

and the particle lagrangian related to the EM field is

$$\mathcal{L}_{int} = -U = -q\Phi + q\vec{A}\cdot\vec{v}$$

The total lagrangian is obtained adding the lagrangian of the free particle

$$\mathcal{L} = -\frac{m_0 c^2}{\gamma} - q\Phi + q\vec{A}\cdot\vec{v} \tag{23}$$

The hamiltonian function is related to the lagrangian function by

$$\mathcal{H}(q_i, P_i) = \sum_i P_i \dot{q}_i - \mathcal{L} \tag{24}$$

with

$$P_i \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = p_i + qA_i$$

The hamiltonian must be a function of q_i and P_i and therefore we must express $\dot{q}_i$ (or v_i) in terms of P_i. From

$$p_i = P_i - qA_i$$

and the relationship between momentum and energy

$$c^2p^2 = E^2 - E_0^2 = m_0^2\gamma^2c^4 - m_0^2c^4$$

we get

$$m_0^2\gamma^2c^2 - m_0^2c^2 = p^2 = (\vec{P} - q\vec{A}) \cdot (\vec{P} - q\vec{A})$$

and therefore

$$\vec{v} = \frac{\vec{p}}{m_0\gamma} = c\frac{\vec{P} - q\vec{A}}{\sqrt{m_0^2c^2 + (\vec{P} - q\vec{A})^2}}$$

Inserting this expression in Eqs. (23) and (24) we finally find the hamiltonian function

$$\mathcal{H}(q_i, P_i) = \sum_i P_i\dot{q}_i - \mathcal{L} = c\sqrt{(\vec{P} - q\vec{A})^2 + m_0^2c^2} + q\Phi$$

18 Some relationships

$$\gamma \equiv \frac{1}{\sqrt{1 - (v/c)^2}} \qquad \beta \equiv \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$m = \gamma m_0 \qquad \vec{p} = \gamma m_0\vec{v} = \frac{m_0\vec{v}}{\sqrt{1 - (v/c)^2}} \qquad \left(\frac{v}{c}\right)^2 = \frac{p^2}{(m_0c)^2 + p^2}$$

$$E = mc^2 \qquad E_0 = m_0c^2 \qquad \frac{E}{E_0} = \frac{m_0\gamma c^2}{m_0c^2} = \gamma$$

$$T = E - E_0 = m_0\gamma c^2 - m_0c^2 = m_0c^2(\gamma - 1)$$

$$E^2 = (T + E_0)^2 = m^2c^4 = m_0^2\gamma^2c^4 = \frac{m_0^2c^4}{1 - (v/c)^2} = \frac{m_0^2c^4}{1 - p^2/(m_0^2c^2 + p^2)}$$

$$= \frac{m_0^2c^4}{m_0^2c^2}(m_0^2c^2 + p^2) = m_0^2c^4 + c^2p^2$$

$$cp = c\gamma m_0v = \frac{E}{E_0}cm_0v = \frac{E}{m_0c^2}cm_0v = \beta E \qquad cp \simeq E \quad \text{for} \quad \beta \to 1$$

A table of relationships between β, γ, momentum and relativistic energy, together with their relative variations, may be found in [8].

References

[1] A. Einstein, "Zur Elektrodynamik bewegter Körper", Ann. Physik, 17, 891 (1905). English translation on the web at https://www.fourmilab.ch/etexts/einstein/specrel/www/.

[2] R. Resnick, "Introduction to Special Relativity", John Wiley & Sons, 1968.

[3] J. C. Hafele, R. E. Keating, "Around-the-World Atomic Clocks: Observed Relativistic Time Gains", Science, Vol. 177, No. 4044 (Jul. 14, 1972), 166-168.

[4] J. D. Jackson, "Classical Electrodynamics", John Wiley & Sons, 1998.

[5] G. N. Lewis and R. C. Tolman, "Contributions from the Research Laboratory of Physical Chemistry of the Massachusetts Institute of Technology: The Principle of Relativity, and Non-Newtonian Mechanics", Proceedings of the American Academy of Arts and Sciences, 44, pp.709-726,1909. See also on the web https://www.ias.ac.in/article/fulltext/reso/024/07/0729-0734.

[6] W. Bertozzi, American Journal of Physics, 32 (7): 551-555 (1964).

[7] H. Goldstein, "Classical Mechanics", Addison-Wesley, 1965.

[8] C. Bovet et al., "A selection of formulae and data useful for the design of A.G. synchrotrons", CERN-MPS-SI-Int-DL-70-4, on the web at http://cds.cern.ch/record/104153.