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On the Possibility of Footprint Compression with One Lens in Nonlinear Accelerator Lattice

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Abstract: Electromagnetic interaction of colliding beams along with other nonlinear fields often limits the beams' lifetimes and luminosities. Nonlinearities result in the spread of betatron frequencies (footprint) and, thus, may enhance dynamic diffusion of particles due to high order resonances. One of the possible ways to eliminate nonlinearities and overcome the corresponding difficulties is compensation of nonlinear forces, but, in practice, it is hardly possible to obtain exact linearity of the system. The compensation with a single nonlinear lens cannot cope with distributed nonlinearities, nonlinearities due to parasitic crossings, etc. In the article, we present a method to compute parameters of nonlinear element (lens) that eliminates both the footprint and resonance strength without achieving full compensation.

Keywords: Accelerator modelling and simulations (multi-particle dynamics, single-particle dynamics); Beam dynamics; Beam optics

[1]this publication is dedicated to the memory of Vyacheslav "Slava" Danilov, who died in 2014. It has originally appeared as Preprint FERMILAB-FN-671 in 1998, when Vyacheslav was a guest scientist at Fermilab, on leave from Budker INP (Novosibirsk, Russia), and appears here with minor modifications and updates.

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Contents

1 Introduction

The use of opposite charge particles for compensation of a kick due to counter bunch in colliders is an attractive idea [1], although there are several issues to be solved for its successful realization (see, e.g., [2]). One of them is provision of proper non-linear electron charge distribution. For example, Fig.1 demonstrates the effect of the ideal "electron compressor" on the 2-D tune diagram in the case of an antiproton-proton collider. The largest "leaf" is the antiproton footprint due to head-on collisions with round Gaussian proton beam with charge distribution $\rho_p(r) = C\exp(-r^2/2\sigma^2)$. The smaller one shows the footprint in the case when electron beam with a charge density profile proportional to $\rho_p(r) = -C\cdot 0.83/(1+(r/\sigma)^8)$ is installed on the antiproton orbit. For convenience of presentation, we have separated the two plots horizontally, as in fact the second one would be around zero tune point $\nu_{(x,y)} \approx 0$. One can see a significant reduction of the tune spread with the electron beam. It was originally thought that the electron beam with Gaussian charge density $\rho_e(r) = -\rho_p(r)$ could lead to complete elimination of the footprint, and thus to compensation of the beam-beam effects.

It was later realized that such an idealized picture of the compensation does not fully reflect the reality. First of all, the beam-beam footprint itself can be significantly distorted by imperfections such as crossing angle at the interaction points, numerous parasitic interactions in multibunch colliders at the locations where the beams are separated and do not actually collide but still interact via their long-range electromagnetic forces, etc. The collider focusing lattice itself is not usually perfectly linear and that must be taken into account, too. An additional difficulty is that the nonlinearities are not usually localized in one element - in contrast, they are distributed over the collider ring - and a single thin lens might not be able to eliminate all nonlinearities from the particle motion even if the field distribution in the lens can be controlled. Also of importance is the ratio of the electron beam length and the beta-function at its location. Finally, creation of an insertion with a predetermined two or three dimensional field map might be quite challenging.

This article is an attempt to investigate whether a single thin nonlinear lens can be added to an arbitrary nonlinear focusing lattice in such a way that the particle motion in the modified structure

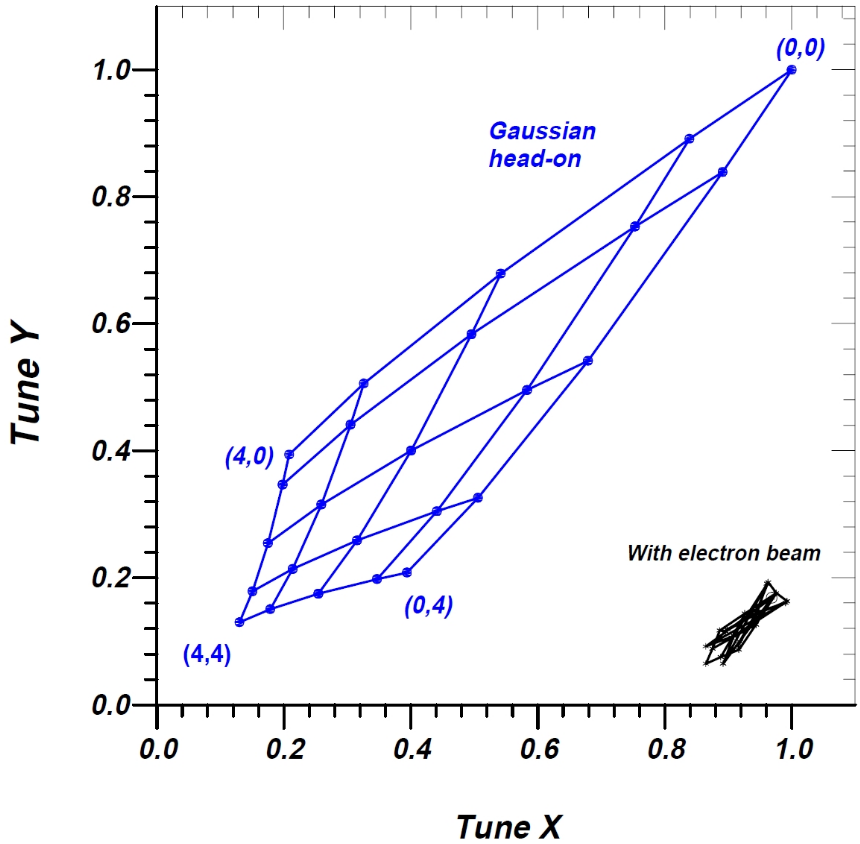

Figure 1. "Electron compression" of head-on footprint of antiprotons: blue lines - horizontal and vertical betatron tune shifts ν_x and ν_y due to collisions with a Gaussian proton beam (in units of the head-on beam-beam parameter ξ); black - the same with an additional electron lens as described in the text. Numbers in parentheses show the horizontal and vertical betatron amplitudes in units of the rms antiproton beam size. The case with electron beam is displaced for clarity.

would become resonance-free, though nonlinear, and the beam of the particles would have zero footprint. We then apply our results to the experimental project of the beam-beam compensation with electron beam (BBCEB) in the Tevatron at FNAL [3].

2 Resonance-Free Nonlinear System

Here we present some of the earlier results on resonance-free nonlinear systems [4]. These systems form a subset of integrable systems, i.e., systems with regular motion, and some of them are applicable to colliding beams. As an example, let us consider nonlinear (de)focusing due to some specially prepared electron beam.

Let us first consider particle motion in a focusing lattice consisting of three elements: 1) a drift space of a unity length (for simplicity); 2) an axially symmetric thin lens, as a representation of the angular-momentum-preserving linear optics in between interactions; and 3) radial beam-beam kick

$k_{rr}(r)$. The 2D map for particle trajectory coordinates x, y and angles x', y' is:

$$\begin{aligned} \bar{x} &= x + x' , \\ \bar{y} &= y + y' , \\ \bar{x'} &= x' + \bar{k_x} , \\ \bar{y'} &= y' + \bar{k_y} , \end{aligned} \tag{2.1}$$

where $k_x = k(r)x/r$, $k_y = k(r)y/r$ and $r = \sqrt{x^2 + y^2}$. Due to conservation of the angular momentum $M = xy' - x'y$, the motion can be reduced to 1D, i.e., to r and $r' = (xx' + yy')/r$. It can be checked directly that if the total radial kick function is equal to:

$$k(r) = -2r - k_{bb}(r) = -2r - \frac{br}{1 + ar^2} , \tag{2.2}$$

where a and b are free parameters, then there is an additional invariant of motion:

$$\mathcal{I}_M(r, r') = (1 + ar^2)\Big((r' + r)^2 + \frac{M^2}{r^2}\Big) + br(r' + r) + r^2 . \tag{2.3}$$

The variables here are changed to r, r', and we used a simple relation $x'^2 + y'^2 2 = ((rr')^2 + (xy' - yx')^2)/r^2 = r'2 + M^2/r^2$. It is easy to find the corresponding electron charge distribution $\rho_e(r)$ which leads to the necessary kick $k_{bb}(r)$. Indeed, if the electron beam length is much smaller than the beta-function at this point, then

$$k_{bb}(r) = \frac{C}{r}\int_0^r \rho_e(r) r dr . \tag{2.4}$$

thus,

$$\rho_e(r) = -\frac{2b}{C}\frac{1}{(1 + ar^2)^2} . \tag{2.5}$$

We do not specify here the value of the constant C, since the constants a, b are arbitrary. Since we provide two invariants of the 2D problem, the resulting motion is regular, though nonlinear (the frequency of oscillations depends on invariant of oscillations) [4].

3 Numerical Algorithm to Eliminate Footprint

In the previous section, we deal with nonlinear integrable system with quite predictable motion. One can use that or a similar system in order to eliminate chaotic particle motion. Difficulties appear if one takes into account nonlinearities and field errors of the real accelerator, which may significantly change the particle motion. It can cause drastic changes if the nonlinear lens brings the system close to higher order resonances. To avoid excitation of these resonances, we need a system with weak dependence of betatron frequency on the oscillation amplitude or with no dependence at all. Such systems are linear in some appropriate variables. Let us consider all 1D classical Hamiltonians

$$H = p^2/2 + U(x) , \tag{3.1}$$

which have the property of constant frequency of oscillations for all initial conditions. (For the sake of simplicity, we assume the mass of the particle is equal to 1.)

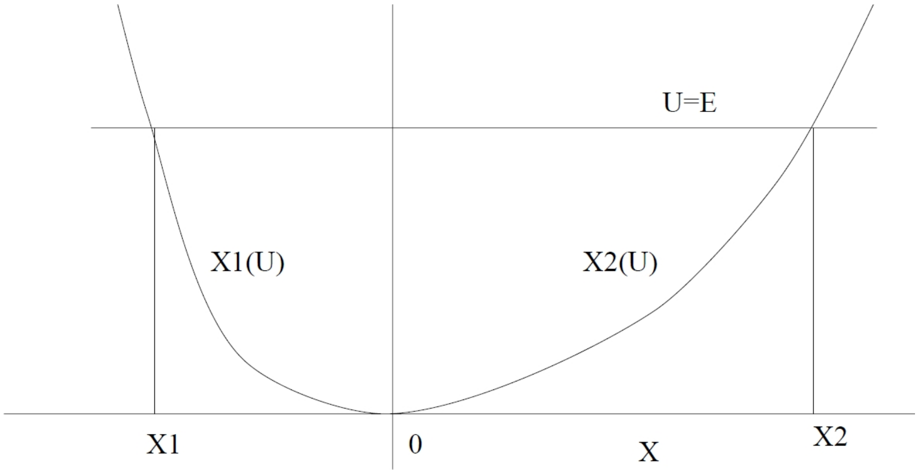

Figure 2. Potential of the classical Hamiltonian vs coordinate (see text).

Under conditions where the potentials $U(x)$ have only one minimum and the oscillation period is a known function $T(E)$ of the energy E (a particular value of the above Hamiltonian), the solution of the problem is given by formula (see e.g. [5]):

$$X_2(U) - X_1(U) = \frac{1}{\sqrt{2}\pi} \int_0^U \frac{T(E)dE}{\sqrt{U-E}} , \tag{3.2}$$

where $X_2(U), X_1(U)$ are left and right boundaries for the particle's motion at the energy $E = U(X1) = U(X2)$ - see Fig.2. Reversion of $X_{1,2}(U)$ gives the potential $U(X)$ vs coordinate for both left and right branches of the potential energy graph. Evidently, we are free to choose only one of the functions $X_1(U)$ or $X_2(U)$, while the second one is to be determined from Eq.(3.2). For a constant period $T(E) = T = const$ in this formula yields:

$$X_2(U) - X_1(U) = \frac{\sqrt{2}}{\pi} T\sqrt{U} . \tag{3.3}$$

This formula has a very simple meaning - while one of the branches, e.g., $X_1(U)$, is determined, the second branch must be adjusted to the first one to keep the same period of oscillations T for all energies. In general, symplectic maps have no appropriate "good" invariant (like energy in the example above), and to find the general form of the maps with constant frequency is a much more difficult problem. We present here a numerical method to construct such maps. It has simple physical meaning but has no reliable mathematical foundation.

The idea is as follows: let us select a frequency (tune of the machine) equal to a resonant one, e.g., the 20th order resonance. This value is chosen for illustration. In fact, for an arbitrary frequency, there is always a rational number in its vicinity such that the difference is negligible. There is a simple way to know whether the motion for every particular initial condition has that frequency or not: one just has to calculate the squared differences of coordinates and momenta at the beginning and at the end of 20 successive map transformations. E.g., in the normalized variables $x = \sqrt{\epsilon} cos(\psi)$ and $x' = \sqrt{\epsilon} sin(\psi)$ (ψ and ϵ are the betatron phase and action, respectively), one can make summation over some region of initial conditions x_i, x'_i and get a special function F:

$$F = \sum_j (x_f - x_i)^2 + (x'_f - x'_i)^2 \,. \tag{3.4}$$

where indices i and f are for initial and final normalized coordinates and angles, respectively, and the index j denotes different phase space elements of initial conditions. When this function is equal to zero $F = 0$, then we have: 1) all the frequencies are equal to the particularly chosen value (1/20 in our case); 2) strength of this resonance is equal to zero. In fact, for resonant islands we get the same average frequency for all phase space elements of the island, but the motion inside the island has its own frequency. It gives the nonzero difference of initial and final conditions after the number of turns equal to the number of resonance (20 turns in our case). When the function F is equal to zero, the motion inside the island is degenerate, so the resonance strength is equal to zero.

We have developed a numerical code to minimize function F. The code can find the solution with very small F in all the important cases when there are enough parameters for optimization. It means that the "number" of integrable systems with constant frequency is rather large, such as for the case of 1D motion with time-independent Hamiltonian, as we have shown above. In the next section we present the results of numerical calculations of the nonlinear lens for the Tevatron.

4 Application to the Tevatron Electron Beam Lens

Fig.3 shows a schematic view of the Tevatron collider, with three beams: circulating high energy proton and antiproton beams and short low-energy high current electron beam.

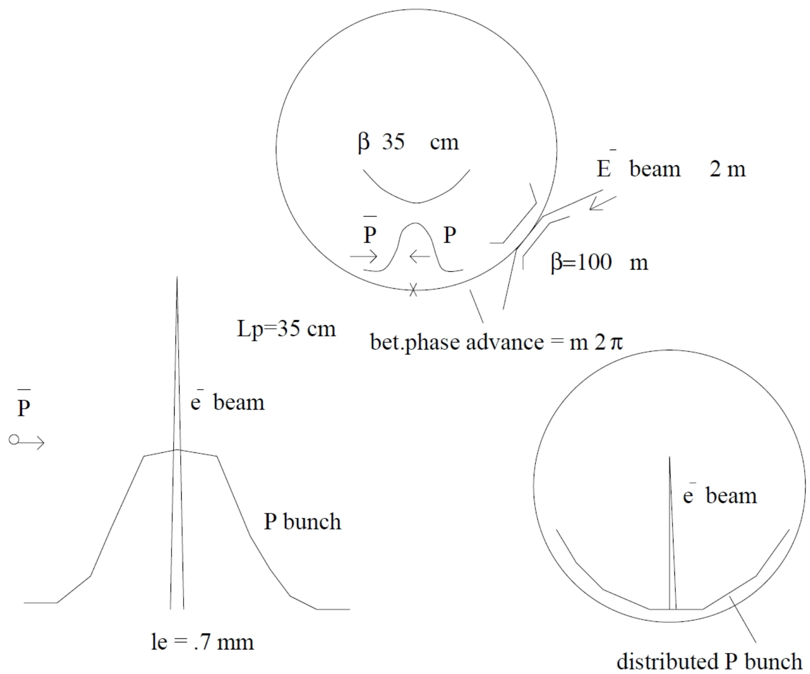

Figure 3. Scheme and some parameters of bunches in the Tevatron.

Evidently, it is beneficial to collide all three beams in one interaction point (IP): if the electron bunch goes along with the proton bunch at the interaction region and has the same size, speed, and charge, then the electromagnetic force due to protons can be compensated for by electrons and the test particles, antiprotons, experience no total kick at the IP. In reality, it is hard to achieve the necessary electron current density (equal to the proton one at the IP) with conventional electron sources. Moreover, often there is no space at the interaction region for necessary additional equipment. Thus, the electron beam has to be placed somewhere else at the ring, preferably in the location where transverse beam size is large. For example, at the location where the beta-function is about β^e = 200m, the Tevatron beam size will be about 1 mm rms, compared to about $\sim 30\mu$m rms transverse beam size at the IP with $\beta^* = 35$ cm. It is desirable to have betatron phase advance between the IP (marked by a cross in the top picture in Fig.3) and the electron beam location to be a multiple of 2π. It was shown in Ref.[1] that a thermionic electron gun can provide necessary beam parameters for compensation of the proton bunch impact on antiprotons.

Nevertheless, there is another difficulty coming from the fact that the proton bunch length of about $\sigma_z \simeq$35cm is comparable to the beta-function at the interaction point β^*. Therefore, the betatron phase advance for antiprotons at the main IP is large $\phi = \int dx/\beta \simeq sigma_z/beta^* \simeq 1$. In contrast, the electron beam length of about 2 m is much less than β^e and the corresponding betatron phase of antiprotons passing the electron beam is very small $\phi \simeq 0.01 - 0.02$. In other words, effectively, the electron beam kick looks like delta-function if transformed to the main IP - see lower left picture in Fig.3. Consequently, this short impact due to electrons contains a lot of resonance harmonics, although the average actions due to proton and electron beams are the same. One can reduce betatron tune spread by such a lens, but this fact alone does not assure the motion is more stable than that with no compensation. For example, if the round proton charge is uniformly distributed along the whole ring - see lower right picture in Fig.3 - than $\bar{p}$ (antiproton) motion is regular in spite of the nonlinear force. Introduction of short electron beam results in time-dependent total force and stochastic dynamics in general. So, the resonance strengths sometimes are more important than the betatron tune spread.

The numerical procedure suggested in the preceding section is used to fix the situation. We operated with transverse electron charge distribution which is the sum of the Gaussian distributions with different rms values:

$$\rho_e(r) = \sum_{n=1}^{6} C_n \exp\Big(-\frac{nr^2}{4\sigma^2}\Big), \tag{4.1}$$

where C_n are variable coefficients for optimization and σ is the rms transverse proton bunch size at the location of the electron beam. In simulations the electron beam produces a delta-function kick because of its short "effective" length, while the proton bunch length is presented as a number of short slices.

We used a somewhat different optimization function than F introduced in previous section, namely:

$$\tilde{F} = \sum_j \frac{(x_f - x_i)^2 + (x'_f - x'_i)^2}{x_i^2 + (x'_i)^2}. \tag{4.2}$$

where j is the index for different initial phase-space elements. The denominator of this expression is added in order to make trajectories with small and large amplitudes the same weight. The numerical code finds coefficients C_n depending on the proton bunch length. The synchrotron amplitude of antiprotons is taken to be zero. In the process of optimization, the value $\tilde{F}$ usually decreases by a factor of ~1000. For example, under conditions of zero length proton bunch, the tune shift due to protons is ξ=0.05, with equal horizontal and vertical beta-functions of $\beta_x^* = \beta_y^*$=35 cm we get $C_1 = 0$, $C_2 = 1.25$, $C_3 = 0$, $C_4 = 0$, $C_5 = 0$, $C_6 = 0$ in the units (for normalized variables). For the proton bunch length of $\sigma_z = 2\beta^*$, we get $C_1 = 0.576$, $C_2 = 0.048$, $C_3 = 0.08$, $C_4 = 0.042$, $C_5 = 0.04$, $C_6 = 0.4$ in the same units. Fig.4 demonstrates the resulting "optimized" distribution and compares it with the Gaussian distribution $C_2 = 1$. Significant difference is clearly seen.

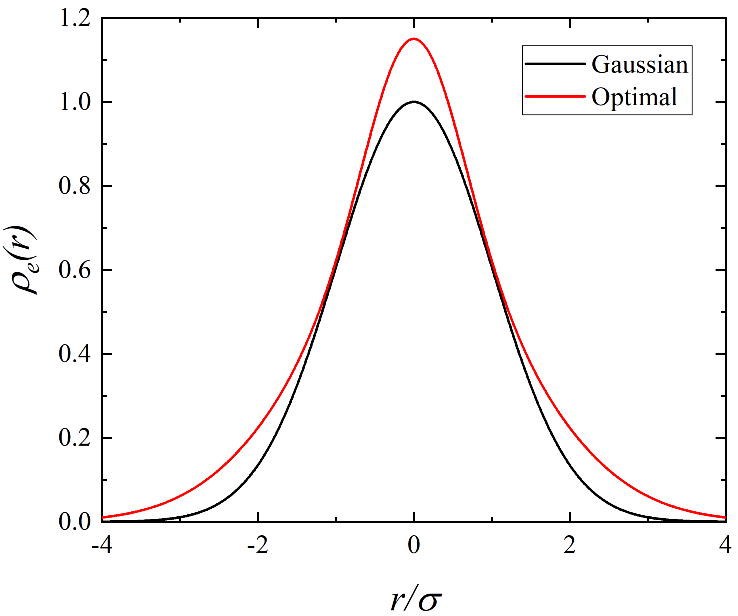

Figure 4. Transverse electron charge distributions necessary for elimination of resonances and tune spread in the antiproton motion: black line is for the Gaussian distribution of the electron charge density to compensate very short proton bunch; red line is for an optimized electron distribution needed to compensate 75-cm long (rms) proton bunch (see text).

5 Conclusion

We suggest a numerical method to calculate the parameters of a single thin lens necessary to compensate for effects of accelerator nonlinearities. It can be applied to a problem of non-linear beam-beam effects in collider rings if a specially prepared electron beam can be used for compensation. The primary studies of the method have shown that it works as soon as there are enough parameters to vary non-linear components. Though the method is valid for 1D case of round beam schemes, it can be generalized for two dimensions.

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Note added [V.S. (2020)]. This work has originally appeared in July 1998 as Preprint FERMILAB-FN-671. Ideas presented in this relatively unknown paper had sprung off previous studies of *integrable systems* [6, 7] and *round colliding beams* [8]. The report has not only played a critical role in establishment of the beam-beam compensation with electron lenses in the Tevatron, but also was a stepping stone toward further development of the concept of nonlinear integrable optics [10] and ongoing experimental studies of such systems with magnets and electron lenses in the IOTA test accelerator at Fermilab [11].

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