Calculations of detuning with amplitude for the McMillan electron lens in the Fermilab Integrable Optics Test Accelerator (IOTA)

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ABSTRACT: The McMillan electron lens will be one of the experimental implementations of nonlinear integrable lattices in the Fermilab Integrable Optics Test Accelerator (IOTA). We describe the physics of the nonlinear McMillan lens and calculate the tune dependence with amplitude from the experimental parameters. Some of the implications for the design of experiments in IOTA are discussed.

KEYWORDS: Accelerator modelling and simulations (multi-particle dynamics; single-particle dynamics); Accelerator Subsystems and Technologies; Beam dynamics; Beam Optics

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1 Introduction

As circular accelerators reach higher intensities, beam instabilities become a design limitation. Instabilities may come from the interaction between the beam and the surrounding environment through wakefields and impedance, or from self fields within the beam. One way to mitigate these instabilities is Landau damping, which arises from having a distribution of betatron tunes. A range of tunes can be generated with nonlinear forces. For instance, octupole magnets are used to create a spread of tunes dependent on the particle’s amplitude. However, octupoles and other nonlinear elements can have a significant drawback, i.e. the reduction of dynamic aperture. There are, however, nonlinear dynamical systems that are integrable and that can be implemented in accelerators [1–6]. For two-dimensional transverse beam dynamics, this requires the existence of two conserved quantities (integrals of motion).

The Integrable Optics Test Accelerator (IOTA) at Fermilab is dedicated in part to the experimental study of novel, integrable, nonlinear focusing lattices [7]. In particular, one straight section is designed to include an electron lens, which will be used for research on nonlinear dynamics, electron cooling, and space-charge compensation [7–9]. Because of their flexibility, electron lenses can be designed to have different effects on the circulating beam [10–14].

In this paper, we focus on the use of electron lenses as nonlinear focusing elements for the implementation of the McMillan integrable system [15] and on some implications for experimental design, such as the measurement of detuning with amplitude.

2 Electron Lenses and the McMillan Integrable System

Electron lenses are based on magnetically confined, low-energy electron beams overlapping with the beam in circular accelerators [13]. The effects on the dynamics of the circulating beam are achieved by setting the intensity and the current-density profile of the low-energy electrons, therefore shaping the electromagnetic field seen by the circulating beam as it traverses the electron lens. Different profiles have been used for different purposes, such as flat for bunch-by-bunch tune shifting and
long-range beam-beam mitigation [10], Gaussian for head-on beam-beam compensation [12], or hollow for beam halo control [11, 14, 16].

A flat current density distribution generates an electric field that is linear with transverse radius. In this case, the electron lens behaves like a linear, radial focusing or defocusing element, depending on the charge of the circulating beam. For an electron lens of length $L$, electron velocity $\beta V$ and current density $j_0$ on axis, the focusing strength $k$ (inverse of the focal length $f$) is

$$k = \frac{2\pi j_0 L (1 \pm \beta \beta_p)}{(B\rho) p \beta \beta_p c^2} \left( \frac{1}{4\pi \epsilon_0} \right) \frac{q_e q_p}{q_e q_p},$$

(2.1)

where $\beta_p$ is the longitudinal velocity of the circulating particles and $(B\rho)_p$ is their magnetic rigidity. The $\pm$-sign represents the two cases of co-propagating (minus) or counter-propagating (plus) beams — for counter-propagating beams, the magnetic force has the same direction as the electrical force. The last term contains the charges of the electron beam $q_e$ and of the circulating particles $q_p$. It indicates a focusing effect ($k < 0$, restoring force) or a defocusing effect ($k > 0$).

In ref. [15], McMillan discussed the stability of dynamical systems and described how to generate a class of symplectic integrable systems in one dimension. In particular, he showed that for the following symplectic mapping of variables $(q, p)$ (representing, for example, a coordinate and its conjugate momentum)

$$\begin{align*}
q_{n+1} &= p_n \\
p_{n+1} &= -q_n + \frac{2\epsilon p_n}{1 + p_n^2},
\end{align*}$$

(2.2)

the quantity $I_{1D} = q^2 p^2 + q^2 + p^2 - 2\epsilon q p$ is invariant. The parameter $\epsilon$ indicates the strength of the nonlinearity.

The McMillan mapping can be extended to the two transverse dimensions of particle motion in an accelerator, provided certain symmetry conditions are met [1–3]. These conditions are achievable for beam-beam forces in colliders, for instance, or with the use of electron lenses.

In the case of the electron lens, the 2D McMillan map is split into two parts: (i) a linear arc with equal amplitude functions $\beta_x = \beta_y = \beta$ at the electron lens and equal phase advances $\mu_x = \mu_y = 2\pi v$ equal to an odd multiple of $\pi/2$, corresponding to a tune $v = 1/4$; (ii) a radial nonlinear kick $kr/[1 + (r/a)^2]$ at the electron lens, where $k$ is the strength (eq. 2.1) and $a$ parameterizes the width of the nonlinear distribution. In Cartesian coordinates $(x, x' = P_x/P_z, y, y' = P_y/P_z)$, where $(x, y)$ is the transverse position and $P$ is the particle momentum (with $P_x \ll P_z$ and $P_y \ll P_z$), the linear arc is represented by the following transformation from initial $(i)$ coordinates to those after the arc $(a)$:

$$\begin{bmatrix}
x' \\
x \\
y' \\
y \end{bmatrix}_a = \begin{bmatrix}
0 & \beta & 0 & 0 \\
-\frac{1}{\beta} & 0 & 0 & 0 \\
0 & 0 & 0 & \beta \\
0 & 0 & -\frac{1}{\beta} & 0 \\
\end{bmatrix}_i \begin{bmatrix}
x \\
x' \\
y \\
y' \end{bmatrix}.$$ 

(2.3)

The nonlinear kicks in the electron lens follow the linear arc and have the following form, which
yields the final coordinates \((f)\) after one period:

\[
\begin{align*}
    x_f &= x_a \\
    x'_f &= x'_a + \frac{k x_a}{1 + \frac{x_a^2}{a^2}} \\
    y_f &= y_a \\
    y'_f &= y'_a + \frac{k y_a}{1 + \frac{y_a^2}{a^2}}.
\end{align*}
\] (2.4)

The required shape of the nonlinear kicks is generated by the following current-density distribution:

\[
j(r) = \frac{j_0}{\left[1 + \left(\frac{r}{a}\right)^2\right]^2}.
\] (2.5)

The design of an electron gun capable of accurately producing such a distribution is one of the critical components of the experimental program.

Combining the linear arc and the McMillan electron lens, the full one-turn map has the following form:

\[
\begin{align*}
    x_{n+1} &= \beta x'_n \\
    x'_{n+1} &= -\frac{x_n}{\beta} + \frac{\beta k x'_n}{1 + \beta^2 \frac{x_n^2 + y_n^2}{a^2}} \\
    y_{n+1} &= \beta y'_n \\
    y'_{n+1} &= -\frac{y_n}{\beta} + \frac{\beta k y'_n}{1 + \beta^2 \frac{x_n^2 + y_n^2}{a^2}}.
\end{align*}
\] (2.6)

This map is symplectic. It is also integrable, having two independent invariants: the longitudinal component of the angular momentum

\[
L_z = P_z \cdot (xy' - yx')
\] (2.7)

and the 2D McMillan invariant

\[
I = \frac{1}{a^2} \left[ \beta^2 \left( x^2 x'^2 + y^2 y'^2 \right) + 2 \beta^2 x x' y y' + a^2 \left( x^2 + y^2 \right) + a^2 \beta^2 \left( x'^2 + y'^2 \right) - k a^2 \beta^2 \left( xx' + yy' \right) \right].
\] (2.8)

Some features of the McMillan system can be deduced from its linearized approximation. Particles near the axis experience a linear force and the maximum detuning. For these particles, the McMillan transformation reduces to the linear map

\[
\begin{pmatrix}
    x \\
    x' \\
    y \\
    y'
\end{pmatrix}_f = \begin{pmatrix}
    0 & \beta & 0 & 0 \\
    -\frac{1}{\beta} & \beta k & 0 & 0 \\
    0 & 0 & \beta & 0 \\
    0 & 0 & -\frac{1}{\beta} & \beta k
\end{pmatrix}
\begin{pmatrix}
    x \\
    x' \\
    y \\
    y'
\end{pmatrix}_o.
\] (2.9)

For stability of the iterated linear map, one must have \(|\beta k| < 2\). The tune of particles on axis is

\[
y = \frac{1}{2\pi} \arccos \left( \frac{\beta k}{2} \right) \quad \text{(on axis)}.
\] (2.10)
Because the effect of the electron lens diminishes with radius, the tunes of particles at large amplitudes tends to the value of the bare arc, \( v = 1/4 \). Therefore, when \( |\beta k| < 2 \), the maximum detuning depends on the strength of the electron lens as follows:

\[
(\Delta \nu)_{\text{max}} = \frac{1}{4} - \frac{1}{2\pi} \arccos \left( \frac{\beta k}{2} \right).
\]

For the experimental demonstration of nonlinear integrable optics and for the study of the beam behavior near the integer resonance, the strength of the electron lens should span at least the full range \( 0 \leq |k| < 2/\beta \).

In IOTA, experiments on nonlinear integrable optics (NIO) with electron lenses will be done first with a circulating beam of 100–150 MeV electrons [7]. The beam is cooled by synchrotron radiation damping in 1–2 s, reaching horizontal and vertical geometrical rms emittances of about 40 nm with round optics. Electrons were chosen as a ‘pencil beam’ to approximate single-particle nonlinear dynamics, under experimental conditions such that the beam sizes are small compared to the typical sizes of the nonlinear elements, so that different areas of phase space can be explored.

This technique was used for experiments on NIO with magnets [17–19]. When protons at 2.5 MeV become available (at the end of 2021, according to current plans), the interplay between nonlinear integrable optics and space charge will be explored, demonstrating whether this technique is viable for reaching higher intensities and brightnesses in high-energy accelerators.

### 3 Analytical Expressions for the Detuning

Because of the symmetry of the McMillan system and because it is separable in polar coordinates, it is convenient to use radial and azimuthal coordinates and momenta

\[
\begin{align*}
    r &= \sqrt{x^2 + y^2}, \\
    p_r &= \frac{xx' + yy'}{\sqrt{x^2 + y^2}}, \\
    \theta &= \arctan2(y, x), \\
    p_\theta &= \frac{xy' - yx'}{\sqrt{x^2 + y^2}},
\end{align*}
\]

and the conserved component of the angular momentum (divided by the constant longitudinal momentum):

\[
L = L_z / P_z = rp_\theta.
\]

In polar coordinates, the McMillan invariant is

\[
I = \frac{\beta^2}{a^2} \left( \frac{r^2 p_r^2}{a^2} + \frac{r^2}{\beta^2} + p_\theta^2 + p^2 - k r p_r \right).
\]

The tune distribution generated by the nonlinear McMillan map was studied analytically in ref. [20]. The theoretical treatment predicts the radial and angular Poincaré rotation numbers (i.e., the betatron tunes of a particle) from the two constants of motion.\(^1\) The radial tune \( \nu_r \) can be calculated from the following expression:

\[
\nu_r (I, L) = \frac{F[\arcsin (\xi), \kappa]}{2K[\kappa]},
\]

---

\(^1\)A slightly different notation was used in ref. [20], based on the dimensionless variables \( r = \sqrt{x^2 + y^2}/a \), \( p_r = (\beta/a)(xx' + yy')/\sqrt{x^2 + y^2} \), and \( p_\theta = \beta(xy' - yx')/a^2 \).
Table 1. Summary of simulation parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Betatron function at kick, $\beta$</td>
<td>3.0 m</td>
</tr>
<tr>
<td>Arc phase advance, $\phi$</td>
<td>0.25 $\cdot$ 2$\pi$</td>
</tr>
<tr>
<td>Geometrical rms emittance, $\epsilon$</td>
<td>39.5 nm</td>
</tr>
<tr>
<td>Beam size, $\sigma$</td>
<td>0.34 mm</td>
</tr>
<tr>
<td>Electron lens size, $a$</td>
<td>2.0 mm</td>
</tr>
<tr>
<td>Electron lens strength (weak case), $k_w$</td>
<td>0.538 m$^{-1}$</td>
</tr>
<tr>
<td>Electron lens strength (strong case), $k_s$</td>
<td>1.07 m$^{-1}$</td>
</tr>
</tbody>
</table>

where $K[k]$ is the complete elliptic integral of the first kind and $F[\phi, \kappa]$ is the incomplete elliptic integral of the first kind. The variables $\xi$ and $\kappa$ are defined as

$$\xi = \sqrt{\frac{\lambda_3 - \lambda_1}{\lambda_3 + 1}} , \quad \kappa = \sqrt{\frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}}.$$  

(3.5)

where the quantities $\lambda_1 < 0 < \lambda_2 < \lambda_3$ are the roots of the polynomial

$$P_3(\lambda) = -\lambda^3 + \left[ I + \left( \frac{\beta k}{2} \right)^2 - 1 \right] \lambda^2 + \left( I - \frac{\beta^2 L^2}{a^4} \right) \lambda - \frac{\beta^2 L^2}{a^4}.$$  

(3.6)

The expression for the angular tune $\nu_\theta$ is

$$\nu_\theta(I, L) = \frac{1}{2\pi} \left[ \nu_\gamma \Delta - \Delta' + \arctan \left( \frac{2L \lambda_3 + 1}{ka^2 - \lambda_3} \right) \right],$$  

(3.7)

with:

$$\Delta = \frac{2\beta L}{a^2 \lambda_3 \sqrt{\lambda_3 - \lambda_1}} \Pi[k|x], \quad \Delta' = \frac{\beta L}{a^2 \lambda_3 \sqrt{\lambda_3 - \lambda_1}} \Pi[\arcsin(\xi), k|x], \quad \chi = \frac{\lambda_3 - \lambda_2}{\lambda_3}.$$  

(3.8)

Here, $\Pi[k|\omega]$ is the complete elliptic integral of the third kind, and $\Pi[\phi, k|\omega]$ is the incomplete elliptic integral of the third kind. These analytical expressions are extremely useful in the design of the experiments and in the interpretation of the results.

4 Numerical Tracking Simulations for IOTA Experiments

Numerical simulations were used to connect the experimental parameters with the theoretical detunings and to provide insights on experiment design.

Here, we used typical parameters achievable in IOTA [7]. The arc was represented by a linear map with $\pi/2$ phase advance, as in eq. 2.3. The amplitude function in both planes was $\beta = 3$ m.

The transverse geometrical rms emittances were equal to $\epsilon = 39.5$ nm, yielding an rms beam size $\sigma = 0.34$ mm. The size $a$ of the McMillan electron lens should be much larger than the beam size (to satisfy the ‘pencil beam’ conditions mentioned above) and also much smaller than both the aperture radius of the machine ($A = 25$ mm in IOTA) and the good field region of the magnets ($R \approx 12$ mm), so that the full range of detuning with amplitude can be explored. Here we show results for $a = 2$ mm. As regards the strength of the electron lens, we chose a ‘weak’ case with
Figure 1. Horizontal and vertical tune spectra for a sample particle in the strong McMillan lens ($k = k_x$).

$k_w = 0.538$ m$^{-1}$ ($\beta k_w = 1.61$) and a ‘strong’ case with $k_x = 1.07$ m$^{-1}$ ($\beta k_x = 3.21$). The nonlinear system is integrable and therefore stable beyond the linear stability limit ($|\beta k| < 2$). However, for a given initial emittance, the equilibrium beam size in general grows with the strength of the electron lens. The simulation parameters are summarized in table 1.

Initial particle coordinates were generated with uncorrelated Gaussian distributions in $x$, $x'$, $y$ and $y'$, for different values of the emittance $\epsilon$. The map of eq. 2.6 was then applied iteratively, using $k_w$ or $k_x$ as electron-lens strengths. A fast Fourier transform (FFT) was applied to the time series of particle coordinates to analyze their frequency spectra.

Figure 1 shows the tune spectrum of a sample particle. Horizontal and vertical spectra are identical because of the symmetry of the system. The starting coordinates were $(x, x', y, y') = (0.2$ mm, 0.1 mrad, $-0.4$ mm, 0.3 mrad) and 100,000 map iterations were applied. For this particle, the theoretical tunes from eqs. 3.4 and 3.7 are $\nu_r = 0.2515$ and $\nu_\theta = 0.1134$. As $\nu_\theta$ measures the oscillations of a particle’s angular position, this tune is the strongest frequency in the horizontal and vertical position spectra. Figure 1 also shows other linear combinations of $\nu_r$ and $\nu_\theta$. (As angular tunes can be positive or negative, we use absolute values in the plot for generality.) The line at $\nu_r - |\nu_\theta| = 0.1381$ is quite prominent, for instance.

The numerical tracking calculations show that the observed peaks in the horizontal and vertical frequency spectra can be identified with linear combinations of the theoretical radial and angular tunes. This feature should be taken into account in the interpretation of experimental results.

In a typical experiment on nonlinear integrable optics [17–19], after the focusing lattice is set up, the electron pencil beam is kicked in a single turn to various amplitudes to measure losses and tunes and to reconstruct the value of the invariants. Because of decoherence (mostly due to momentum spread and nonlinear detuning) and noise, the length of the available turn-by-turn centroid position data from the beam position monitors (BPMs) is limited (from a few hundred to a few thousand turns). This of course affects the accuracy of the measurements and widens the tune...
Figure 2. Frequency spectrum of centroid motion for a beam that was initially displaced in the horizontal plane, for the case of a strong McMillan lens \((k = k_s)\). The initial beam emittance was 39.5 nm and the horizontal displacement was 4 mm. The black vertical line represents the theoretical value of \(a\) for a single particle starting at the same initial position.

lines, with possible overlap of radial and angular modes. A simulated example is shown in figure 2. The tune of the displaced pencil beam and can be estimated from the frequency spectrum.

Figure 3 illustrates the effect of the finite spread in coordinates in a pencil beam on the spectra of centroid motion. For a beam with an initially realistic Gaussian emittance of 39.5 nm (box plot), the spectral lines are wide. However, a robust statistical analysis can yield good estimates of the detuning with amplitude. For a beam with a very small initial emittance (0.295 nm), the effect of the spread in coordinate values only shows up close to the axis, generating small deviations from the single-particle theory. In this case, the peak of the frequency spectrum is sufficiently accurate. Besides giving the magnitude of the detuning, this type of curves is useful to determine which values of the electron-lens size \(a\) should be used to satisfy the criteria outlined at the beginning of this section — larger than the pencil-beam size, but small enough to generate the full detuning within the available aperture.

Figure 4 shows some numerical values of the angular tune \(\nu_\theta\) as a function of the invariants \(I\) and \(L\). In the case of the weak lens (left) the maximum detuning is achieved when \(I = L = 0\), corresponding to the expression of eq. 2.10, with \(\arccos (\beta k_w/2)/(2\pi) = 0.101\). In the strong lens case (right), the fractional tune goes to 0 when \(L = 0\) and \(I \leq 0\). Unlike for the weak lens, this tune shift can be achieved with a range of coordinates. Using expression 3.3 of the McMillan invariant in polar coordinates, one can find the radius \(\tilde{r}\) below which this range of integer tunes is possible.

Setting \(p_\theta = 0\) and fixing \(r\), the minimum \(I\) is obtained for \(p_r = kr/(1 + r^2/a^2)/2\), yielding

\[
\tilde{r} = a \sqrt{\frac{\beta^2 k^2}{4} - 1}. \quad (4.1)
\]

For all values of \(r \leq \tilde{r}\), the angular tune can reach 0. In figure 5, we used an artificially large
Figure 3. Numerical simulations of the reconstructed detuning vs. transverse horizontal displacement in a strong McMillan lens ($k = k_x$). The box-and-whisker plots summarize the frequency spectra (such as the one in fig. 2) of an initially Gaussian beam with a realistic emittance of 39.5 nm; the segments are placed at the 10%, 25%, median (orange), 75% and 90% quantiles. The blue dots are the positions of the spectrum peaks for a beam with much smaller emittance (0.295 nm). The blue curve is the theoretical $v_{y}$ prediction (eq. 3.7) for a single particle with the same initial position $x$ and vanishing $x'$, $y$ and $y'$ ($L = 0$). The tick mark on the horizontal axis represents the value of the size parameter $a = 2$ mm.

Figure 4. The value of the angular tune as a function of the invariants $I$ and $L$ for a weak McMillan system (left, $\beta k_w = 1.61$) and for the strong case (right, $\beta k_x = 3.21$).
emittance of 2.95 μm for the initial Gaussian beam to illustrate the spread of single-particle tunes in a strong McMillan lens and to verify eq. 4.1. For \( r \leq \tilde{r} \), the spread in \( p_r \) and \( p_\theta \) generates tunes that cover the full range from 0 to \( 1/4 \).

5 Conclusions and Outlook

An electron lens for the IOTA storage ring at Fermilab is being designed. One of its main functions is the implementation of integrable optics to improve the stability of intense beams against space charge and impedances and, in general, to study the experimental feasibility of these dynamical systems. The nonlinear McMillan lens is part of this program. Based on the theoretical formulation of the nonlinear system, we described some of the implications for experimental design, such as the required strength and size of the electron lens, the achievable tune spreads, and the systematics in the measurements of detuning. Further design work is ongoing. It includes, for instance, spreading the kicks over the length of the actual lens, adding the effect of the solenoid, studying the sensitivity to the design parameters (such as the shape of the McMillan kick or the phase advance in the arc), calculating the magnitude of nonlinear beam size adjustments as a function of focusing strength, and predicting the response of the beam to magnet errors and other perturbations.

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