Measurement of undulator radiation power noise and comparison with \textit{ab initio} simulation

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Generally, turn-to-turn fluctuations of synchrotron radiation power in a storage ring depend on a 6D phase-space distribution of the electron bunch. This effect is related to the interference of fields, radiated by different electrons. Changes in the relative electron positions and velocities inside the bunch result in fluctuations in the total emitted energy per pass in a synchrotron radiation source. This effect has been previously described assuming constant and equal electron velocities before entering the synchrotron radiation source. In this paper, we present a generalized formula for the fluctuations with a non-negligible beam divergence. Further, we corroborate this formula in a dedicated experiment with undulator radiation in the Integrable Optics Test Accelerator (IOTA) storage ring at Fermilab. Lastly, possible applications in beam instrumentation are discussed.

I. INTRODUCTION

Full understanding of the radiation, generated by accelerating charged particles, is crucial for accelerator physics and electrodynamics in general. The predictions of classical electrodynamics for pulse-by-pulse average characteristics of synchrotron radiation, such as the total radiated power, spectral composition, angular intensity distribution and brightness [1], are supported by countless observations. In fact, they are confirmed every day by routine operations of synchrotron radiation user facilities. On the other hand, the pulse-to-pulse statistical fluctuations of synchrotron radiation have not been studied at the same level of detail yet, although substantial progress has been made in the past few decades. The turn-to-turn intensity fluctuations of incoherent spontaneous bending-magnet, wiggler and undulator radiation in storage rings have been studied in Refs. [2–7], both theoretically and experimentally. The statistical properties of the Free-Electron Laser (FEL) radiation have been studied in Refs. [8–14]. Moreover, Refs. [15, 16] claimed to observe a non-classical Sub-Poissonian photon statistics in the seventh coherent spontaneous harmonic of an FEL, although it could have been an instrumentation effect [17]. In any case, more experimental studies into statistical properties of synchrotron radiation are required to ascertain good understanding of it. In this paper, we will describe our observation of turn-to-turn power fluctuations of incoherent spontaneous undulator radiation in the Integrable Optics Test Accelerator (IOTA) storage ring at Fermilab [18]. Also, we will extend the existing theoretical description [5, 6] of such fluctuations. Namely, in Refs. [5, 6], only the effect of spatial distribution of the electrons inside the bunch on the turn-to-turn fluctuations is considered. However, in general, the distribution of electrons’ velocities affects the fluctuations as well. We present a generalized formula for the fluctuations in the case of non-negligible beam divergence in this paper. It agrees better with the measurements in IOTA.

Fluctuations and noise do not always degrade the results of an experiment. There are numerous examples when noise is used to measure the parameters of a specific system, or even fundamental constants. Some examples are the pioneering determination of the elementary charge $e$ by the shot noise [19], and the determination of the Boltzmann constant $k_B$ by the Johnson-Nyquist noise [20]. An example, relevant to the field of accelerator physics, is the use of Schottky noise pick-ups in storage rings [21–23] to measure transverse rms emittances, the momentum spread, the number of particles, etc. In fact, we will see that the fluctuations in synchrotron radiation are similar to the Schottky noise. Both effects are related to the existence of discrete point-like charges as opposed to a continuous charge fluid. Therefore, measurements of electron bunch parameters via synchrotron radiation fluctuations have been reported too. However, they were mostly focused on the longitudinal bunch length [3, 4, 6]. Only Ref. [7] reported an order-of-magnitude measurement of a transverse emittance. In this paper, we will present one example of a measurement of an unknown

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small vertical emittance of a flat beam in IOTA, given a known horizontal emittance, a longitudinal bunch shape and ring focusing functions, using our new formula for the fluctuations. We will see that taking into account the beam divergence is critical in this specific measurement in IOTA. For more results regarding this beam diagnostics via fluctuations in IOTA, we refer the reader to our separate publication [24].

II. THEORETICAL DESCRIPTION

Let us assume that we have a detector that can measure the number of detected synchrotron radiation photons $N$ at each revolution in a storage ring. Then, according to [1, 5], the variance of this number can be expressed as

$$
\text{var}(N) = \langle (N - \langle N \rangle)^2 \rangle = \langle N \rangle + \frac{1}{M}\langle N \rangle^2,
$$

(1)

where the linear term represents the photon shot noise, related to the quantum discrete nature of light. This effect would exist even if there was only one electron. Indeed, the electron would radiate photons with a Poisson statistics [25–27]. The quadratic term corresponds to the interference of fields radiated by different electrons. Changes in relative electron positions and velocities, inside the bunch, result in fluctuations of the radiation power, and, consequently, of the number of detected photons. In a storage ring, the effect arises because of the betatron motion, synchrotron motion, radiation induced diffusion, etc. The dependence of $\text{var}(N)$ on the electron bunch parameters is introduced through the parameter $M$, which will be called the number of coherent modes, following the notation of [1, 2, 5]. In [5], we derived an equation for $M$ for an electron bunch with a Gaussian transverse density distribution and an arbitrary longitudinal density distribution, assuming an rms bunch length much longer than the radiation wavelength and a negligible electron beam divergence. In this paper, we present an equation for $M$, extended to an arbitrary beam divergence,

$$
\frac{1}{M} = (1 - 1/n_e)\sqrt{\pi} \int \frac{d\mathbf{k}d\mathbf{\phi}_1d\mathbf{\phi}_2d\mathbf{x}'P_k(\mathbf{x}', \mathbf{\phi}_1 - \mathbf{\phi}_2)I_k(\mathbf{\phi}_1, \mathbf{x}')I_k^*(\mathbf{\phi}_2, \mathbf{x}')}{\langle N_{s.c.} \rangle^2},
$$

(2)

with

$$
P_k(\mathbf{x}', \mathbf{\phi}_1 - \mathbf{\phi}_2) = \frac{1}{4\pi\sigma_x\sigma_y} e^{\frac{\mathbf{x}'^2}{\sigma_x^2} + \frac{(\mathbf{\phi}'_2 - \mathbf{\phi}'_1)^2}{\sigma_y^2}} e^{-ik\Delta_x(\mathbf{\phi}_1, \mathbf{\phi}_2)\mathbf{x}' - ik\Delta_y(\mathbf{\phi}_1, \mathbf{\phi}_2)\mathbf{y}'} e^{-k^2\Sigma_2^2(\mathbf{\phi}_1, \mathbf{\phi}_2)^2 - k^2\Sigma_y^2(\mathbf{\phi}_1, \mathbf{\phi}_2)^2},
$$

(3)

where $s = 1, 2$ numerates the polarization component, $n_e$ is the number of electrons in the bunch, $k = 2\pi/\lambda$ is the magnitude of a wave vector; $\phi_1 = (\phi_{1x}, \phi_{1y})$, $\phi_2 = (\phi_{2x}, \phi_{2y})$ represent angles of direction of the radiation in the paraxial approximation, hereinafter, $x$ and $y$ refer to the horizontal and the vertical axes, respectively,

$$
\sigma_{z}^{\text{eff}} = 1/(2\sqrt{\pi} \int \rho^2(z)dz),
$$

(6)

where $\rho(z)$ is the electron bunch longitudinal density distribution function, $\int \rho(z)dz = 1$, and $\sigma_{z}^{\text{eff}}$ is equal to the rms bunch length $\sigma_z$ for a Gaussian bunch; $\mathbf{x}' = (x', y')$ represents the direction of motion of an electron at the radiator center, relative to a reference electron; $\sigma_{x'}$ and $\sigma_{y'}$ are the rms beam divergences, $\sigma_{x'}^2 = \gamma_x\epsilon_x + D_{x'}^2\sigma_p^2$,

$$
\sigma_{y'}^2 = \gamma_y\epsilon_y; \Sigma_2^2 = \epsilon_x/\gamma_x + (\gamma_xD_x + D_x'\alpha_x)^2\beta_x\epsilon_x\sigma_p^2/\sigma_x^2, \Sigma_4^2 = \epsilon_y/\gamma_y, \Delta_x = (\alpha_x\epsilon_x - D_xD_{x'}\sigma_p^2/\sigma_x^2, \Delta_y = \alpha_y/\gamma_y, \alpha_x, \beta_x, \gamma_x, \alpha_y, \beta_y, \gamma_y$ are the Twiss parameters of an uncoupled focusing optics in the synchrotron radiation source; $D_x, D_{x'}$ are the horizontal dispersion and its derivative, the vertical dispersion is assumed to be zero; $\epsilon_x, \epsilon_y$ are the unnormalized rms emittances; $\sigma_p$ is the relative rms momentum spread. The following two useful relations exist, $\sigma_z^2 = \Sigma_2^2 + \sigma_z^2\Delta_x^2$, $\sigma_y^2 = \Sigma_y^2 + \sigma_y^2\Delta_y^2$. The complex radiation field amplitude $E_{k,\nu}(\mathbf{x})$, generated by a reference electron, is given by the following expression, see [5, 26], [1, p. 38],

$$
E_{k,\nu}(\mathbf{x}) = \sqrt{\frac{\alpha k}{2(2\pi)^3}} \int dt\nu(t)e^{ikt - ik\cdot v(t)},
$$

(7)

where $\mathbf{v} = (\phi_x, \phi_y)$, $k = k(\phi_x, \phi_y, 1 - \phi_z^2/2 - \phi_y^2/2)$, $\alpha = 1/137$, $\nu(t)$ is the considered polarization vector ($s = 1, 2$), $r(t)$ is the trajectory of the reference electron in the synchrotron radiation source, $\nu(t)$ is the velocity of the reference electron as a function of time.

The parameter $\langle N_{s.c.} \rangle$ in Eq. (2) is the average number of detected photons per turn for a single electron circulating in the ring. We consider the case of an incoherent
radiation ($\sigma_z k \gg 1$), therefore, the average number of detected photons for the entire bunch can be obtained as

$$\langle N \rangle = n_e \langle N_{\text{acc}} \rangle.$$  \hfill (8)

The integrals in Eqs. (2) and (5) are taken from minus to plus infinity over all integration variables except for $k$, which goes from zero to plus infinity. The spectral sensitivity and the aperture of the detector are assumed to be included in the detection efficiency $\eta_{k,s}(\phi)$, which is a function of $k$ and $\phi$ for that reason. The derivation of Eq. (2) is largely analogous to [5] and is outlined in Appendix A. Appendix B provides an illustrative closed-form expression for $M$, based on Eq. (2) in the approximation of a Gaussian spectral-angular distribution of the radiation.

In IOTA, we study the undulator radiation, because the quadratic term in Eq. (1), sensitive to bunch parameters, is greater for undulators and wigglers than for dipole magnets. The complex field amplitude $E_{k,s}(\phi)$, generated by a single electron, can be numerically calculated by our computer code [28], based on the equations from [29], or by using the SRW package [30]. Then, the integrals in Eqs. (2) and (5) can be calculated by a Monte-Carlo algorithm. Our C++ code with Python bindings for calculation of Eqs. (2) and (5) is provided in the repository [31].

### III. APPARATUS

In our experiment, a single electron bunch circulated in the IOTA ring, see Fig. 1(a), with a revolution period of 133.3 ns and the beam energy of 96.4 ± 1 MeV. We studied two transverse focusing configurations in IOTA: (1) strongly coupled, resulting in approximately equal transverse mode emittances and (2) uncoupled, resulting in two drastically different emittances. Henceforth, we will refer to the beams in these configurations as round and flat beams, respectively. In both cases, the bunch length and the emittances depend on the beam current due to intra-beam scattering [32, 33], beam interaction with its environment [34], etc. The longitudinal bunch density distribution $\rho(z)$ was measured and recorded by a high-bandwidth wall-current monitor [35]. It was not exactly Gaussian, but this fact was properly accounted for by Eq. (6) for $\sigma_z^{\text{eff}}$, which works for any longitudinal bunch shape. The IOTA rf cavity operated at 30 MHz (4th harmonic of the revolution frequency) with a voltage amplitude of about 360 V. The rms momentum spread $\sigma_p$ was calculated from the known rf voltage amplitude, the design ring parameters and the measured rms bunch length $\sigma_z$. In our experiments, the relation was

$$\sigma_p \approx 9.1 \times 10^{-5} \times \sigma_z \text{[cm].}$$ \hfill (9)

It is an approximate equation, because of the bunch-induced rf voltage (beam loading) and a small deviation of $\rho(z)$ from the Gaussian shape. However, the effect of $\sigma_p$ in Eq. (2) in IOTA was almost negligible. Therefore, such estimation was acceptable.

For the round beam, the IOTA transverse focusing functions (4D Twiss functions) were chosen to produce approximately equal mode emittances at zero beam current, $\epsilon_x \approx \epsilon_y \approx 12 \text{ nm (rms, unnormalized). It was empirically confirmed that they remained equal at all beam currents with a few percent precision. The expected zero-current emittances for a flat beam were $\epsilon_x \approx 50 \text{ nm, } \epsilon_y \approx 0 \text{ (unknown). The expected zero-current rms bunch length and the rms momentum spread for both round and flat beams were } \sigma_z = 9 \text{ cm, } \sigma_p = 8.3 \times 10^{-5}. \text{ In our experiment, the electron beam sizes were monitored and recorded by visible-light synchrotron radiation image monitors (SLMs) in seven dipole bend locations, at M1L-M4L and at M1R-M3R, see Fig. 1(a). The smallest reliably resolvable emittance by the SLMs is about } 20 \text{ nm. Figure 2 illustrates the bunch parameters of the round beam as a function of current. Below, we will present measurements with a flat beam at only one value of beam current, } 2.66 \text{ mA, measured with a precision DCCT current monitor. The parameters of the flat beam at this current value are } \sigma_z = 31.9 \text{ cm, } \sigma_z^{\text{eff}} = 29.5 \text{ cm, } \sigma_p = 3.0 \times 10^{-4}. \text{ The horizontal emittance is } \epsilon_x = 0.66 \mu\text{m, as measured by the SLMs with a monitor-to-monitor variation of } \pm 50 \text{ nm. The small vertical emittance of the flat beam was unresolvable by the SLMs. However, in Sec. V, we will demonstrate how it can be reconstructed using the fluctuations measurements for the flat beam.}

At the center of the undulator, in the uncoupled optics, the Twiss parameters are $\beta_x = 204 \text{ cm, } \beta_y = 98 \text{ cm, } \alpha_x = 1.25, \alpha_y = -0.87$, the horizontal dispersion $D_x = 101 \text{ cm, its derivative } D_x' = -4.22$. The strongly coupled optics was created by adding a skew quadrupole to the uncoupled optics. The coupling parameter $u$ [36] was about 0.5.
everywhere in the ring. Therefore, the following is correct for the coupled case 4D Twiss functions, $\beta_{1x} \approx \beta_{2x}$, $\beta_{1y} \approx \beta_{2y}$. Moreover, their sums, $\beta_{1x} + \beta_{2x}$, $\beta_{1y} + \beta_{2y}$, are approximately equal to the Twiss beta functions in the uncoupled case, $\beta_x$, $\beta_y$. Equation (2) assumes an uncoupled optics. However, this specific strongly coupled optics used in IOTA can be approximated as the uncoupled optics, but with equal horizontal and vertical emittances $\epsilon_x = \epsilon_y = \epsilon$. More specifically, what is used in the derivation of Eq. (2) (see Appendix A) is the 6D phase-space distribution of the electrons, Eq. (A6). This distribution, for the round beam, when calculated using the approximation of uncoupled optics with equal emittances $\epsilon_x = \epsilon_y = \epsilon$, and the distribution, calculated using the exact 4D Twiss functions and equal mode emittances, $\epsilon_1 = \epsilon_2 = \epsilon$, are almost indistinguishable. This was intended by the initial design of the coupled optics in IOTA.

The undulator strength parameter (peak) $K_u = 1.0$ with the number of periods $N_u = 10.5$ and the period length $\lambda_u = 5.5\text{cm}$. A photodetector was installed in a dark box above the M4R dipole magnet, see Fig. 1(b). The light produced in the undulator was directed to the photodetector (3.5 m away) by a system of two mirrors ($\varphi 2\text{in}$). Then, it was focused by a lens ($\varphi 2\text{in}$, focal distance $F = 150\text{mm}$) into a spot, smaller than the sensitive area of the detector ($\varphi 1.0\text{mm}$). Because of the two round mirrors, which are at 45° to the direction of propagation of the radiation, the angular aperture takes an elliptical shape with the vertical axis smaller than the horizontal by a factor of $\sqrt{2}$. Namely, the horizontal and the vertical semi-axes are 7.3 mrad and 5.1 mrad, respectively. We studied the fundamental of the undulator radiation, $\lambda_1 = 1.16\text{nm}$. The characteristic angular radiation size is $\gamma^{-1}\sqrt{1 + K_u^2}/N_u = 2.0\text{mrad}$, see [1, p. 50]. Therefore, most of the radiation was accepted by the aperture. The spectrum of the fundamental is rather broad (about 0.2 pm FWHM) due to the small number of periods ($N_u = 10.5$) in our undulator. As a photodetector we used an InGaAs PIN photodiode [37], which has high quantum efficiency (about 80%) around the fundamental.

Using the elliptical angular aperture mentioned above and the manufacturers’ specifications for the spectral transmission of the vacuum chamber window at the M4R dipole magnet, the two mirrors, the focusing lens, and the quantum efficiency of the InGaAs photodiode, we constructed the detection efficiency function $\eta_{k,s}(\phi)$ for our system. The lens’ spectral transmission had to be linearly extrapolated for a small range outside of the range, provided in the manufacturer’s specifications. There were no free adjustable parameters. We pre-calculated the field amplitude $E_{k,s}(\phi)$, generated by a single electron, for the parameters of our undulator on a 3D grid ($k$, $\phi_x$, $\phi_y$) by our code [28]. Figure 3(a) shows the simulated spectrum, where the intensity is integrated over the elliptical aperture,

$$
\frac{d\langle N_{s,e} \rangle}{dk} = \sum_{s=1,2} \int d\phi |\eta_{k,s}(\phi) E_{k,s}(\phi)|^2.
$$

The blue line is calculated for an ideal detection system, where $\eta_{k,s}(\phi) = 1$ inside the elliptical aperture, and zero outside. The red line is calculated with $\eta_{k,s}(\phi)$, constructed using the manufacturers’ specifications of the optical elements in our system. Figure 3(b) shows the
Angular distribution with \( \eta_{k,s}(\phi) \) of our system,

\[ \frac{d\langle N_{s,e} \rangle}{d\phi_x d\phi_y} = \sum_{s=1.2} \int dk \eta_{k,s}(\phi) \left| \mathcal{E}_{k,s}(\phi) \right|^2. \]  

(11)

Figure 4 illustrates our full photodetector circuit. First, the radiation pulse is converted into a photocurrent pulse by the photodiode, see Fig. 4(a). Then, the photocurrent pulse is integrated by an op-amp-based RC integrator, which outputs a longer pulse of a comfortably measured voltage amplitude. The op-amp was capable of driving a 50-Ω input load of a fast digitizing scope, located \( \approx 100 \) m away. The resistor \( R_0 = 580 \) kΩ in the circuit in Fig. 4(a) was used to remove the offset in the integrator output signal (about 0.3 V), produced by the op-amp input bias current and the photodiode leakage current. The output voltage pulse of the integrator at \( i \)th IOTA turn can be represented as \( A_i f(t) \), where \( A_i \) is the signal amplitude at the \( i \)th turn and \( f(t) \) is the average signal for one turn, normalized so that its maximum value is 1, see Fig. 4(a). The time \( t \) in \( f(t) \) is in the range 0–133.3 ns, i.e., within one IOTA revolution. The number of photoelectrons, generated by the light pulse at \( i \)th turn, \( N_i \), can be calculated as the time integral of the output pulse of the integrator divided by the electron charge \( e \) and the resistance \( R_\text{f} \), i.e.,

\[ N_i = \int A_i f(t) dt / (e R_\text{f}), \]  

(12)

where

\[ \chi = \int f(t) dt / (e R_\text{f}) = 2.08 \times 10^7 \text{ photoelectrons/V}, \]  

with a \( \pm 5 \% \) uncertainty, because of the uncertainty in \( R_\text{f} \). We verified Eq. (13) empirically at different voltage amplitudes \( A_i \) and different bunch lengths, which define the lengths of the input light pulses. During our experiments at different beam currents, \( A_i = 0–1.2 \) V.

Since we also knew the empirical linear relation between the beam current and the integrator voltage amplitude, we could use it in Eq. (13) to find the average number of detected photons (photoelectrons) per one electron of the electron bunch. The result of this calculation was \( 8.8 \times 10^{-3} \) photoelectrons/electron. This value is rather close to the result obtained in our simulation,

\[ \sum_{s=1.2} \int dk \eta_{k,s}(\phi) \left| \mathcal{E}_{k,s}(\phi) \right|^2 \]  

(15)

In our experiment, the expected relative fluctuation of \( A_i \) was \( 10^{-4}–10^{-3} \) (rms), which is considerably lower than the digitization resolution of our 8-bit broad-band oscilloscope. To overcome this problem, we used a comb (notch) filter [39], shown in Fig. 4(b). The signal splitter splits the integrator output into two identical signals. Further, the lengths and the characteristics of the cables in the two arms were chosen such that one of the signals is delayed by exactly one IOTA revolution and, at the same time, the losses and dispersion in both arms are approximately equal. The time delay could be adjusted with a 0.1 ns precision. Therefore, the time delay error was negligible, because the input pulses of the comb filter were about 30 ns long (FWHM). Finally, a passive hybrid [40] produces the difference and the sum of the signals in the two arms — its output channels \( \Delta \) and \( \Sigma \),
respectively. For an ideal comb filter,
\[
\Delta_i(t) = \xi(A_i - A_{i-1})f(t), \quad (16)
\]
\[
\Sigma_i(t) = \xi(A_i + A_{i-1})f(t), \quad (17)
\]
where we assume that the pulse shape of input and output signals of the comb filter is the same — \( f(t) \). This means that we assume a negligible dispersion in the comb filter, which is a very good approximation according to our comparison of input and output pulses with the oscilloscope. Also, as a result of this comparison, we determined the parameter \( \xi = 0.31 \). Of course, our comb filter is not perfect. There is some cross-talk between \( \Delta \)- and \( \Sigma \)-channels, some noise in the signals, a small undesirable reflection in one of the arms, resulting in a small satellite pulse about 85 ns away from the main pulse, see Fig. 4(c). In addition, the hybrid is AC-coupled.

With these effects taken into account Eqs. (16) and (17) take the form
\[
\Delta_i(t) = \xi(A_i - A_{i-1})f(t) + \mu_\Delta \Sigma_i(t) + \delta_r A_i f(t - t_r) + \nu_{\Delta i}(t) - \Delta_{\text{AC}}, \quad (18)
\]
\[
\Sigma_i(t) = \xi(A_i + A_{i-1})f(t) + \mu_\Sigma \Delta_i(t) + \delta_r A_i f(t - t_r) + \nu_{\Sigma i}(t) - \Sigma_{\text{AC}}, \quad (19)
\]
where \( t \) is within one IOTA turn (0–133.3 ns), \( \mu_\Delta \) and \( \mu_\Sigma \) describe the cross-talk between \( \Delta \)- and \( \Sigma \)-channels (< 1%), \( \delta_r A_i f(t - t_r) \) describes the reflected pulse in one of the arms (perhaps the short one), \( t_r = 85 \) ns, \( \delta_r \approx 1.5 \times 10^{-3} \); and it is assumed that the noise contributions \( \nu_{\Delta i}(t) \) and \( \nu_{\Sigma i}(t) \) enter the equations as sum terms, independent of the signal amplitude; the constants \( \Delta_{\text{AC}} \) and \( \Sigma_{\text{AC}} \) come from the fact that the hybrid is AC-coupled and the averages of \( \Delta_i(t) \) and \( \Sigma_i(t) \) over a long time have to be zero.

For each measurement, we recorded 1.5 ms-long waveforms (about \( n_{\text{rev}} = 11 \) 250 IOTA revolutions) of \( \Delta \)- and \( \Sigma \)-channels with the oscilloscope at 20 GSa/s. The beam current decay is negligible during this 1.5 ms.

In Eq. (19), the noise, the cross-talk term, and the reflection term are negligible. \( \Sigma \)-channel can be used to measure the photoelectron count mean \( \langle N \rangle \) during the 1.5 ms. Using Eq. (13) and the non-negligible part of Eq. (19),
\[
\langle N \rangle = \frac{\chi (\Sigma(t_{\text{peak}})) + \Sigma_{\text{rev}}}{2\xi}, \quad (20)
\]
where we introduced \( t_{\text{peak}} \) — the time within each turn, corresponding to the peak of the signal, \( f(t_{\text{peak}}) = 1 \),
\[
\langle \Sigma(t_{\text{peak}}) \rangle = \frac{1}{n_{\text{rev}}} \sum_{i=1}^{n_{\text{rev}}} \Sigma_i(t_{\text{peak}}). \quad (21)
\]

The idea of using the comb filter, is that in an ideal comb filter, see Eq. (16), the \( \Delta \)-channel would provide the exact difference between two consecutive turns in IOTA. In this case we would be able to look directly at the turn-to-turn fluctuations. The offset would be removed, and the oscilloscope could be used with the appropriate scale setting, with negligible digitization noise. In our non-ideal comb filter, see Eq. (18), the additional terms have impact on the \( \Delta \)-signal, see Fig. 4(c). Nonetheless, with a proper analysis of the \( \Delta \)-signal it is possible to determine \( \text{var}(\mathcal{N}) \) from it with a good precision.

Namely, if we take the variance of Eq. (18) with respect to \( i \) at a fixed time \( t \), then we obtain
\[
\text{var}(\Delta(t)) = 2\xi^2 \text{var}(A)f^2(t) + \text{var}(\nu_\Delta(t)), \quad (22)
\]
where the contribution from \( \nu_{\Delta i}(t) \) and \( \delta_r A_i f(t) \) could be dropped, because the fluctuations of \( \Sigma_i(t) \) and \( A_i \) are strongly attenuated by the factors \( \mu_\Delta \) and \( \delta_r \), respectively. Also, \( \text{var}(\Delta_{\text{AC}}) = 0 \) since \( \Delta_{\text{AC}} \) is constant during 1.5 ms. The left-hand side of Eq. (22), as a function of \( t \in [0, 133.3 \text{ ns}] \), could be obtained from the collected waveforms of \( \Delta \)-channel as
\[
\text{var}(\Delta(t)) = \frac{1}{n_{\text{rev}}} \sum_{i=1}^{n_{\text{rev}}} \Delta_i^2(t) - \left[ \frac{1}{n_{\text{rev}}} \sum_{i=1}^{n_{\text{rev}}} \Delta_i(t) \right]^2. \quad (23)
\]

The results of such calculation for 2000 moments of time \( t \) within an IOTA revolution are shown in Fig. 5. These data are for the round beam. The blue, orange, and green lines correspond to three significantly different values of beam current within the range studied in our experiment, the brown line corresponds to zero beam current.

\[\text{FIG. 5. Variance of } \Delta \text{-signal as a function of time (see Eq. (22)) within one IOTA revolution. Round beam data.}\]

Figure 5 suggests that there is a constant noise level, independent of time and independent of the signal amplitude. Specifically, it suggests that the noise term in Eq. (22) is
\[
\text{var}(\nu_\Delta(t)) = \text{var}(\nu_\Delta) = 8.8 \times 10^{-8} \text{ V}^2. \quad (24)
\]

The observed rms noise amplitude (≈ 0.3 mV) was analyzed using the noise model for the detector electrical
schematic, Figs. 4(a) and (b), as well as the typical electrical characteristics of the photodiode [37] and the operational amplifier [38]. The three main contributions to the rms noise in the $\Delta$-channel are the following: the oscilloscope input amplifier noise, 0.21 mV; the operational amplifier input voltage noise, 0.18 mV; and the operational amplifier input current noise, 0.037 mV. When added in quadrature, these three sources explain the measured noise.

The peaks rising above the noise level in Fig. 5 can be fitted well with $f^2(t)$ (fits not shown). Thus, their shape is in agreement with Eq. (22) as well.

Therefore, using Eqs. (13) and (22), the photoelectron count variance $\text{var}(N)$ can be determined as

$$\text{var}(N) = \chi^2 \text{var}(A) = \chi^2 \frac{\text{var}(\Delta(t_{\text{peak}})) - \text{var}(\nu_\Delta)}{2\sigma^2}, \quad (25)$$

see Eq. (23) for definition of $\text{var}(\Delta(t_{\text{peak}}))$. The noise level in terms of $\text{var}(N)$ is

$$\frac{\chi^2 \text{var}(\nu_\Delta)}{2\sigma^2} = 2.0 \times 10^8. \quad (26)$$

We verified this method of measurement of $\langle N \rangle$ and $\text{var}(N)$ with an independent test light source with known fluctuations in Appendix C, where we also estimated the statistical error of the measurement of $\langle N \rangle$ by our apparatus, namely, $\pm 2.7 \times 10^8$ — it is approximately constant in the range of $\text{var}(N)$, observed with the undulator radiation in IOTA.

**IV. MEASUREMENT RESULTS**

The measured fluctuation data points for the round beam at different values of beam current are shown in Fig. 6(a) (blue points). The blue dashed straight line, $\text{var}(N) = \langle N \rangle$, represents the photon shot noise contribution to the fluctuations — the first sum term in Eq. (1). The values of $M$ extracted from the fluctuation data points using the equation,

$$M = \langle N \rangle^2 / (\text{var}(N) - \langle N \rangle), \quad (27)$$

are shown in Fig. 6(c) (blue points). The error bars of the blue data points in Figs. 6(a),(c) represent the statistical error of our technique, see Appendix C for details. Furthermore, Fig. 6(c) has a curve for $M$, simulated by Eq. (2) (red line), and, for comparison, a curve for $M$, simulated by [5, Eq. (49)] (dashed black line), which neglects beam divergence. Corresponding curves for simulated $\text{var}(N)$ are shown in Fig. 6(a). The shaded light red areas in Figs. 6(a),(c) show the uncertainty range of our simulation by Eq. (2).

For this simulation, we needed the values of the following four bunch parameters, entering Eq. (2), $\epsilon_x, \epsilon_y, \sigma_p, \sigma_z$, at all beam currents. Further, we needed the values of Twiss functions in the undulator. Also, the parameters of the undulator and the detection system. All these aspects were described in Sec. III. We had no free parameters in this simulation. Numerical calculation of the integrals in Eq. (2) and [5, Eq. (49)] was performed by the Monte-Carlo algorithm on the Midway2 cluster at the University of Chicago Research Computing Center using our computer code [28, 31].

The simulation uncertainty range (shaded light red area) primarily comes from the uncertainty in the beam energy $96.4 \pm 1$ MeV. The next by magnitude source of uncertainty is the SLMS’ $\pm 8$ nm monitor-to-monitor error of $\epsilon$ (for the round beam we use $\epsilon_x = \epsilon$ and $\epsilon_y = \epsilon$ in Eq. (2)). It is approximately two times smaller. The uncertainties of other parameters ($\sigma_p, \sigma_z$, Twiss functions in the undulator, etc.) had negligible effect. The manufacturers’ specifications for the optical elements of our system did not provide any uncertainties. Therefore, they were not considered.

We also collected fluctuations data for another experiment configuration. Figure 6(b) shows fluctuation data points for a flat beam (uncoupled focusing) at a fixed beam current 2.66 mA. The corresponding reconstructed values of $M$ are shown in Fig. 6(d). In this measurement, the photoelectron count mean (horizontal axis) was varied by using different optical neutral density filters (one point without a filter and four points with filters). Neutral density filters are filters that have constant attenuation in a certain wavelength range, in our case, around the fundamental harmonic of the undulator radiation. A new bunch was injected into the ring for each measurement. The oscilloscope waveforms for $\Delta$- and $\Sigma$-channels were recorded when the beam current decayed to 2.66 mA. The error bars in Figs. 6(b),(d) correspond to the statistical error of our technique, estimated in Appendix C. The red curve in Fig. 6(b) is a fit with a constant $M$. A corresponding horizontal line is shown in Fig. 6(d). The value of $M$ in this fit is $M_{\text{fit}} = 4.38 \times 10^9 \pm 1.0 \times 10^8$. This value was calculated as the average of the five values of $M$ in Fig. 6(d), and the error was calculated as the standard deviation of these five values.

We do not present any simulation results for the fluctuations $\text{var}(N)$ at this measurement configuration, because the SLMS provided very inconsistent estimates for the very small vertical emittance $\epsilon_y$ of the flat beam — the max-to-min variation reached a factor of eight. We believe this happened because the beam images were close to the resolution limit (one pixel $\approx 50 \mu$m in terms of beam size), therefore, the monitor-to-monitor variation primarily came from the Twiss beta-function variation ($\beta^{'\text{max}}_y / \beta^{'\text{min}}_y \approx 12$). Additional effects limiting the resolution include the diffraction limit, the point spread function of the cameras, calibration errors of the SLMS. Without $\epsilon_y$ we could not use Eq. (2) to make a prediction for $M$ and $\text{var}(N)$.

However, we have attempted the reverse of this procedure, namely, the reconstruction of the unknown $\epsilon_y$
FIG. 6. (a) The fluctuations measurement for the round electron beam in IOTA as a function of beam current, a prediction by Eq. (2) (red solid line) and a prediction by [5, Eq. (49)] (black dashed line), which does not account for the beam divergence. (b) The fluctuations measurement for the flat electron beam at a fixed beam current 2.66 mA with 4 different optical neutral density filters and one point without any filters, as well as a parabolic fit (a prediction could not be made, because the vertical emittance of the flat beam was unknown). (c) and (d) present the data of (a) and (b) in terms of the number of coherent modes $M$.

via the measured fluctuations for the flat beam shown in Figs. 6(b),(d). Indeed, the measured value of the number of coherent modes $M_{\text{meas}}$ is a function of four bunch parameters,

$$M_{\text{meas}} = M(\epsilon_x, \epsilon_y, \sigma_p, \sigma_{\text{eff}}),$$

the full form of the right-hand side is given by Eq. (2). The horizontal emittance $\epsilon_x$ of the flat beam at beam current 2.66 mA could still be reliably measured via the SLMs, 0.66 \(\mu\)m. The effective bunch length $\sigma_{\text{eff}}$ could still be determined from $\rho(z)$ and the ring parameters, $\sigma_p = 3.0 \times 10^{-4}$. The only unknown in Eq. (28) is $\epsilon_y$. Equation (28) can be solved for $\epsilon_y$ by a simple bisection method. The result is $\epsilon_y = 8.4 \pm 1.5$ nm, where the uncertainty corresponds to the statistical error of $M_{\text{fit}}$, mentioned above. For comparison, we also used [5, Eq. (49)] in Eq. (28), which neglects beam divergence. In this case, we obtained $\epsilon_y = 18.3 \pm 1.3$ nm.

It should be noted, that in this reconstruction of $\epsilon_y$, there is also a systematic error due to the uncertainty of the beam energy ($\pm 1$ MeV) and the systematic error of $\epsilon_x$ measurement by the SLMs ($\pm 50$ nm monitor-to-monitor). We estimated these two contributions to the systematic error of $\epsilon_y$ for the reconstruction by Eq. (2). They are $\pm 2.5$ nm and $\pm 1.6$ nm, respectively. These systematic errors are rather significant. However, they are not directly linked to our measurement technique. They are related to the fact that the beam energy and the horizontal emittance of a flat beam in IOTA are not known with better precision at the moment. Further improvements in beam characterization in IOTA will reduce the systematic error of our fluctuations-based technique of $\epsilon_y$ measurement.

V. DISCUSSION AND CONCLUSIONS

In Figs. 6(a),(c), our simulation by Eq. (2) and the measurement agree within the simulation and the measurement uncertainties. In IOTA, the bunch parameters $\epsilon_x$, $\epsilon_y$, $\sigma_p$, $\sigma_{\text{eff}}$ depend on the beam current because of various intensity dependent effects, e.g., intra-beam scattering [41], beam interaction with its environment [34], etc. Therefore, $M$ is a function of the beam current as well, it can be seen in Fig. 6(c).

In Figs. 6(b),(d), all data points correspond to one value of the beam current, 2.66 mA. The photoelectron count mean is varied by using neutral density filters with different attenuation factors $\eta_{\text{ND}}$. Such filters linearly scale down the photoelectron count mean, $\langle N \rangle \rightarrow \eta_{\text{ND}} \langle N \rangle$. However, they do not change $M$, because if $\eta_{k,s}(\phi)$ is replaced by $\eta_{\text{ND}} \eta_{k,s}(\phi)$ in Eq. (2), then $\eta_{\text{ND}}$ cancels out in the numerator and the denominator. This is consistent with Fig. 6(d). Indeed, all measured values of $M$ are equal within the uncertainty range. We reconstructed the value of the vertical emittance $\epsilon_y$ of the flat beam from $M_{\text{fit}}$ via Eq. (2) and via [5, Eq. (49)], which neglects beam divergence. We obtained very different results, $\epsilon_y = 8.4 \pm 1.5$ nm and $\epsilon_y = 18.3 \pm 1.3$ nm, re-
spectively. This shows that, for the flat beam, accounting for the beam divergence is critical. Indeed, in this measurement, the horizontal beam divergence was 0.94 mrad, which is rather close to the characteristic radiation angle $\gamma^{-1}\sqrt{(1 + K^2/2)/N_a}$ = 2.0 mrad, which gives an estimate of the angular size of $\mathcal{E}_{h\perp}(\Phi)$. Clearly, in this case it will have a significant effect on the integral in the numerator of Eq. (2). This is why we use Eq. (2) in our Letter [24], focused on emittance measurements via fluctuations, as opposed to [5, Eq. (49)], which is simpler, but neglects beam divergence.

In addition, we made an independent estimate for the vertical emittance $\epsilon_y$ of the flat beam based on the beam lifetime, see Appendix D for details. Namely, at the beam current 2.66 mA, the beam lifetime is solely determined by Touschek scattering [36, 42]. The Touschek lifetime is a function of beam emittances and bunch length. Therefore, since we knew the horizontal emittance $\epsilon_x$ of the flat beam, measured by the SLMs, the bunch length, measured by the wall-current monitor, and we knew the measured beam lifetime $[I/(dI/dt)]$, we could find $\epsilon_y$. The result was $\epsilon_y = 9.6$ nm, which is rather close to the fluctuations-based measurement, $\epsilon_y = 8.4 \pm 1.5$ nm.

In the round beam, at the same beam current 2.66 mA, the beam divergence in the undulator is about 0.43 mrad (both $x$ and $y$). It is smaller than the horizontal beam divergence of the flat beam, and, hence, is further away from the characteristic radiation angle 2.0 mrad. Therefore, the effect of beam divergence on the fluctuations simulation in Figs. 6(a),(c) is not as dramatic. However, the deviation of the simulation by [5, Eq. (49)], neglecting beam divergence, from the measurement is certainly noticeable. Whereas the simulation by Eq. (2) agrees well with the measurement.

In the future, it would be beneficial to repeat these fluctuation measurements with a longer and brighter undulator. Currently, the undulator in IOTA has only 10.5 periods and, hence, the radiation power is relatively low and the spectrum of the fundamental harmonic is relatively broad (about 0.2 pm FWHM). To achieve a signal of a comfortably measured amplitude, we had to avoid using a monochromator or restricting the angular aperture. We had to collect all available radiation. Therefore, the integrals in Eqs. (2) and (5) had to be calculated over a broad range of angles and wavelengths. If we could use a monochromator, a slit or a pinhole detector, these integrals could be significantly simplified. Further, if our undulator had more periods and if we were able to use a monochromator, we could slowly vary the beam energy and find the energy, at which the detected power is at maximum, i.e., when we are centered on the peak of the fundamental harmonic. In this case, the systematic error of $\epsilon_y$ measurement via the fluctuations, related to the uncertainty of the beam energy, would disappear. Moreover, it can be shown [24] that if we could place a narrow vertical slit in front of the detector, then the fluctuations magnitude would only depend on $\epsilon_y$, i.e., the systematic error of $\epsilon_y$ measurement, related to the uncertainty of $\epsilon_z$, measured by the SLMs, would disappear too. Finally, for a brighter undulator, the statistical error of measured fluctuations magnitude would be lower as well.

Furthermore, we are considering using our fluctuations measurement apparatus, or similar, as a tool for the diagnostics of Optical Stochastic Cooling in IOTA [43–45]. In this experiment, the beam emittance will be even smaller and the existing diagnostic tools may not have sufficient resolution. Even though we cannot measure transverse emittances and bunch length individually in this way, the fluctuations may serve as an indicator of the cooling process. Indeed, when the cooling process starts, the electron bunch shrinks, which, in turn, makes the fluctuations of the number of detected photons increase.

To conclude, we presented Eq. (2), which, together with Eq. (1), describes the turn-to-turn fluctuations in the number of detected synchrotron radiation photons, produced by an electron bunch, which has a Gaussian transverse density distribution, an arbitrary longitudinal density distribution and non-negligible rms beam divergences, the rms bunch length is assumed to be significantly larger than the radiation wavelength. This equation is presented for the first time, beam divergence has been neglected in all previous considerations [5, 6]. Beam divergence can be neglected if it is significantly smaller than the characteristic radiation angle — $\gamma^{-1}\sqrt{(1 + K^2/2)/N_a}$ for the fundamental of undulator radiation. We presented the results of an experiment with a round beam in IOTA, where beam divergence had an impact on the fluctuations of the undulator radiation. We showed that our new Eq. (2) agrees better with the measurements, than [5, Eq. (49)], which neglects beam divergence. Finally, we proposed a non-invasive technique of measurement of a small unknown vertical emittance of a flat beam in IOTA via the fluctuations. This new technique is described in detail in [24].

ACKNOWLEDGMENTS

We would like to thank the entire FAST/IOTA team at Fermilab for helping us with building and installing the experimental setup and taking data, especially Mark Obrycki, James Santucci, and Wayne Johnson. We also thank Greg Saewert for constructing the photodetector circuit and for providing the test light source. David Johnson and Todd Johnson provided the test equipment and assisted during our detector tests. Numerous pieces of advice, given by Daniil Frolov, are greatly appreciated as well. This work was completed in part with resources provided by the University of Chicago Research Computing Center. This research is supported by the University of Chicago and the US Department of Energy under contracts DE-AC02-76SF00515 and DE-AC02-06CH11357. This manuscript has been authored by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the US Department of Energy, Office of Science, Office of High Energy Physics.
Appendix A: Derivation of fluctuations with considerable beam divergence

Previously, we derived an equation for $M$ [5, Eq. (49)] for the case of a monoenergetic beam, zero beam divergence and temporally incoherent radiation. Below we outline the steps to extend this result to the case of a considerable beam divergence (Eq. (2)).

One can start from [5, Eq. (21)], but written in a form that accounts for beam divergence, namely,

$$
\text{var}(\mathcal{N}) = \langle \mathcal{N} \rangle - \langle \mathcal{N} \rangle^2
\geq \int d\xi \varphi(p) \left[ \sum_{s=1,2} \int dk d\phi_{k,s}(\phi) \left| \sum_m \mathcal{E}^{(m)}_{k,s}(\phi) \right|^2 \right]^2,
\tag{A1}
$$

where $\xi$ describes the states in the 6D phase-space of all the electrons in the center of the radiator,

$$
\xi = \tilde{x}_1, x_1', y_1, y_1', t_1, \delta p_1
\ldots
\tilde{x}_n, x_n', y_n, y_n', t_n, \delta p_n,
\tag{A2}
$$

where $t_m$ is the time when $m$th electron passes the center of the synchrotron light source, $\varphi(p)$ represents the density function for the probability to have the state $\xi$,

$$
\varphi(p) = p(\tilde{x}_1, x_1', y_1, y_1', t_1, \delta p_1)
\ldots
p(\tilde{x}_n, x_n', y_n, y_n', t_n, \delta p_n),
\tag{A3}
$$

$x_m$ and $x'_m$ refer to the monoenergetic component of the motion, because there is also a contribution from the horizontal dispersion, so that

$$
x_m = \tilde{x}_m + D_x \delta p_m, \quad x'_m = \tilde{x}'_m + D_x' \delta p_m,
$$

the vertical dispersion is assumed to be zero. According to [1, Eq. (2.93)], the complex field amplitude of $m$th electron $\mathcal{E}^{(m)}_{k,s}(\phi)$ can be expressed through the amplitude of the reference electron $\mathcal{E}_{k,s}(\phi - x'_m)$ (see Eq. (7)) as

$$
\mathcal{E}^{(m)}_{k,s}(\phi) = e^{ix_m - ik_m x_m - ik_y y_m} \mathcal{E}_{k,s}(\phi - x'_m)
$$

where $x'_m = (x'_m, y'_m)$ and it is assumed that $\mathcal{E}^{(m)}_{k,s}(\phi)$ does not depend on $\delta p_m$. For the reference electron $\tilde{x}, \tilde{x}', y, y', t, \delta p$ are equal to zero.

We assume an electron bunch that is Gaussian in transverse plane and has a Gaussian distribution for $\delta p$. The longitudinal density distribution $\rho(z)$ is arbitrary. The beam focusing optics is assumed to be uncoupled. In this case, the probability density function for one electron takes the following form,

$$
p(\tilde{x}, \tilde{x}', y, y', t, \delta p) = \frac{1}{4\pi^2 \sigma_x \sigma_y} \exp \left[ -\frac{1}{2\sigma_x^2} C_x(\tilde{x}, \tilde{x}') - \frac{1}{2\sigma_y^2} C_y(y, y') \right] \rho(-ct) \frac{1}{\sqrt{2\pi\sigma_p}} \exp \left[ -\frac{\delta p^2}{2\sigma_p^2} \right],
$$

with

$$
C_x(\tilde{x}, \tilde{x}') = \gamma_x \tilde{x}^2 + 2\alpha_x \tilde{x} \tilde{x}' + \beta_x (\tilde{x}')^2,
$$

$$
C_y(y, y') = \gamma_y y^2 + 2\alpha_y y y' + \beta_y (y')^2.
$$

Given Eqs. (A2) to (A8), and assuming the regime of longitudinal incoherence ($k \sigma_z \gg 1$), the integration in Eq. (A1) is solely a mathematical procedure. It is analogous to the derivation in [5] where $\xi$ included only $x_m, y_m$ and $t_m$. The only difference is the additional integration over $\tilde{x}_m'$, $y_m'$ and $\delta p_m$, with $m = 1 \ldots n_e$.

$$
\mathcal{E}_{k,s}(\phi) = \sqrt{\frac{C}{(2\pi)^{3/2} \sigma_k \sigma_{r,x} \sigma_{r,y}}} \exp \left( \frac{(k - k_0)^2}{2\sigma_k^2} - \frac{\phi_x^2}{2\sigma_{r,x}^2} - \frac{\phi_y^2}{2\sigma_{r,y}^2} \right)
$$

where $k_0$ refers to the center of the spectrum of the Gaussian radiation, $\sigma_k$ is the spectral rms width, $\sigma_{r,x}$ and $\sigma_{r,y}$ are the angular rms radiation sizes, $\sigma \ll$
for a Gaussian electron bunch, the following result can be obtained from Eq. (2),

\[ M = (1 - 1/n_r)^{-1} \sqrt{1 + 4\sigma_k^2 \sigma_i^2} \]

\[ \times \left[ \frac{1 + 4k_0^2 (\sigma_r^2 \sigma_{r',x'}^2 + \sigma_{r',y'}^2 \Sigma^2_{r,r'})^{2} + \frac{\sigma_{x'}^2}{\sigma_{r,r'}}^2}{1 + 4k_0^2 (\sigma_y^2 \sigma_{r,r'}^2 + \sigma_{y',r'}^2 \Sigma^2_{r,y'}) + \frac{\sigma_{y'}^2}{\sigma_{r,y'}}^2} \right] \]

which corresponds to the rms value 1.82 \times 10^{-3}. We believe that these fluctuations primarily come from the jitter in the pulse generator amplitude.

Further, we used neutral density filters to lower the number of photons detected by our apparatus. Neutral density filters are filters that have constant optical density in the wavelength region of interest. As they lower \( \langle N \rangle \) for the test light source, \( \text{var}(N) \) is lowered in the following known way,

\[ \text{var}(N) = \langle N \rangle + \theta \langle N \rangle^2, \quad (C2) \]

i.e., the relative fluctuations stay practically constant \( \text{var}(N)/\langle N \rangle^2 \approx \theta \), because they are caused by the pulse generator amplitude jitter, but at a very low \( \langle N \rangle \) the photon shot noise term (the first term in Eq. (C2)) may have a noticeable contribution, this is similar to Eq. (1). By using many different neutral density filters and their combinations we were able to record \( \Delta \)- and \( \Sigma \)-channel waveforms for a wide range of \( \text{var}(N) \), see Fig. 7(a), including the range observed in our experiment in IOTA, shown in Fig. 7(b) and highlighted by a red rectangle in Fig. 7(a).

In Figs. 7(a) and (b), the parameter \( \theta \) of the red prediction curve was obtained in a configuration without any neutral density filters, when the detector noise and the photon shot noise were negligible, see Eq. (C1). The blue fluctuations data points, obtained from the \( \Delta \)- and \( \Sigma \)-channel waveforms using Eqs. (20) and (25), agree with the red prediction curve in the entire studied range of \( \text{var}(N) \), including the range of Fig. 5(b) corresponding to the measurements in IOTA. This means that the method of extracting \( \text{var}(N) \) from the waveforms, described in Fig. 5 and Eq. (25), actually works well, and that the instrumental noise \( \text{var}(\nu_{\Delta}) = 2.0 \times 10^5 \) is indeed independent of the signal amplitude.

We estimated the statistical error of our measurement of photoelectron count variance in IOTA as the rms deviation of the fluctuation data points for the test light source from the prediction curve in Fig. 7(b). The error is \( \pm 2.7 \times 10^6 \). It is used in the error bars in Fig. 6 and

\[ \theta = \frac{\text{var}(N)}{\langle N \rangle^2} = 3.31 \times 10^{-6}, \quad \text{(C1)} \]
Appendix D: Vertical emittance estimation for the flat beam by Touschek beam lifetime

The beam lifetime could be reliably determined from the measured beam current $I$ as a function of time, namely, as $|I/(dI/dt)|$. During all of our measurements the beam current was measured with a precision DCCT with a time step of one second. At the beam current values from our experiment (1–3 mA), the beam lifetime is determined solely by Touschek scattering. In general, the momentum acceptance is a function of the position along the ring. It is limited by the longitudinal bucket size, $\delta_{rf}$, and by the dynamic momentum aperture. A constant effective momentum acceptance $\delta^{\text{eff}}_{\text{acc}}$ can be used [46] to describe the losses due to Touschek scattering. It is equal to or smaller than $\delta_{rf}$. We used the approach described in [47, 48] to calculate the Touschek lifetime. Figure 8 shows the measured beam lifetime for the round beam, a calculation with the momentum acceptance limited only by the rf bucket size ($\delta_{rf} = 2.8 \times 10^{-3}$ in IOTA), and a calculation with an effective momentum acceptance $\delta^{\text{eff}}_{\text{acc}} = 2.0 \times 10^{-3}$.

The calculation with $\delta^{\text{eff}}_{\text{acc}} = 2.0 \times 10^{-3}$ almost perfectly agrees with the measurement. The emittances and the beam lifetime of the round beam are known to us with good accuracy, the only unknown in this Touschek lifetime calculation for the round beam is $\delta^{\text{eff}}_{\text{acc}}$. We believe that Fig. 8 illustrates that in IOTA $\delta^{\text{eff}}_{\text{acc}} = 2.0 \times 10^{-3}$.

Further, we decided to apply this Touschek lifetime calculation (with $\delta^{\text{eff}}_{\text{acc}} = 2.0 \times 10^{-3}$) to the flat beam at beam current 2.66 mA, where we know the measured lifetime (559 sec) and the horizontal emittance, but we do not know the small vertical emittance. We found the following value for the vertical emittance, $\epsilon_y = 9.6$ nm, which is rather close to the fluctuations-based measurement, $\epsilon_y = 8.4 \pm 1.5$ nm.

FIG. 8. Beam lifetime of round beam as a function of beam current.
