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Gravitational Contact Interactions and the Physical Equivalence of Weyl Transformations in Effective Field Theory

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Abstract

Theories of scalars and gravity, with non-minimal interactions, $\sim (M_P^2 + F(\phi_i))R + L(\phi_i)$, have graviton exchange induced contact terms. These terms arise in single particle reducible diagrams with vertices, $\propto q^2$, that cancel the Feynman propagator denominator, $1/q^2$, and are familiar in various other physical contexts. In gravity these lead to additional terms in the action such as $\sim F(\phi_i)T_\mu^\mu(\phi_i)/M_P^2$ and $F(\phi_i)\partial^2 F(\phi_i)/M_P^2$. The contact terms are equivalent to induced operators obtained by a Weyl transformation that removes the non-minimal interactions, leaving a minimal Einstein-Hilbert gravitational action. This demonstrates explicitly the equivalence of different representations of the action under Weyl transformations, both classically and quantum mechanically. To avoid such "hidden contact terms" one is compelled to go to the minimal Einstein-Hilbert representation.

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I. INTRODUCTION

In recent years there has been considerable interest in scale invariant theories that, by way of spontaneous scale symmetry breaking or "inertial symmetry breaking," dynamically generate the Planck mass and associated phenomena of inflation and hierarchies [1–3]. A feature many of these approaches have in common is the notion of some pre-Planckian era, in which fundamental scalars exist and couple to gravity through non-minimal interactions, $\sim F(\phi_i)R$. The scalars then acquire VEV's that lead to a Planck mass, $\sim M_P^2 R + F'(\phi_i)R$, where F' contains residual active scalar fields that couple non-minimally.

A key tool in the analysis of these models is the Weyl transformation, [4]. This involves a redefinition of the metric, $g' = \Omega(\phi_i)g$, in which g comingles with scalars. Ω can be chosen to lead to a new effective theory, typically one that is pure Einstein-Hilbert, $\sim M_P^2 R + L'(\phi_i)$, in which the non-mimimal interactions have been removed.[1]

The Weyl transformation is classically exact. However, it is often difficult to discern how the original non-minimal interaction theory is physically equivalent to the pure Einstein-Hilbert form. There may be apparent advantages in using the transformed theory that are not evident in the original, or vice versa. These apparent advantages, however, may not really be present when all effects are taken into account. It is also unclear how the Weyl transformation is compatible with a full quantum theory [5].

In the present paper we will address these questions. We will work to first order in $1/M_P^2$ in a linearized version of a theory with Planck mass and non-minimal interactions. We will not perform a Weyl redefinition of the metric. Nonetheless, we will demonstrate how the Weyl transformation form of $L'(\phi_i)$ necessarily arises perturbatively by way of Feynman diagrams involving graviton exchange.

This happens by way of *contact terms* that are generated by the graviton exchange amplitudes. These are bona fide physical effects that occur in various venues in physics and, though they arise in tree approximation, they must be included into the effective action of the theory at the given order of perturbation theory. Moreover, this represent essentially "integrating out" the vertices that lead to the contact terms. The result is that the non-minimal interactions will disappear from the theory at any given order in perturbation theory and are replaced by new, pointlike interactions from the contact terms.

[1] Alternatively, one might partially remove a subset of scalars from the non-minimal interactions $\sim M_P^2 R + F''(\phi_i)R$ where F'' is optimized for some particular model application.

Perhaps not surprisingly, the form of the contact term interactions corresponds identically with the Weyl transformation that takes the theory to the pure Einstein-Hilbert form. We argue that, once the Planck scale is generated in the theory, by spontaneous or "inertial" symmetry breaking [3], then the action should be "diagonalized," in analogy to diagonalizing the kinetic terms, so that the contact terms do not appear perturbatively. This mandates a Weyl transformation to a pure Einstein-Hilbert action which is a unique specification of the theory.

We will be computing potentials that arise from graviton exchange. This will require gauge fixing, and we will use the standard De Donder gauge in a first pass, following Donoghue, et.al. [6]. However, we will also find it illuminating to consider a different gauge choice which separates a traceless metric from it's trace. The trace metric has a ghost signature, but it uniquely controls the relevant contact terms associated with the Weyl transformation. Otherwise, both gauges give the same results, as they must.

We turn presently to a brief discussion of contact terms in general and a toy model that will be structurally similar to the gravitational case.

A. Contact Interactions

Generally, single particle irreducible (1PI) Feynman diagrams describe perturbative corrections (or renormalizations) of a Lagrangian based field theory action. On the other hand, reducible diagrams, those that break into two disconnected diagrams upon cutting a line, are the radiative effects that one computes from the given action [7]. There is, however, an exception: sometimes single particle reducible diagrams correspond to "contact term" interactions. These then become part of the action.

Contact terms arise in a number of phenomena. Diagrammatically they arise when a vertex for the emission of, e.g., a massless quantum, of momentum q_μ, is proportional to q^2. This vertex then cancels the $1/q^2$ from the propagator when the quantum is exchanged. This q^2/q^2 cancellation leads to an effective pointlike operator from an otherwise single-particle reducible diagram.

For example, in electroweak physics a vertex correction by a W-boson to a gluon emission induces a quark flavor changing operator, e.g., describing $s \to d$+gluon, where s (d) is a strange (down) quark. This has the form of a local operator:

$$g\kappa\bar{s}\gamma_\mu T^A d_L D_\nu G^{A\mu\nu} \tag{1}$$

where $G^{A\mu\nu}$ is the color octet gluon field strength and $\kappa \propto G_{Fermi}$. This implies a vertex for an emitted gluon of 4-momentum q and polarization $\epsilon^{A\mu}$, of the form $g\kappa\bar{s}\gamma_\mu T^A d_L \epsilon^{A\mu} \times q^2 + ...$ However, the gluon propagates, $\sim 1/q^2$, and couples to a quark current $\sim g\epsilon^{A\mu}\bar{q}\gamma_\mu T^A q$. This results in a contact term:

$$g^2\kappa\left(\frac{q^2}{q^2}\right)\bar{s}\gamma^\mu T^A d_L \bar{q}\gamma_\mu T^A q \;\sim\; g^2\kappa\bar{s}\gamma^\mu T^A d_L \bar{q}\gamma_\mu T^A q \tag{2}$$

The result is a 4-body local operator which mediates electroweak transitions between, e.g., kaons and pions [8], also known as "penguin diagrams" [9]. Note the we can rigorously obtain the contact term result by use of the gluon field equation within the operator of eq.(1),

$$D_\nu G^{A\mu\nu} = g\bar{q}\gamma^\mu T^A q. \tag{3}$$

This is justified as operators that vanish by equations of motion, known as "null operators," will generally have gauge noninvariant anomalous dimensions and are unphysical [10].

Another example of a contact term occurs in the case of a cosmic axion, described by an oscillating classical field, $\theta(t) = \theta_0\cos(m_a t)$, interacting with a magnetic moment, $\vec{\mu}(x)\cdot\vec{B}$, through the electromagnetic anomaly $\kappa\theta(t)\vec{E}\cdot\vec{B}$. A static magnetic moment emits a virtual spacelike photon of momentum $(0,\vec{q})$. The anomaly absorbs the virtual photon and emits an on-shell photon of polarization $\vec{\epsilon}$, inheriting energy $\sim m_a$ from the cosmic axion. The Feynman diagram, with the exchanged virtual photon, yields an amplitude, $\propto (\theta_0\mu^i\epsilon_{ijk}q^j)(1/\vec{q}^{\,2})(\kappa\epsilon^{k\ell h}q_\ell m_a\epsilon_h) \sim (\kappa\theta_0 m_a\vec{q}^{\,2}/\vec{q}^{\,2})\vec{\mu}\cdot\vec{\epsilon}$. The $\vec{q}^{\,2}$ factor then cancels the $1/\vec{q}^{\,2}$ in the photon propagator, resulting in a contact term which is an induced, parity violating, oscillating electric dipole interaction: $\sim \kappa\theta(t)\vec{\mu}\cdot\vec{E}$. This results in cosmic axion induced *electric dipole* radiation from any magnet, including an electron [11].

B. Illustrative Toy Model of Contact Terms

In preparation for the analysis of gravitational contact terms we first present a schematic discussion of a simple toy model that illustrates the emergence of contact terms and is structurally similar to what we encounter in gravity.[2] Consider a single real scalar field ϕ and operators A and B, which can be functions of other fields, with the action given by:

$$S = \int \frac{1}{2}\partial\phi\partial\phi - A\partial^2\phi - B\phi \tag{4}$$

[2] Here Lorentz indices have been suppressed and the contraction of indices understood. $\widehat{T}$ refers to the time ordered product, where T is the trace of the stress tensor.

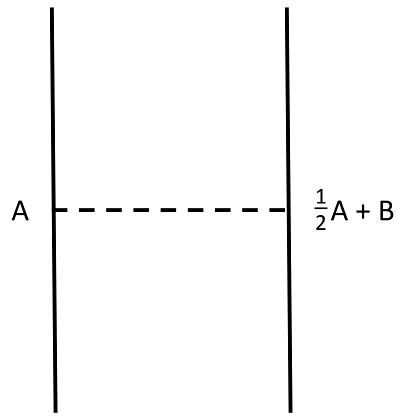

FIG. 1: Contact terms in the toy model are generated by diagrams with exchange of ϕ (dashed). In gravity, with non-minimal term $\sim \int \sqrt{-g} F(\phi_i) R$ and matter field Lagrangian $\sim \int \sqrt{-g} L(\phi_i)$ then A is replaced by $F(\phi)$ and B is replaced by $L(\phi)$, and the dashed line is a graviton propagator.

Here ϕ has a propagator $\frac{i}{q^2}$, but the vertex of a diagram involving A has a factor of $\partial^2 \sim -q^2$. This yields a pointlike interaction, $\sim q^2 \times \frac{i}{q^2}$, in a single particle exchange of ϕ, and therefore implies contact terms:

$$\begin{aligned} \widehat{T}\ i\int A\partial^2\phi \ \ i\int B\phi &\to -\frac{i}{q^2}\left(-q^2\right)AB &= i\int AB \\ \frac{1}{2}\widehat{T}\ i\int A\partial^2\phi \ \ i\int A\partial^2\phi &\to -\frac{i}{2q^2}A^2\left(-q^2\right)^2 &= \frac{i}{2}\int A\partial^2 A. \end{aligned} \tag{5}$$

This also produces a nonlocal interaction $-\frac{i}{2q^2}BB$.

Exponentiating these operators we see that we have diagrammatically obtained a local effective action:

$$S = \int \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}A\partial^2 A + AB + \text{ long distance terms} \tag{6}$$

Of course, we can see this straightforwardly by "solving the theory," by defining a shifted field:

$$\phi = \phi' - \frac{1}{\partial^2}\left(\partial^2 A + B\right) \tag{7}$$

Substituting and integrating by parts, this yields:

$$S = \int \frac{1}{2}\partial\phi'\partial\phi' + \frac{1}{2}A\partial^2 A + AB + \frac{1}{2}B\frac{1}{\partial^2}B \tag{8}$$

An equivalent effective local action that describes both short and large distance is then,

$$S = \int \frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}A\partial^2 A + AB - B\phi \tag{9}$$

The contact terms have become pointlike components of the effective action, while the long distance effects are produced by ϕ exchange. Note that the derivatively coupled operator A has no long distance interactions due to ϕ exchange. Moreover, in the effective action of eq.(9) we have implicitly "integrated out" the $A\partial^2\phi$, which is no longer part of the action and is replaced by new operators $\frac{1}{2}A\partial^2 A + AB$. One can also adapt the use of equations of motion to simplify the action but this requires care. For example, the insertion of the ϕ equation of motion into $A\partial^2\phi$ correctly gives the AB term but misses the factor of $1/2$ in the $A\partial^2 A$ term.

II. GRAVITATIONAL CONTACT TERMS

We will consider a general theory involving scalar fields ϕ_i, an Einstein-Hilbert term and a non-minimal interaction:

$$S = \int \sqrt{-g}\left(\frac{1}{2}M_P^2 R\left(g_{\mu\nu}\right) + \frac{1}{2}F\left(\phi_i\right) R\left(g_{\mu\nu}\right) + L\left(\phi_i\right)\right). \tag{10}$$

where we use the metric signature and curvature tensor conventions of [12]. In parallel with the general discussion we will quote the results for a simple model,

$$S = \int \sqrt{-g}\left(\frac{1}{2}M_P^2 R\left(g_{\mu\nu}\right) + \frac{1}{2}\xi\phi^2 R\left(g_{\mu\nu}\right) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - W(\phi)\right). \tag{11}$$

The matter lagrangian has the stress tensor and stress tensor trace:

$$\begin{aligned} T_{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int \sqrt{-g}L\left(\phi_i\right) \\ T &= g^{\mu\nu}T_{\mu\nu} \end{aligned} \tag{12}$$

which, in the simple model, take the form,

$$\begin{aligned} T_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - W(\phi)\right). \\ T &= -\partial^\sigma\phi\partial_\sigma\phi + 4W(\phi) \end{aligned} \tag{13}$$

This is the usual matter stress tensor and it is conserved by the ϕ equations of motion to leading order in $1/M_P^2$ in a linearized gravity approximation, and we can neglect the contribution of the non-minimal term (S_2 below) to the stress tensor conservation at this order.[3]

[3] This is not the "improved stress tensor" of [12], where "improvement terms" are separately explicitly conserved and come from an assumed conformal non-minimal coupling of ϕ to gravity, $\frac{1}{2}\xi\phi^2 R$, with $\xi = \frac{1}{6}$, and is not relevant in the present discussion.

We treat the theory perturbatively, expanding around flat space. Hence we linearize gravity with a weak field $h_{\mu\upsilon}$:

$$g_{\mu\upsilon} \approx \eta_{\mu\upsilon} + \frac{h_{\mu\upsilon}}{M_P}, \qquad g^{\mu\upsilon} \approx \eta^{\mu\upsilon} - \frac{h^{\mu\upsilon}}{M_P} + O(h^2), \qquad \sqrt{-g} \approx 1 + \frac{1}{2}\frac{h}{M_P}, \qquad h = \eta^{\mu\nu} h_{\mu\nu} \quad (14)$$

The scalar curvature is then:

$$\begin{aligned} R &= R_1 + R_2 \\ M_P R_1 &= \left(\partial^2 h \; - \partial^\mu \partial^\nu h_{\mu\nu}\right) \\ M_P^2 R_2 &= -\frac{3}{4}\partial^\rho h^{\mu\nu}\partial_\rho h_{\mu\nu} - \frac{1}{2} h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2} h^{\mu\nu}\partial_\mu\partial_\nu h \\ &\quad + \partial_\nu\left(h^{\nu\mu}\partial^\rho h_{\mu\rho}\right) - \frac{1}{2}\partial_\nu\left(h^{\nu\mu}\partial_\mu h\right) + h^{\mu\nu}\partial_\rho\left(\partial_\mu h^\rho_\nu \; - \frac{1}{2}\partial^\rho h_{\mu\nu}\right) \\ &\quad + \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu_\rho - \frac{1}{2}\partial_\mu h \partial^\nu h^\mu_\nu \; + \frac{1}{4}\partial_\mu h \partial^\mu h \end{aligned} \quad (15)$$

Using this the action is given by $S = S_1 + S_2 + S_3$ where S_1 is the Fierz-Pauli action:

$$\begin{aligned} S_1 &= \frac{1}{2} M_P^2 \int \sqrt{-g} R = \frac{1}{2} M_P^2 \int \left(R_2 + \frac{1}{2}\frac{h}{M_P} R_1\right) \\ &= \frac{1}{2}\int h^{\mu\nu}\left(\frac{1}{4}\partial^2 \eta_{\mu\nu}\eta_{\rho\sigma} - \frac{1}{4}\partial^2\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{2}\partial_\rho\partial_\sigma\eta_{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\rho\eta_{\nu\sigma}\right) h^{\rho\sigma} \end{aligned} \quad (16)$$

Note that the leading term in the first order expansion $\frac{1}{2} M_P^2 R_1$ is a total divergence and is zero in the Einstein-Hilbert action. What remains is the Fierz-Pauli action written in a factorized form $h\,(...)\,h$.

On the other hand the non-minimal interaction, S_2, takes the form:

$$S_2 = \frac{1}{2}\int \sqrt{-g} F(\phi_i)\, R(g_{\mu\nu}) = \int \frac{1}{2M_P} F(\phi_i)\; \Pi^{\mu\nu} h_{\mu\nu} \quad (17)$$

where it is useful to introduce the transverse derivative,

$$\Pi^{\mu\nu} = \partial^2 \eta^{\mu\nu} \; - \partial^\mu \partial^\nu. \quad (18)$$

Finally, S_3 is the matter action and coupling to the gravitational weak field:

$$S_3 = \int \; L(\phi_i) - \frac{h^{\mu\nu}}{2M_P} T_{\mu\nu} \quad (19)$$

Due to the conservation of $T^{\mu\nu}$ and the transverse derivative, the full action S possesses the local gauge invariance,

$$\delta h_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu. \quad (20)$$

Since S_2 involves derivatives, the Feynman diagrams involving S_2 and S_3 will generate contact terms in the gravitational potential generated by single graviton exchange. This will closely parallel the toy model.

A. Graviton Propagator

We are interested in the gravitational potential amongst the operators that comprise S_2 and S_3. This is mediated by a single graviton exchange, as in Figure 1, effectively integrating out the S_2 term in analogy to the $A\partial^2\phi$ term in the toy model. For this we require the graviton propagator and, due to the underlying gauge invariance, it is necessary first to gauge-fix.

A conventional choice of gauge is the De Donder gauge:[4]

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h \tag{21}$$

which is defined by the condition,

$$0 = g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu h^\lambda_\nu \ + \eta^{\mu\nu}\partial_\nu h^\lambda_\mu - \partial^\lambda \eta^{\mu\nu} h_{\mu\nu}\right). \tag{22}$$

The De Donder gauge is a member of a one-parameter family of gauges defined by $\partial_\mu h^{\mu\nu} = w\partial^\nu h$, where $w = \frac{1}{2}$ in the De Donder case. In Section IV we discuss an alternative gauge, $w = \frac{1}{4}$, which is somewhat more transparent for our present application but, of course, yields the same results.

The Fierz-Pauli action in De Donder gauge, by substituting eq.(21) into S_1, takes the form:

$$S_1 = \frac{1}{2}\int \frac{1}{8} h^{\mu\nu}\left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma}\right)\partial^2 h^{\rho\sigma} = \frac{1}{2}\int h^{\mu\nu}\left(\frac{1}{8}P_{\mu\nu\ \rho\sigma}\right)\partial^2 h^{\rho\sigma} \tag{23}$$

where,

$$P_{\mu\nu\ \rho\sigma} = \eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\alpha\sigma} \tag{24}$$

and $P_{\mu\nu\ \rho\sigma}$ is the spin-2 projection operator.

The inverse of the kinetic term operator is $A^{\mu\nu\ \rho\sigma}$, given by:

$$\begin{aligned}
&\frac{1}{8}\left(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}\right)A^{\alpha\beta\ \rho\sigma} = \delta^{\rho\sigma}_{\mu\nu} \\
&\delta^{\rho\sigma}_{\mu\nu} = \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\right) \\
&\text{hence,}\quad A^{\mu\nu\ \rho\sigma} = 2P^{\mu\nu\ \rho\sigma}.
\end{aligned} \tag{25}$$

Note that the normalization follows from our choice of scale, $\sim h_{\mu\nu}/M_P$, in the linear gravity expansion, eq.(14). This gives the propagator in a path integral with action S_1:

$$\langle 0|\widehat{T}\ h_{\rho\sigma}(x)\ h_{\mu\nu}(y)\ |0\rangle = \int Dg\ e^{iS_1}\left(h_{\rho\sigma}(x)h_{\mu\nu}(y)\right) = iA_{\mu\nu\ \rho\sigma}D(x-y) \tag{26}$$

[4] We presently follow the lecture notes of Donoghue *et. al.*, [6], though we differ in normalization; note the correspondence of our normalization to Donoghue's [6] is $\kappa = 2/M_P$.

where

$$D(x-y) = \frac{1}{\partial^2} = \int \frac{-1}{q^2+i\epsilon} e^{iq\cdot(x-y)} \frac{d^4q}{(2\pi)^4} \tag{27}$$

is a time ordered scalar field Green's function satisfying $\partial^2 D(x-y) = \delta^4(x-y)$.[5] The momentum space Feynman propagator for gravitons is then,

$$\langle 0|\widehat{T}\ h_{\rho\sigma}(q)\ h_{\alpha\beta}(-q)\ |0\rangle \ = \ \frac{-i}{q^2+i\epsilon} A_{\rho\sigma,\alpha\beta}. \tag{28}$$

The procedure of substituting the gauge condition into the action, then inverting, is analogous in electrodynamics to substituting $\partial_\mu A^\mu = 0$ into the action, which yields the photon propagator in Feynman gauge, $\sim -ig_{\mu\nu}/q^2$. In analogy to using Feynman gauge, we must take care to tie the graviton propagator, eq(26), onto conserved currents, such as the stress tensor or the transverse derivative, which then guarantees gauge invariance of a given tree amplitude.

B. Newtonian Potential

Let us first consider the Newtonian potential. This can be computed from a Feynman diagram for graviton exchange. Equivalently, the action is determined by simply shifting the graviton field. Using the truncated action:

$$S = \int \frac{1}{2} h^{\mu\nu} \left(\frac{1}{8} P_{\mu\nu\ \rho\sigma}\right) \partial^2 h^{\rho\sigma} - \frac{h^{\mu\nu}}{2M_P} T_{\mu\nu} \tag{29}$$

we can define a shifted $h'^{\rho\sigma}$:

$$h^{\rho\sigma} = h'^{\rho\sigma} + \frac{1}{2M_P}\frac{1}{\partial^2} A^{\mu\nu\ \rho\sigma} T_{\mu\nu} \tag{30}$$

Hence,

$$S \ = \ \frac{1}{2}\int h'^{\mu\nu} \left(\frac{1}{8} P_{\mu\nu\ \rho\sigma}\right) \partial^2 h'^{\rho\sigma} - \frac{1}{2}\left(\frac{1}{2M_P}\right)^2 \int\int d^4x\ d^4y\ T^{\mu\nu}(x) A_{\mu\nu\rho\sigma} D(x-y) T^{\rho\sigma}(y). \tag{31}$$

For stationary masses, located at $x = 0$ and $x = r$ the stress tensor is pure 00,

$$T^{00}(x) = m_1 \delta^3(\vec{x}) + m_2 \delta^3(\vec{x}-\vec{r}). \tag{32}$$

[5] We use the shorthand $\frac{1}{\partial^2} f(x) = \int D(x-y) f(y) d^4y$, and $\frac{1}{\partial^2\partial^2} f(x) = \int D(x-y) D(y-z) f(z) d^4y d^4z$, etc.

Insert this into the second term of eq.(31), and note the time integrated stationary Green's function becomes,

$$\int\int d^4x\; d^4y\; \delta^3(\vec{x})\; \delta^3(\vec{y}-\vec{r})\; D(x-y) = \int dt\; \frac{1}{4\pi r} \tag{33}$$

and $A_{0000} = -2$, which yields the effective action,

$$\int \frac{2}{(2M)^2}\frac{1}{4\pi r} m_1 m_2\; dt = \int \frac{G_N m_1 m_2}{r} dt \tag{34}$$

where $M_P^2 = (8\pi G_N)^{-1}$ and implies an attractive Newtonian gravitational potential.

The Feynman propagator yields the graviton exchange amplitude in momentum space,

$$\frac{1}{2}\frac{1}{(2M_P)^2}\,(i)^2\,\frac{-i}{q^2+i\epsilon} T^{\mu\nu} A_{\mu\nu\;\rho\sigma} T^{\rho\sigma} = \frac{1}{4M_P^2}\frac{-i}{q^2+i\epsilon}\left(2T^{\rho\sigma}T_{\rho\sigma} - TT\right) \tag{35}$$

where $T = \eta^{\rho\sigma}T_{\rho\sigma}$ is the trace of the stress tensor. This operator corresponds to the second term of the action, eq.(31), with the amplitude factor of i (a combinatorial factor of 2 will arise in a matrix element of this operator in states such as $\langle m_1 m_2|...|m_1 m_2\rangle$, and reproduces the potential of eq.(34)),

C. Contact Terms from Single Graviton Exchange

Here we evaluate the operators in the Feynman diagrams of Figure 1 arising from single graviton exchange between S_2 and S_3. In classical background fields, ϕ_i, graviton exchange between the pair $\langle S_2 S_3\rangle$ gives:

$$\begin{aligned} -i\langle S_2 S_3\rangle \;&=\; -i(i^2)\int\int d^4y\; d^4x\; \frac{1}{(2M_P)^2} F\,(x)\,(-T^{\rho\sigma}(y))\langle 0|\widehat{T}\;\Pi^{\mu\nu}h_{\mu\nu}(x)\;h_{\rho\sigma}(y))\;|0\rangle \\ &=\; \int\int d^4y\; d^4x\; \frac{1}{(2M_P)^2} F\,(x)\,\Pi^{\mu\nu}A_{\mu\nu\rho\sigma}D(x-y)T^{\rho\sigma}(y) \end{aligned} \tag{36}$$

where we have:

$$\Pi^{\mu\nu}A_{\mu\nu\rho\sigma} = 2\partial^2\eta_{\rho\sigma} + 4\partial_\rho\partial_\sigma. \tag{37}$$

Rearranging and integrating by parts:

$$-\,i\langle S_2 S_3\rangle = \int\int d^4y\; d^4x\; \frac{F(x)}{2M_P^2}\left(\partial^2 D(x-y)T(y) - 2D(x-y)\partial_\rho\partial_\sigma T^{\rho\sigma}\right) \tag{38}$$

and we note that $\partial_\rho\partial_\sigma T^{\rho\sigma}$ vanishes by the conservation of the stress tensor. The first term involving the trace, $T(y)$, is a contact term arising from $\partial^2 D(x-y) = \delta^4(x-y)$. Hence the gravitational potential generates a contact term interaction in the effective action of the form:

$$\int d^4x\; \frac{F\,(\phi_i(x))}{2M_P^2} T(\phi_i(x)) \qquad \rightarrow \qquad \int d^4x\; \frac{\xi\phi^2}{2M_P^2}\left(-\partial^\mu\phi\partial_\mu\phi + 4W(\phi)\right) \tag{39}$$

where we quote the general result and that of the simple model.

Furthermore, we have the exchange of a graviton involving the pair $\langle S_2 S_2\rangle$:

$$\begin{aligned} -i\langle S_2 S_2\rangle &= -\frac{1}{2} i\left(i^2\right) \int\int d^4y\, d^4x\, \frac{1}{(2M_P)^2} F(x)\, F(y)\, \langle 0|\widehat{T}\ \Pi^{\mu\nu} h_{\mu\nu}(x)\ \Pi^{\rho\sigma} h_{\rho\sigma}(y)|0\rangle \\ &= -\frac{1}{2}\int\int d^4y\, d^4x\, \frac{1}{4M_P^2} F(x)\, \Pi^{\mu\nu} A_{\mu\nu\rho\sigma}\ \Pi^{\rho\sigma} D(x-y) F(y)\, . \end{aligned} \tag{40}$$

Note the factor of $\frac{1}{2}$ coming from the second order perturbative expansion. Here we have,

$$\Pi^{\mu\nu} A_{\mu\nu\rho\sigma}\ \Pi^{\rho\sigma} = 6\partial^2\partial^2 \tag{41}$$

leading to the result:

$$-i\langle S_2 S_2\rangle = -\int d^4x\, \frac{3}{4M_P^2}\, F\left(\phi_i\left(x\right)\right) \partial^2 F\left(\phi_i\left(x\right)\right). \tag{42}$$

This is the analogy of the $\frac{1}{2}A\partial^2 A$ term in the toy model.

In summary the gravitational potential amongst S_2 and S_3 terms mediated by a single graviton exchange diagram yields contact terms that are an effective action, S_{CT}, and represents the effect of integrating out the S_2 term:

$$S_{CT} = -\int d^4x\, \frac{3}{4M_P^2} F\left(\phi_i\right) \partial^2 F\left(\phi_i\right) + \int d^4x\, \frac{1}{2M_P^2} F\left(\phi_i\right) T\left(\phi_i\right) \tag{43}$$

In the simple model case, we can rearrange the $F\partial^2 F$ term to obtain,

$$S_{CT} = \int d^4x\, \frac{3\xi^2}{M_P^2}\, \phi^2 \partial\phi\partial\phi + \int d^4x\, \frac{\xi\phi^2}{2M_P^2} \left(-\partial^\sigma\phi\partial_\sigma\phi + 4W(\phi)\right) \tag{44}$$

Note the sign of the $F\partial^2 F$ is opposite (repulsive) to that of the toy model, a point that we will clarify below.

III. WEYL TRANSFORMATION

In the previous section we directly evaluated the effective action by calculating a single graviton exchange potential and separating the contact terms, which must be interpreted as parts of the effective action. There is, however, another route, which is to perform a Weyl transformation.

We can define:

$$g_{\mu\nu}(x) = \Omega^{-2} g'_{\mu\nu}(x), \qquad g^{\mu\nu}(x) = \Omega^2 g^{\mu\nu\prime}(x), \qquad \sqrt{-g} = \sqrt{-g'}\Omega^{-4} \tag{45}$$

and use:

$$\begin{aligned} R(\Omega^{-2}g') &= \Omega^2 R(g) + 6\Omega^3 D\partial\Omega^{-1} \\ L\left(g_{\mu\nu}(x), \phi_i(x)\right) &= L\left(\Omega^{-2} g'_{\mu\nu}(x), \phi_i(x)\right) \end{aligned} \tag{46}$$

With the choice $\Omega^2 = \left(1 + \frac{F(\phi_i)}{M_P^2}\right)$ we have:

$$\begin{aligned} S &\equiv \int \sqrt{-g}\left(\frac{1}{2}M_P^2 R\left(g_{\mu\nu}\right) + \frac{1}{2}F\left(\phi_i\right)R\left(g_{\mu\nu}\right) + L\left(\phi_i\right)\right) \\ &\to \int \sqrt{-g'}(\frac{1}{2}M_P^2 R\left(g'_{\mu\nu}\right) + 6\Omega D\partial\Omega^{-1} + \Omega^{-4}L\left(\Omega^{-2}g'_{\mu\nu}(x), \phi_i(x)\right)) \end{aligned} \tag{47}$$

and we obtain:

$$\begin{aligned} S = \int \sqrt{-g'}(\frac{1}{2}M_P^2 R\left(g'_{\mu\nu}\right) - 3M_P^2\partial_\mu\left(1 + \frac{F\left(\phi_i\right)}{M_P^2}\right)^{+1/2}\partial^\mu\left(1 + \frac{F\left(\phi_i\right)}{M_P^2}\right)^{-1/2} \\ + \left(1 + \frac{F}{M_P^2}\right)^{-1}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \left(1 + \frac{F\left(\phi_i\right)}{M_P^2}\right)^{-2}W(\phi,\chi)) \end{aligned} \tag{48}$$

Keeping terms to $O(\frac{1}{M_P^2})$ and integrating by parts we have:

$$S = S_1 + \int\left(L\left(\phi_i(x)\right) - \frac{3F\left(\phi_i\right)\partial^2 F\left(\phi_i\right)}{4M_P^2} + \frac{F\left(\phi_i\right)T\left(\phi_i\right)}{2M_P^2}\right) \tag{49}$$

The Weyl transformed action is identically consistent with the contact terms of eq.(43) above, to first order in $1/M_P^2$.

Hence, contact terms arise in gravity with non-minimal couplings to scalar fields due to graviton exchange. Their form is equivalent to a Weyl redefinition of the theory to one with a pure Einstein-Hilbert action and reinforces their role as induced components of the effective action. Hence working in any theory with a non-minimal interaction $\sim F(\phi)R$ will lead to these contact terms at order $1/M_P^2$. The contact terms can be avoided in perturbation theory by going to the pure Einstein-Hilbert action with a Weyl tranformation.

The Weyl transformation is nonperturbative. It is technically simpler than the gravitational potential calculation, and it confirms the tricky normalization factors and phases in the graviton exchange calculation. As the Weyl transformation makes no reference to a gauge choice, a calculation of the the contact terms in other gauges should yield the equivalent results. To check the invariance we turn now to a calculation in an alternative gauge which sheds further light on the origin of their structure.

IV. ANOTHER GAUGE

Presently we will choose a gauge that will more clearly show what is going on in the contact term equivalence with Weyl transformations. In particular, we obtained a negative sign for the analogy to the positive sign $A\partial^2 A$ of the toy model, which becomes clear in the present gauge choice.

We begin by defining traceless and trace fields for the weak field metric:

$$s_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h \qquad\qquad t_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu}h = \eta_{\mu\nu}t \tag{50}$$

hence $h_{\mu\nu} = s_{\mu\nu} + \eta_{\mu\nu}t$ and $h = 4t$. The Fierz-Pauli action and non-minimal terms in these variables become,

$$\begin{aligned} S_1 &= \frac{1}{2}\int \frac{3}{2}t\partial^2 t - \frac{1}{4}s^{\mu\nu}\partial^2 s_{\mu\nu} + \frac{1}{2}s^{\mu\nu}\partial_\mu\partial^\rho s_{\rho\sigma} - 3t\partial_\mu\partial_\nu s^{\mu\nu} \\ S_2 &= \int \frac{1}{M_P}F(\phi)\left(3\partial^2 t \; - \partial_\lambda\partial^\beta s^\lambda_\beta\right) \end{aligned} \tag{51}$$

The coupling to gravity is:

$$-\frac{h^{\mu\nu}}{2M_P}T_{\mu\nu} = -\frac{s^{\mu\nu}}{2M_P}T_{\mu\nu} - \frac{t}{2M_P}T \tag{52}$$

Note that t can be viewed as a small shift in the trace of the metric; $4\delta t = \delta h$, and $\delta s = 0$ and it therefore exclusively couples to the trace of the matter field stress tensor.

Under a gauge transformation we have:

$$\begin{aligned} \delta s_{\mu\nu} &= \partial_\mu A_\nu + \partial_\nu A_\mu - \frac{1}{2}\eta_{\mu\nu}\partial_\rho A^\rho \\ \delta t_{\mu\nu} &= \frac{1}{2}\eta_{\mu\nu}\partial_\rho A^\rho \end{aligned} \tag{53}$$

Things simplify considerably if we can impose the gauge condition,

$$\partial^\mu s_{\mu\nu} = 0. \tag{54}$$

Note that this is different from the condition $\partial^\mu h_{\mu\nu} = 0$ owing to the tracelessness of $s_{\mu\nu}$. However, with $\partial^\mu s_{\mu\nu} = 0$ we see that $s_{\mu\nu}$ exclusively contains the propagating modes of gravitational waves. For a gravitational wave propagating in the $z-$direction in empty space the modes are $h_{xy} = s_{xy}$ and $h_{xx} - h_{yy} = s_{xx} - s_{yy}$ and $t = 0$.

Indeed, we can find a gauge transformation to fix $\partial^\mu s_{\mu\nu} = 0$. Given any arbitrary configuration $s^0_{\mu\nu}$ and $t^0_{\mu\nu}$ we can choose,

$$\partial^\mu s_{\mu\nu} = \partial^\mu s^0_{\mu\nu} + \partial^2 A_\nu + \frac{1}{2}\partial_\nu(\partial\cdot A) = 0 \tag{55}$$

and we find (see footnote 5):

$$A_\nu = -\frac{1}{\partial^2}\partial^\mu s^0_{\mu\nu} + \frac{1}{3}\frac{\partial_\nu\partial^\rho}{\partial^2\partial^2}\partial^\mu s^0_{\mu\rho} \tag{56}$$

Verifying we see that,

$$\partial^\mu s_{\mu\nu} = \partial^\mu s^0_{\mu\nu} + \partial^2\left(-\frac{1}{\partial^2}\partial^\mu s^0_{\mu\nu} + \frac{1}{3}\frac{\partial_\nu\partial^\rho}{\partial^2\partial^2}\partial^\mu s^0_{\mu\rho}\right) - \frac{1}{3}\frac{\partial_\nu\partial^\rho}{\partial^2}\partial^\mu s^0_{\mu\rho} = 0 \tag{57}$$

Note that the gauge transformation also preserves the traceless of $s_{\mu\nu}$ as,

$$\delta\eta^{\mu\nu}s_{\mu\nu} = 2\partial^\nu A_\nu - 4\times\frac{1}{2}\partial_\rho A^\rho = 0 \tag{58}$$

Under this transformation we also redefine t:

$$t_{\mu\nu} = t^0_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\partial\cdot A = t^0_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\frac{1}{\partial^2}\partial^\rho\partial^\sigma s^0_{\rho\sigma} \tag{59}$$

We remark that this gauge choice is one of a single parameter, w, family of gauge choices,

$$\partial_\mu h^{\mu\nu} = w\partial^\nu h \tag{60}$$

The De Donder gauge corresponds to $w=\frac{1}{2}$ while the present gauge choice, $\partial_\mu s^{\mu\nu}=0$, corresponds to $w=\frac{1}{4}$.

In the $w=\frac{1}{4}$ gauge the Fierz Pauli action simplifies to:

$$\begin{aligned} S_1 &= \frac{1}{2}\int\left(-\frac{3}{2}\partial t\partial t + \frac{1}{4}\partial s^{\mu\nu}\partial s_{\mu\nu}\right) \\ &= \frac{1}{2}\int\left(\frac{3}{2}t\partial^2 t - \frac{1}{8}s^{\alpha\beta}\left(\eta_{\rho\alpha}\eta_{\sigma\beta}+\eta_{\sigma\alpha}\eta_{\rho\beta}\right)\partial^2 s^{\rho\sigma}\right). \end{aligned} \tag{61}$$

The inverse of the kinetic term tensor is then,

$$\begin{aligned} -\frac{1}{8}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}\right)B^{\alpha\beta\ \rho\sigma} &= \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\right) \\ B^{\alpha\beta\ \rho\sigma} &= -2\left(\eta^{\alpha\rho}\eta^{\beta\sigma}+\eta^{\beta\rho}\eta^{\alpha\sigma}\right) \end{aligned} \tag{62}$$

The propagator for $s_{\rho\sigma}$ is now,

$$\langle 0|\widehat{T}\ s_{\rho\sigma}\ s_{\alpha\beta}\ |0\rangle = \frac{-i}{q^2+i\epsilon}B_{\rho\sigma,\alpha\beta} \tag{63}$$

The gauge invariance of amplitudes is controlled by the conserved traceless tensors on the vertices. Hence, we must explicitly ensure that $s^{\mu\nu}$ couples to conserved *and traceless* tensors only. Note that any conserved field $s_{\mu\nu}$ can be made traceless, and maintain conservation, by applying the projection,

$$s_{\mu\nu}\to s_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\nu\partial_\mu}{\partial^2}\right)\eta^{\rho\sigma}s_{\rho\sigma}. \tag{64}$$

Applying this to the energy momentum tensor the appropriate $s^{\mu\nu}$ coupling to a conserved and traceless stress tensor is given by,

$$-\frac{s^{\mu\nu}}{2M_P}\widetilde{T}_{\mu\nu} - \frac{t}{2M_P}T \qquad \text{where} \qquad \widetilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu}T - \frac{\partial_\nu\partial_\mu}{\partial^2}T\right). \tag{65}$$

We now repeat our calculation of the gravitational potential in this gauge. From the exchange of the $s^{\mu\nu}$ field with the momentum space projection operator on the vertices,

$$\widetilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) T \tag{66}$$

we have the amplitude,

$$\frac{1}{2}\left(\frac{1}{2M_P}\right)^2 \frac{-i(i)^2}{q^2+i\epsilon}\widetilde{T}^{\rho\sigma}\left(B_{\rho\sigma\alpha\beta}\right)\widetilde{T}^{\alpha\beta} = \frac{-i}{q^2+i\epsilon}\left(\frac{1}{2M_P}\right)^2\left(2T^{\rho\sigma}T_{\rho\sigma} - \frac{2}{3}TT\right) \tag{67}$$

The exchange of the t field which, *c.f.* eq(61), has a noncanonical, and wrong sign for a scalar. normalization and yields,

$$\frac{-i}{q^2+i\epsilon}(i)^2\frac{1}{3}\left(\frac{1}{2M_P}\right)^2 TT \tag{68}$$

and the sum of the s and t contributions is:

$$\frac{-i}{q^2+i\epsilon}\left(\frac{1}{2M_P}\right)^2\left(2T^{\rho\sigma}T_{\rho\sigma} - TT\right) \tag{69}$$

as obtained previously in the De Donder gauge. The repulsive scalar term, owing to the wrong sign kinetic term of t, is absorbed into the full gauge invariant result, and this reduces back to the De Donder gauge result which yields the Newtonian gravitational potential. We therefore see explicitly that in two different gauges, $w = \frac{1}{2}$ (De Donder) and $w = \frac{1}{4}$ ($\partial_\mu s^{\mu\nu} = 0$ and $\eta_{\mu\nu}s^{\mu\nu} = 0$) the physical results are equivalent. These are the two most interesting cases due to the simplifications detailed above.

From eq.(61) we see that the field t is a ghost, however it is not produced radiatively. If we consider $F(\phi_i) = 0$, then the equation of motion of t is $3\partial^2 t = \frac{2}{M_P}T(\phi_i)$. However, we can always write $T(\phi_i)$ as a divergence of a current (the local scale or Weyl current, $\partial^\mu K_\mu = T$ [13]) and therefore $3\partial_\mu t = \frac{2}{M_P}K_\mu$ and t is coupled in first order to the Weyl current and becomes a "tracker solution," $t = \int^x \frac{2}{3M_P}K_\mu dz^\mu$. There is no radiative wave, however there will be, e.g., cosmological solutions where t describes an expanding or shrinking universe.

However, the ghost field t propagates off shell and will produce a contact interaction. In this $w = \frac{1}{4}$ gauge the non-minimal term now depends only upon the trace field t:

$$\frac{1}{2}\int\sqrt{-g}FR = \int\frac{3}{2M_P}F\partial^2 t \tag{70}$$

Consider the t part of the action,

$$S_t = \frac{1}{2}\int -\frac{3}{2}\partial_\mu t\partial^\mu t + \frac{3F}{2M_P}\partial^2\frac{t}{M_P} - \frac{t}{2M_P}T. \tag{71}$$

We define a normalized field, χ, by,

$$t = Z\chi \qquad \frac{3}{4}Z^2 = \frac{1}{2} \qquad Z = \sqrt{\frac{2}{3}} \tag{72}$$

and,

$$S_\chi = \int \frac{1}{2}\chi\partial^2\chi + \sqrt{\frac{3}{2}}\frac{F}{M_P}\partial^2\frac{\chi}{M_P} - \frac{\chi}{2M_P}\sqrt{\frac{2}{3}}T \tag{73}$$

We see we essentially have the toy model action of eq(4) but with the ghost sign for the χ kinetic term. We can solve by shifting χ:

$$\chi = \chi' - \sqrt{\frac{3}{2}}\frac{F}{M_P} + \frac{1}{\partial^2}\sqrt{\frac{2}{3}}\frac{1}{2M_P}T \tag{74}$$

to obtain the contact terms,

$$S \to \int -\frac{1}{2}\partial\chi'\partial\chi' + \frac{3}{4M_P}F\partial^2F - \frac{F}{2M_P}T - \frac{1}{6M_P^2}T\frac{1}{\partial^2}T. \tag{75}$$

where the large distance piece was computed above and combines with the s exchange to give the usual Newtonian potential. Restoring the original normalization the effective action is therefore:

$$S = \int -\frac{3}{2}\partial t\partial t + \frac{1}{4}\partial s^{\mu\nu}\partial s_{\mu\nu} - \frac{3F\partial^2F}{4M_P^2} + \frac{FT}{2M_P^2} + L(s,\phi_i) - \frac{t}{2M_P}T(\phi_i) - \frac{s^{\mu\nu}}{2M_P}\widetilde{T}_{\mu\nu} \tag{76}$$

The contact terms are the same as those found in eq(43) in the De Donder gauge, demonstrating their gauge invariance as expected from the Weyl transformation structure.

V. CONCLUSIONS

We have provided some insight into the physical meaning and equivalence of actions related by a Weyl transformation. Our analysis confirms that contact term effects are operant and that Weyl equivalent representations with non-minimal terms yield explicitly equivalent physics to a pure minimal Einstein-Hilbert form.

The Weyl transformation to the minimal Einstein-Hilbert form is, in a sense, inevitable. If one didn't know about the Weyl transformation one would discover it in the induced contact terms in the single graviton exchange potential involving non-minimal couplings. However the Weyl transformation is more powerful as it is fully non-perturbative. Technically it provides a powerful check on the normalization and implementation of the graviton propagators in various gauges, which can otherwise be somewhat confusing.

The non-minimal form of the action is incomplete without including the contact terms into the action. The theory then becomes identical to the Weyl transformed form with a pure minimal

Einstein-Hilbert action. This implies that there are pitfalls in directly interpreting the physics in the non-minimal form since the contact terms must be included.

The minimal Einstein-Hilbert action is special and does not generate these contact terms. In a sense, by going to the minimal Einstein-Hilbert form we are diagonalizing the graviton derivative terms throught the action. Our analysis required an Einstein-Hilbert term with a Planck mass and we expand perturbatively in inverse powers of M_P^2. A Weyl invariant theory, where $M_P = 0$, is nonperturbative and our analysis is then inapplicable. Indeed, there is no conventional gravity in this limit since the graviton kinetic term does not then exist. In this sense we view the formation of the Planck mass by, e.g., inertial symmetry breaking, as a dynamical phase transition, similar to a disorder-order phase transition in a material medium [13].

As an exploration of the gauge invariance of our result we have shown explicitly that, instead of the $w = 1/2$, De Donder gauge, we can use the $w = 1/4$, $\partial_\mu s^{\mu\nu} = 0$ gauge employing a traceless $s_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h$ metric together with a separate trace field, t. A gauge transformation exists that takes arbitrary s and t to the $\partial_\mu s^{\mu\nu} = 0$ gauge. Then we find that t exclusively controls the non-minimal term and the contact interations. t has a wrong sign (ghost) kinetic term, however it is not produced as a propagating, on shell gravitational wave. It nonetheless appears virtually and, together with $s_{\mu\nu}$, produces the Newtonian potential and the equivalent contact terms as obtained in De Donder gauge.

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