BETATRON FREQUENCIES AND THE POINCARÉ ROTATION NUMBER

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0. Danilov theorem

Consider a symplectic map of the plane (corresponding to a one-turn map of an accelerator), $\mathrm{M} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$(q', p') = \mathrm{M}\,(q, p),$$

where the prime symbols ($'$) indicate the transformed phase space coordinates. Let R_n be the rotation angle in the phase space (q, p) around a stable fixed point between two consecutive iterations (q_n, p_n) and (q_{n+1}, p_{n+1}). Then, the limit, when it exists,

$$\nu = \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{n=0}^{N} R_n$$

is called the **rotation number** (the **betatron frequency** of the one-turn map) for that particular orbit of the map M.

Theorem (Danilov theorem)

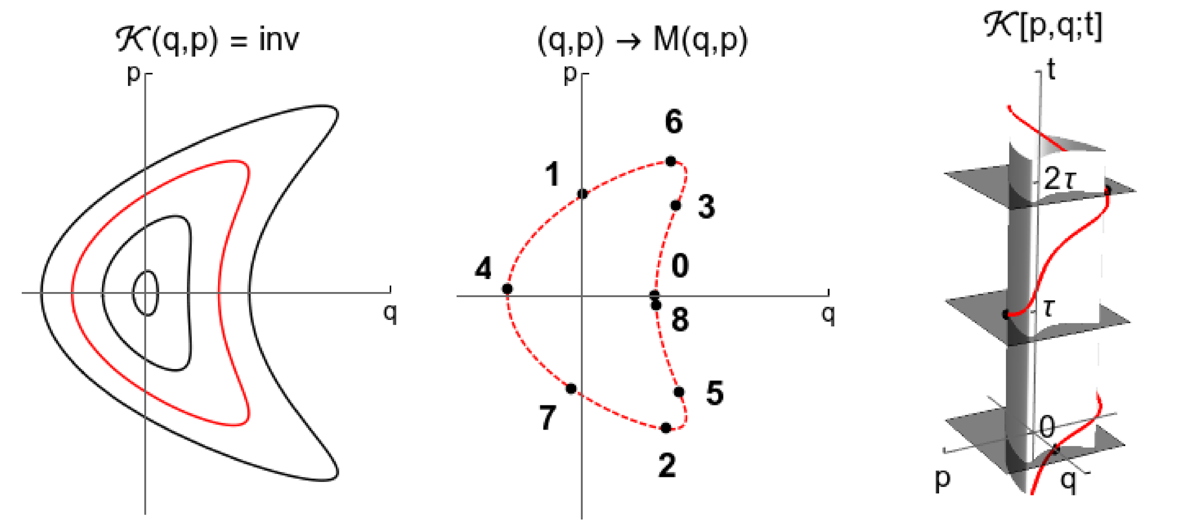
Suppose a symplectic map of the plane,

$$(q', p') = \mathrm{M}\,(q, p),$$

is integrable with the invariant (integral) $\mathcal{K}(q, p)$, then its Poincaré rotation number is

$$\nu(\mathcal{K}) = \int_q^{q'} \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q \Bigg/ \oint \left(\frac{\partial \mathcal{K}}{\partial p}\right)^{-1} \mathrm{d}q,$$

where the integrals are evaluated along the invariant curve.



Corollary (1)

$$\nu(\mathcal{K}) = \frac{\mathrm{d}J'}{\mathrm{d}J} \qquad \text{where} \qquad J'(\mathcal{K}) = \frac{1}{2\pi} \int_q^{q'} p(\mathcal{K}, q)\,\mathrm{d}q,$$

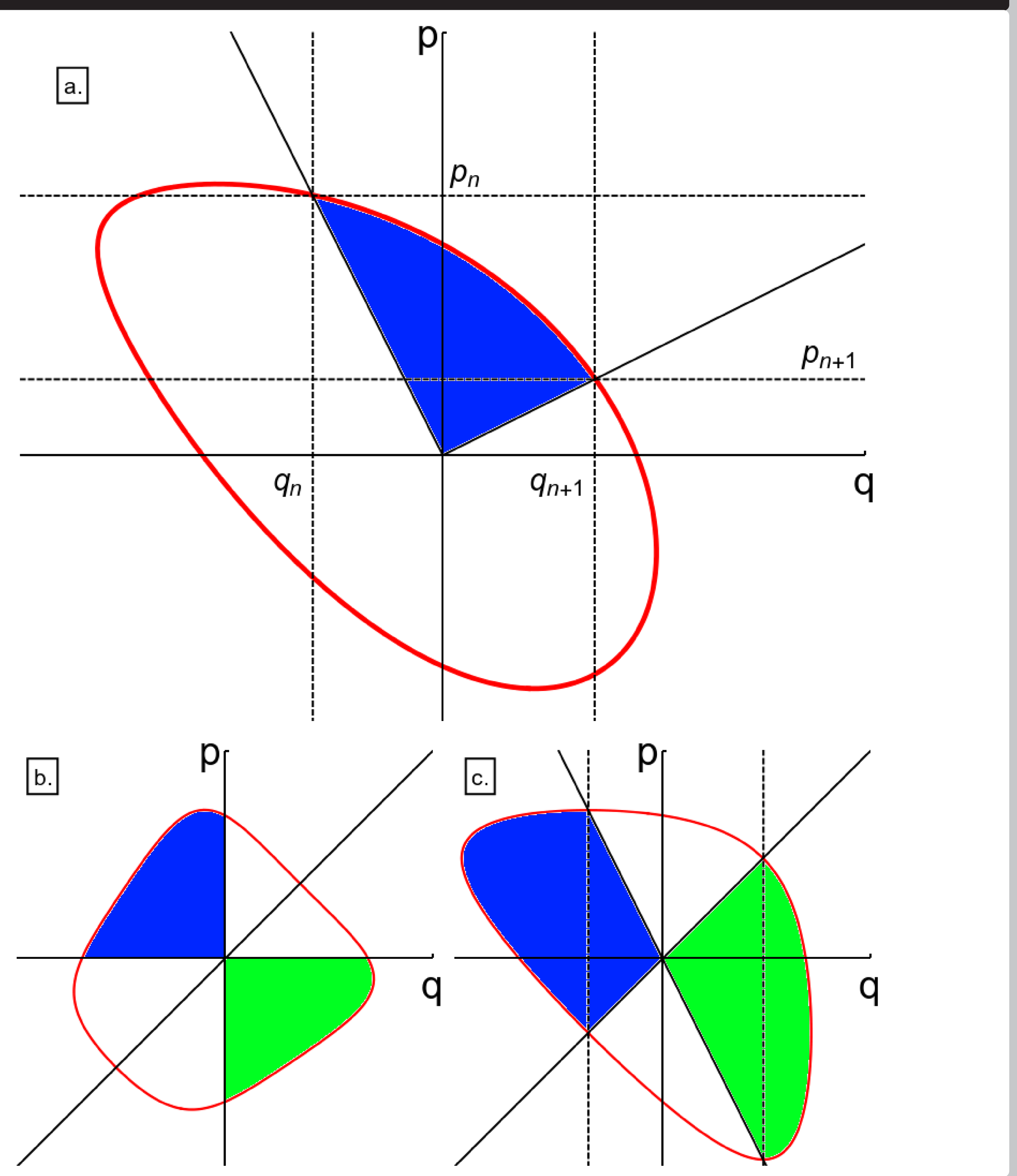
is the partial action calculated as a sector integral around the stable fixed point. For a linear map ($\nu = \mathrm{const}$),

$$\nu = J'/J.$$

Corollary (2)

The Hamiltonian function corresponding to the map M is

$$\mathrm{H}(\mathcal{K}) = J'(\mathcal{K}).$$



1. Nonlinear Integrable Map With Polygon Invariants

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} p \\ -q + p + \frac{|p-1|-|p+1|}{2} \end{bmatrix}.$$

The map has polygon invariants: squares for $|q, p| < 1$ and hexagons otherwise. When $|q, p| < 1$ the map is linear and betatron frequency is

$$\nu = \frac{1}{4}.$$

The rotation number in nonlinear layer can be found using

$$S_{total} = (2\,\alpha + 2)^2 - \alpha^2 = 3\,\alpha^2 + 8\,\alpha + 4$$
$$S_{1/4} = (\alpha + 1)^2 - \tfrac{1}{2}\alpha^2 = \tfrac{1}{2}\alpha^2 + 2\,\alpha + 1$$

and then

$$\nu = \frac{\mathrm{d}J'}{\mathrm{d}J} = \frac{\mathrm{d}S_{1/4}/\mathrm{d}\alpha}{\mathrm{d}S_{total}/\mathrm{d}\alpha} = \boxed{\frac{2 + \alpha}{8 + 6\,\alpha}}.$$

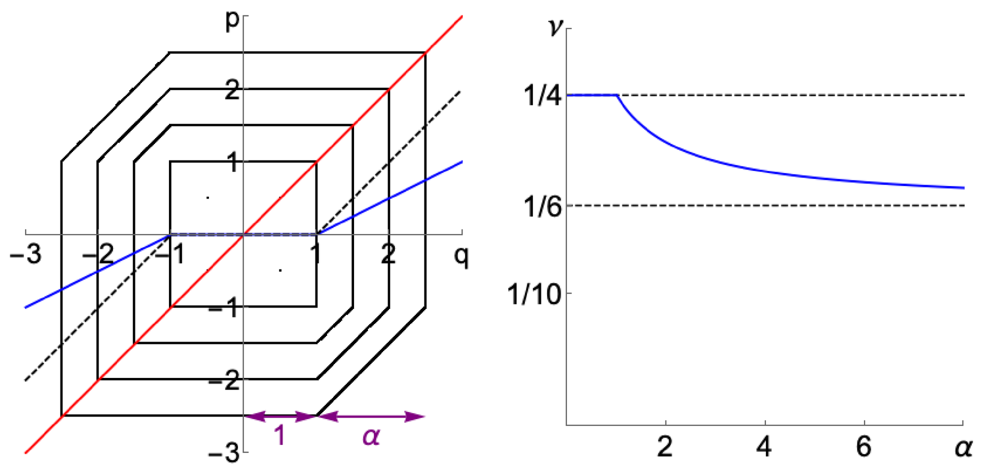


Figure 1: The left plot contains iterations of the map (black dots). The blue and the red lines are the symmetry lines for the map, $p = f(q)/2$ and $p = q$ respectively. The black dashed line shows the force function $f(q) = p + \frac{|p-1|-|p+1|}{2}$. Right plot is the rotation number as a function of α, where α is the invariant of motion defined through initial conditions on the invariant curve as $(q_0, p_0) = (\alpha, 0)$.

2. McMillan Map

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} p \\ -q + a\,p/(b\,p^2 + 1) \end{bmatrix}.$$

McMillan map has the following integral:

$$\mathcal{K}[q, p] = b\,q^2 p^2 + q^2 + p^2 - a\,q\,p,$$

which is non-negative for $b > 0$ and $|a| < 2$. The rotation number can be expressed through Jacobi elliptic functions as follows:

$$\nu(\mathcal{K}) = \frac{1}{4\,\mathrm{K}(w)} \mathrm{arcds}\left(\left(d(\mathcal{K})^2 + 4\,\mathcal{K}\,b\right)^{-1/4}, w\right),$$

where $\mathrm{K}(w)$ is the complete elliptic integral of the first kind and the inverse Jacobi function, $\mathrm{arcds}(x, w)$, is defined as follows

$$\mathrm{arcds}(x, w) = \int_x^{\infty} \frac{\mathrm{d}t}{\sqrt{(t^2 + w^2)(t^2 + w^2 - 1)}}.$$

$$2\,w^2(\mathcal{K}) = 1 + d(\mathcal{K})/\sqrt{d(\mathcal{K})^2 + 4\,\mathcal{K}\,b},$$

where $d(\mathcal{K}) = a^2/4 + \mathcal{K}\,b - 1$.

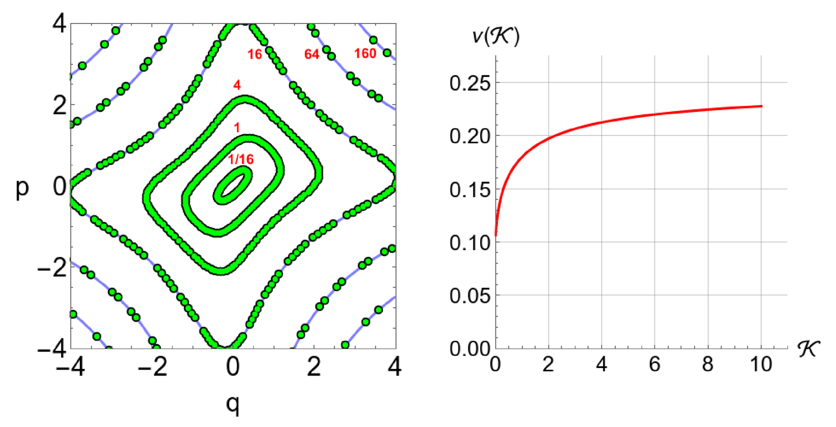


Figure 2: The left plot contains iterations (green dots) of the McMillan map ($a = 1.6$, $b = 1$). Constant level sets of the invariant are shown with blue lines and the corresponding value of the invariant $\mathcal{K}$ is shown in red. Right plot is the rotation number as a function of its integral.

3. Nonintegrable Map (Symplectic Octupole Map)

As our third example, we will consider a non-integrable Hénon cubic map:

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} p \\ -q + a\,p + \epsilon\,p^3 \end{bmatrix}.$$

At small amplitudes this map is linear and the rotation number is $\nu \approx \arccos(a/2)/(2\,\pi)$. At large amplitudes this map becomes chaotic and unstable. Let us propose an approximate integral (the exact integral does not exist since it is a non-integrable map).

$$\mathcal{K}_{\text{approx}}[q, p] = p^2 + q^2 - a\,p\,q - \frac{\epsilon}{a} p^2 q^2 + \frac{7\,\epsilon}{5\,a\,(4 - a^2)} \left(p^2 + q^2 - a\,p\,q\right)^2 + O\left(\epsilon^2\right).$$

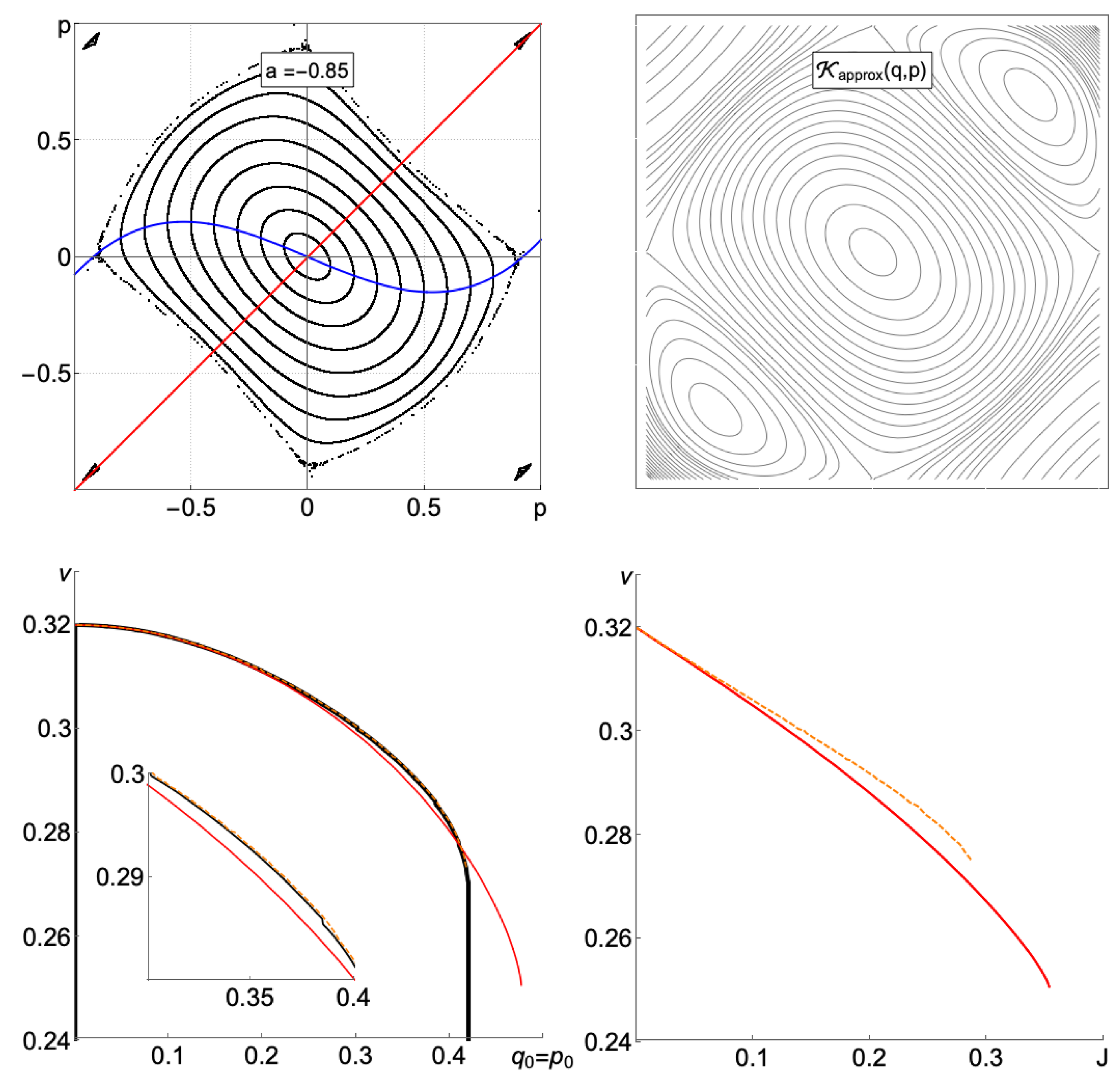


Figure 3: The top row: phase-space trajectories of a cubic map, obtained by tracking with $a = -0.85$ (left plot) and level sets of the approximate invariant (right plot), on the same scale. The red and blue lines in the top left plot corresponds to symmetry lines $p = q$ and $p = (a\,q + \epsilon\,q^3)/2$ respectively. The bottom row: the left plot shows the rotation number as a function of initial conditions in the form $q_0 = p_0$, by using averaging (black solid line), and by using the Danilov theorem numerically (orange dashed) from the tracking data. The red solid line corresponds to the rotation number obtained from the approximate invariant using the Danilov theorem as well. The right bottom plot shows the dependence of ν as a function of action J, from tracking (orange dashed) and from the approximate invariant (red solid).

4. 4D McMillan Map Integrable in polar coordiantes

Consider the 4D map which can be realized in accelerators by employing the electron lens:

$$\begin{bmatrix} x' \\ p_x' \\ y' \\ p_y' \end{bmatrix} = \begin{bmatrix} \alpha_x x + \beta\,p_x \\ -\gamma_x x - \alpha_x\,p_x + \frac{a\,x'}{b\,r'^2+1} \\ \alpha_y y + \beta\,p_y \\ -\gamma_y y - \alpha_y\,p_y + \frac{a\,y'}{b\,r'^2+1} \end{bmatrix} \quad \to \quad \begin{bmatrix} r' \\ p_r' \\ \theta' \\ p_\theta' \end{bmatrix} = \begin{bmatrix} \sqrt{p_r^2 + \frac{p_\theta^2}{r^2}} \\ -p_r \frac{r}{r'} + \frac{a\,r'}{b\,r'^2+1} \\ \theta + \arctan \frac{p_\theta}{r\,p_r} \\ p_\theta \end{bmatrix}$$

with two integrals of motion

$$\mathcal{K}[r, p_r, p_\theta] = b\,r^2 p_r^2 + r^2 - a\,r\,p_r + p_r^2 + \frac{p_\theta^2}{r^2} \qquad \text{and} \qquad p_\theta = x\,p_y - y\,p_x.$$

Using Danilov theorem the radial and angular rotation numbers are

$$\nu_r(\mathcal{K}, p_\theta) = \mathrm{F}\left[\arcsin\sqrt{\frac{\zeta_3 - \zeta_1}{\zeta_3 + 1}}, \kappa\right] \Big/ \left(2\,\mathrm{K}\,(\kappa)\right), \qquad \kappa = \sqrt{\frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_1}}$$

$$\nu_\theta(\mathcal{K}, p_\theta) = \frac{\Delta}{2\,\pi} \left[\nu_r - \frac{\Delta'}{\Delta} + \frac{\arctan\left(\frac{2\,p_\theta}{a}\,\frac{\zeta_3+1}{\zeta_3}\right)}{\Delta}, \right]$$

where

$$\Delta = \frac{2\,p_\theta}{\zeta_3\sqrt{\zeta_3 - \zeta_1}}\,\Pi\left[\kappa \,\middle|\, \frac{\zeta_3 - \zeta_2}{\zeta_3}\right], \qquad \Delta' = \frac{p_\theta}{\zeta_3\sqrt{\zeta_3 - \zeta_1}}\,\Pi\left[\arcsin\sqrt{\frac{\zeta_3 - \zeta_1}{\zeta_3 + 1}}, \kappa \,\middle|\, \frac{\zeta_3 - \zeta_2}{\zeta_3}\right],$$

$\mathrm{K}(\kappa)$ and $\mathrm{F}(\phi, \kappa)$, and, $\Pi(\kappa\,|\alpha)$ and $\Pi(\phi, \kappa\,|\alpha)$ are the complete and incomplete elliptic integrals of the first and the third kinds respectively. $\zeta_1 < 0 < \zeta_2 < \zeta_3$ are the roots of the polynomial

$$\mathcal{P}_3(\zeta) = -\zeta^3 + \left[\mathcal{K} + \left(\frac{a}{2}\right)^2 - 1\right] \zeta^2 + (\mathcal{K} - p_\theta^2)\,\zeta - p_\theta^2.$$

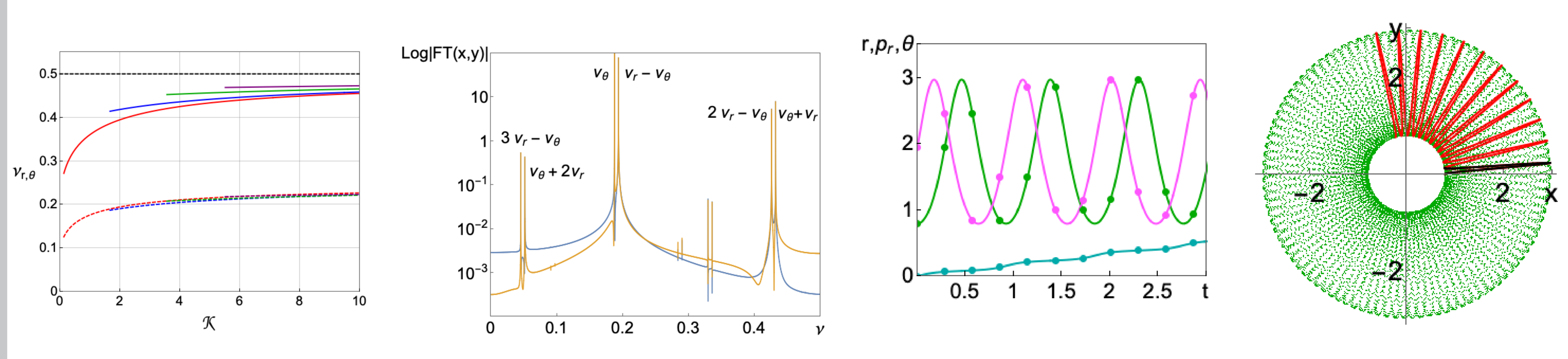


Figure 4: Left plots show the radial ν_r (solid lines) and angular ν_θ (dashed lines) rotation numbers as a function of first invariant $\mathcal{K}$. Different curves correspond to different values of the second invariant $p_\theta = 0, 0.5, 1, 2$ (shown with red, blue, green and purple). The second plot shows the $\log$ of absolute value of Fourier transform for Cartesian coordinates. Two right plots shows the parametrization of map. The first one contains r, p_r and θ as functions of continuous parameter t. When sampled with $t = \int_r^{r'} (\partial p_r/\partial \mathcal{K})\,\mathrm{d}r$ continuous functions correspond to application of map (shown with dots). The second plot shows the continuous x-y trajectory (red line) and iterations of the mapping (green dots). The black part of the red curve represents one radial oscillation. The map parameter is $a = 1.6$.

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