FERMILAB-SLIDES-18-029-T

Stephen Parke, Fermilab 1/31/2018

arXiv:1604.08167v1 [hep-ph] 27 Apr 2016

Compact Perturbative Expressions For Neutrino Oscillations in Matter

Peter B. Denton[a,b] Hisakazu Minakata[c,d] Stephen J. Parke[a]

Addendum to "Compact Perturbative Expressions for Neutrino Oscillations in Matter"

1801.06514

Peter B. Denton,[a] Hisakazu Minakata,[b] Stephen J. Parke[c]

This manuscript has been authored by Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the U.S. Department of Energy, Office of Science, Office of High Energy Physics

doi: 10.5281/zenodo.1163591

Exact Analytic Solution Issue:

- Solve Cubic Characteristic Eqn.

$$\begin{aligned} &\lambda^3 - \left(a + \Delta m^2_{21} + \Delta m^2_{31}\right)\lambda^2 \\ &+[\Delta m^2_{21}\Delta m^2_{31} + a\left\{(c^2_{12} + s^2_{12}s^2_{13})\Delta m^2_{21} + c^2_{13}\Delta m^2_{31}\right\}]\,\lambda \\ &- c^2_{12}c^2_{13}a\Delta m^2_{21}\Delta m^2_{31} \qquad = 0 \end{aligned}$$

IF

- $a = 0$
- or $\Delta m^2_{21} = 0$
- or $\sin\theta_{12} = 0$
- or $\sin\theta_{13} = 0$

THEN characteristic Eqn

FACTORIZES !

BUT

See Zaglauer & Schwarzer, Z. Phys. C 1988

$$\lambda_1 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\left[u + \sqrt{3(1-u^2)}\right],$$

$$\lambda_2 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2-3t}\left[u - \sqrt{3(1-u^2)}\right],$$

$$\lambda_3 = \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2-3t},$$

$$s = \Delta_{21} + \Delta_{31} + a,$$

$$t = \Delta_{21}\Delta_{31} + a[\Delta_{21}(1 - s^2_{12}c^2_{13}) + \Delta_{31}(1 - s^2_{13})],$$

$$u = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c^2_{12}c^2_{13}}{2(s^2-3t)^{3/2}}\right)\right],$$

here $\Delta_{ij} \equiv \Delta m^2_{ij}$

DOES NOT TRIVIALLY SIMPLIFY !

2 flavor mixing in matter

$$ax^2 + bx + c = 0$$

simple, intuitive, useful

3 flavor mixing in matter

$$ax^3 + bx^2 + cx + d = 0$$

complicated, counter intuitive, ...

Neutrino Evolution in Matter:

$$i\frac{d}{dx}\nu = H\nu \quad \text{with} \quad \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$(2E)\,H = U_{PMNS}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31} \end{bmatrix} U^\dagger_{PMNS} + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a = 2\sqrt{2}G_F N_e E$$

$$U_{PMNS} \equiv U_{23}(\theta_{23},0)\, U_{13}(\theta_{13},-\delta)\, U_{12}(\theta_{12},0) :=: U_{23}(\theta_{23},\delta)\, U_{13}(\theta_{13},0)\, U_{12}(\theta_{12},0)$$

$:=:$ means equal after multiplying by a diagonal phase matrix on the left and/or right hand side.

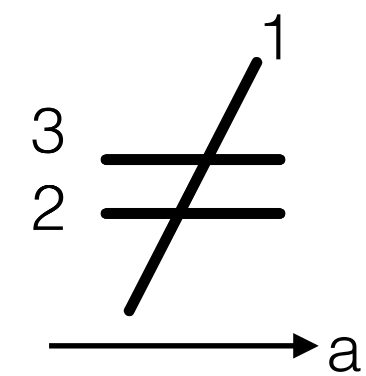
$$i\frac{d}{dx}\,\nu' = \boxed{U^\dagger_{23}(\theta_{23},\delta)\,H\,U_{23}(\theta_{23},\delta)}\,\nu' \qquad \text{with } \nu' = U^\dagger_{23}(\theta_{23},\delta)\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$$

Neutrino Evolution in Matter (conti):

$$U_{23}^{\dagger}(\theta_{23},\delta)\, H\, U_{23}(\theta_{23},\delta) = H_D + H_{OD}$$

D=diagonal   OD= off-diagonal

$$(2E)\, H_D = \begin{bmatrix} a + s_{13}^2 \Delta m_{ee}^2 & & \\ & (c_{12}^2 - s_{12}^2)\Delta m_{21}^2 & \\ & & c_{13}^2 \Delta m_{ee}^2 \end{bmatrix}$$



$$\Delta m_{ee}^2 \equiv \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2$$

$$(2E)\, H_{OD}/\Delta m_{ee}^2 = s_{13}c_{13} \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix}$$

0.15

$$+\, c_{13}\ s_{12}c_{12} \left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix}$$

0.015

$$-\, s_{13}\ s_{12}c_{12} \left(\frac{\Delta m_{21}^2}{\Delta m_{ee}^2}\right) \begin{bmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \end{bmatrix}$$

0.002

Rotation by $U_{13}(\widetilde{\theta}_{13})$

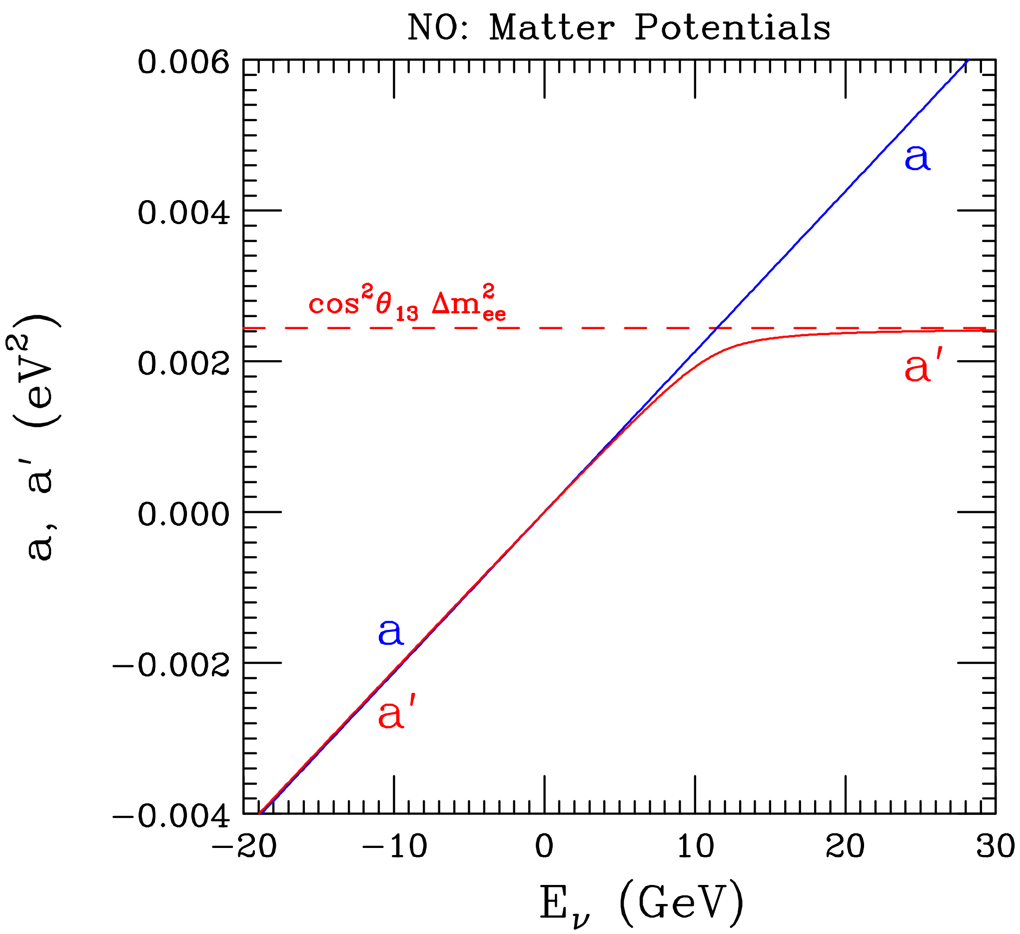
then $U_{12}(\widetilde{\theta}_{12})$

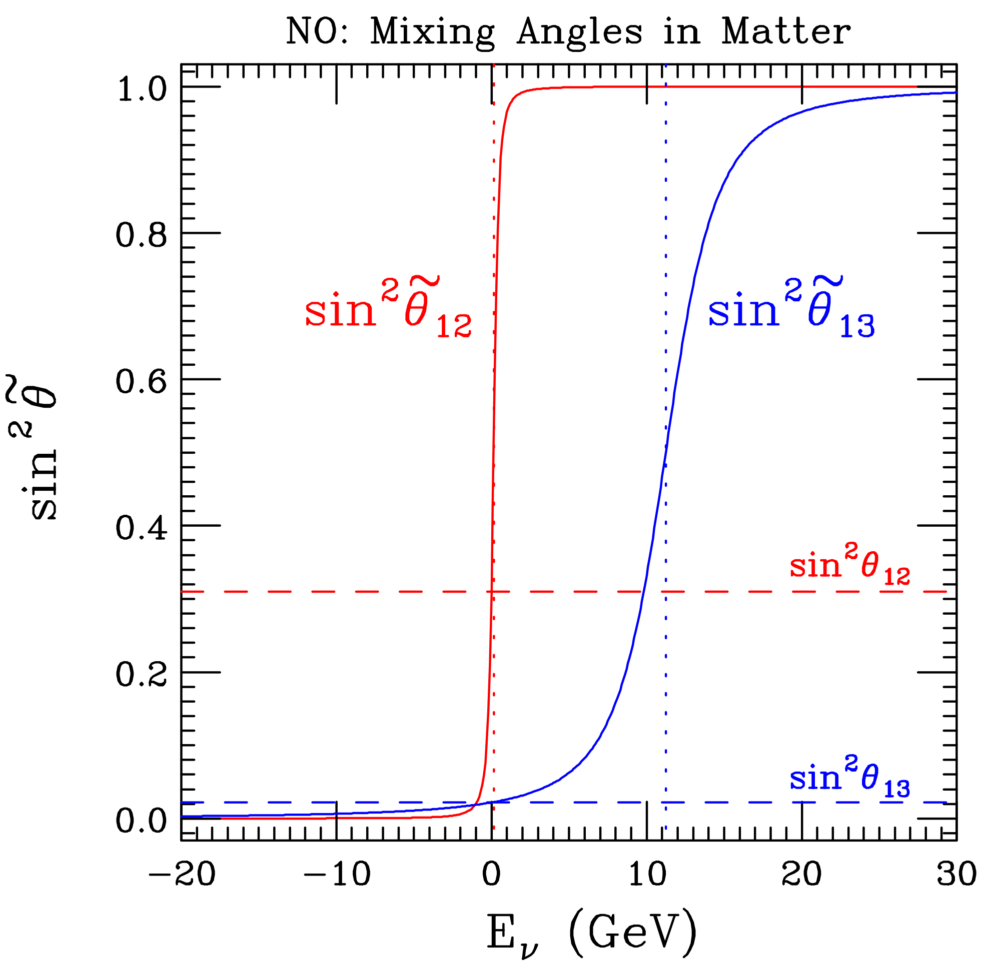
$$a = 2\sqrt{2} G_F N_e E$$

$$\cos 2\widetilde{\theta}_{13} = \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}$$

$$a' \equiv a \cos^2 \widetilde{\theta}_{13} + \Delta m^2_{ee} \sin^2(\widetilde{\theta}_{13} - \theta_{13})$$

$$\cos 2\widetilde{\theta}_{12} = \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12} \underbrace{\cos^2(\widetilde{\theta}_{13} - \theta_{13})}_{\approx 1}}}$$





Masses Squared:

$$(2E)\,H_D = \mathrm{diag}(\,\widetilde{m^2}_1,\ \widetilde{m^2}_2,\ \widetilde{m^2}_3\,)$$

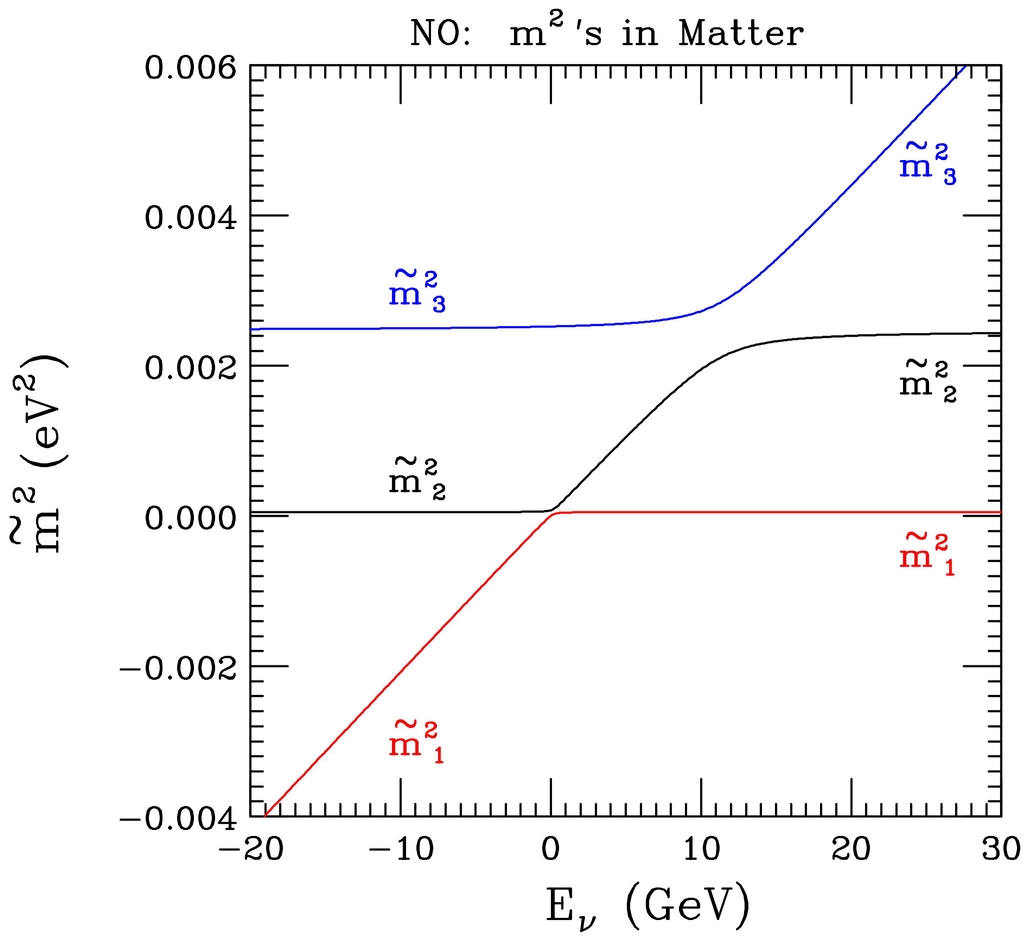
$$\widetilde{m_1^2} = \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a')$$

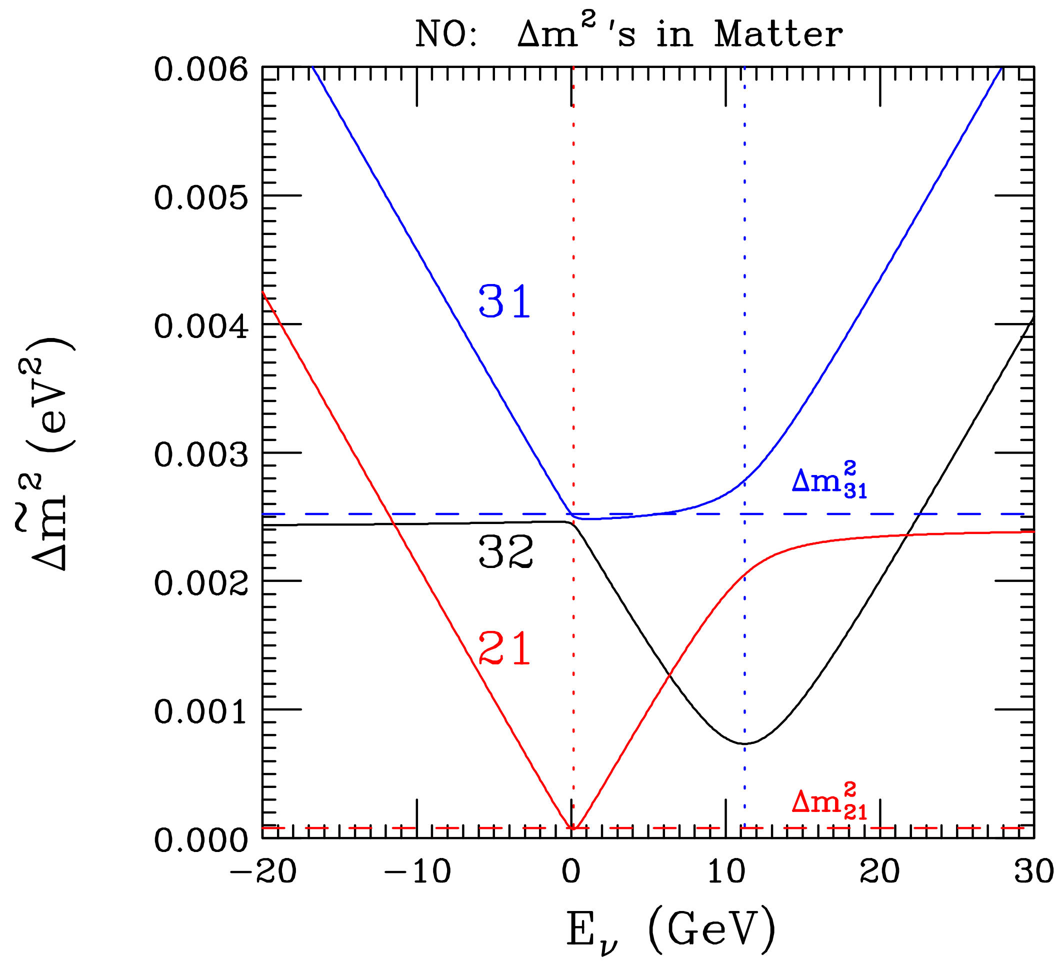
$$\widetilde{m_2^2} = \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a')$$

$$\widetilde{m_3^2} = \Delta m^2_{31} + (a - a')$$

$$\Delta\widetilde{m^2}_{21} = \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})} \approx |\,\Delta m^2_{21}\,\cos 2\theta_{12} - a'\,|$$

when $|a'| \gg \Delta m^2_{21}$





vacuum $\Rightarrow$ matter

$$\Delta m^2_{jk} \rightarrow \Delta \widetilde{m^2}_{jk}$$
$$\theta_{13} \rightarrow \widetilde{\theta}_{13}$$
$$\theta_{12} \rightarrow \widetilde{\theta}_{12}$$
$$\theta_{23} \rightarrow \theta_{23}$$
$$\delta \rightarrow \delta$$

$$P^{vac}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta m^2_{31}, \Delta m^2_{21}, \theta_{13}, \theta_{12}, \theta_{23}, \delta)$$

$$\Rightarrow P^{mat}_{\nu_\alpha \rightarrow \nu_\beta}(\Delta \widetilde{m^2}_{31}, \Delta \widetilde{m^2}_{21}, \widetilde{\theta}_{13}, \widetilde{\theta}_{12}, \theta_{23}, \delta)$$

0th order !

What about H_{OD} ?

$\widetilde{s}_{12} \equiv \sin \widetilde{\theta}_{12}$, etc

$$(2E)\, H_{OD}/\Delta m^2_{ee} \quad = \quad \sin(\widetilde{\theta}_{13} - \theta_{13})\ s_{12}c_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \right) \begin{bmatrix} & & -\widetilde{s}_{12} \\ & & \widetilde{c}_{12} \\ -\widetilde{s}_{12} & \widetilde{c}_{12} & \end{bmatrix}$$

$$\sin(\widetilde{\theta}_{13} - \theta_{13}) \approx s_{13}c_{13} \left(\frac{a}{\Delta m^2_{ee}} \right)$$

4×10^{-4}

for $E = 2$ GeV and $\rho = 3$ g.cm^{-3}

Perturbation Theory !!!

1st order:

- $\Delta \widetilde{m^2}_{jk}$ are unchanged since $[\,H_{OD}\,]_{jj} \equiv 0$

- $U^M_{PMNS} \Rightarrow U^M_{PMNS}(1 + W_1)$

defn: $U^M_{PMNS} \equiv U_{23}(\theta_{23}, \delta)\, U_{13}(\widetilde{\theta}_{13}) U_{12}(\widetilde{\theta}_{12})$

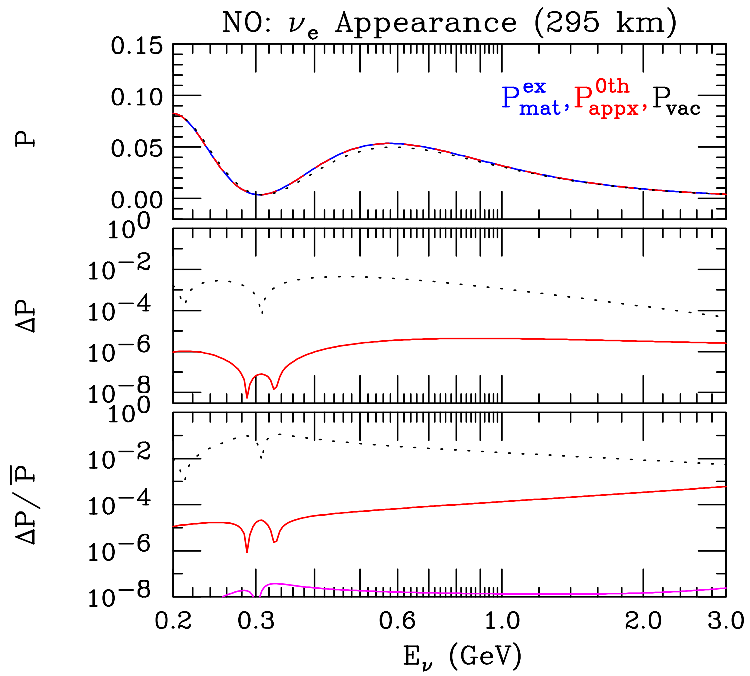
where:

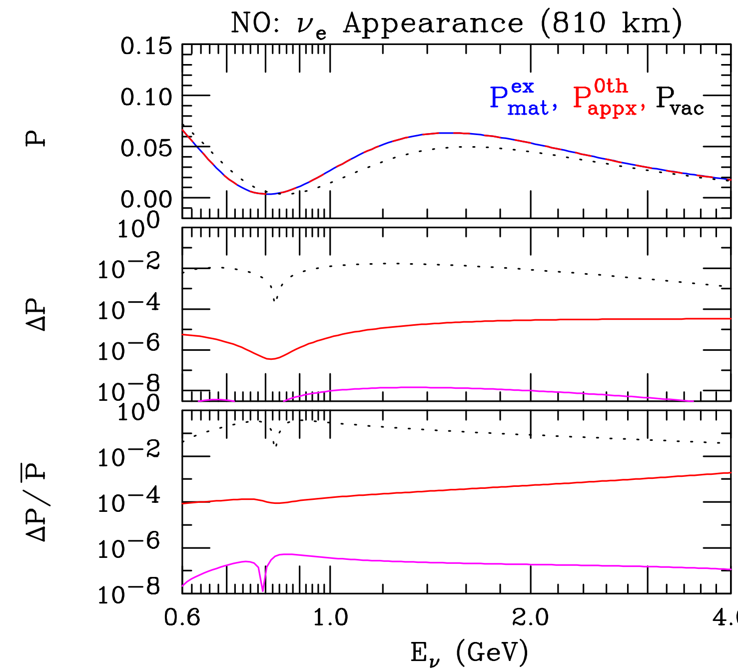
$$W_1 = \sin(\widetilde{\theta}_{13} - \theta_{13})\; s_{12} c_{12}\; \Delta m^2_{21} \begin{pmatrix} 0 & 0 & -\widetilde{s}_{12}/\Delta\,\widetilde{m^2}_{31} \\ 0 & 0 & +\widetilde{c}_{12}/\Delta\,\widetilde{m^2}_{32} \\ +\widetilde{s}_{12}/\Delta\,\widetilde{m^2}_{31} & -\widetilde{c}_{12}/\Delta\,\widetilde{m^2}_{32} & 0 \end{pmatrix}$$

$$0.002 \leq |W_1| \leq \; 0.01$$

2nd order: see paper

T2K/HK





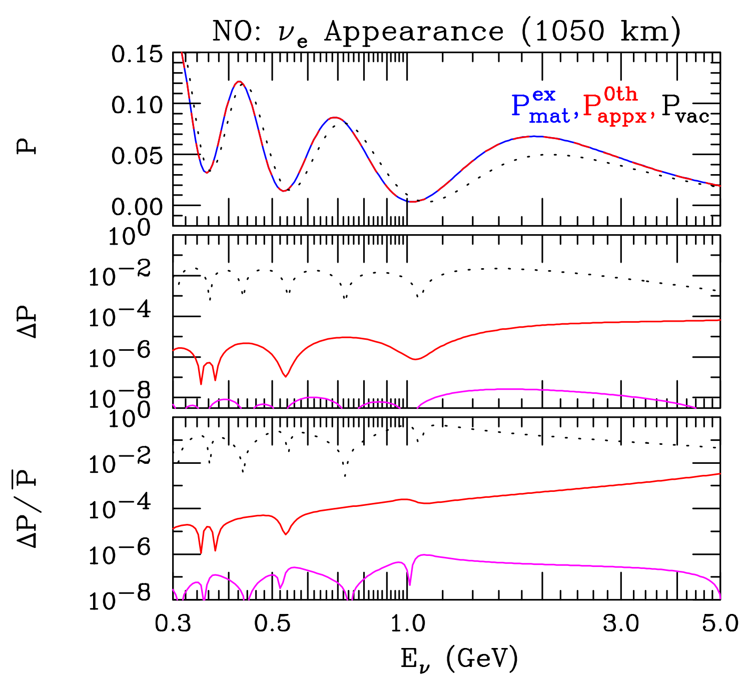
NOvA

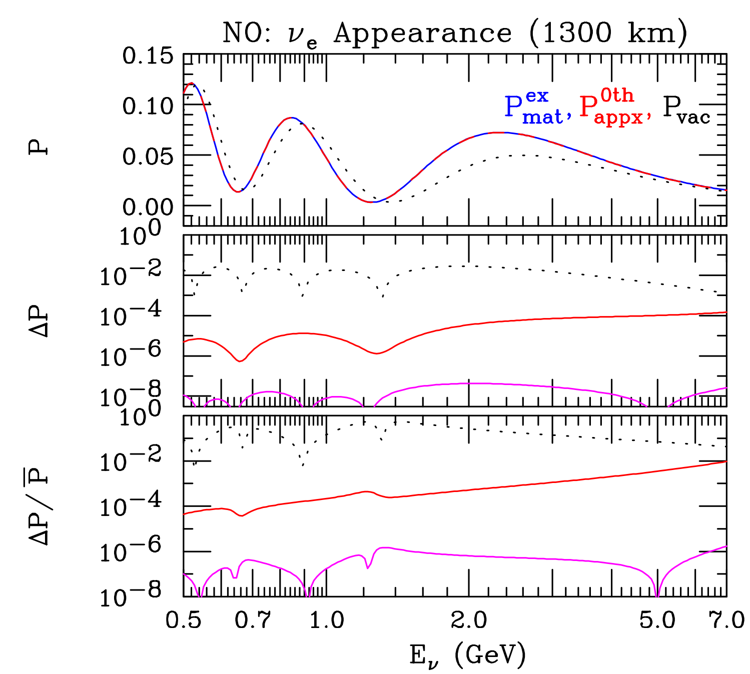
0th order

1st order

2nd order

T2HKK





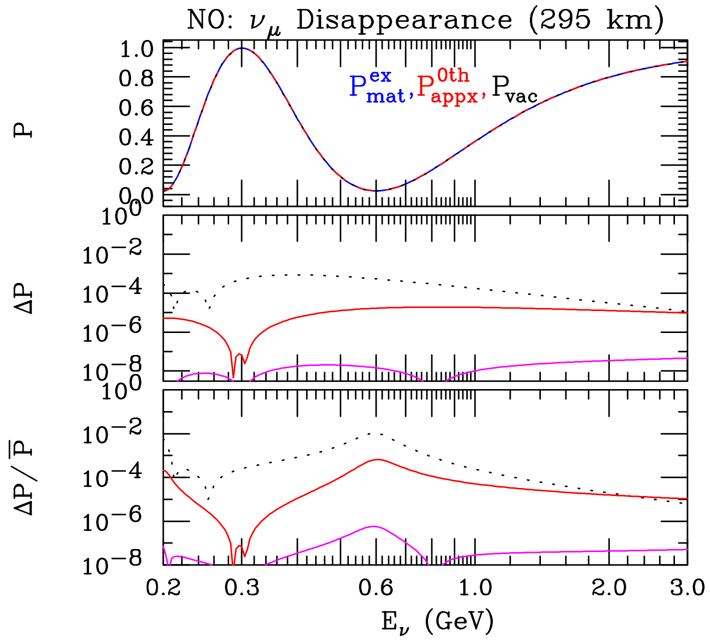
DUNE

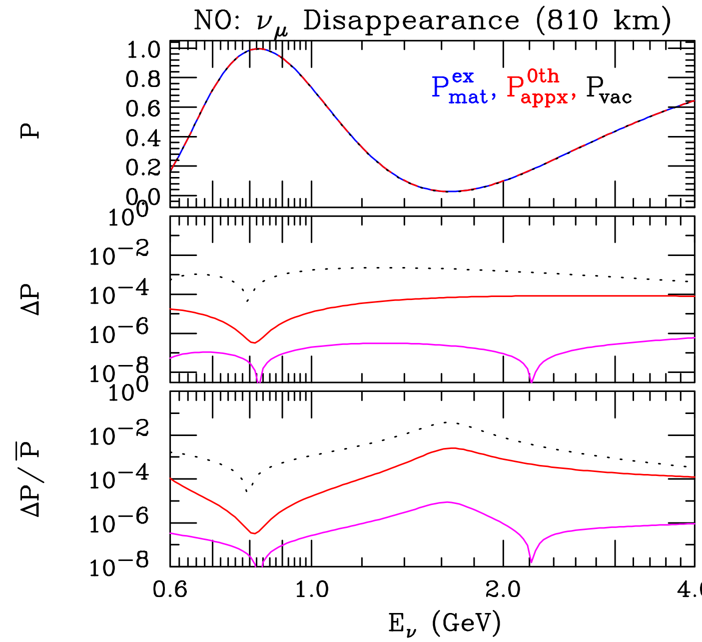
Top panel: P^{ex}_{mat}, P^{0th}_{appx} and P_{vac}

Middle and bottom panels:

black dotted lines	red solid lines	magenta solid lines
$\Delta P = \lvert P^{ex}_{mat} - P_{vac}\rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{0th}_{appx}\rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{1st}_{appx}\rvert$
$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P_{vac})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{0th}_{appx})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{1st}_{appx})$

T2K/HK





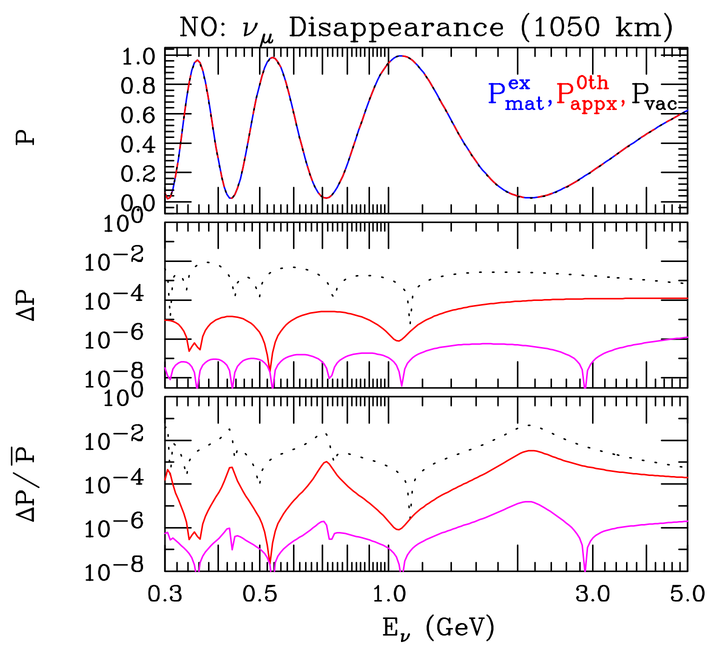
NOvA

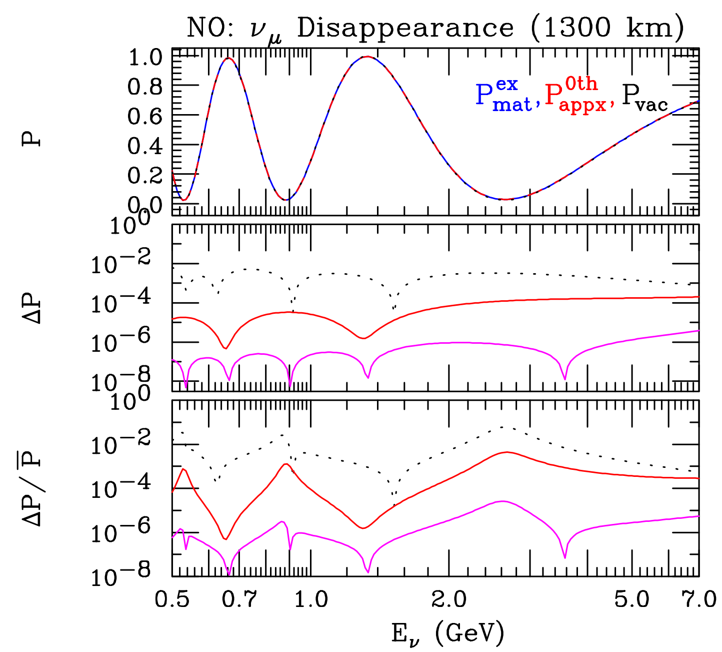
0th order

1st order

2nd order

T2HKK





DUNE

Top panel: P^{ex}_{mat}, P^{0th}_{appx} and P_{vac}

Middle and bottom panels:

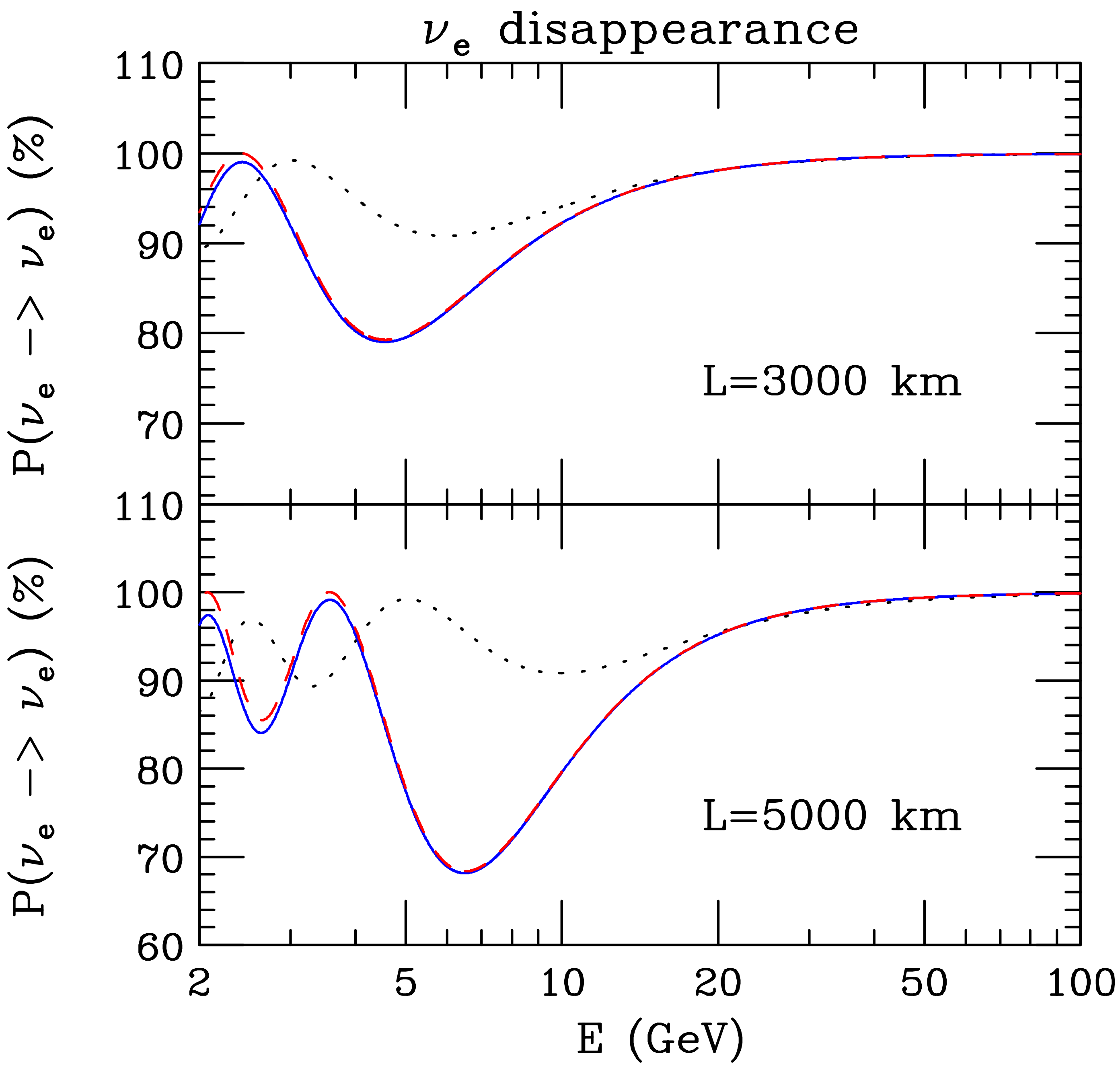
black dotted lines	red solid lines	magenta solid lines
$\Delta P = \lvert P^{ex}_{mat} - P_{vac} \rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{0th}_{appx} \rvert$	$\Delta P = \lvert P^{ex}_{mat} - P^{1st}_{appx} \rvert$
$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P_{vac})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{0th}_{appx})$	$\overline{P} = \frac{1}{2}(P^{ex}_{mat} + P^{1st}_{appx})$

arXiv:1505.01826

ν_e Survival Probability:

$\widetilde{\theta}_{13}$

$$P(\nu_e \to \nu_e) \; = \; 1 - \sin^2 2\widetilde{\theta}_{13} \; \sin^2 \frac{\Delta \widetilde{m^2}_{ee} L}{4E} - ...$$



$$\sin 2\widetilde{\theta}_{13} = \frac{\sin^2 2\theta_{13}}{[(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}]}$$

$$\Delta \widetilde{m^2}_{ee} = \Delta m^2_{ee} \sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}$$

depth of first minimum

$$\sin 2\theta_{13} \quad \to \quad \sin 2\widetilde{\theta}_{13}$$

energy at first minimum

$$\frac{\Delta m^2_{ee} L}{2\pi} \quad \to \quad \frac{\Delta \widetilde{m^2}_{ee} L}{2\pi}$$

exact - approx - vaccum

Summary:

- DMP (1604.08167) gives us a SYSTEMATIC EXPANSION for the oscillation probabilities in matter

- 0th order is VERY SIMPLE and SUFFICIENT for most accelerator experiments

- the expansion parameter is SMALL: $\sin(\widetilde{\theta}_{13} - \theta_{13})\, s_{12} c_{12} \left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \right) \sim 0.0004$ for $E = 2$ GeV and $\rho = 3$ g.cm^{-3} and proportional to ρE

- by construction:
  - the 0th order REPRODUCES vacuum oscillation probabilities exactly
  - at 1st order ONLY the mixing matrix is modified, implying that at 0th order the neutrino masses in matter are very accurate

- at high orders, BOTH neutrino masses in matter and mixing matrix in matter are modified

- this DMP perturbative expansion gives an ENHANCED understanding of oscillation probabilities in matter backup

Translator:

13 sector:

$$\begin{aligned}
\lambda_c - \lambda_a(a=0) &= \Delta m^2_{ee}\cos 2\theta_{13} \\
\lambda_a - \lambda_a(a=0) &= a \\
\lambda_c - \lambda_a &= \Delta m^2_{ee}\cos 2\theta_{13} - a \\
\lambda_+ - \lambda_- &= \Delta m^2_{ee}\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}} \\
eq:(2.3.5) \Rightarrow \quad \cos 2\phi &= \cos 2\widetilde{\theta}_{13} \quad \Rightarrow \phi = \widetilde{\theta}_{13}
\end{aligned}$$

12 sector:

$$\begin{aligned}
\lambda_0 - \lambda_-(a=0) &= \Delta m^2_{21}\cos 2\theta_{12} \\
\lambda_- &= \lambda_a c^2_\phi - 2 s_\phi c_\phi s_{12} c_{12}\Delta m^2_{ee} + \lambda_c s^2_\phi \\
\lambda_- - \lambda_-(a=0) &= a c^2_\phi + \Delta m^2_{ee}\sin^2(\phi - \theta_{13}) = a' \\
\lambda_0 - \lambda_- &= \Delta m^2_{21}\cos 2\theta_{12} - a' \\
\lambda_2 - \lambda_1 &= \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 + \sin^2 2\theta_{12}\cos^2(\phi - \theta_{13})} \\
&\equiv \Delta\widetilde{m^2}_{21} \\
\cos 2\psi &= \cos 2\widetilde{\theta}_{12} \quad \Rightarrow \psi = \widetilde{\theta}_{12} \quad eq:(2.4.9)
\end{aligned}$$

$\widetilde{m^2}_j$:

$$\begin{aligned}
\lambda_+ + \lambda_- = \lambda_a + \lambda_c &= \Delta m^2_{31} + a + s^2_{12}\Delta m^2_{21} \\
\lambda_2 + \lambda_1 = \lambda_0 + \lambda_- &= \Delta m^2_{21} + a' \\
\lambda_2 - \lambda_1 &= \Delta\widetilde{m^2}_{21} \\
\lambda_1 &\equiv \widetilde{m^2}_1 = \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a') \\
\lambda_2 &\equiv \widetilde{m^2}_2 = \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a') \\
\lambda_3 = \lambda_+ &\equiv \widetilde{m^2}_3 = \Delta m^2_{31} + (a - a')
\end{aligned}$$

DMP Summary:

The mixing angles in matter, which we denote by a $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ here, can also be calculated in the following way, using $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, as follows, see Addendum[5]:

$\widetilde{\theta}_{13}$

$$\cos 2\widetilde{\theta}_{13} = \frac{(\cos 2\theta_{13} - a/\Delta m^2_{ee})}{\sqrt{(\cos 2\theta_{13} - a/\Delta m^2_{ee})^2 + \sin^2 2\theta_{13}}}, \tag{6}$$

where $\boxed{a \equiv 2\sqrt{2}G_F N_e E_\nu}$ is the standard matter potential, and

$\widetilde{\theta}_{12}$

$$\cos 2\widetilde{\theta}_{12} = \frac{(\cos 2\theta_{12} - a'/\Delta m^2_{21})}{\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \; + \; \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}}, \tag{7}$$

where $\boxed{a' \equiv a\,\cos^2\widetilde{\theta}_{13} + \Delta m^2_{ee}\sin^2(\widetilde{\theta}_{13} - \theta_{13})}$ is the θ_{13}-modified matter potential for the 1-2 sector. In these two flavor rotations, both $\widetilde{\theta}_{13}$ and $\widetilde{\theta}_{12}$ are in range $[0, \pi/2]$.

θ_{23} and δ are unchanged in matter for this approximation.

From the neutrino mass squared eigenvalues in matter, given by

$$\begin{aligned}
\widetilde{m^2_3} &= \Delta m^2_{31} + (\,a - a'\,), \\
\widetilde{m^2_2} &= \frac{1}{2}(\Delta m^2_{21} + \Delta\widetilde{m^2}_{21} + a'\,), \\
\widetilde{m^2_1} &= \frac{1}{2}(\Delta m^2_{21} - \Delta\widetilde{m^2}_{21} + a'\,),
\end{aligned} \tag{8}$$

it is simple to obtain the neutrino mass squared differences in matter, i.e. the Δm^2_{jk} in matter, which we denote by $\Delta\,\widetilde{m^2}_{jk}$, which are given by

$$\begin{aligned}
\Delta\,\widetilde{m^2}_{21} &= \Delta m^2_{21}\sqrt{(\cos 2\theta_{12} - a'/\Delta m^2_{21})^2 \; + \; \sin^2 2\theta_{12}\cos^2(\widetilde{\theta}_{13} - \theta_{13})}\;, \\
\Delta\,\widetilde{m^2}_{31} &= \Delta m^2_{31} + (\,a - \frac{3}{2}a'\,) + \frac{1}{2}\left(\Delta\widetilde{m^2}_{21} - \Delta m^2_{21}\right), \\
\Delta\,\widetilde{m^2}_{32} &= \Delta\,\widetilde{m^2}_{31} - \Delta\,\widetilde{m^2}_{21}.
\end{aligned} \tag{9}$$