Inertial Symmetry Breaking

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We review and expand upon recent work demonstrating that Weyl invariant theories can be broken “inertially,” which does not depend upon a potential. This can be understood in a general way by the “current algebra” of these theories, independently of specific Lagrangians. Maintaining the exact Weyl invariance in a renormalized quantum theory can be accomplished by renormalization conditions that refer back to the VEV’s of fields in the action. We illustrate the computation of a Weyl invariant Coleman-Weinberg potential that breaks a U(1) symmetry together with scale invariance.

I. INTRODUCTION

The discovery of the Higgs boson, a fundamental, pointlike, scalar field, unaccompanied by a natural custodial symmetry, has led many authors to turn to scale symmetry in search of new organizing principles. Weyl symmetry [1] provides a natural setting for scalars with gravity and may provide a foundational symmetry for physics [2][3][4][5][6][7]. Though Weyl symmetry is technically distinct from scale (diffeomorphism) symmetry (as we discuss in Section II), the two have many features in common, and we will refer casually to “Weyl” and “scale” symmetries interchangeably.

Weyl symmetry, like many candidate symmetries seen in nature, must be broken. Breaking is generally treated spontaneously, implemented via potentials. However, for Weyl symmetry such potentials cannot contain any explicit mass scales. Thus, one cannot sculpt a conventional “Mexican hat” potential since an $m^2 \phi^2$ term is disallowed. However, potentials that can break Weyl symmetry have general properties that are both interesting and restrictive.

For example, if we have a multi-field scale invariant potential, $W(\phi_i)$, then scale invariance implies $\sum \phi_i \delta W/\delta \phi_i = 4W$. It follows that, if the fields develop nontrivial vacuum expectation values (VEV’s), $\langle \phi_i \rangle$, such that $\delta W/\delta \langle \phi_i \rangle = 0$, then necessarily $W(\langle \phi_i \rangle) = 0$, i.e., the cosmological constant is zero at the symmetry breaking minimum. It immediately follows that there must be a flat direction, since if $\langle \phi_i \rangle \rightarrow \lambda \langle \phi_i \rangle$, then $W(\lambda \langle \phi_i \rangle) = \lambda^4 W(\langle \phi_i \rangle) = 0$, hence the potential energy is zero along the flat direction. A dilaton then arises as the Nambu-Goldstone boson of spontaneously broken Weyl symmetry. Such potentials will be considered in Sections IV and V.

Our main goal in the present note, however, is to summarize a new way to break Weyl and other symmetries that does not employ a potential. This mechanism is implicit in many of the approaches taken to spontaneously generating the Planck scale, but it has not to our knowledge been codified prior to ref. [7][8]. The resulting “current algebra” is interesting and powerful, allowing general statements to be made for a wide class of Weyl invariant theories. This is a direct consequence of the structure of the Weyl current and the presence of inflation in the early universe. It unifies the inflationary universe with the formation of the Einstein-Hilbert effective action and the Planck scale itself. It can break other symmetries in the gauge sector of the standard model and its extensions, and can lead to large hierarchies. We call this mechanism “inertial spontaneous symmetry breaking” [8].

Most authors generally construct scale invariant theories, starting in the “Jordan frame,” which is manifestly scale invariant. Typically one then performs a Weyl transformation to the “Einstein frame,” where the conventional mass scales (e.g., Planck mass, cosmological constant, etc.) are introduced as parameters of the transformation. The ensuing inflationary dynamics is usually discussed in this Einstein frame, where the dynamics is most familiar.

But, various questions then arise: Where did the Einstein frame mass parameters, such as $M_{Planck}$, come from? If Weyl symmetry is spontaneously broken what is the “order parameter” of the broken phase? For example, in the standard model there is a clear distinction between the symmetric (Higgs VEV zero) and broken (Higgs VEV nonzero) phases. What is the analogue distinction between the symmetric initial (Jordan frame) theory and the ultimate Einstein frame theory?

Presently we follow ref. [7][8] and observe that, by remaining in the original Jordan frame and treating the full evolution dynamics there, one can observe how the theory evolves and spontaneously breaks scale symmetry. The key ingredient is the Weyl scale symmetry current, $K_{\mu}$.

We find that the expansion of the universe in a pre

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inflationary phase drives the current charge density, $K_0$, to zero. This is much as any conserved current density, such as the density of electric charge, dilutes to zero by general expansion. However, the Weyl current, $K_\mu$, is the derivative of a scalar, $K_\mu = \partial_\mu K$, and in particular $K_0 = \partial_0 K$. We refer to $K$ as the “kernel” of the current. Hence, as the $K_\mu$ current density is diluted away, $K_0 \to 0$, the kernel $K$ therefore evolves to an arbitrary constant; $K \to \bar{K}$ constant. In a Weyl invariant theory this implies that scale symmetry is broken, and a Planck mass is generated that is $\propto \sqrt{\bar{K}}$.

Hence, any random initial values of the fields and their derivatives will inflate to a universe that is then described with a Planck mass, and other mass scales. $K$ plays the role of the symmetry breaking order parameter and directly defines $M^2_{\text{Planck}}$. A potential may then be needed to engineer the final vacuum, and determine the ratios of individual fields $\langle \phi_i \rangle$, yet it plays no direct role in the inertial Weyl symmetry breaking phenomenon.

With a little thought, one can guess the structure of the order parameter $K$. Consider a set of $N$ scalar fields $\{\phi_i\}$. The fields are non-minimally coupled to gravity as $(-1/12) \sum_i \phi_i \partial_i^2 R(g)$. Therefore, if any of the $\phi_i$ should develop a VEV, we would expect scale breaking, and a nonzero $K$. Hence, we expect that the order parameter takes the form, $K \sim c \sum_i \phi_i^2$. However, if any $\phi_i$ has $\alpha_i = 1$, then we can remove it from the action by a local Weyl transformation, absorbing it into the metric. We therefore expect $K = c' \sum_i (1 - \alpha_i) \phi_i^2$. Indeed, we will find that $K_\mu = \partial_\mu K$, with $c' = 1/2$, by a Noether variation of the Jordan frame theory under a Weyl transformation, thus confirming our guess. More elegantly, we can find $K_\mu = \partial_\mu K$ and $D_\mu K^\mu = 0$ directly from the trace of the Einstein equations in the Jordan frame, combined with the Klein-Gordon (KG) equations. We emphasize that $K_\mu$ is conserved “on-shell,” a consequence of both the Einstein and KG equations.

The result that $K \to \bar{K}$ constant as the universe expands implies that $N - 1 \{\phi'_i\}$ will ultimately satisfy the constraint $\bar{K} = (1/2) \sum_i (1 - \alpha_i) \phi_i'^2$. The constrained $\{\phi'_i\}$ lie on an ellipsoid in field space. This leaves one field unconstrained that becomes the dilaton, and it is intimately related to the $K_\mu$ current. If we perform a field redefinition,

$$\phi_i = \exp(\sigma/f)\phi'_i \quad g_{\mu\nu} = \exp(-2\sigma/f)g'_{\mu\nu}. \quad (1)$$

we will find for the Weyl invariant action:

$$S(\phi, g) = S(\phi', g') + \int \sqrt{-g'} \partial_\mu K(\phi') \partial^\mu(\sigma/f) + K(\phi')(\partial\sigma/f)^2 \quad (2)$$

Now using the constraint that $K(\phi') = \bar{K}$ constant, we have:

$$S(\phi, g) = S(\phi', g') + \frac{1}{2} \int \sqrt{-g} (\partial\sigma)^2 \quad (3)$$

Here we identify $f^2 = 2\bar{K}$ so the dilaton is canonically normalized. From this we see that the dilaton, $\sigma$, describes a dilation of the ellipse, and moves in field space orthogonally to the $N - 1 \{\phi'_i\}$ fields. Hence the dilaton decouples from everything except gravity (this holds true for fermions and gauge bosons as well).

We further see that the current written in the unconstrained fields is equivalent to one written in the constrained fields by: $K_\mu = \partial_\mu K(\phi) = \partial_\mu (K(\phi')e^{2\sigma/f})$. Hence in the broken phase (Einstein frame) limit $K(\phi') \to \bar{K}$ constant, $K_\mu \to 2\bar{K}\partial_\mu \sigma/f = f\partial_\mu \sigma$. where $f = \sqrt{2\bar{K}}$. This is as we expect for a Nambu-Goldstone boson, e.g., the axial current of the pion takes the analogous form $f\partial_\mu \pi$.

Why is this formulation important? Results following from the “current algebra” of Weyl invariant theories are general statements that are true, independent of the specific structure of the Lagrangian. The particular structure of $K_\mu$ and $K$ is independent of the form of any scale invariant potential, but the detailed structure of $K$ does depend upon the choice of the non-minimal couplings (and also any higher derivative gravitational terms can modify the simple forms we just discussed).

The behavior of the current algebra will remain intact, since $K_\mu = \partial_\mu K$ is conserved, but e.g., the ellipsoid defined by $\bar{K}$ could become a more general locus.

The survival of the general feature of inertial breaking with a stable groundstate, e.g., a stable $M_{\text{Planck}}$, requires that the quantum theory does not break Weyl symmetry through loops. Here it is important that one does not conflate the procedure of regularization, which generally introduces arbitrary mass scales, with renormalization, which introduces counterterms to define the final theory and its symmetries. Though it is convenient, one need not deploy a regulator that is consistent with the symmetries of a renormalized theory. In fact, such a regulator may not exist, though the symmetries can exist in the final renormalized theory. Furthermore, physics should not depend upon the choice of regulator, and even scalar masses in a theory with a momentum space cutoff can be viewed as technically natural, as emphasized by Bardeen [9]. The nonexistence of a symmetry in the regulator does not imply the nonexistence of the symmetry in the renormalized theory: only anomalies do.

Presently we must confront the meaning of scale anomalies which could seemingly appear in quantum loops and spoil $D_\mu K^\mu = 0$. Scale anomalies occur in an effective theory and reflect the need for the specification of input scales, usually at higher energies. For example, the QCD scale anomaly reflects the need for a definition of $g_3$ at some high energy scale, which can serve as the boundary

\footnote{Even dimensional regularization cannot by itself determine the chiral anomalies of a theory such as the standard model, which are ultimate positioned in the relevant currents by the choice of counterterms.}
condition of the running coupling $g_3(\mu)$. We can view the origin of $\Lambda_{QCD}$ in the low energy world as “predicted” via the solution to the RG (Renormalization Group) equation, given a high energy input parameter. At one-loop order:

$$\frac{\Lambda_{QCD}}{M} = \exp \left( \frac{1}{8\pi b_0 \alpha_3(M)} \right)$$

(4)

where $b_0 = -(11/3 - 2n_f/3)$. Here $\alpha_3(M)$ is the input value of $g_3^2/\alpha_3$ in the QCD coupling defined at the scale $M$, and $\Lambda_{QCD}$ defines the particular RG trajectory of the running $\alpha_3(\mu)$. The low energy scale $\Lambda_{QCD}$ is “inherited” from the implicit high energy scale $M$ by the RG evolution. Here we have explicitly broken scale symmetry by injecting the hard input mass scale, $M$, into the theory.\footnote{One could allow $\Lambda_{QCD}$ to define $\alpha_3(\mu)$ if one is only considering QCD, but this is otherwise a “bottoms up” view of the world in which, e.g., grand unification makes no sense. Here we assume a “top-down” understanding where low energy physics is determined by ultra-high energy dynamics at $M \gg \Lambda_{QCD}$. This implies the notion of “dimensional transmutation,” whereby a dimensionless coupling constant, $\alpha_3(\mu)$, defines a mass-scale $\Lambda_{QCD}$ with no other inputs, is not applicable. $\alpha_3(\mu)$ will always require some high energy boundary condition $\alpha_3(M)$ with implicit scale $M$.}

To maintain the scale symmetry, $M$ must be replaced by something in the action, such as the VEV of some high energy field, $\Phi$, such as the $(24)$ in $SU(5)$, or some other dynamical scale. As we’ll see in our effective potential calculation the most natural choice of $\Phi = \sqrt{K}$, the order parameter. The RG of QCD generates a large hierarchy, but the strong scale injecting the hard input mass scale, $\theta$, of computation the most natural choice of $3$ is $\theta$. The potential is then muted out of nothing: “dimensional inheritance” may be a better moniker for this phenomenon.

This illustrates the general result that in Weyl invariant theories there are no hard input mass scales, and mass is only defined by ratios of VEV’s. The QCD scale is inherited from the scale $\Phi$, and not dimensionally transmitted out of nothing: “dimensional inheritance” may be a better moniker for this phenomenon.

If no additional mass scales, beyond of the dynamical VEV’s of the fields present in the action are introduced into the final renormalized action, then Weyl symmetry can remain intact. This is contrary to the usual method of computation: usually an external mass scale $M$ is introduced when a regularized amplitude, e.g., such as a Coleman-Weinberg potential \cite{10}, is renormalized. This leads directly to the Weyl current anomaly (or scale current trace anomaly) as we discuss in V.A. If instead, one replaces $M$ by $\sqrt{K}$, then the anomaly is cancelled, and the Weyl symmetry is maintained. The potential is then a function of fields only and contains dimensionless ratios of fields as the arguments of logs. We will demonstrate how this works with an explicit calculation of a scale symmetric Coleman-Weinberg potential in V.B. This will also demonstrate the inertial breaking of a “flavor” $U(1)$ symmetry.

We begin with a brief overview of Weyl symmetry in Section II. While we mainly focus on globally Weyl invariant theories, we sketch how the approach can be implemented in both global and local Weyl invariant theories, where the latter introduces Weyl’s “photon” $A_\mu$. We then discuss the phenomenon of inertial symmetry breaking in Sections III and IV. Finally we turn to the effects of quantum mechanics in Section V and conclusions in Section VI.

## II. WEYL SYMMETRY IN A NUTSHELL

Many years ago Hermann Weyl had the idea that, since coordinates are merely numbers invented by humans to account for events in space-time, they should not carry length scale \cite{1}. Rather, the concept of length should be accounted for events in space-time, they should not carry coordinate dierentials are scale free. Therefore, under a local Weyl scale transformation we would have:

$$\sqrt{g} \rightarrow e^{-4\epsilon} \sqrt{g}$$

(5)

Here we’ve included the local transformation of a scalar field which transforms like a mass (length)$^{-1}$. The contravariant metric must transform as (length)$^{-2}$ to preserve the condition: $g_{\mu\nu} g^{\mu\rho} = \delta_\rho^\mu$. Weyl transformations are distinct from coordinate dieromorphisms that define scale transformations on coordinates, as $\delta x^\mu = \epsilon(x) x^\mu$, which we discuss below. The global Weyl symmetry corresponds as usual to $\epsilon = \text{constant in spacetime}$.

It is straightforward to construct a list of local Weyl invariants:

$$\phi^2(x) g_{\mu\nu}(x): \quad \phi^{-2}(x) g^{\mu\nu}(x): \quad \sqrt{-g}(x) \phi^4(x): \quad$$

$$R(\phi^2 g_{\mu\nu}) = \phi^{-2} R(g_{\mu\nu}) + 6 \phi^{-3} D^\rho (\partial_\rho \phi)$$

(6)

$$\sqrt{-g} \phi^4 R(\phi^2 g_{\mu\nu}) = \sqrt{-g} \phi^2 R(g_{\mu\nu}) + 6 \phi D^\rho (\partial_\rho \phi)$$

Note that the computation of $R(\phi^2 g_{\mu\nu})$ above requires that any Christoffel symbols used in the definition of $R$ be evaluated in the metric $\phi^2 g_{\mu\nu}$. Using these identities we can construct an action that is locally Weyl invariant:

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{12} \phi^4 R(\phi^2 g) - \frac{\lambda}{4} \phi^4 \right)$$

$$= \int d^4x \sqrt{-g} \left( \frac{\lambda}{12} \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} \phi^2 R(g) - \frac{\lambda}{4} \phi^4 \right)$$

(7)

where we substituted the relationship of eq.[6] and integrated by parts using the divergence rule $D_\mu V^\mu = \sqrt{-g} \partial_\mu (\sqrt{-g} V^\mu)$. Here we obtain the famous locally
Weyl invariant theory in which the nonminimal coupling of scalars to gravity is fixed by the coefficient 1/12, needed to canonically normalize the φ kinetic term. This is a special and somewhat degenerate theory, since we can revert to the metric \( \tilde{g}_{\mu\nu} = g^2 g_{\mu\nu} \) and φ disappears from the action. The theory has a vanishing Weyl current [11].

We note that covariant gauge fields, such as the electromagnetic vector potential, \( A_\mu \), do not transform under the local Weyl transformation, since they are associated with derivatives \( \partial_\mu \rightarrow \partial_\mu - i e A_\mu \), which, like coordinates, do not transform. The electromagnetic fields that have the usual engineering scale \( \sim (\text{mass})^2 \), \( \tilde{E} \) and \( \tilde{B} \), are contained in the field strength with one covariant and one contravariant index, \( F_\mu^\nu \), e.g., \( \tilde{E}_\mu = F^0_\mu \).

We can construct a covariant derivative of a scalar field under local Weyl transformations by introducing the “Weyl photon,” \( \tilde{A}_\mu \), as

\[
\tilde{D}_\mu \phi = \partial_\mu \phi - \tilde{A}_\mu \phi
\]

where \( \phi(x) \rightarrow e^{i \epsilon(x)} \phi(x) \) and \( \tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) + \partial_\mu \epsilon(x) \) (note the major difference from electrodynamics in the absence of a factor of \( i \) in the coefficient of \( \tilde{A}_\mu \)). Armed with this we can construct another local Weyl invariant:

\[
\sqrt{-g} g^{\mu\nu} \tilde{D}_\mu \phi(x) \tilde{D}_\nu \phi(x).
\]

This is a locally invariant kinetic term. We can combine it with the previous invariants to define an action in which the Weyl symmetry is local, yet the nonminimal coupling of scalars to \( R \) is arbitrary:

\[
S = \int d^4x \sqrt{-g} \left( \frac{1}{2} (1 - \alpha) g^{\mu\nu} \tilde{D}_\mu \phi \tilde{D}_\nu \phi - \frac{\alpha}{12} \phi^4 \right)
\]

\[
= \int d^4x \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \alpha \frac{\phi^2 R(g)}{12} - \frac{\lambda}{4} \phi^4 \right)
\]

From the action of eq.(10) we see that the Weyl current is:

\[
K_\mu = -\frac{1}{\sqrt{-g}} \delta S \delta A^\mu = (1 - \alpha) \left( \phi \partial_\mu \phi - \tilde{A}_\mu \phi^2 \right)
\]

\[
= (1 - \alpha) \phi \tilde{D}_\mu \phi.
\]

By setting \( \tilde{A}_\mu = 0 \) we obtain a globally invariant theory, and this current becomes the conserved Noether current for the global Weyl invariant theory:

\[
K_\mu = (1 - \alpha) \phi \partial_\mu \phi.
\]

As stated in the introduction, we then have:

\[
K_\mu = \partial_\mu K; \quad K = \frac{1}{2} (1 - \alpha) \phi^2
\]

The structure and conservation law of \( K_\mu \) also follows directly from the Einstein and Klein-Gordon equations in the Jordan frame [2], which we demonstrate in Section III.

We further remark that, in the local Weyl photon theory, we can include a kinetic term: \(-1/(4g^2) F_\mu^\nu \tilde{F}^{\mu\nu} \) where \( F_\mu^\nu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \). When the theory inertially breaks, the dilaton will be eaten by \( \tilde{A}_\mu \), giving a massive Weyl photon, of mass \( 2g\sqrt{R} \).

How does scale symmetry differ from Weyl symmetry? Scale symmetry is generally codified as a diffeomorphism. In a flat spacetime, where we hold the metric fixed \( g_{\mu\nu} = \eta_{\mu\nu} \), the coordinates and fields transform under a local Noether variation \( \epsilon(x) \) as:

\[
\delta x^\mu = \epsilon(x) x^\mu
\]

\[
\delta \phi(x) = -\epsilon(x) \phi(x) + \epsilon(x) x^\mu \partial_\mu \phi(x).
\]

If a theory with action \( S \) is scale invariant, then we find \( J^\mu = -\delta S / \delta \partial_\mu \epsilon = \epsilon x^\mu \partial_\mu \) is the scale current, where \( T^\mu_\nu \) is the canonical stress tensor. Taking the divergence we immediately have \( \partial_\mu J^\mu = T^\mu_\mu \), which is the trace of stress tensor.

However, there are traditional difficulties in defining the scale current in the case of a formally massless scalar field, \( \phi \). Even in a scale invariant classical theory we obtain \( T^\mu_\mu \neq 0 \), unless we construct the “improved stress tensor,” \( T^\mu_\nu \) of Callan-Coleman-Jackiw (CCJ) [12]. The improved stress tensor can be obtained by allowing an arbitrary metric, \( g_{\mu\nu} \), with the addition of a nonminimal coupling of \( \phi \) to curvature, \(-1/(12) \phi^2 R \). We then compute \( T^{\mu\nu} \) by variation of the action wrt \( g_{\mu\nu} \), and then impose the flat space limit, \( g_{\mu\nu} = \eta_{\mu\nu} \). The result is the improved stress tensor of CCJ, and corresponds to the rhs of our eq.(20) below, with \( \alpha = 1 \):

\[
T^{\mu\nu}_\mu = \frac{2}{3} ( \partial_\mu \phi \partial_\nu \phi - \frac{1}{6} (g_{\mu\nu} \partial_\lambda \phi \partial_\lambda \phi)
\]

\[
+ \frac{1}{2} (g_{\mu\nu} \phi \partial^2 \phi - \phi \partial_\mu \partial_\nu \phi) + g_{\mu\nu} V(\phi).
\]

With the scale current \( J_\mu = x^\nu T^{\mu\nu}_\mu \), the trace is the rhs of eq.(24) below, with \( \alpha = 1 \):

\[
T^{\mu\mu}_\mu = \phi D^2 \phi + 4V(\phi) = -\phi \delta V / \delta \phi + 4V(\phi).
\]

For a scale invariant theory, \( V(\phi) \propto \phi^4 \), hence \( \phi \delta V / \delta \phi = 4V \). The trace then vanishes by the Klein-Gordon equation, eq.(22) below (see also [13]).

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3 Here there is a subtlety. We must define the derivative of any conformal field as a commutator: \( [D_\mu, \Phi] = \partial_\mu \Phi - A_\mu [W, \Phi] \) where \( [W, \phi] = w \phi \) and \( w \) is the conformal charge of \( \Phi \). Hence \( w = 1 \) for \( \phi \). We also require \( w = -2 \) for \( g_{\mu\nu} \), \( w = -4 \) for \( \det g \), etc. Note that \( [D_\mu, g_{\mu\nu}] = D_\mu g_{\rho\sigma} + 2 \tilde{A}_\mu g_{\rho\sigma} = \tilde{A}_\mu g_{\rho\sigma} \) since \( D_\mu g_{\rho\sigma} = 0 \). This insures the invariance of the action with the Weyl covariant derivation under integration by parts. Note that we can alternatively define a restricted “pure gauge theory” with \( A_\mu = \partial_\mu \ln(\chi) \), where \( \chi \) is any massless scalar field.
Hence, the improved stress tensor is that of a Weyl invariant theory in the flat space limit. Not surprisingly, the trace is now identically the divergence of the Weyl current. If Weyl (scale) symmetry is conserved (broken) then scale (Weyl) symmetry will be conserved (broken). For practical purposes, Weyl symmetry of a theory in $D = 4$ contains as much information as the diffeomorphism scale symmetry. The Weyl symmetry is actually more convenient to implement than scale diffeomorphisms since it does not involve shifting coordinate arguments of fields.

### III. TALE OF TWO ACTIONS

Let us now see how the Weyl current emerges and plays a central role by a simple example. Consider a real scalar field theory action together with Einstein gravity and a cosmological constant (our metric signature convention is $(1,-1,-1,-1)$):

$$S = \int \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \Lambda + \frac{1}{2} M_P^2 R \right). \quad (17)$$

This action provides a caricature of the cosmological world we live in.

We imagine an initial, ultra-high-temperature phase in which the massless scalar $\sigma$ has the dominant energy density, $\rho_{\sigma} \propto T^4$. Consider a Friedman-Robertson-Walker (FRW) metric:

$$g_{\mu\nu} = \begin{bmatrix} 1, -a^2(t), -a^2(t), -a^2(t) \end{bmatrix} \quad H = \frac{\dot{a}}{a}. \quad (18)$$

In this theory the universe initially expands in a FRW phase, with the temperature red-shifting as $T \sim 1/a(t)$, and the scale factor growing as $a(t) \sim \sqrt{t}$. Eventually the $\sigma$ thermal energy becomes smaller than the cosmological constant, $\rho_{\sigma} < \Lambda$, and we then enter a deSitter phase with exponential growth, $a(t) \sim e^{\sqrt{\Lambda/3M_P^2}}$. We can model the thermal phase as a pre-inflationary era, and the cosmological constant then represents a potential energy that drives inflation. In any case, the intuition that allows us to readily understand how this works is well-honed.

Now consider a different action:

$$S = \int \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\lambda}{4} \phi^4 - \frac{\alpha}{12} \phi^2 R \right). \quad (19)$$

This action is scale invariant, having no cosmological constant or Planck scale. In fact, this is precisely the action of eq. (10) with $\dot{A}_\mu = 0$ and is globally Weyl invariant.

It turns out, as we see below, that these two theories are mathematically equivalent, provided $\alpha < 1$. Our question then becomes, given that, from our accumulated experience in inflationary cosmology we understand the dynamics of eq. (17) so well, then how can we easily understand the dynamics of eq. (19)? At first this doesn’t look too hard. Indeed, if $\phi$ starts out in some very high-temperature phase, where the energy density is large compared to $\lambda \phi^4$ then we expect the scale factor will increase in a scale invariant way, $a(t) \sim t$. This follows by intuiting that the Hubble constant satisfies $H^2 \sim T^4/\phi^2$, where the $\phi^2$ factor in the denominator replaces $M_P^2$. In thermal equilibrium we expect $\phi^2 \sim T^2$ and thus $H \sim \frac{2}{3} T \sim \frac{1}{t}$. Therefore, $a(t) \sim t$.

However, as the universe cools, we expect $\phi(x)$ to settle into some spatially constant VEV $\langle \phi \rangle$. Our intuition from conventional $M_P^2 R$ gravity tells us that this VEV will slow-roll in the potential, with $\langle \phi \rangle$ eventually becoming zero. However, in eq. (19) this would imply a vanishing $M_P$, and the details of the solution are less clear. It is plausible that the increasing strength of gravity will increase the Hubble damping, and halt the relaxation of $\langle \phi \rangle$, perhaps leading to a nonzero cosmological constant $\lambda \langle \phi \rangle^4$. If true, this would then match the cosmological constant case of eq. (17), and it would imply a spontaneous breaking of scale symmetry. But how can we see that this happens in a simple and intuitive way, without having to puzzle over the solutions of coupled nonlinear differential equations?

Indeed, from eq. (19) we can directly obtain the Einstein equation:

$$\frac{1}{6} \alpha \phi^2 G_{\alpha\beta} = \left( \frac{3 - \alpha}{3} \right) \partial_\alpha \phi \partial_\beta \phi - g_{\alpha\beta} \left( \frac{3 - 2\alpha}{6} \right) \partial^\mu \phi \partial_\mu \phi + \frac{1}{3} \alpha \left( g_{\alpha\beta} \phi D^2 \phi - \phi D_\beta D_\alpha \phi \right) + g_{\alpha\beta} V(\phi). \quad (20)$$

The trace of the Einstein equation becomes:

$$-\frac{1}{6} \alpha \phi^2 R = (\alpha - 1) \partial^\mu \phi \partial_\mu \phi + \alpha \phi D^2 \phi + 4V(\phi). \quad (21)$$

We also have the Klein-Gordon (KG) equation for $\phi$:

$$0 = \phi D^2 \phi + \phi \frac{\delta}{\delta \phi} V(\phi) + \frac{1}{6} \alpha \phi^2 R. \quad (22)$$

We can combine the KG equation, eq. (22), and trace equation, eq. (21), to eliminate the $\alpha \phi^2 R$ term, and obtain:

$$0 = (1 - \alpha) \phi D^2 \phi + (1 - \alpha) \partial^\mu \phi \partial_\mu \phi + \phi \frac{\delta}{\delta \phi} V(\phi) - 4V(\phi). \quad (23)$$

This can be written as a current divergence equation:

$$D^\mu K_\mu = 4V(\phi) - \phi \frac{\partial}{\partial \phi} V(\phi). \quad (24)$$

where $K_\mu = (1 - \alpha) \phi \partial_\mu \phi$ is the Weyl current as given in eq. (12). For the scale invariant potential, $V(\phi) \propto \phi^6$, the $rhs$ of eq. (24) vanishes and the $K_\mu$ current is then covariantly conserved:

$$D^\mu K_\mu = 0. \quad (25)$$
We see that this is an “on-shell” conservation law, i.e., it assumes that the gravity satisfies eq. (20).

We can now understand the behavior of this theory by the spontaneous breaking of the Weyl symmetry. Starting with an arbitrary classical \( \phi \), after a period of general expansion, in some arbitrary patch of space, \( \phi \) becomes approximately spatially constant, but time dependent, hence, \( D^2 \phi = \phi + 3H \phi \). The Einstein current conservation law of eq. (25) then becomes:

\[
\dot{K} + 3HK = 0.
\] (26)

If we take \( \phi \) to be a function of time \( t \) only, we have by eq. (26)

\[
K(t) = c_1 + c_2 \int_0^t \frac{dt'}{a^3(t')}
\] (27)

where \( c_1 \) and \( c_2 \) are constants which are determined by initial values of \( (\phi, \dot{\phi}) \). Therefore, under arbitrary initial conditions:

\[
K(t \to \infty) \to \overline{K} \text{ constant},
\] (28)

and it follows that:

\[
\phi(t \to \infty) \equiv \phi_0, \text{ hence, } \overline{K} = \frac{1}{2}(1 - \alpha)\phi_0^2.
\] (29)

\( \overline{K} \) is the constant asymptotic value of the order parameter, and defines the broken Weyl symmetry phase.

The Planck mass and cosmological constant are then determined:

\[
M_P^2 = -\frac{\alpha}{3(1 - \alpha)} \overline{K}, \quad \Lambda = \frac{\lambda \overline{K}^2}{(1 - \alpha)^2}.
\] (30)

Then the (00) Einstein equation, with \( G_{00} = -3H^2 \), gives

\[
H^2 = -\frac{\lambda \phi_0^2}{2\alpha} = -\frac{\lambda \overline{K}}{\alpha(1 - \alpha)} = \frac{\Lambda}{3M_P^2}.
\] (31)

This gives eternal inflation if \( \alpha < 1 \) with constant \( K = \overline{K} \), which matches the Einstein frame conclusions.

We have thus understood how the behavior of eq. (19) matches that of eq. (17) by way of the \( \overline{K} \)-current! We never had to solve the complicated eqs. (20, 22), as the behavior is dictated by the “current algebra.” We have also gained insight into how the Weyl symmetry is broken, as the Jordan frame theory of eq. (17) flows into the Einstein frame of eq. (17) under the general pre-inflationary expansion, and the relaxation of \( K_0 \to 0 \), and \( K \to \overline{K} \).

The asymptotic field \( \phi(t \to \infty) \equiv \phi_0 \) is a particular solution, and is subject to “small fluctuations.” In eq. (19) we identify \( \phi \) with the asymptotically constant field VEV, \( \phi_0 \), and include a small fluctuation field, \( \sigma/f \), where \( f = \sqrt{2K} \), as:

\[
\phi = \phi_0 \exp(\sigma/f),
\] (32)

and perform the Weyl metric transformation:

\[
g_{\mu\nu} = \exp(-2\sigma/f)\tilde{g}_{\mu\nu},
\] (33)

we obtain [7]:

\[
S = \int \sqrt{-g} \left( \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \Lambda + \frac{1}{2} M_P^2 R(\tilde{g}) \right). \] (34)

Therefore, “small deformations” of \( \phi_0 \) are described by \( \sigma \). We also see that the scale invariant theory, eq. (19), defined in the “Jordan frame” is equivalent to the “Einstein frame” theory eq. (17) in the broken symmetry phase, where we have traced the origin of the mass scales to the inertial spontaneous symmetry breaking order parameter \( K \).

The massless field \( \sigma \) is, of course, the dilaton, with “decay constant” \( f \). The variation of the action of eq. (34) with respect to \( \sigma/f \) yields the form of the current in the broken phase, \( K_\mu = f \partial_\mu \sigma \), the analogue of the pion axial current, \( f_\mu \partial_\mu \pi \). The dilaton reflects the fact that the exact underlying Weyl symmetry remains intact, though it is hidden in the Einstein frame. We can rescale both the VEV \( \phi_0 \to e^\epsilon \phi_0 \) and the Hubble constant \( H_0 \to e^\epsilon H_0 \) while their ratio remains fixed:

\[
\frac{H_0^2}{\phi_0^2} = \frac{\lambda}{2|\alpha|}.
\] (35)

It is straightforward to extend this to include matter fields [14]. If all ordinary masses arise only from the spontaneous breaking of the Weyl symmetry, then the dilaton only couples directly to gravity. There are then no Brans-Dicke-like constraints, [14].

IV. TWO SCALAR THEORY

We now consider a more realistic \( N = 2 \) model, with scalars \( (\phi, \chi) \), and the potential:

\[
W(\phi, \chi) = \frac{\lambda}{4} \phi^4 + \frac{\xi}{4} \chi^4 + \frac{\delta}{2} \phi^2 \chi^2.
\] (36)

The action takes the form:

\[
S = \int \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - W(\phi, \chi) - \frac{1}{12} \alpha_1 \phi^2 R - \frac{1}{12} \alpha_2 \chi^2 R \right). \] (37)

This has been studied extensively in [2] [3] [7].

We presently follow the approach of [7] and work directly in the defining Jordan frame of eq. (37), and then just follow the dynamics. The sequence of steps follows those of the previous single scalar case. The trace of the Einstein equation immediately becomes:

\[
M_P^2 R = (\alpha_1 - 1) \partial^\mu \phi \partial_\mu \phi + (\alpha_2 - 1) \partial^\mu \chi \partial_\mu \chi + \alpha_1 \phi D^2 \phi + \alpha_2 \chi D^2 \chi + 4W(\phi, \chi). \] (38)
where,
\[ M_p^2 = -\frac{1}{6}(\alpha_1 \phi^2 + \alpha_2 \chi^2). \]  
(39)

The Klein-Gordon equations for the scalars are:
\[
0 = D^2 \phi + \delta \phi \chi^2 + \lambda \phi^3 + \frac{1}{6} \alpha_1 \phi R \\
0 = D^2 \chi + \delta \chi \phi^2 + \xi \chi^3 + \frac{1}{6} \alpha_2 \chi R 
\]  
(40)
and we again use the trace equation and the Klein-Gordon equations to eliminate \( R \). We obtain:
\[
D_\mu K^\mu = 4W - \phi \frac{\delta W}{\delta \phi} - \chi \frac{\delta W}{\delta \chi}, 
\]  
(41)
where now:
\[
K_\mu = (1 - \alpha_1) \phi \partial_\mu \phi + (1 - \alpha_2) \chi \partial_\mu \chi. 
\]  
(42)

The current can be written in terms of a kernel as \( K_\mu = \partial_\mu K \) and is conserved for a scale invariant potential, \( W(\phi, \chi) \).

This leads to a realistic cosmological evolution as illustrated in Fig.\[\text{[3]}\] During an initial pre-inflationary phase, a small patch of the universe will redshift from arbitrary initial field values and velocities, \((\phi, \phi; \chi, \chi)\). The redshifting will cause \((\phi, \chi) \to 0\) and the \( K_0 \) charge density will dilute away as \( \sim a(t)^{-3} \). As in the simple single scalar case this leads to a state with constant kernel \( K \to K \), which again determines the Planck mass, \( K \sim M^2_p \), and spontaneously breaks scale symmetry.

When a constant \( K \approx K \) is attained, the fields VEV’s \((\langle \phi \rangle, \langle \chi \rangle)\) are constrained to lie on an ellipse defined by:
\[
K = \frac{1}{2} \left[(1 - \alpha_1) \langle \phi \rangle^2 + (1 - \alpha_2) \langle \chi \rangle^2 \right].
\]  
(43)
(this ellipse condition is, to our knowledge, first discussed in \[\text{[3]}\]). The pre-expansion has caused the fields to fall to approximate initial locations on the ellipse, \((\langle \phi \rangle_0, \langle \chi \rangle_0)\), which are random. This trapping of the fields on the ellipse leads to inflation.

We now make an additional assumption that the potential of eq.\[\text{(36)}\] has a flat direction. For example, we can take the specific form:
\[
W(\phi, \chi) = \frac{\lambda}{4} (\chi^2 - \eta \phi^2)^2. 
\]  
(44)

The flat direction \( \chi = \sqrt{\eta} \phi \) defines a ray in the \((\phi, \chi)\) plane that intersects the ellipse. The random initial values of \((\langle \phi \rangle_0, \langle \chi \rangle_0)\) on the ellipse would generally not be expected to coincide with the flat direction.

For a significant region of the values on the ellipse, the fields then slow-roll, migrating toward the intersection of the potential and generating a finite period of inflation. If we assume \( \eta \ll 1 \) this intersection occurs near the right-most end of the ellipse where \( \langle \chi \rangle \ll \langle \phi \rangle \) in the

![FIG. 1: Plot of the Hubble parameter, \( H \), \( \phi \), \( \chi \) and the ratio of the two components of the effective Planck mass, \( M^2_\phi \) and \( M^2_{\chi} \), as a function of \( a \); we have normalized the \( a \)-axis to the scale factor at the end of inflation, \( a_* \).](image)

V. QUANTUM THEORY

A. Weyl Invariant Effective Potentials

For the scenario of inertial spontaneously broken Weyl symmetry to work, and lead to a stable Planck mass, it is essential that the Weyl current be identically conserved:
\[
D_\mu K^\mu = 0. 
\]  
(45)
Failure of this is not an option! In what follows we will refer to nonzero contributions coming from loops to the
theories of renormalizable quantum field theories, e.g., as “Weyl anomalies.” As we’ve seen in Section II, the trace anomalies of the scale current defined by diffeomorphisms are identical to those of $K_\mu$.

Scale and Weyl symmetry of a theory appears ab initio to be broken by arbitrary divergences of quantum loops. Loop divergences are subtle, however, and are often confused with physics. Here we adopt an operating principle that has been espoused by W. Bardeen [9]: The allowed symmetries of a renormalized quantum field theory are determined by anomalies, (or absence thereof). Other quantum loop divergences are essentially unphysical artifacts of the method of calculation.

Our problem of maintaining Weyl symmetry requires that we build a theory that has no anomaly in $K_\mu$. To understand this problem, and its solution, we turn the the Coleman-Weinberg potential [10].

In computing Coleman-Weinberg potentials for massless scalar fields we encounter an infrared divergence that must be regularized. To do so we often introduce explicit “external” mass scales into the theory by hand. These are mass scales that are not part of the defining action of the theory, and essentially define the RG trajectories of coupling constants. These externally injected mass scales lead directly to the Weyl anomaly.

We can see this in the famous paper of Coleman and Weinberg [10]. In their eq.(3.7), to renormalize the quartic scalar coupling constant in an effective potential at one-loop level, $V(\phi)$, they introduce a mass scale $M$. Once one injects $M$ into the theory, scale and Weyl symmetries are broken, and the effective potential in the leading $\ln(\phi/M)$ limit then takes the form:

$$V(\phi) = \frac{\beta_1}{4} \phi^4 \ln(\phi/M).$$  \hspace{1cm} (46)$$

Here $\beta_1$ is the one-loop renormalization group $\beta$-function, $d\lambda(\mu)/d\ln \mu = \beta_1$. We emphasize that this is not an issue of regularization, but rather stems from the process of renormalization.

The manifestation of this is seen in the Weyl anomaly, (and trace of the improved stress tensor [12]), i.e., the divergence of the $K_\mu$ current:

$$\partial_\mu K^\mu = 4V(\phi) - \phi \frac{\partial}{\partial \phi} V(\phi) = -\frac{\beta}{2} \phi^4.$$ \hspace{1cm} (47)$$

The anomaly is directly associated with the $\beta$-function of the potential coupling constant $\lambda$. The coupling constant $\lambda(\phi)$ is viewed to run with the field (VEV) as determined by the RG equation. A specification of the boundary condition of the RG trajectory has injected $M$ into the theory, e.g., $\lambda(\phi = M) = 0$. We clearly see in eq.(47) that the anomaly is then generated. This is similar to the QCD case described in Section I.

Of course, depending upon the application, there’s nothing wrong with the Coleman-Weinberg potential defined this way. If one is only treating the effective potential as a subsector of the larger theory, where $M$ is truly external to the subsector, then we can simply defer the question of the true origin of $M$ in the larger theory. If, however, Weyl symmetry is to be maintained as an exact invariance of the overall theory, then $M$ must be replaced by an internal mass scale that is part of action, i.e., $M$ must then be (the VEV of) a dynamical entity, such as a field $\chi$ (or $K$) appearing in the extended action. We would then have the Coleman-Weinberg potential:

$$V(\phi, \chi) = \frac{\beta_1}{4} \phi^4 \ln(\phi/\chi).$$ \hspace{1cm} (48)$$

(or $\chi \rightarrow cK$) and then we have the Weyl anomaly:

$$\partial_\mu K^\mu = \left(4 - \phi \frac{\partial}{\partial \phi} - \phi \frac{\partial}{\partial \chi}\right) V(\phi, \chi) = 0.$$ \hspace{1cm} (49)$$

The $\beta$-function contributions have now cancelled. To see how the argument of the log might be specified more precisely we turn to an explicit calculation in the next section.

**B. U(1) Model**

Inertial symmetry breaking yields a spontaneously generated mass scale which becomes the Planck mass and can produces other scales in the theory. It can also drive the breaking of other symmetries. To see this, let us revisit the two scalar theory with the nonminimal coupling to gravity of the form:

$$S = \int \sqrt{-g} g^{\mu\nu} \left(\frac{1}{2} \partial_\mu \phi_1 \partial_\nu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial_\nu \phi_2 \right) - \int d^4 x \sqrt{-g} \left(\frac{\alpha}{12} (\phi_1^2 + \phi_2^2) R + W\right)$$ \hspace{1cm} (50)$$

where $W = W(\phi_1, \phi_2)$. We now have a common overall value of $\alpha$. Apart from the potential, this theory thus has a $U(1) = SO(2)$ symmetry, and we can introduce a complex field:

$$\Phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \hspace{1cm} \Phi^\dagger \Phi = \frac{1}{2} (\phi_1^2 + \phi_2^2)$$ \hspace{1cm} (51)$$

and the action becomes:

$$S = \int \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \Phi^\dagger \partial_\nu \Phi - \frac{\alpha}{6} \Phi^\dagger \Phi R - W\right).$$ \hspace{1cm} (52)$$

The $K_\mu$ current and kernel become:

$$K_\mu = \partial_\mu K \hspace{1cm} K = (1 - \alpha) \Phi^\dagger \Phi.$$ \hspace{1cm} (53)$$

The inertial symmetry breaking implies that dynamically $K \rightarrow \mathcal{K}$, hence:

$$\Phi^\dagger \Phi \rightarrow f^2/(1 - \alpha) \hspace{1cm} f^2 = 2\mathcal{K}.$$ \hspace{1cm} (54)$$

However, now this also defines the spontaneously broken phase of the $U(1)$ symmetry, We can therefore rewrite:

$$\Phi = \frac{f}{\sqrt{2}} \exp(i\pi/f + \sigma/f)).$$ \hspace{1cm} (55)$$
In this representation we now have:

\[ S = \int \sqrt{-g} \left( g^{\mu \nu} e^{2\sigma/f} \left( \frac{1}{2} \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} \partial_\mu \bar{\pi} \partial_\nu \bar{\pi} \right) - \frac{\alpha}{12} e^{2\sigma/f} f^2 R - \frac{f^4}{4} e^{4\sigma/f} \widetilde{W} \left( \frac{\pi}{f} \right) \right) \]  \quad (56)

where \( \widetilde{W} = e^{-4\sigma}/W \). The Weyl transformation is now implemented, as expected, by shifting \( \sigma \) and rescaling the metric:

\[ \sigma \rightarrow \sigma + \epsilon f \quad g_{\mu \nu} \rightarrow e^{-2\epsilon} g_{\mu \nu} \]  \quad (57)

while \( \tilde{\pi}(x) \) and \( f \) are held fixed. This is the usual view of the dilaton as a Nambu-Goldstone boson and the transformation implemented on field variables.

However, now we can see from eq.(56) that an equivalent covariant global scale transformation is also a symmetry:

\[ \sigma \rightarrow e^{\epsilon} \sigma \quad \tilde{\pi}(x) \rightarrow e^{\epsilon} \tilde{\pi}(x) \]

\[ f \rightarrow e^{\epsilon} f \quad g_{\mu \nu} \rightarrow e^{-2\epsilon} g_{\mu \nu} \]  \quad (58)

Here we are doing a scale transformation on all dimensional quantities, including the “constant” \( f^2 \) together with the metric transformation.

It is important to distinguish in a theory between transforming dynamical quantities, i.e., fields, vs. transforming fixed input parameters. The latter is not a symmetry: only dynamical fields can represent the symmetry group and have Noether currents. However, while \( f \) appears to be a parameter of the low energy (Einstein frame) theory, it is in fact a dynamical quantity, \( f^2 = 2K \propto \Phi \Phi \) in the underlying (Jordan frame) theory, and eq.(58) represents the covariant Weyl transformation in that underlying theory.

Now, performing the metric redefinition, \( g_{\mu \nu} \rightarrow e^{-2\epsilon} g_{\mu \nu} \) as in the simple single field example the action of eq.(56) becomes:

\[ S = \int \sqrt{-g} g^{\mu \nu} \left( \frac{1}{2} \partial_\mu \sigma \partial_\nu \sigma + \frac{1}{2} \partial_\mu \bar{\pi} \partial_\nu \bar{\pi} \right) - \int d^4x \sqrt{-\bar{g}} \left( \frac{\alpha}{12} f^2 R + \frac{f^4}{4} \widetilde{W} \left( \frac{\pi}{f} \right) \right) \]  \quad (59)

The original Weyl invariance is hidden in the action of eq.(59) since the new metric \( e^{-2\epsilon} g_{\mu \nu} \) has been made Weyl invariant under eq.(57). However, the transformation of eq.(58) remains nontrivial and reflects the Weyl invariance of eq.(59). Geometrically, the Weyl transformation rescales the radius of the circle, \( f \), while holding the angle variable \( \tilde{\pi}(x)/f \) fixed. While the transformation can be implemented by shifting \( \sigma \), as in eq.(57), it is convenient to treat \( \sigma \) as transforming covariantly as in eq.(58) with the accompanying rescaling of \( f \).

The \( U(1) \) symmetry is also spontaneously broken. The field \( \tilde{\pi}/f \) has disappeared from the nonminimal gravitational coupling. It is the Nambu-Goldstone boson (NGB) of the broken \( U(1) \) symmetry. However, by contrast to the usual Higgs mechanism, e.g., with a Mexican hat potential, the Higgs boson is generally heavy with a mass given by \( m^2 \sim f^2 \lambda \) where \( \lambda \) is the quartic potential defining the hat. Here the the Higgs boson is the dilaton, but it is massless as it too is a Nambu-Goldstone boson of broken Weyl symmetry.

The model can be generalized by including more scalars. With \( N \) scalars and an \( SO(N) \) invariant nonminimal coupling to gravity, the inertial symmetry spontaneously breaks \( SO(N) \) to \( SO(N-1) \) by giving the kernel \( \propto \sum \phi_i^2 \) a VEV. The fields are then confined to \( S_{N-1} \) in theory space. The inertial symmetry breaking is independent of the potential, and the potential can explicitly break this symmetry. An arbitrary point on \( S_{N-1} \) will dynamically move during the inflationary phase. This motion is controlled by the potential, \( W(\phi) \). It is interesting to speculate that there maybe nontrivial topological configurations associated with inertial symmetry breaking, in multiscalar theories, corresponding to homotopy groups, such as \( \Pi_3(SU(2)) \), etc.

Thus far we have considered only the inertial symmetry breaking. Let us now consider the role of potential. Here the existence of flat directions is fundamentally important if we want the final vacuum to have a small cosmological constant. Perhaps the simplest potential in our original \( (\phi_1, \phi_2) \) two scalar model with a flat direction consists of a single field quartic interaction

\[ W(\phi_1, \phi_2) = \frac{\lambda}{4} \phi_1^4 = \lambda (\Phi - \Phi^1)^4 \]  \quad (60)

where the flat direction is the \( \phi_1 \) axis. An arbitrary field configuration after an initial period of expansion is a point on \( S_1 \). This will then dynamically evolve during the inflationary phase toward the minimum. In the above example, an arbitrary point on the \( (\phi_1, \phi_2) \) circle will move toward the \( \phi_2 = 0 \) minimum of \( W \).

In our \( U(1) \) model, after performing the Weyl redefinition to remove \( e^{\sigma/f} \) factors, the potential is:

\[ \widetilde{W} = \frac{\lambda f^4}{4} \sin^4 (\tilde{\pi}/f) \]  \quad (61)

The flat direction corresponds to a shift symmetry, \( \Phi \rightarrow \Phi + \epsilon \) for real \( \epsilon \). The potential of eq.(61) breaks \( U(1) \rightarrow Z_2 \), since the potential is invariant under shifts of \( \tilde{\pi}(x)/f \rightarrow \tilde{\pi}(x)/f + \pi N \). Quantum loops will necessarily lead to a running of the coupling \( \lambda \) and modifications of the potential.

We can focus on the real scalar field theory Lagrangian in flat space:

\[ L = \frac{1}{2} (\partial \tilde{\pi})^2 - \frac{1}{4} \lambda f^4 \sin^4 (\tilde{\pi}/f) \]  \quad (62)

To compute the potential, we introduce a classical source term in the lagrangian, \( -J \tilde{\pi} \). Through equations of motion, \( J \) induces the shift in the field,

\[ \tilde{\pi} = \tilde{\pi}_c + \frac{f^2}{2} \tilde{\pi} \]  \quad (63)
where $\tilde{\pi}_c$ satisfies the renormalized equation of motion, 
$\partial^2 \tilde{\pi}_c + \lambda f^3 \sin^3 (\tilde{\pi}_c/f) \cos (\tilde{\pi}_c/f) + J = 0$. The Lagrangian becomes:

$$L = L_0(\tilde{\pi}_c) + [\hbar] \tilde{L}(\tilde{\pi}_c, \tilde{\pi})$$

(64)

where, the on-shell $\tilde{\pi}_c$ with the source term leads to cancellation of tadpoles (linear $\tilde{\pi}$ terms). Hence to $O(\hbar)$:

$$L_0(\tilde{\pi}_c) = \frac{1}{2} (\partial \tilde{\pi}_c)^2 - \frac{1}{4} \lambda f^4 \sin^4 (\tilde{\pi}_c/f)$$

(65)

and:

$$\tilde{L}(\tilde{\pi}_c, \tilde{\pi}) = \frac{1}{2} (\partial \tilde{\pi})^2 - \frac{1}{2} G \tilde{\pi}^2 + \ldots$$

(66)

where:

$$G = \lambda f^2 (3 \sin^2 (\tilde{\pi}_c/f) - \sin^4 (\tilde{\pi}_c/f))$$

$$\sim 3 \lambda f^2 \sin^2 (\tilde{\pi}_c/f)$$

(67)

where we drop the subleading term for small $\tilde{\pi}_c/f$. The effective theory is non-renormalizable, but, like chiral perturbation theory, it becomes renormalizable in an expansion in large $f$, and we have displayed the leading behavior for small fixed $\tilde{\pi}_c$ and large $f$. This expression, better than a polynomial expansion in $\tilde{\pi}_c/f$, protects the global $Z$ shift symmetry $\tilde{\pi}_c/f \to \tilde{\pi}_c/f + 2\pi N$. There is no wave-function renormalization constant at one loop order.

We integrate out the quantum fluctuations, $\tilde{\pi}$. The effective Lagrangian takes the form:

$$L = L_0(\tilde{\pi}_c) - V_{eff}$$

$$V_{reg} = i\hbar \int \frac{d^4 \ell}{(2\pi)^4} \ln(\ell^2 - 3 \lambda f^2 \sin^2 (\tilde{\pi}_c/f)).$$

(68)

Let $y = \sin (\tilde{\pi}_c/f)$ and, up to additive constants $\propto \Lambda^4$ we obtain a regularized expression:

$$V_{reg} = \frac{1}{32 \pi^2} \left( (3 \lambda f^2 y^2) \Lambda^2 - \frac{1}{2} (3 \lambda f^2 y^2)^2 \left( \ln \left( \frac{\Lambda^2}{3 \lambda f^2 y^2} \right) + \frac{1}{2} \right) \right) \left( \frac{1}{\ell^2 - \Lambda^2} \right).$$

(69)

Not surprisingly, we have quadratic and log divergences. It doesn’t matter how we define $\Lambda$ under Weyl transformations, as this is only a regularized expression.

We now introduce a set of three counterterms:

$$V_{ct} = \delta_0 f^4 + \frac{1}{2} \delta_1 f^4 \sin^2 (\tilde{\pi}_c/f)$$

$$+ \frac{1}{4} \delta_2 f^4 \sin^4 (\tilde{\pi}_c/f).$$

(70)

The counterterms are defined by imposing renormalization conditions. With a vanishing $\sin^2 (\tilde{\pi}_c/f)$ term we will require a definition of the quartic counterterm at a nonzero scale $y = \sin (\tilde{\pi}_c/f) = c$, due to the infrared singularity. Note that this is a Weyl invariant specification since $\tilde{\pi}_c/f$ is Weyl invariant, by eq.(58). The counterterms are determined by:

$$0 = (V_{reg} + f^4 \delta_0)_{y=0}$$

$$0 = \left( \frac{\partial^2}{\partial y^2} V_{reg} + f^4 \delta_1 \right)_{y=0}$$

$$0 = \left( \frac{\partial^4}{\partial y^4} V_{reg} + 3 f^4 \delta_2 \right)_{y=c}. \quad (71)$$

From the regularized expression we have:

$$\delta_0 = 0$$

$$\delta_1 = -\frac{1}{16 \pi^2} \left( 3 \Lambda^2 \right)$$

$$\delta_2 = -\frac{9}{16 \pi^2} \left( \lambda^2 \ln \left( \frac{\Lambda^2}{3 \lambda f^2 c^2} \right) - \frac{11}{3} \lambda^2 \right).$$

Hence the renormalized potential is:

$$V_{reg} + V_{ct} = -\frac{1}{4} \lambda f^4 \sin^4 (\tilde{\pi}_c/f)$$

$$- \frac{9 f^4}{64 \pi^2} \left( \lambda^2 \sin^4 (\tilde{\pi}_c/f) \left( \ln \left( \frac{\sin^2 (\tilde{\pi}_c/f)}{c^2} \right) - \frac{25}{6} \right) \right).$$

(72)

This reproduces the usual CW result for $\lambda^4$ theory with $\tilde{\pi}_c \to f \sin (\tilde{\pi}_c/f)$.

The leading log limit is of the form:

$$V = -\frac{9 f^4 \lambda^2}{64 \pi^2} \sin^4 (\tilde{\pi}_c/f) \ln \left( \frac{\sin^2 (\tilde{\pi}_c/f)}{c^2} \right)$$

(73)

and the sub-leading (non-log) terms can be subsumed into the definition of $c^2$. This provides a calculation of $\beta_1$ (see [13] for discussion of the direct RG approach to Coleman-Weinberg potentials).

The results of eqs.(72,73) are Weyl covariant quantum loop induced potentials, and lead to invariant actions, $\sim \int \sqrt{-g} V$, by the transformation of eq.(58). There is no requirement of Weyl invariance in the intermediate steps of the regularized calculation.

VI. CONCLUSIONS

We have discussed how inflation and Planck scale generation emerge as a unified phenomenon from a dynamics associated with global Weyl symmetry [14][15][16]. We have placed particular emphasis upon the Weyl current, $K_\mu$. In the pre-inflationary universe, the Weyl current density, $K_\mu$, is driven to zero by general expansion. However, $K_\mu$ has a kernel structure, i.e.,

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4 The effective action is: $-i\hbar \ln(Z) \sim -i\hbar \int \ln(1/(\ell^2 - \mu^2)) = i\hbar \int \ln(\ell^2 - \mu^2)$. The potential is $-i\hbar \int \ln(\ell^2 - \mu^2)$ . The integral can be done by performing a Wick rotation ($\ell_0 \to i \ell_0$, $\ell^2 \to -\ell^2_0 - \ell^2$, and $d^4 \ell \to id \ell_0 d^3 \ell$) and we use a Euclidean momentum space cut-off, $\Lambda$. 

$K_{\mu} = \partial_{\mu} K$, and as $K_0 \to 0$, the kernel evolves as $K \to K$ constant. This resulting constant $K$ is the order parameter of the Weyl symmetry breaking, indeed, $K$ directly defines $M^2_{\phi}$.

This defines “inertial spontaneous symmetry breaking” \[8\]. This mechanism involves a new form of dynamical scale symmetry breaking, driven by the formation of a nonzero kernel, $K$. The scale breaking has nothing to do with any potential in the theory, but is solely dynamically generated by gravity. In addition, a scale invariant potential with a flat-direction ultimately determines the vacuum of the theory and relative VEV’s of the scalar fields contributing to $K$. There is a harmless dilaton associated with the dynamical symmetry breaking which decouples from everything except gravity \[14\].

We illustrated this phenomenon in a single scalar field theory, $\phi$, with nonminimal coupling. We see how, starting in the Jordan frame, we smoothly evolve into the Einstein frame. The theory has a conserved current, $K_{\mu} = (1 - \alpha)\phi\partial_{\mu}\phi$. The scale current charge density dilutes to zero in the pre-inflationary phase $K_0 \sim (a(t))^{-3}$. The kernel, $K = (1 - \alpha)\phi^2/2$, and hence the VEV of $\phi$, are driven to constants. With $\alpha < 0$, this induces a positive Planck (mass)$^2$. The resulting inflation in this simple model is eternal.

In multi-scalar-field theories we see that we will have a generalized $K = \sum_i (1 - \alpha_i)\phi_i^2/2$. As this is driven to a constant, it defines an ellipsoidal constraint on the scalar field VEV’s, and the Planck scale is again generated $\propto K$. An inflationary slow-roll is then associated with the field VEV’s migrating along the ellipse, ultimately flowing to an intersection of the flat direction with the ellipsoid. In a two-scalar scheme the terminal phase of inflation is similar to standard $\phi^4$ inflation, since the effective theory is now essentially Einstein gravity with a fixed $M^2_{\phi}$. The final cosmological constant vanishes with exact flatness of the potential.

Any Weyl symmetry breaking effect is intolerable in these schemes and will show up as a nonzero divergence in the $K_{\mu}$ current (a Weyl anomaly). We show how a Weyl invariant Coleman-Weinberg action is computable when renormalization masses are defined by field (VEV’s) contained in the action. It is natural to renormalize the theory with the scale $f^2 = 2K$ itself.

We demonstrated in this calculation how a $U(1)$ symmetry is broken solely by the inertial symmetry breaking mechanism. The $U(1)$ scalar ends up in a Higgs phase, but the Higgs boson is now the massless dilaton. There is considerable work to continue along these lines exploring how gauge symmetries can be broken, how topological vacua might arise, and possible applications to the real world.

There remain challenges. For example, if one imbeds this in “A-gravity” \[10\] then the $\alpha$ terms can feedback, via quantum loops, upon gravity on scales above $M_{P}$, and, e.g., violate a flat direction in perturbation theory. This poses a potential problem for the program as it leads to an uncontrollably large induced $U(1)$ invariant potential $\sim \lambda(\Phi^4\Phi)^2$, which, in turn, would lead to an uncontrollably large cosmological constant in the Einstein frame theory.

Our philosophy here, however, may differ from that of A-gravity. We view the $R^2$ terms to be induced, in analogy to Coleman-Weinberg potentials, and therefore, taking a form schematically like $\sim \epsilon R^2 \log(K/R)$ etc. As such, we wouldn’t treat these terms as part of the propagator, $i.e.$, we would not include $1/\epsilon p^4$ terms for small, perturbative $\epsilon$. Our theory is then effectively cut-off at the Planck scale because the $M^2_{\phi} R$ term becomes an interaction $\alpha \phi^2 R$ at trans-Planckian energies. Gravity might be viewed as a collective phenomenon in this view. Alternatively, we may seek resolution with A-gravity by inclusion of $e.g.$, the Weyl photon, or other generalizations of the present models.

Possible resolutions to these and other issues are currently under study and will be discussed elsewhere.

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