

# Heavy-Meson Spectrum Tests of the Oktay–Kronfeld Action

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## Fermilab Lattice, MILC, and SWME Collaborations

We present heavy-meson spectrum results obtained using the Oktay–Kronfeld (OK) action on MILC asqtad lattices. The OK action was designed to improve the heavy-quark action of the Fermilab formulation, such that heavy-quark discretization errors are reduced. The OK action includes dimension-6 and -7 operators necessary for tree-level matching to QCD through order  $O(\Lambda^3/m_Q^3)$  for heavy-light mesons and  $O(v^6)$  for quarkonium, or, equivalently, through  $O(a^2)$  with some  $O(a^3)$  terms with Symanzik power counting. To assess the improvement, we extend previous numerical tests with heavy-meson masses by analyzing data generated on a finer ( $a \approx 0.12$  fm) lattice with the correct tadpole factors for the  $c_5$  term in the action. We update the analyses of the inconsistency parameter and the hyperfine splittings for the rest and kinetic masses.

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## 1. Introduction

The parameter  $\varepsilon_K$  quantifies indirect CP violation in the neutral kaon system. At present, the tension between the Standard Model (SM) and experimental values of  $|\varepsilon_K|$  is  $3.4\sigma$  [1] with the value of  $|V_{cb}|$  from the exclusive decay  $B \rightarrow D^* \ell \nu$  [2]. This value of  $|V_{cb}|$ , the most precise from exclusive decays to date, is  $3\sigma$  away from the value from inclusive decays [3]. The largest error in the  $\varepsilon_K$  determination in the SM comes from  $|V_{cb}|$ , so it is crucial to improve the precision of exclusive

The dominant error of exclusive  $|V_{cb}|$  comes from the heavy-quark discretization error in the form-factor calculation of the semi-leptonic decay  $B \rightarrow D^* \ell \nu$  [2]. Hence, the SWME Collaboration plans to use the Oktay–Kronfeld (OK) action [4] in the upcoming calculation in order to reduce it efficiently. This action is an improved version of the Fermilab action [5], which incorporates the dimension-6 and -7 bilinear operators needed for tree-level matching to QCD through order  $O(\Lambda^3/m_Q^3)$  for heavy-light mesons and  $O(v^6)$  for quarkonium. We expect that the bottom- and charm-quark discretization errors could be reduced below the current 1% level. A similar error for the charm-quark could also be achieved with other highly-improved actions, such as HISQ [6].

For the heavy-meson spectrum, we present results for the inconsistency parameter [7, 8] and hyperfine splittings, all of which test how well the Fermilab and OK actions perform in practice. For this purpose, we follow the strategy of our previous work [9], in which the  $c_5$  term was not completely tadpole-improved. In this work, we now implement the tadpole improvement for  $c_5$  completely. We also extend the data analysis to data sets produced on a finer ( $a \approx 0.12$  fm) MILC asqtad lattice.

## 2. Meson Correlators

We use a subset of the MILC  $N_f = 2 + 1$  asqtad ensembles at  $a = 0.12$  fm and  $0.15$  fm [11], summarized in Table 1. We compute meson correlators  $C(t, \mathbf{p})$

$$C(t, \mathbf{p}) = \sum_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}} \langle \mathcal{O}^\dagger(t, \mathbf{x}) \mathcal{O}(0, \mathbf{0}) \rangle. \quad (2.1)$$

The interpolating operators  $\mathcal{O}(x)$  are

$$\mathcal{O}_t(x) = \bar{\psi}_\alpha(x) \Gamma_{\alpha\beta} \Omega_{\beta t}(x) \chi(x) \quad (\text{heavy-light meson}), \quad (2.2)$$

$$\mathcal{O}(x) = \bar{\psi}_\alpha(x) \Gamma_{\alpha\beta} \psi_\beta(x) \quad (\text{quarkonium}), \quad (2.3)$$

where the heavy-quark field  $\psi$  is that of the OK action, while the light-quark field  $\chi$  is that of the asqtad action. The spin structure is  $\Gamma = \gamma_5$  for the pseudoscalar and  $\Gamma = \gamma_i$  for the vector meson.

$a(\text{fm})$	$N_L^3 \times N_T$	$\beta$	$am_l$	$am_s$	$u_0$	$a^{-1}(\text{GeV})$	$N_{\text{conf}}$	$N_{\text{src}}$
0.12	$20^3 \times 64$	6.79	0.02	0.05	0.8688	$1.683_{-16}^{+43}$	484	6
0.15	$16^3 \times 48$	6.60	0.029	0.0484	0.8614	$1.350_{-13}^{+35}$	500	4

Table 1: Parameters of the MILC asqtad ensembles with  $N_f = 2 + 1$  flavors [10].

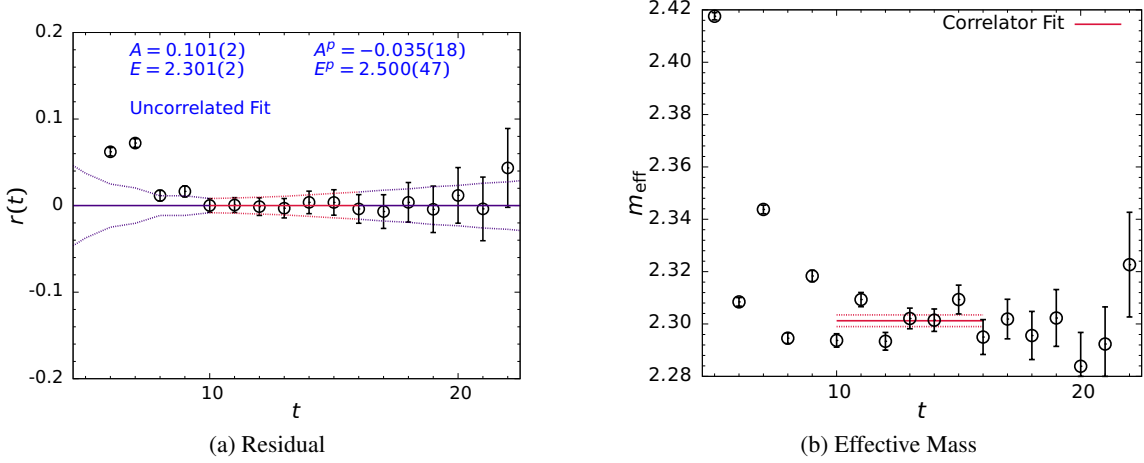


Figure 1:  $r(t)$  and  $m_{\text{eff}}(t)$  for a pseudoscalar heavy-light meson correlator at  $\kappa = 0.038$  and  $\mathbf{p} = \mathbf{0}$ , obtained using the uncorrelated fit.

The taste degree of freedom for the staggered fermion is obtained from the 1-component field  $\chi$  with  $\Omega(x) = \gamma_1^{x_1} \gamma_2^{x_2} \gamma_3^{x_3} \gamma_4^{x_4}$  [12, 10].

We compute 2-point correlators with 4 different values of hopping parameter:  $\kappa = 0.038, 0.039, 0.040, 0.041$ . We fix the valence light-quark mass to  $am_s$  in Table 1. We choose 11 meson momenta,  $a\mathbf{p} = (2\pi/N_L)\mathbf{n}$ :  $\mathbf{n} = (0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0), (2,1,0), (2,1,1), (2,2,0), (2,1,1), (3,0,0), (4,0,0)$ . To increase statistics, correlators are computed at 6 different source time slices on each gauge configuration.

Each correlator is folded in half, and then fit to the function

$$f(t) = A\{e^{-Et} + e^{-E(T-t)}\} + (-1)^t A^P\{e^{-E^P t} + e^{-E^P(T-t)}\}, \quad (2.4)$$

where  $A, A^P, E$ , and  $E^P$  are determined by fitting. Figure 1 shows the correlator fit results with fit residual  $r(t)$  and effective mass  $m_{\text{eff}}(t)$  for a pseudoscalar heavy-light meson data:

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}, \quad m_{\text{eff}}(t) = \frac{1}{2} \ln \left( \frac{C(t)}{C(t+2)} \right). \quad (2.5)$$

We exclude the largest momentum,  $\mathbf{n} = (4,0,0)$ , from the dispersion relation fits, because these data are very noisy, and the correlator fits are poor.

### 3. Dispersion Relation

Once we obtain the ground state energy at each momentum, we fit them to the non-relativistic dispersion relation [5],

$$E = M_1 + \frac{\mathbf{p}^2}{2M_2} - \frac{(\mathbf{p}^2)^2}{8M_4^3} - \frac{a^3 W_4}{6} \sum_i p_i^4, \quad (3.1)$$

where  $M_1$  ( $M_2$ ) is the rest (kinetic) mass. In the Fermilab formulation, the kinetic mass is chosen to be the physically relevant mass [5], because that choice minimizes discretization errors in matrix elements and in mass splittings.

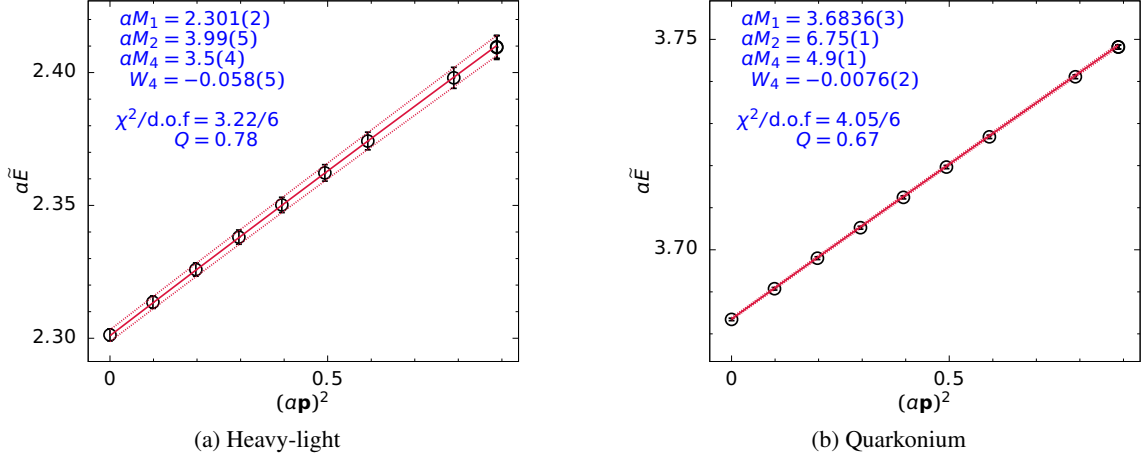


Figure 2: Fit results of pseudoscalar meson spectrum to dispersion relation at  $\kappa = 0.038$ .

When fitting the data to the dispersion relation, we use the full covariance matrix and no Bayesian prior information. In Fig. 2, we plot results after subtracting from the data the  $W_4$  term, which parametrizes the breaking of  $O(3)$  rotational symmetry. Here,  $\tilde{E}$  is defined to be

$$\tilde{E} = E + \frac{a^3 W_4}{6} \sum_i p_i^4. \quad (3.2)$$

Note that the two data points at momenta  $\mathbf{n} = (2, 2, 1)$  and  $(3, 0, 0)$  lie on top of each other due to the removal of the  $W_4$  term. As one can see from the plots, fits to Eq. (3.1) are good enough to determine the kinetic mass reliably.

#### 4. Inconsistency Parameter

The inconsistency parameter  $I$ , Eq. (4.1), is designed to examine the improvements by  $O(\mathbf{p}^4)$  terms in the action [7, 8]. This is, in particular, good for probing the improvement by the OK action, because it isolates those improvement terms by construction.

$$I \equiv \frac{2\delta M_{\bar{Q}q} - (\delta M_{\bar{Q}Q} + \delta M_{\bar{q}q})}{2M_{2\bar{Q}q}} = \frac{2\delta B_{\bar{Q}q} - (\delta B_{\bar{Q}Q} + \delta B_{\bar{q}q})}{2M_{2\bar{Q}q}}, \quad (4.1)$$

where

$$\delta M_{\bar{Q}q} \equiv M_{2\bar{Q}q} - M_{1\bar{Q}q} \quad (4.2)$$

is the difference between the kinetic ( $M_2$ ) and rest ( $M_1$ ) masses. By construction,  $I$  vanishes in the continuum limit, and it should be closer to 0 for more improved actions.

The meson masses  $M$  can be written as a sum of the perturbative quark masses  $m_1$  or  $m_2$  and the binding energy  $B$  as follows:

$$M_{1\bar{Q}q} = m_{1\bar{Q}} + m_{1q} + B_{1\bar{Q}q}, \quad M_{2\bar{Q}q} = m_{2\bar{Q}} + m_{2q} + B_{2\bar{Q}q}. \quad (4.3)$$

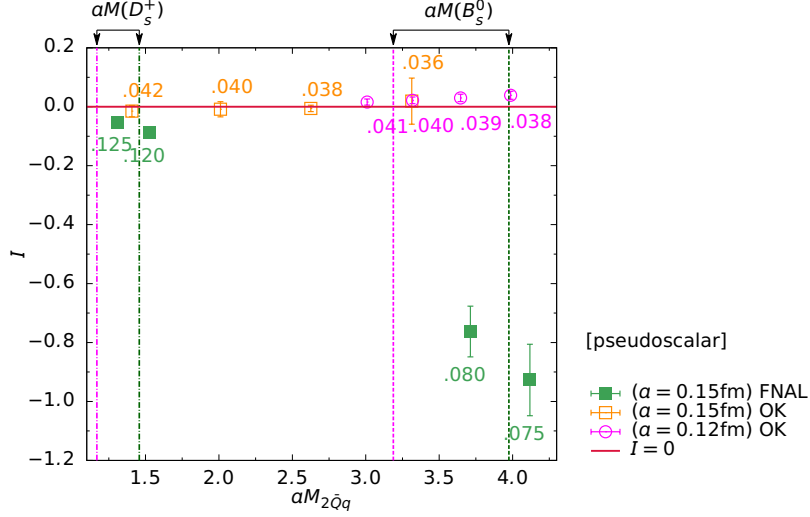


Figure 3: Inconsistency Parameter  $I$ . Data labels denote  $\kappa$  values. The square (green and orange) represents old data sets at  $a = 0.15$  fm and circle (magenta) represents new data sets at  $a = 0.12$  fm. Vertical lines represent physical masses of  $B_s^0$  (dotted) and  $D_s^+$  (dash-dotted) mesons. Near the  $B_s^0$  meson mass,  $I$  almost vanishes for the OK action, but for the Fermilab action it does not. This behavior suggests the OK action is significantly closer to the continuum limit.

These formulas define  $B_1$  and  $B_2$ . Then, substituting them into Eq. (4.1), the quark masses cancel out, and the inconsistency parameter becomes a relation among the binding energies

$$\delta B_{\bar{Q}q} = B_{2\bar{Q}q} - B_{1\bar{Q}q}. \quad (4.4)$$

The corresponding quantities for Eqs. (4.2), (4.3), and (4.4) for heavy ( $\bar{Q}Q$ ) and light ( $\bar{q}q$ ) quarkonium are defined similarly. Because light quarks always have  $ma \ll 1$ , the  $O((ma)^2)$  distinction between rest and kinetic mass is negligible. We therefore omit  $\delta M_{\bar{q}q}$  (or  $\delta B_{\bar{q}q}$ ) when forming  $I$ .

Considering the non-relativistic limit of quark and antiquark system, for  $S$ -wave case, the spin-independent binding-energy difference can be expressed as follows [8, 10]:

$$\delta B_{\bar{Q}q} = \frac{5}{3} \frac{\langle \mathbf{p}^2 \rangle}{2\mu_2} \left[ \mu_2 \left( \frac{m_{2\bar{Q}}^2}{m_{4\bar{Q}}^3} + \frac{m_{2q}^2}{m_{4q}^3} \right) - 1 \right] + \frac{4}{3} a^3 \frac{\langle \mathbf{p}^2 \rangle}{2\mu_2} \mu_2 \left[ w_{4\bar{Q}} m_{2\bar{Q}}^2 + w_{4q} m_{2q}^2 \right] + O(\mathbf{p}^4), \quad (4.5)$$

where  $\mu_2^{-1} = m_{2\bar{Q}}^{-1} + m_{2q}^{-1}$ , and  $m_2$ ,  $m_4$ , and  $w_4$  are defined through the quark dispersion relation analog of Eq. (3.1). Equation (4.5) holds for the quarkonium  $\delta B_{\bar{Q}Q}$  too. The leading contribution of  $O(\mathbf{p}^2)$  in  $\delta B$  vanishes when  $m_4 = m_2$  and  $w_4 = 0$ , also for orbital angular momenta beyond the  $S$  wave [10]. The OK action matches  $m_4 = m_2$  and  $w_4 = 0$ , so the two expressions in square brackets vanish (at the tree level), leaving  $I \sim \mathbf{p}^4 \approx 0$ .

The result for the pseudoscalar channel is shown in Fig. 3. We find that  $I$  is close to 0 for the OK action even in the mass region where the Fermilab action produces very large  $|I| \approx 1$ . This outcome provides good numerical evidence for the improvement expected with the OK action. It also shows that the new data set with the coarse ( $a = 0.12$  fm) ensemble data covers the  $B_s^0$  mass.

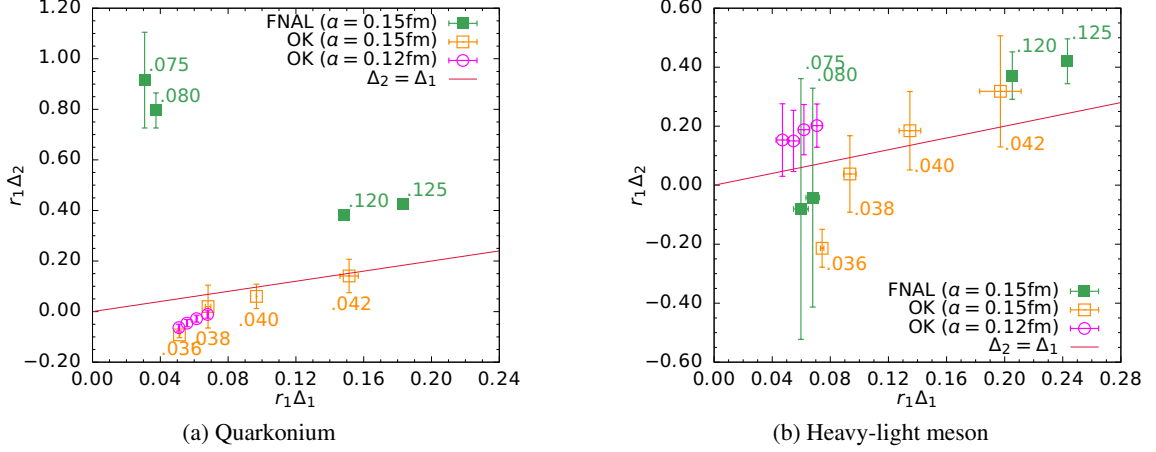


Figure 4: Hyperfine splitting obtained from the kinetic mass vs. that obtained from the rest mass.

## 5. Hyperfine Splittings

The hyperfine splitting  $\Delta$  is defined to be the difference in mass between the vector ( $M^*$ ) and pseudoscalar ( $M$ ) mesons:

$$\Delta_1 = M_1^* - M_1, \quad \Delta_2 = M_2^* - M_2. \quad (5.1)$$

The hyperfine splitting of the kinetic mass ( $\Delta_2$ ) has a larger error than that of the rest mass ( $\Delta_1$ ), mainly because the kinetic mass requires correlators with  $\mathbf{p} \neq \mathbf{0}$ , which are noisier than  $\mathbf{p} = \mathbf{0}$ . Interestingly, with the OK action the statistical error is about 1/6 of that with the Fermilab action, as one can see in Fig. 4. Thus, the OK action is not only more accurate in the sense of improved action but also statistically more precise. From Eq. (4.4), we have

$$\Delta_2 = \Delta_1 + \delta B^* - \delta B. \quad (5.2)$$

Spin-independent contributions to the binding energies cancel, so the difference in hyperfine splittings  $\Delta_2 - \Delta_1$  diagnoses the improvement of spin-dependent  $O(\mathbf{p}^4)$  terms. As one can see in Fig. 4, the OK action shows clear improvement for quarkonium: the OK results lie close to the continuum limit  $\Delta_2 = \Delta_1$  (the red line). The heavy-light results do not deviate much from the line  $\Delta_2 = \Delta_1$  even with the clover action, and remain in good shape with the OK action.

## 6. Conclusion and Outlook

The results for the inconsistency parameter show that the OK action improves the  $O(\mathbf{p}^4)$  effects, in practice as well as in theory. The hyperfine splitting shows that the OK action significantly improves the higher-dimension chromomagnetic effects on the quarkonium spectrum. For the heavy-light system, the data for the hyperfine splittings at 0.15 fm suffer from statistical errors that are too large to draw any firm conclusion.

The SWME Collaboration plans to determine  $|V_{cb}|$  by calculating  $B \rightarrow D^{(*)} \ell \nu$  semi-leptonic form factors with the OK action and commensurately improved currents. For this purpose, a project

to obtain the improved current relevant to the decay  $B \rightarrow D^* \ell \nu$  at zero recoil is underway [13]. Another component of this plan is to calculate the 1-loop coefficients for  $c_B$  and  $c_E$  in the OK action. A highly optimized conjugate gradient inverter using QUDA is under development [14].

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