Possible Alternatives to Tri-bimaximal Mixing

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Abstract

Possible alternatives to tri-bimaximal mixing are presented based on other symmetry principles, and their predictions for $|U_{e3}|$, $\sin^2 \theta_{12}$ and $\sin^2 \theta_{23}$ are compared to the present neutrino mixing data. In some cases perturbations are required to give better agreement with the data, and the use of a minimal approach is illustrated. Precise experimental determinations of the mixing parameters will be required to decipher the correct mixing pattern and to pin down the appropriate flavor symmetry.

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1   Introduction

The first discoveries of neutrino oscillations arose from observations of the depletions of atmospheric muon-neutrinos [1] and solar electron-neutrinos [2], relative to their expected predictions. In efforts to understand these findings, many theorists adopted top-down approaches in attempts to construct models which would explain the data. For this purpose, various forms of the neutrino and charged lepton mass matrices were postulated, some applied directly to the light left-handed neutrino mass matrix, while other more ambitious efforts invoked the seesaw mechanism partly using also the framework of grand unified models. Examples to constrain the mass matrices involved the assignment of texture zeros, the use of a vertical family symmetry group, and/or the selection of a horizontal flavor symmetry, usually of a continuous type such as $U(1)$, $SU(2)$ or $SU(3)$. The more complete models and their predictions differed by the choice of family and flavor symmetries, and the fermion and Higgs representation assignments made in the construction of the unknown Yukawa interactions needed to extend the Standard Model.

As the oscillation data became more accurate with refinements in the atmospheric [3] and solar [4] neutrino experiments and introduction of land-based reactor [5] and long baseline neutrino [6] experiments, bottom-up approaches to construct models became more feasible. Among the first to realize the mixing data were pointing to a rather simple construction were Harrison, Perkins and Scott [7], who coined the phrase "tri-bimaximal mixing". In this scheme the atmospheric neutrino mixing angle (θ_{23}) is maximal $45°$, the reactor neutrino mixing angle (θ_{13}) vanishes, while the solar neutrino mixing angle is $\theta_{12} \simeq 35.3°$, such that $\sin^2 \theta_{12} = \frac{1}{3}$. With this tri-bimaximal mixing (TBM) texture in mind, many models have been constructed based on the discrete symmetry groups such as S_3, A_4, S_4, T', etc., with a vast majority using A_4 (see Refs. [8, 9] for reviews on flavor symmetries, in particular A_4, and Ref. [10] for a classification of all existing (50+) type I seesaw, type II seesaw and non-seesaw A_4 models). While the tri-bimaximal mixing pattern lies within 1σ of the present experimental fits, the best-fit points require some deviation from that pattern.

It is fair to say that TBM dominates the theoretical literature in flavor model building[1]. We remind the reader that attempts to explain the mixing data based on grand unified models using continuous flavor symmetry groups were also reasonably successful in explaining the mixing data (see Ref. [11] for a list of 13 valid $SO(10)$ models in agreement with current data). This raises the issue whether there indeed exists some hidden flavor symmetry, such as A_4, or whether the nearly observed TBM mixing is accidental in nature. Reference [11] tried to attack this issue from the point of perturbing the neutrino mass matrix m_ν^{TBM} corresponding to TBM. It was argued that when relative corrections to the mass matrix entries are applied, the value of $|U_{e3}|$ can be crucial to distinguish TBM from grand unified theories. A very recent paper [12] has shown that mass matrices which are significantly different from m_ν^{TBM} are also allowed. It is thus important not to focus solely

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[1]The original suggestion of tri-bimaximal mixing was a purely phenomenological Ansatz and only later shown to be obtainable in dedicated flavor models.

on one particular mixing scheme, such as TBM, but to look for other options as well. In any case, it is apparent that very accurate experimental determinations of the neutrino mixing parameters will be required in order to pin down the source of the flavor mixing.

In the spirit of the above considerations, we point out in this letter the existence of a plethora of alternatives to TBM and explore a number of other possibilities for the neutrino mixing matrix. We wish to stress that many of the mixing scenarios that we describe are allowed by the current data equally well. Some of them have been obtained in models with the flavor symmetry specified at the outset, and very often the choice of symmetry group is motivated by geometrical considerations. Good examples here are the two golden ratio possibilities for the solar neutrino mixing angle. Among the other examples we give is trimaximal mixing, where only the second column of the tri-bimaximal mixing matrix with equal flavor contributions is postulated. Variations of this theme make the invariant assumption for the first or third column or one of the three rows. Yet another hypothesis involves quark-lepton complementarity where the quark and neutrino mixing matrices are related. Obviously, one should try to disentangle the huge number of proposed flavor models in order to sort out the correct one, or at least rule out many of the incorrect ones [13].

We should also mention that it is not unlikely that corrections to mixing schemes may apply. Radiative corrections, effects of charged lepton rotations, soft breaking, or "NLO" effects of the underlying flavor models are possibilities. The magnitude of the corrections relies heavily on the models which realize the respective scenarios, and depend on a number of unknown parameters, such as neutrino masses or CP phases. Let us mention, however, that radiative corrections are small for a normal hierarchy of neutrino masses, and that charged lepton rotations play no role if the symmetry basis coincides with the charged lepton mass basis. In principle one could perform for each scenario to be discussed in the following a dedicated analysis of perturbations in analogy, e.g., to the model-independent study for TBM in Ref. [14], or to studies for concrete models in Refs. [15]. In the present letter we neglect the study of these aspects, and rather focus on pointing out the existence of a variety of alternatives to TBM, their possible physics motivation, and the "unperturbed" predictions of the scenarios. In principle, for each scenario considered, one can use the bottom-up approach to determine the neutrino mass matrix and presumably to construct a model based on some discrete flavor symmetry which yields the desired mixing. This is well illustrated, for instance, in the case of tri-bimaximal mixing for which an extensive literature exists in which models based on one of the discrete symmetries mentioned above have been proposed.

The plan of the paper is as follows: for each mixing scenario considered in Section 2, we have plotted the allowed mixing angle ranges and compared them with the present mixing data. Conclusions are drawn in Section 3. Some of the schemes display a high amount of symmetry but require moderate perturbations in order to bring them into compliance with the data, and we have treated the various possibilities for doing so in several cases in the Appendix.

Parameter	Best-fit$^{+1\sigma}_{-1\sigma}$	2σ	3σ
$\sin^2 \theta_{12}$	$0.318^{+0.019}_{-0.016}$	0.29-0.36	0.27-0.38
$\sin^2 \theta_{23}$	$0.500^{+0.070}_{-0.060}$	0.39-0.63	0.36-0.67
$\sin^2 \theta_{13}$	$0.013^{+0.013}_{-0.009}$	≤ 0.039	≤ 0.053

Table 1: Mixing angles and their 1σ, 2σ and 3σ ranges [16].

2 Lepton Mixing Schemes

We begin with the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix, which in general is given by

$$U = U_\ell^\dagger U_\nu \,, \tag{1}$$

where U_ℓ (U_ν) stems from diagonalization of the charged lepton (neutrino) mass matrix. The standard form of the PMNS matrix is

$$U = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{i\delta} & c_{23}\, c_{13} \end{pmatrix} P \,, \tag{2}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with δ the unknown CP-violating Dirac phase. The two equally unknown Majorana phases appear in $P = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$. While the phases are currently unconstrained, the present best-fit values of the mixing angles and their 1σ, 2σ and 3σ ranges [16] are presented in Table 1 (other groups obtain very similar results [17]). The above parameterization of U is obtained by three consecutive rotations:

$$U = R_{23}(\theta_{23})\, \tilde{R}_{13}(\theta_{13}; \delta)\, R_{12}(\theta_{12}) \,, \text{where e.g.,}$$

$$R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_{13}(\theta_{13}; \delta) = \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{i\delta} & 0 & c_{13} \end{pmatrix}. \tag{3}$$

The most popular mixing scenario approximating the current data is the tri-bimaximal one [7, 18]:

$$U_{\mathrm{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \tag{4}$$

corresponding to[2]

$$\sin^2\theta_{12} = \frac{1}{3} \ , \quad \sin^2\theta_{23} = \frac{1}{2} \ , \quad |U_{e3}| = 0 \,. \tag{5}$$

The overwhelming majority of the plethora of models (see [8–10] for a list of references) invokes the symmetry group A_4. One reason is that A_4 is rather economical: it is the smallest discrete group containing a three dimensional irreducible representation (IR). Furthermore, in the flavor basis it can be generated by two generators[3] S and T, one of which is diagonal and leaves the charged lepton mass matrix diagonal, while the other one leaves m_ν^{TBM} invariant [8], where

$$m_\nu^{\mathrm{TBM}} = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B-D) \end{pmatrix} \tag{6}$$

is the most general neutrino mass matrix leading to TBM. A geometrical motivation is provided by noting that A_4 is the symmetry group of the regular tetrahedron, and the angle between two faces is $2\theta_{\mathrm{TBM}}$, where $\sin^2\theta_{\mathrm{TBM}} = \frac{1}{3}$. Models can be constructed in such a way that the Yukawa couplings, and hence the mass matrices, are invariant under certain group elements, which are generated by S and T, which in turn are connected to the symmetry of the geometrical object the group describes. In this way the connection between geometry and flavor physics can arise.

Tri-bimaximal mixing is a variant of the more general μ–τ symmetry, which leaves solar neutrino mixing unconstrained:

$$U_{\mu-\tau} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\frac{\sin\theta_{12}}{\sqrt{2}} & \frac{\cos\theta_{12}}{\sqrt{2}} & -\sqrt{\frac{1}{2}} \\ -\frac{\sin\theta_{12}}{\sqrt{2}} & \frac{\cos\theta_{12}}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \,, \tag{7}$$

corresponding to

$$\sin^2\theta_{23} = \frac{1}{2} \ , \quad |U_{e3}| = 0 \,. \tag{8}$$

From a theoretical point of view, θ_{12} is unconstrained by μ–τ symmetry and hence can be expected to be a number of order one. This is indeed in good agreement with data. A simple Z_2 or S_2 exchange symmetry acting on the neutrino mass matrix suffices to generate μ–τ symmetry. In fact, any symmetry having Z_2 or S_2 as a subgroup can be used, for instance, D_4 [19].

We now turn to other mixing scenarios which serve as alternatives to the tri-bimaximal one. First consider trimaximal mixing and its variants [20–23] (see also [24]). Here a

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[2]To obtain this form of U, it is necessary to insert $\theta_{23} = -\pi/4$ in the standard parameterization (2) of the PMNS matrix. Compared to $\theta_{23} = +\pi/4$, the difference is unphysical, of course. In the following we will use $\theta_{23} = -\pi/4$ whenever we speak about maximal atmospheric mixing.

[3]Some groups require 3 generators.

given row or column of U takes the same form as for tri-bimaximal mixing. The term "trimaximal" was originally used for the case of the second column of the PMNS matrix being identical to the TBM case. The analogous possibilities for the other rows and columns go under the same banner "trimaximal". The notation is such that if the ith column (row) of U has the same form as for TBM, then the scenario is called TM_i (TM^i). In case this applies to the first column of U, the condition is:

$$TM_1 : \quad \begin{pmatrix} |U_{e1}|^2 \\ |U_{\mu1}|^2 \\ |U_{\tau1}|^2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/6 \\ 1/6 \end{pmatrix}. \tag{9}$$

The implications of this Ansatz are [23]

$$\sin^2 \theta_{12} = \frac{1}{3} \frac{1 - 3 |U_{e3}|^2}{1 - |U_{e3}|^2} \simeq \frac{1}{3} \left(1 - 2 |U_{e3}|^2 \right) \tag{10}$$

and

$$\cos \delta \, \tan 2\theta_{23} = -\frac{1 - 5 |U_{e3}|^2}{2\sqrt{2} |U_{e3}| \sqrt{1 - 3 |U_{e3}|^2}} \simeq \frac{-1}{2\sqrt{2} |U_{e3}|} \left(1 - \frac{7}{2} |U_{e3}|^2 \right). \tag{11}$$

For the second column the originally-named trimaximal condition is

$$TM_2 : \quad \begin{pmatrix} |U_{e2}|^2 \\ |U_{\mu2}|^2 \\ |U_{\tau2}|^2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \tag{12}$$

leading to [21, 23]

$$\sin^2 \theta_{12} = \frac{1}{3} \frac{1}{1 - |U_{e3}|^2} \geq \frac{1}{3} \tag{13}$$

and

$$\cos \delta \, \tan 2\theta_{23} = \frac{2 \cos \theta_{13} \cot 2\theta_{13}}{\sqrt{2 - 3 \sin^2 \theta_{13}}} = \frac{1 - 2 |U_{e3}|^2}{|U_{e3}| \sqrt{2 - 3 |U_{e3}|^2}}$$
$$\simeq \frac{1}{\sqrt{2}} \frac{1}{|U_{e3}|} \left(1 - \frac{5}{4} |U_{e3}|^2 \right). \tag{14}$$

If we would insist that the third column of U_{TBM} remains invariant instead, i.e., $|U_{e3}|^2 = 0$, $|U_{\mu3}|^2 = |U_{\tau3}|^2 = \frac{1}{2}$, then $\theta_{13} = 0$, $\theta_{23} = \pi/4$, while θ_{12} is a free parameter and δ is arbitrary. This case (TM_3 in our notation) is nothing other than μ–τ symmetry.

It was argued [20] that models based on flavor symmetries which have A_4 as a subgroup should be possible for TM_2. For TM_1 and TM_3, these groups are S_4 and S_3, respectively. Models based on flavor symmetry groups $\Delta(27)$ [21] and S_3 [22] have been constructed for the trimaximal scenario TM_2.

Now consider the case where one of the rows of the tri-bimaximal mixing matrix remains invariant [23]. We start with the case of the first row in U_{TBM} remaining invariant, denoting this by TM^1,

$$TM^1 : \quad \left(|U_{e1}|^2, \, |U_{e2}|^2, \, |U_{e3}|^2 \right) = \left(\frac{2}{3}, \, \frac{1}{3}, \, 0 \right). \tag{15}$$

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Here θ_{23} is a free parameter, while $\sin^2\theta_{12} = \frac{1}{3}$, as well as $\theta_{13} = \delta = 0$.

If we consider only the second or third row invariant, we can again correlate all four mixing parameters. Starting with the second row, i.e.,

$$\text{TM}^2: \qquad \left(|U_{\mu 1}|^2,\ |U_{\mu 2}|^2,\ |U_{\mu 3}|^2\right) = \left(\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2}\right), \tag{16}$$

one immediately finds from $|U_{\mu 3}|^2 = \frac{1}{2}$:

$$\sin^2\theta_{23} = \frac{1}{2\left(1 - |U_{e3}|^2\right)} \simeq \frac{1}{2}\left(1 + |U_{e3}|^2\right) \geq \frac{1}{2}, \tag{17}$$

with atmospheric neutrino mixing on the "dark side" ($\theta_{23} \geq \pi/4$). The second correlation among the mixing parameters is

$$\sin^2\theta_{12} \simeq \frac{1}{3} - \frac{2\sqrt{2}}{3}\,|U_{e3}|\,\cos\delta\ + \frac{1}{3}\,|U_{e3}|^2\,\cos 2\delta\,. \tag{18}$$

On the other hand, with the third row remaining invariant,

$$\text{TM}^3: \qquad \left(|U_{\tau 1}|^2,\ |U_{\tau 2}|^2,\ |U_{\tau 3}|^2\right) = \left(\frac{1}{6},\ \frac{1}{3},\ \frac{1}{2}\right), \tag{19}$$

the atmospheric neutrino mixing is now predicted on the "bright side," ($\theta_{23} \leq \pi/4$):

$$\sin^2\theta_{23} = \frac{1 - 2\,|U_{e3}|^2}{2\left(1 - |U_{e3}|^2\right)} \simeq \frac{1}{2}\left(1 - |U_{e3}|^2\right) \leq \frac{1}{2}, \tag{20}$$

while the solar neutrino mixing is correlated with $|U_{e3}|$ and δ according to

$$\sin^2\theta_{12} \simeq \frac{1}{3} + \frac{2\sqrt{2}}{3}\,|U_{e3}|\,\cos\delta\ + \frac{1}{3}\,|U_{e3}|^2\,\cos 2\delta\,. \tag{21}$$

We also note the recently proposed tetramaximal mixing scheme (T^{4}M) [25]. Its name stems from the fact that it can be obtained by four consecutive rotations, each having a maximal angle of $\pi/4$, and properly chosen phases associated with the rotations:

$$U_{\text{tetra}} = R_{23}(\pi/4; \pi/2)\, R_{13}(\pi/4; 0)\, R_{12}(\pi/4; 0)\, R_{13}(\pi/4; \pi)\,. \tag{22}$$

The notation of the rotation matrices is defined in Eq. (3). The definite predictions are[4]

$$\delta = \pi/2, \quad \sin^2\theta_{23} = \tfrac{1}{2}\,, \quad \sin^2\theta_{12} = (\tfrac{5}{2} + \sqrt{2})^{-1} \simeq 0.255\,,$$
$$|U_{e3}|^2 = \tfrac{1}{4}\,(\tfrac{7}{4} - \sqrt{2})\ \sin^2\theta_{12} = \tfrac{1}{4}\,(\tfrac{3}{2} - \sqrt{2}) \simeq 0.021\,. \tag{23}$$

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[4]By multiplying a fifth maximal rotation $R_{12}(\pi/4; 2\pi/3)$ to the right of U_{tetra} one could obtain "quintamaximal mixing", which has more complicated predictions: $\sin^2\theta_{12} = (3 + \sqrt{2})/(10 + 4\sqrt{2}) \simeq 0.282$, $|U_{e3}|^2 = (3 - 2\sqrt{2})/8 \simeq 0.021$, $\sin^2\theta_{23} = \tfrac{1}{2}$ and $J_{\text{CP}} = (3\sqrt{2} - 2)/256 \simeq 0.0088$. Here $J_{\text{CP}} = \text{Im}\{U_{e1}\, U_{\mu 2}\, U_{e2}^*\, U_{\mu 1}^*\}$ is the usual measure for CP violation.

Another interesting possible property of U is that it might be symmetric: $U = U^T$. One can show that there follows one constraint on the mixing parameters [26]:

$$|U_{e3}| = \frac{\sin\theta_{12} \sin\theta_{23}}{\sqrt{1 - \sin^2\delta \cos^2\theta_{12} \cos^2\theta_{23}} + \cos\delta \cos\theta_{12} \cos\theta_{23}} \ . \tag{24}$$

Scenarios in which this happens at lowest order, for instance, reflect "Quark-Lepton Universality" [27]. Here it is proposed that down quarks and charged leptons are diagonalized by the same matrix V and that the down quark mass matrix is hermitian. Furthermore, $m_D = m_{\rm up} = m_{\rm up}^T$, and M_R is also diagonalized by V, where m_D (M_R) is the Dirac (Majorana) mass matrix in the type I seesaw mechanism. With these assumptions it follows that the PMNS matrix is symmetric. In general, U is symmetric if $U_\ell = S U_\nu^\dagger$, where S is a symmetric and unitary matrix. Moreover, if m_ν^* and (symmetric) m_ℓ are diagonalized by the same matrix, again the PMNS matrix is symmetric.

Several proposed mixing matrices single out the solar mixing angle for special treatment. In the case of bimaximal mixing (BM), $\sin^2\theta_{12} = 1/2$, with the same atmospheric and reactor neutrino mixing angles as in the case of tri-bimaximal mixing or μ–τ symmetry. Hence the mixing matrix has the form [28]

$$U_{\rm BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{25}$$

In [29] it has been shown that for instance one can use the discrete symmetry S_3 to construct such a mixing matrix. While the value $\sin^2\theta_{12} = \frac{1}{2}$ is ruled out by close to 10σ, this mixing scenario has recently been revived in the form of a model based on S_4 [30]. Here the two generators of the group are chosen such that one is diagonal and the other one leaves $m_\nu^{\rm BM}$ invariant, where $m_\nu^{\rm BM}$ is the most general mass matrix leading to bimaximal mixing, which is obtained from Eq. (6) by removing B. Bimaximal mixing can be corrected by charged lepton corrections, leading to QLC scenarios (see below).

Another possibility proposed here is "hexagonal mixing" (HM), where $\theta_{12} = \pi/6$, or $\sin^2\theta_{12} = 1/4$. In this case, again with maximal atmospheric and vanishing reactor neutrino mixings, the mixing matrix is given by

$$U_{\rm HM} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{26}$$

Here D_{12} is an appropriate discrete flavor symmetry. The angle $\theta_{12} = \pi/6$ is obviously the external angle of the dodecagon, whose symmetry group is D_{12}. One can also use D_6, where the external angle is $\pi/3$. Both BM and HM require corrections to bring them into agreement with current global fits. A strategy to do this is given in the Appendix. Note

that this requires a larger correction for bimaximal mixing than for hexagonal mixing, where the necessary correction is moderate.

There are two proposals which link solar neutrino mixing with the golden ratio angle $\varphi = (1 + \sqrt{5})/2$:

$$\varphi_1 : \quad \cot \theta_{12} = \quad \varphi \Rightarrow \sin^2 \theta_{12} = \frac{1}{1 + \varphi^2} \simeq 0.276 \,, \tag{27}$$

$$\varphi_2 : \quad \cos \theta_{12} = \quad \frac{\varphi}{2} \Rightarrow \sin^2 \theta_{12} = \frac{1}{4} \left(3 - \varphi\right) \simeq 0.345 \,. \tag{28}$$

The observation that the first relation is allowed has been made in Refs. [31]. Interestingly, the first relation may be obtained with the choice of A_5 as the flavor symmetry group, as noted in Ref. [32]. This follows since A_5 is isomorphic to the symmetry group of the icosahedron whose 12 vertices separated by edge-length 2 have Cartesian coordinates specified by $(0, \pm 1, \pm \varphi)$, $(\pm 1, \pm \varphi, 0)$ and $(\pm \varphi, 0, \pm 1)$. Indeed, one can write the generators of one of the three-dimensional IRs of A_5 in terms of φ [32]. One could in principle assign the values $\sin^2 \theta_{23} = \frac{1}{2}$ and $U_{e3} = 0$ to the two golden ratio relations.

The second golden ratio relation was proposed first in [33]. In Ref. [34] a model based on the discrete flavor symmetry D_{10} has been applied to obtain this angle. Believe it or not, $\cos \theta_{12} = \frac{\varphi}{2}$ implies nothing other than $\theta_{12} = \pi/5$, and therefore arguments similar to those given above for hexagonal mixing apply: the angle $\pi/5$ is the external angle of a decagon and D_{10} is its rotational symmetry group.

The final class of alternative mixing scenarios we consider deals with Quark-Lepton Complementarity (QLC), which can be used to relate the quark and lepton mixing matrices. The most naive form relates the solar neutrino mixing angle, θ_{12}, to the quark Cabibbo angle, θ_{12}^q, by [35, 36]

$$\mathrm{QLC}_0 : \quad \theta_{12} = \frac{\pi}{4} - \theta_{12}^q \Rightarrow \sin^2 \theta_{12} \simeq 0.280 \,. \tag{29}$$

One may assume a similar relation for the 23-sector, $\theta_{23} = \frac{\pi}{4} - \theta_{23}^q$, leading to $\sin^2 \theta_{23} \simeq 0.459$.

These QLC relations can be approximately obtained by multiplying a bimaximal matrix, see Eq. (25), with the CKM (or a CKM-like) matrix. For definiteness, we stick to the CKM matrix in what follows. It is given in the Wolfenstein parametrization [37] by

$$V = \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 \left(\rho - i \eta\right) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\ A \lambda^3 \left(1 - \rho + i \eta\right) & -A \lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4). \tag{30}$$

In analogy to the PMNS matrix it is a product of two unitary matrices, $V = V_{\mathrm{up}}^\dagger V_{\mathrm{down}}$, where V_{up} (V_{down}) is associated with the diagonalization of the up- (down-) quark mass

matrix. As reported in [38] the best-fit values and the 1σ, 2σ and 3σ ranges of the parameters $\lambda, A, \bar{\rho}, \bar{\eta}$ are

$$\lambda = \sin\theta_C = 0.2272^{+0.0010,\,0.0020,\,0.0030}_{-0.0010,\,0.0020,\,0.0030} \,,$$

$$A = 0.809^{+0.014,\,0.029,\,0.044}_{-0.014,\,0.028,\,0.042} \,,$$

$$\bar{\rho} = 0.197^{+0.026,\,0.050,\,0.074}_{-0.030,\,0.087,\,0.133} \,,$$

$$\bar{\eta} = 0.339^{+0.019,\,0.047,\,0.075}_{-0.018,\,0.037,\,0.057} \,,$$

(31)

where $\bar{\rho} = \rho\,(1 - \lambda^2/2)$ and $\bar{\eta} = \eta\,(1 - \lambda^2/2)$. From the relation $U = V^\dagger U_{\rm BM}$ one finds[5]

$$\text{QLC}_1: \quad \sin^2\theta_{12} \simeq \tfrac{1}{2} - \tfrac{\lambda}{\sqrt{2}}\cos\phi + \mathcal{O}(\lambda^3) \,, \quad |U_{e3}| \simeq \tfrac{\lambda}{\sqrt{2}} + \mathcal{O}(\lambda^3) \,,$$

$$\sin^2\theta_{23} \simeq \tfrac{1}{2} - \tfrac{\lambda^2}{4}(1 + 4\,A\,\cos(\phi - \omega)) + \mathcal{O}(\lambda^4) \,,$$

(32)

where λ is the sine of the leading 12-entry in V, i.e., the sine of the Cabibbo angle. The phases ϕ and ω are not related to the phase in the CKM matrix but are relative phases [39] between $U_\ell = V$ and $U_{\rm BM}$, with ϕ corresponding to the Dirac phase in neutrino oscillations. With the Jarlskog invariant serving as the measure of leptonic CP violation,

$$J_{\rm CP} = \text{Im}\{U_{e1}\,U_{\mu 2}\,U_{e2}^*\,U_{\mu 1}^*\} \simeq \frac{\lambda}{4\sqrt{2}}\sin\phi + \mathcal{O}(\lambda^3) \,,$$

(33)

numerically one finds $|U_{e3}| \simeq 0.160$, $\sin^2\theta_{12} \gtrsim 0.339$, and $|J_{\rm CP}| \lesssim 0.0274$, since $\phi \lesssim \pi/4.25$ for $\sin^2\theta_{12}$ to be in its allowed 3σ range.

To obtain this scenario in a seesaw framework[6], an approach somewhat similar to that for Quark-Lepton Universality discussed above is possible [35,36]: diagonalization of m_ν is achieved via $m_\nu = U_{\rm BM}^*\,m_\nu^{\rm diag}\,U_{\rm BM}^\dagger$ and produces exact bimaximal mixing. The U_ℓ matrix diagonalizing the charged lepton mass matrix m_ℓ corresponds to the CKM matrix V. With $m_\ell = m_{\rm down}^T$, where $m_{\rm down}$ is the down-quark mass matrix, it follows that the up-quark mass matrix $m_{\rm up}$ is real and diagonal. It is assumed to correspond to the Dirac mass matrix in the type I seesaw formula, and this in turn fixes M_R.

Then there is the second QLC scenario, in which the PMNS matrix is given by $U_{\rm BM}^*\,V^\dagger$. One finds

$$\text{QLC}_2: \quad \sin^2\theta_{12} \simeq \tfrac{1}{2} - \lambda\cos\phi + \mathcal{O}(\lambda^3) \,, \quad |U_{e3}| \simeq \tfrac{A\lambda^2}{\sqrt{2}} + \mathcal{O}(\lambda^3) \,,$$

$$\sin^2\theta_{23} \simeq \tfrac{1}{2} + \tfrac{A\lambda^2}{\sqrt{2}}\cos\phi' + \mathcal{O}(\lambda^3) \,,$$

(34)

where λ is the 12-entry, and $A\,\lambda^2$ the 23-entry of V. Again the phases ϕ and ϕ' are unrelated to the phase in the CKM matrix. Note that there is now a correlation between leptonic CP violation and quark CKM mixing:

$$J_{\rm CP} \simeq \frac{A\,\lambda^2}{4\sqrt{2}}\sin\phi' + \mathcal{O}(\lambda^4) \,.$$

(35)

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[5]Sometimes a Georgi-Jarlskog factor of $\frac{1}{3}$ appears in model realizations of QLC, in which case the results to be presented can be obtained approximately by replacing λ with $\lambda/3$.

[6]Seesaw realizations of QLC scenarios are studied in detail in [40].

Scenario	$\sin^2\theta_{12}$		$\sin^2\theta_{23}$		$\sin^2\theta_{13}$	
	min	max	min	max	min	max
TBM	0.333		0.500		0.000	
$\boldsymbol{\mu-\tau}$	—		0.500		0.000	
TM$_1$	0.296	0.333	**		—	
TM$_2$	0.333	0.352	**		—	
TM$_3$	—		0.500		0.000	
TM^1	0.333		—		0.000	
TM^2	**		0.500	0.528	—	
TM^3	**		0.472	0.500	—	
T^{4}M	0.255		0.500		0.021	
U=U$^\mathbf{T}$	0.000	0.389	0.000	0.504	0.0343	0.053
BM	0.500		0.500		0.000	
HM	0.250		0.500		0.000	
$\boldsymbol{\varphi_1}$	0.276		0.500		0.000	
$\boldsymbol{\varphi_2}$	0.345		0.500		0.000	
QLC$_0$	0.280		0.459		—	
QLC$_1$	0.331	0.670	0.442	0.534	0.023	0.029
QLC$_2$	0.276	0.726	0.462	0.540	0.0005	0.0016

Table 2: Predictions for $\sin^2\theta_{12}$, $\sin^2\theta_{23}$, and $\sin^2\theta_{13} = |U_{e3}|^2$ for the different mixing scenarios considered. The appearance of the symbol — indicates a free parameter of the model, while the symbol ** indicates a prediction which depends upon the unknown $|U_{e3}|$ and phase δ. The min and max values listed are determined from the presently allowed 3σ range for $|U_{e3}|$.

Here the type I seesaw realization goes as follows [36]: diagonalization of m_ν is achieved via $m_\nu = U_\nu^* m_\nu^{\text{diag}} U_\nu^\dagger$ and U_ν is related to V (in the sense that $U_\nu = V^\dagger$). The charged leptons are diagonalized by $U_\ell = U_{\text{BM}}^T$. This in turn can be achieved when $V_{\text{up}} = V^\dagger$, therefore V_{down} must be the unit matrix. With the definition of $M_R = V_R^* M_R^{\text{diag}} V_R^\dagger$, where $V_R = V_{\text{up}}^*$, we have $m_{\text{up}} = m_D = V_{\text{up}} m_{\text{up}}^{\text{diag}} V$, and since $V_{\text{up}} = V^\dagger$ the neutrino mass matrix $m_\nu = -m_D^T M_R^{-1} m_D$ is diagonalized by the CKM matrix. Note that QLC1, QLC2 and Quark-Lepton Universality require that the eigenvalues of the fermion mass matrices differ even though some of the mixing angles are the same. Such mass matrices may, e.g., be "form diagonalizable" ones [41], which means that the mixing matrix which diagonalizes the mass matrix is independent of the values of the eigenvalues (such as for bimaximal or TBM).

We summarize the numerical values of all scenarios considered here in Table 2. For

some of the scenarios, all three mixing angles are predicted, while in others one or two of the mixing parameters remain free parameters (indicated by the symbol −) and are not determined by the models in question. Aside from the simple μ–τ symmetry case also realized with the TM$_3$ scenario, these situations arise when the presently unknown experimental reactor neutrino angle, θ_{13}, appearing in the mixing element $|U_{e3}| = \sin\theta_{13}$ remains unpredicted. Where possible, minimum and maximum values of the mixing parameters are determined by adopting the present experimental 3σ range for the mixing element $|U_{e3}|$, and in the cases of the QLC$_1$ and QLC$_2$ models, also for the Wolfenstein parameters. For four of the models, one of the mixing angles is constrained by the other mixing parameters, cf. Eqs. (11), (14), (18), and (21), but the actual numerical value relies not only on one knowing $|U_{e3}|$ but also the unknown phase δ. Such constrained predictions are indicated by the symbol ** in Table 2.

In Figs. 1, 2 and 3 we plot the ranges or values of the three mixing variables, $\sin^2\theta_{12}$, $\sin^2\theta_{23}$, and $|U_{e3}|$, respectively that can be obtained for each of the scenarios by varying, if necessary, the other variables over their present 3σ experimental range. The experimentally allowed best-fit values, 1σ and 3σ ranges of the variables are indicated by solid or broken horizontal lines as shown in the figures. Two-dimensional plots are given in Figs. 4, 5, and 6 as functions of $\sin^2\theta_{12}$ vs. $|U_{e3}|$, $\sin^2\theta_{23}$ vs. $|U_{e3}|$, and $\sin^2\theta_{23}$ vs. $\sin^2\theta_{12}$, respectively. The correlations between those observables can be crucial to distinguish scenarios with similar predictions.

It is clear from the figures that most of the models cover the presently allowed ranges of the mixing angles, with the notable exceptions of the bimaximal and hexagonal mixing models, BM and HM. For these models, one needs to make perturbations on the zeroth order results given in Table 2. We present in Appendix A a simple procedure to perturb the hexagonal and bimaximal mixing matrices, as well as the relevant procedure for the quark-lepton complementarity models, in order to bring their results into better agreement with the data.

3   Conclusions

With more refined neutrino mixing data available, it is clear that TBM gives a reasonably accurate lowest order approximation to the PMNS mixing matrix. With this in mind, many authors have constructed top-down models based on some discrete flavor symmetry group which yield TBM mixing as a natural consequence. Of the possible choices, the A_4 group appears to be the most favored choice based on its simplicity.

We have argued in this paper, however, that other possible approximations to the mixing matrix exist such as trimaximal mixing or its variants, tetramaximal mixing, a symmetric mixing matrix, bimaximal and hexagonal mixings, and mixings based on the golden ratio angle or quark-lepton complementarity. Many of these scenarios have already been discussed in the literature, but we have compiled this list in order to make easy comparisons of their predictions. For those requiring perturbations to bring them into better agreement with the data, we have illustrated how triminimal perturbations of the

bimaximal, and hexagonal mixings or quark-lepton complementarity, for example, can accomplish this. For each one of the starting mixing matrix assumptions, one can then use a bottom-up approach to determine the appropriate neutrino mass matrix from which a suitable discrete flavor symmetry will presumably reproduce the observed mixing matrix.

The theoretical literature focusses heavily on TBM, and it would be dangerous to avoid looking for and studying alternatives. We hope that the present paper contributes to the required attention on alternatives.

Acknowledgments

A Perturbing Hexagonal and Bimaximal Mixing

We begin by discussing the hexagonal mixing Ansatz, defined by

$$\theta_{12}^\ell = \frac{\pi}{6} = 30° \Rightarrow \sin^2\theta_{12}^\ell = \frac{1}{4}\,, \tag{A1}$$

together with maximal θ_{23}^ℓ and $\theta_{13}^\ell = 0$. From now on we denote lepton (quark) mixing angles with a superscript ℓ (q). For this scenario the unperturbed mixing matrix in the lepton mass basis reads

$$U_{\text{HM}} = \begin{pmatrix} \sqrt{\frac{3}{4}} & \frac{1}{2} & 0 \\ -\frac{1}{2\sqrt{2}} & \sqrt{\frac{3}{8}} & -\sqrt{\frac{1}{2}} \\ -\frac{1}{2\sqrt{2}} & \sqrt{\frac{3}{8}} & \sqrt{\frac{1}{2}} \end{pmatrix} P\,, \tag{A2}$$

where the Majorana phases are contained in $P = \text{diag}(1, e^{i\alpha}, e^{i\beta})$. The mass matrix in the charged lepton basis is given by $m_\nu^0 = U^* m_\nu^{\text{diag}} U^\dagger$ and has the texture

$$m_\nu^0 = \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A + \sqrt{\frac{8}{3}}B + D) & \frac{1}{2}(A + \sqrt{\frac{8}{3}}B - D) \\ \cdot & \cdot & \frac{1}{2}(A + \sqrt{\frac{8}{3}}B + D) \end{pmatrix}\,, \tag{A3}$$

where the masses and Majorana phases are contained in

$$A - \sqrt{\frac{2}{3}}B = m_1\,, \quad A + \sqrt{6}B = m_2\, e^{-2i\alpha}\,, \quad D = m_3\, e^{-2i\beta}\,. \tag{A4}$$

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We can also write

$$m_\nu^0 = \frac{m_1}{4} \begin{pmatrix} 3 & -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} \\ \cdot & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \frac{1}{2} \end{pmatrix} + \frac{m_2\,e^{-2i\alpha}}{4} \begin{pmatrix} 1 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \\ \cdot & \frac{3}{2} & \frac{3}{2} \\ \cdot & \cdot & \frac{3}{2} \end{pmatrix} + \frac{m_3\,e^{-2i\beta}}{2} \begin{pmatrix} 0 & 0 & 0 \\ \cdot & 1 & -1 \\ \cdot & \cdot & 1 \end{pmatrix}$$

$$= m_1\,\Phi_1\,\Phi_1^T + m_2\,e^{-2i\alpha}\,\Phi_2\,\Phi_2^T + m_3\,e^{-2i\beta}\,\Phi_3\,\Phi_3^T\,,$$

where $\Phi_{1,2,3}$ are the columns of the mixing matrix. In this limit the ee element of m_ν^0, whose magnitude governs the rate of neutrino-less double beta decay vanishes when the Majorana phase is such that $e^{-2i\alpha} = -1$ and in addition the relation $m_1 = \frac{1}{3}m_2$, or $m_1^2 = \Delta m_\odot^2/8$ holds.

Independent on the source of perturbation, the most general way to describe deviations from hexagonal mixing is [42] (see also [43])

$$U = R_{23}(-\pi/4)\,U_\epsilon\,R_{12}(\pi/6)\,, \quad \text{where } U_\epsilon = R_{23}(\epsilon_{23}^\ell)\,\tilde{R}_{13}(\epsilon_{13}^\ell;\delta^\ell)\,R_{12}(\epsilon_{12}^\ell)\,. \tag{A5}$$

Note that the order of the small rotations in U_ϵ is chosen such that it corresponds to the order of rotations in the usual description of a mixing matrix. This "triminimal" [42] parametrization implies that each small parameter is responsible for only one observable[7] . The observables are obtained from Eq. (A5) as follows:

$$\sin^2\theta_{12}^\ell = \frac{1}{4}\left(\cos\epsilon_{12}^\ell + \sqrt{3}\,\sin\epsilon_{12}^\ell\right)^2 \simeq \frac{1}{4}\left(1 + 2\sqrt{3}\,\epsilon_{12}^\ell + 3\,(\epsilon_{12}^\ell)^2\right)\,,$$
$$\sin^2\theta_{23}^\ell = \frac{1}{2} - \cos\epsilon_{23}^\ell\,\sin\epsilon_{23}^\ell \simeq \frac{1}{2} - \epsilon_{23}^\ell\,, \tag{A6}$$
$$U_{e3} = \sin\epsilon_{13}^\ell\,e^{-i\delta^\ell}\,.$$

Note that U_{e3} agrees with its form in the usual parameterization and that the deviation from maximal atmospheric mixing is to very good precision given by ϵ_{23}^ℓ. Regarding solar neutrino mixing, the values $\sin^2\theta_{12}^\ell$ of 0.318, 0.302, 0.337, 0.27, 0.38 and $\frac{1}{3}$ are obtained for $\epsilon_{12}^\ell = 0.076, 0.058, 0.096, 0.023, 0.141$, and 0.092.

In the same way we can perturb the bimaximal mixing matrix, given by Eq. (25). The triminimally perturbed bimaximal mixing matrix can be written as

$$U = R_{23}(-\pi/4)\,U_\epsilon\,R_{12}(\pi/4)\,, \tag{A7}$$

with U_ϵ the same as in Eq. (A5). The observables are obtained as

$$\sin^2\theta_{12}^\ell = \left(\frac{1}{2} + \sin\epsilon_{12}^\ell\,\cos\epsilon_{12}^\ell\right) \simeq \frac{1}{2} + \epsilon_{12}^\ell\,,$$
$$\sin^2\theta_{23}^\ell = \left(\frac{1}{2} - \sin\epsilon_{23}^\ell\,\cos\epsilon_{23}^\ell\right) \simeq \frac{1}{2} - \epsilon_{23}^\ell\,, \tag{A8}$$
$$U_{e3} = \sin\epsilon_{13}^\ell\,e^{-i\delta^\ell}\,.$$

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[7]A similar strategy may be applied to tetra-maximal mixing, where θ_{12} lies slightly below the current 3σ range.

Compared to the hexagonal mixing scenario, the values $\sin^2\theta_{12}^\ell$ of 0.318, 0.302, 0.337, 0.27, 0.38 and $\frac{1}{3}$ are obtained for $\epsilon_{12}^\ell = -0.186, -0.204, -0.166, -0.239, -0.121$, and -0.170.

Returning to hexagonal mixing, one may discuss a related parametrization for the CKM matrix in the spirit of QLC. Namely, with the requirement that the 12-mixing angles of the quark and lepton sector add up to 45 degrees, it follows automatically that

$$(\theta_{12}^q)^0 = 15° = \frac{\pi}{12} \Rightarrow \sin(\theta_{12}^q)^0 = \frac{\sqrt{3}-1}{2\sqrt{2}} = 0.2588 , \tag{A9}$$

Note that at zeroth order $\theta_{12}^\ell = 2\theta_{12}^q$. There are models in the literature leading to this angle $(\theta_{12}^q)^0$ [44]. In the spirit of triminimality, we can describe the necessary but small deviations of this scheme with

$$V = R_{23}(\epsilon_{23}^q) \, \tilde{R}_{13}(\epsilon_{13}^q; \delta^q) \, R_{12}(\epsilon_{12}^q) \, R_{12}(\pi/12) . \tag{A10}$$

The sine of the 12-mixing angle is given by

$$\sin\theta_{12}^q = \frac{1}{2}\sqrt{2 - \sqrt{3}\cos 2\epsilon_{12}^q + \sin 2\epsilon_{12}^q} \simeq \frac{\sqrt{3}-1}{2\sqrt{2}}\left(1 + (2+\sqrt{3})\,\epsilon_{12}^q\right). \tag{A11}$$

Note that the last expression is equivalent to $\sin\theta_{12}^q \simeq \sin(\theta_{12}^q)^0 + \epsilon_{12}^q \cos(\theta_{12}^q)^0$. Numerically we have $\sin\theta_{12}^q \simeq 0.2588 + 0.9659\,\epsilon_{12}^q$, so that ϵ_{12}^q can be almost directly identified with the deviation of the sine of Cabibbo angle from $\frac{\sqrt{3}-1}{2\sqrt{2}}$. In order to bring $\sin\theta_{12}^q$ into the observed 1σ or 3σ range given in Eq. (31) one requires

$$\epsilon_{12}^q = -0.0326^{+0.00102,0.00308}_{-0.00102,0.00308} . \tag{A12}$$

Note that here ϵ_{12}^q is negative, while ϵ_{12}^ℓ (see Eq. (A6)) is positive. Choosing the tempting value $\epsilon_{12}^\ell = -\epsilon_{12}^q$ gives $\sin^2\theta_{12}^\ell \simeq 0.279$.

We finish by noting an interesting observation made in Ref. [45]: taking the golden ratio relation φ_1 ($\tan\theta_{12}^\ell = 1/\varphi$) at face value, and assuming QLC ($\theta_{12}^\ell + \theta_{12}^q = \pi/4$) gives

$$\tan\theta_{12}^q = \tan(\pi/4 - \theta_{12}^\ell) = \frac{1 - 1/\varphi}{1 + 1/\varphi} = \frac{1}{\varphi^3} , \tag{A13}$$

or $\sin\theta_{12}^q \simeq 0.2298$. Hence, the golden ratio may appear in the quark sector as well.

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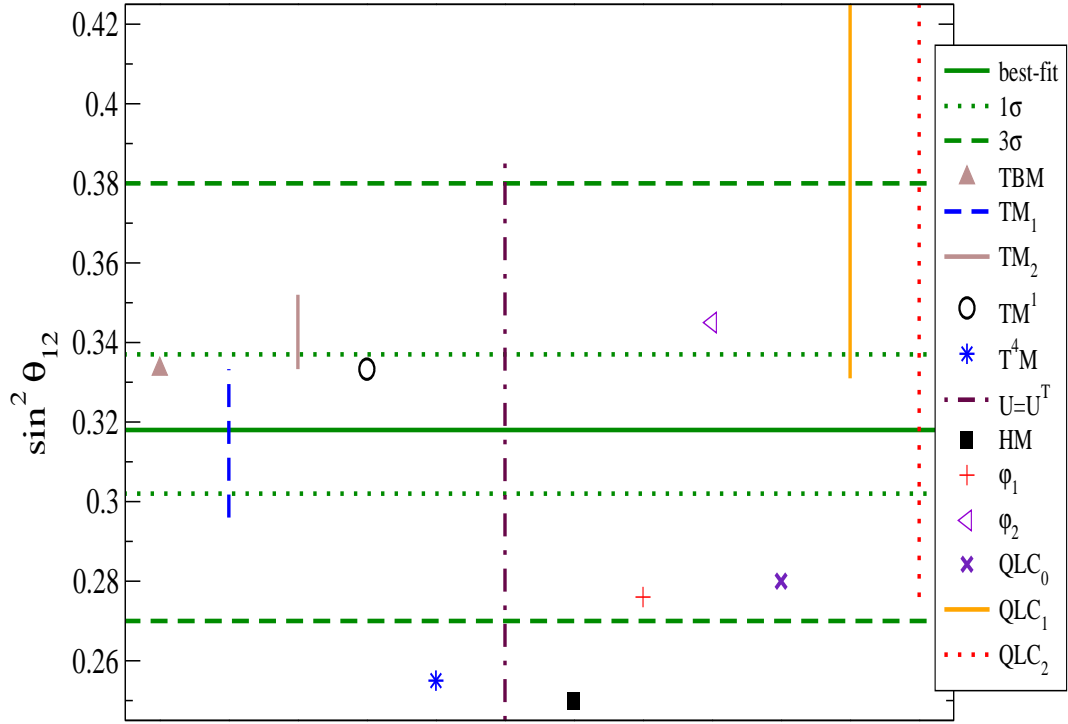
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Figure 1: Predictions for $\sin^2 \theta_{12}$ of the mixing scenarios discussed in the text. For some of the scenarios $\sin^2 \theta_{12}$ depends on the other mixing parameters. Varying them in their experimentally allowed 3σ ranges gives the plotted ranges of $\sin^2 \theta_{12}$.
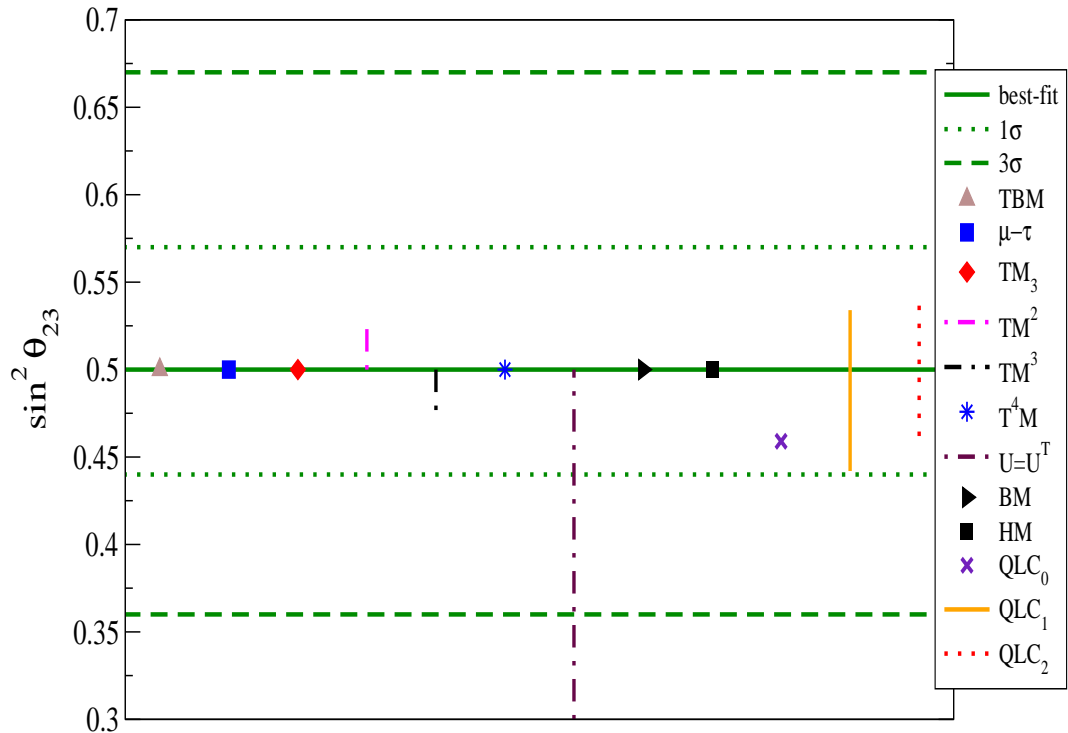

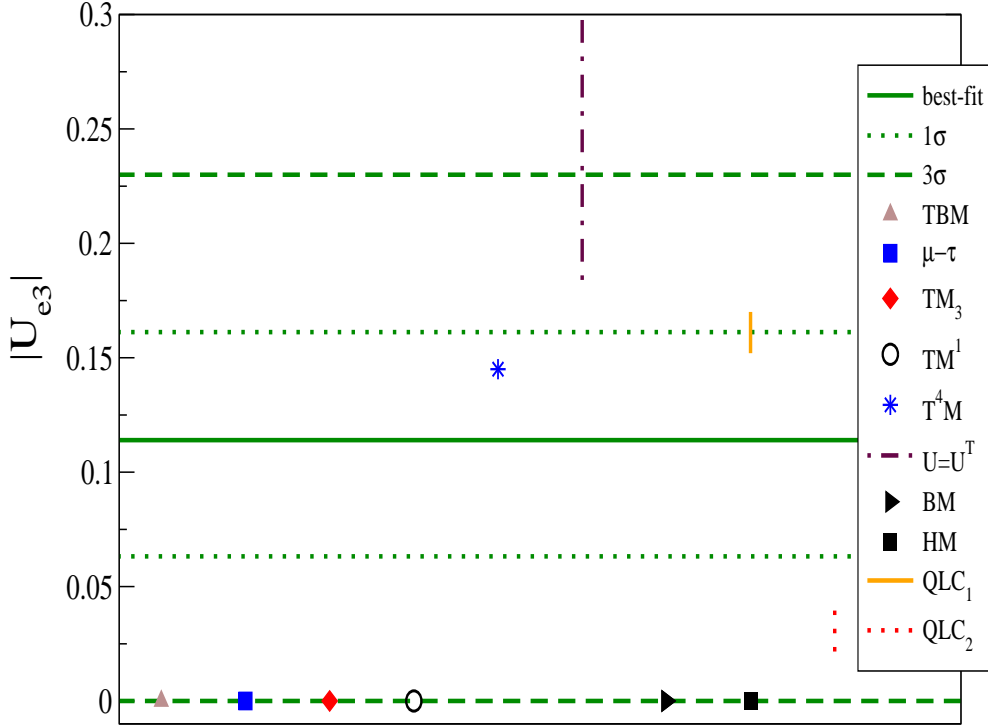Figure 2: Same as Fig. 1, but now for $\sin^2 \theta_{23}$.
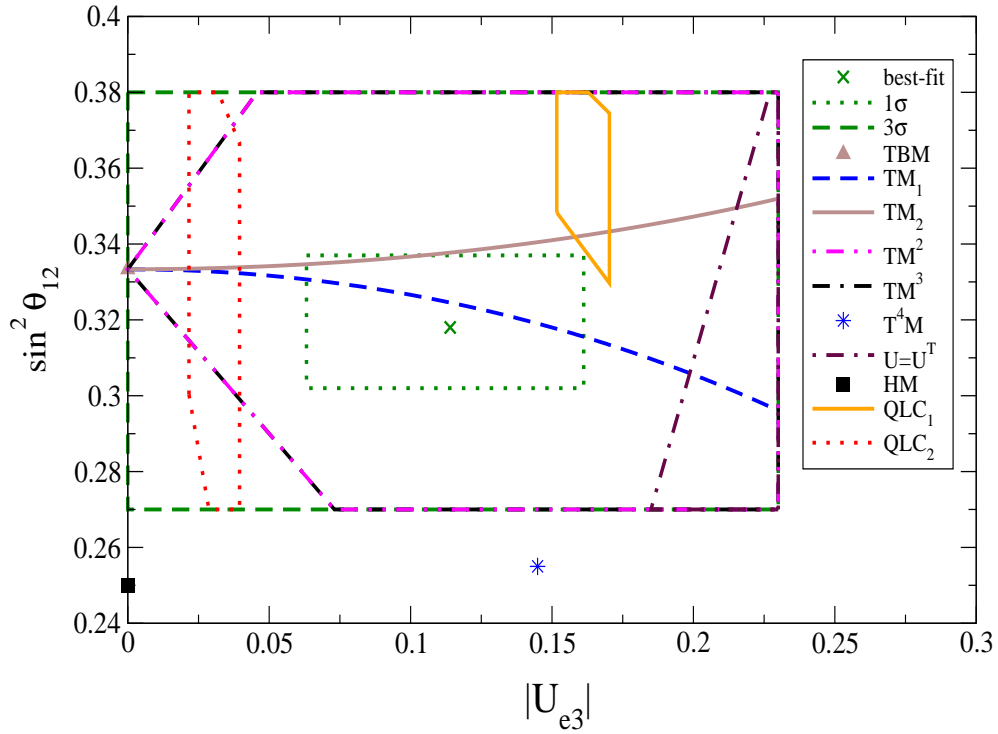
Figure 3: Same as Fig. 1, but now for $|U_{e3}|$.

Figure 4: Correlations between $\sin^2\theta_{12}$ and $|U_{e3}|$ constrained by the experimental 3σ ranges of the mixing parameters. For scenarios where $\sin^2\theta_{12}$ depends also on the unknown Dirac phase δ the whole area inside the corresponding lines is possible, while in the case of $TM_{1,2}$ only parameter combinations lying on the dashed (blue) and continuous (brown) lines, respectively, are allowed. TM^2 and TM^3 are here indistinguishable.
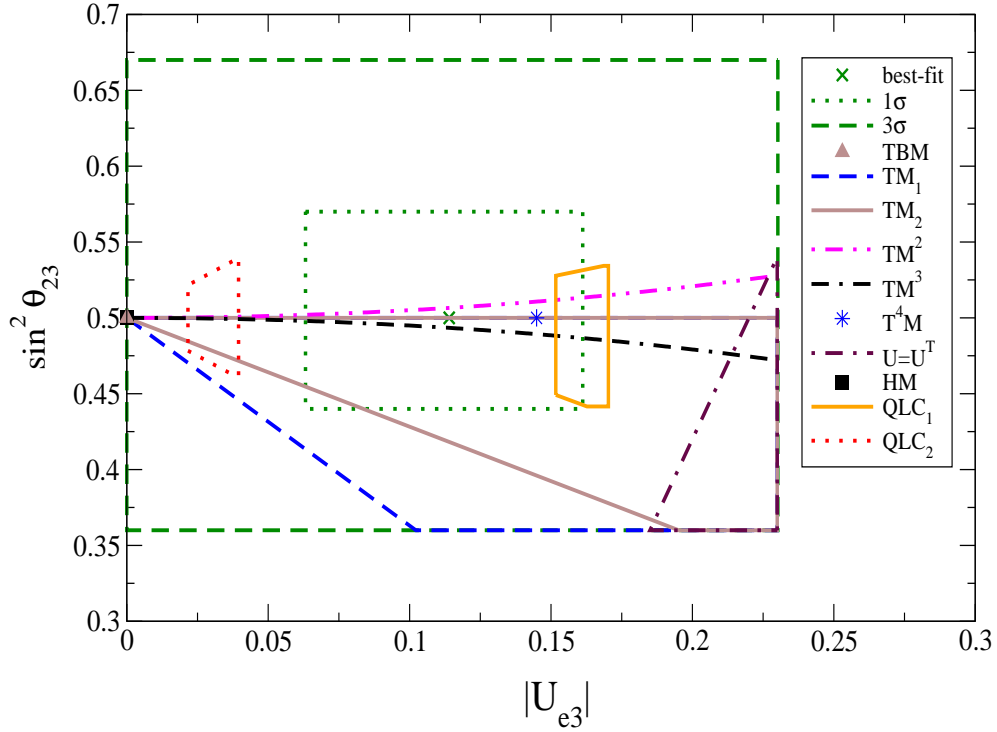
Figure 5: Same as Fig. 4, but now for $\sin^2\theta_{23}$. Like $\sin^2\theta_{12}$ in the $TM_{1,2}$ scenarios, in the $TM^{2,3}$ scenarios (magenta and black line, respectively) $\sin^2\theta_{23}$ depends only on $|U_{e3}|$ and not on the Dirac phase δ.
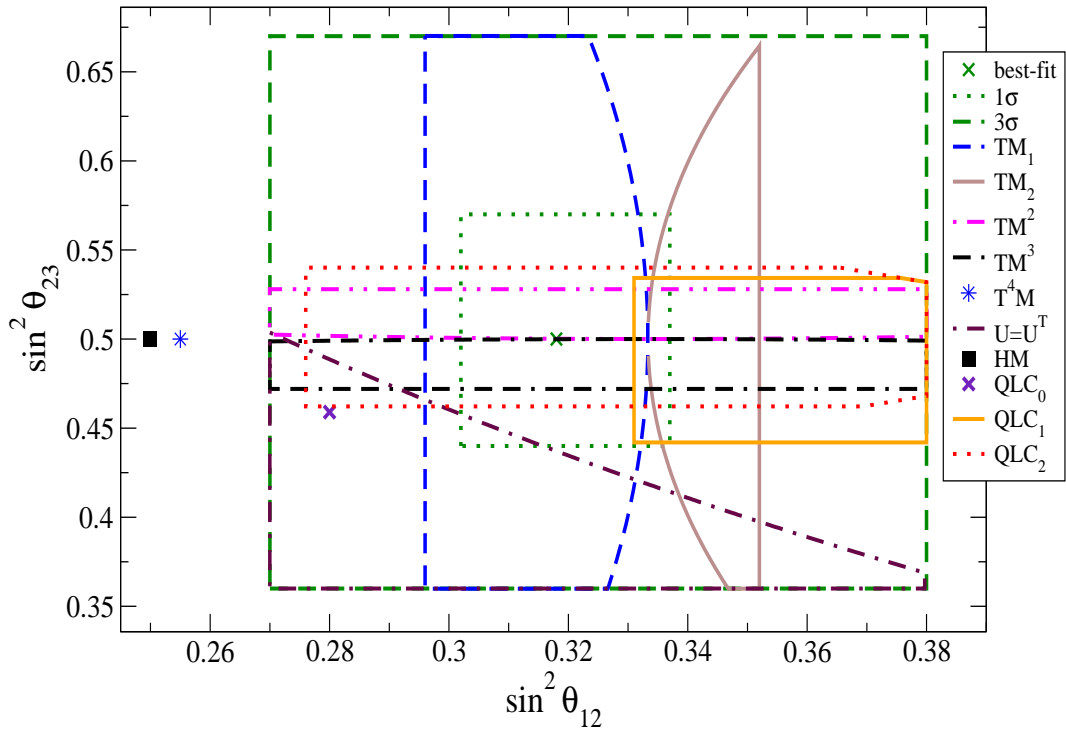

Figure 6: Same as Figs. 4 and 5, but now the correlations between $\sin^2\theta_{23}$ and $\sin^2\theta_{12}$ are plotted.