

Masses, fermions and generalized D -dimensional unitarity

R. Keith Ellis* and Walter T. Giele†

Fermilab, Batavia, IL 60510, USA

Zoltan Kunszt‡

Institute for Theoretical Physics, ETH, CH-8093 Zürich, Switzerland

Kirill Melnikov§

*Department of Physics and Astronomy, University of Hawaii,
2505 Correa Rd. Honolulu, HI 96822*

(Dated: June 20, 2008)

Abstract

We extend the generalized D -dimensional unitarity method for numerical evaluation of one-loop amplitudes by incorporating massive particles. The issues related to extending the spinor algebra to higher dimensions, treatment of external self-energy diagrams and mass renormalization are discussed within the context of the D -dimensional unitarity method. To validate our approach, we calculate in QCD the one-loop scattering amplitudes of a massive quark pair with up to three additional gluons for arbitrary spin states of the external quarks and gluons.

PACS numbers: 13.85.-t,13.85.Qk

*Electronic address: ellis@fnal.gov

†Electronic address: giele@fnal.gov

‡Electronic address: kunszt@itp.phys.ethz.ch

§Electronic address: kirill@phys.hawaii.edu

I. INTRODUCTION

Good understanding of background and signal processes will be necessary to interpret data from the Large Hadron Collider (LHC) and observe physics beyond the Standard Model. In particular, large multiplicity final states are of interest [1]. Reliable predictions for such processes require computations of next-to-leading order (NLO) QCD corrections. Traditional methods for NLO calculations have difficulties in dealing with processes of such complexity; as a result, many new approaches to one-loop computations have been suggested in recent years [1].

Among those approaches, generalized unitarity stands out [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. The key feature of this method is that it allows calculation of one-loop scattering amplitudes directly from tree amplitudes leading to a computational algorithm of polynomial complexity [25]. The efficiency of generalized unitarity for NLO calculations for processes with high multiplicity final states has been explicitly demonstrated in Refs. [26, 27].

Until recently, generalized unitarity was mostly used to compute the cut-constructible parts [28] of scattering amplitudes, while calculations of the rational parts proved to be challenging. In Refs. [10, 11] the four-dimensional boot-strap method was developed to evaluate the rational part. Another approach developed to generate the rational part uses generalized D -dimensional unitarity [15, 16].

In a recent paper [24], we extended the method of Refs. [23, 29] in such a way that *both* cut-constructible and rational parts are obtained within a single formalism using integer-dimensional on-shell cuts. This method leads to a computational algorithm of polynomial complexity, as shown in Ref. [27].

Up to now, generalized unitarity has been mainly studied in the context of multi-gluon scattering amplitudes which simplifies the problem significantly. In the general case, one has to deal with two additional issues – different types of particles that participate in the scattering process and the fact that massive particles can be involved. It is necessary to address these issues before generalized unitarity becomes a practical tool for NLO calculations of phenomenological interest. The goal of this paper is to do exactly that and extend the applicability of generalized D -dimensional unitarity by considering one-loop amplitudes involving gluons and massive quarks. The computational method developed in Ref. [24] can

handle both extensions easily.

Dealing with particles of different flavors requires more sophisticated bookkeeping, but is otherwise straightforward. However, the presence of massive particles introduces new conceptual issues. An obvious consequence of having virtual particles with non-zero masses contributing to one-loop scattering amplitudes is that in addition to quadruple, triple and double cuts, we also have to deal with single-particle cuts. Such an extension is straightforward; the necessary details have already been given in Ref. [23]. A more interesting consequence of massive particles present in the scattering process is that generalized unitarity applied to certain double- and single-particle cuts becomes more subtle. This is closely related to external wave function renormalization constants which originate from Feynman diagrams with self-energy insertions on external lines¹. We will show that this complication can be circumvented without encumbering the formalism.

To validate the method, we focus on the calculation of one-loop amplitudes with a massive quark anti-quark pair and up to three gluons. These one-loop amplitudes have been calculated using more traditional methods. The one-loop corrections to $t\bar{t} + 2$ and $t\bar{t} + 3$ partons scattering have been first calculated in Ref. [30, 31] and Ref. [32], respectively.

The outline of the paper is as follows. In Section II we discuss the modification of the D -dimensional generalized unitarity method required to include massive fermions. Section III describes the subtleties that arise when massive particles are involved in the one-loop scattering amplitude. In Section IV we present numerical results for the one-loop amplitudes $0 \mapsto t\bar{t} + 2$ gluons and $0 \mapsto t\bar{t} + 3$ gluons. The conclusions and outlook are given in Section V.

II. ONE-LOOP AMPLITUDES AND DIMENSIONALITY OF SPACE-TIME

One-loop calculations in quantum field theory are divergent and require regularization at intermediate stages of the calculations. The conventional choice is dimensional regularization where momenta and polarization vectors of unobserved virtual particles are continued to arbitrary dimensions [33, 34]. By keeping the momenta and polarization vectors of all observable external particles in four dimensions, one can define the one-loop helicity ampli-

¹ Similar problems appear due to diagrams that can be interpreted as one-loop expectation values of quantum fields.

tudes to be used in NLO parton-level generators [35]. Once the dependence of a one-loop amplitude on the dimensionality of space-time is established, the dimensionality D can be interpolated to the non-integer value $D = 4 - 2\epsilon$. The divergences of one-loop amplitudes are regularized by the parameter ϵ .

While the analytical implementation of the dimensional regularization procedure is well-established (see for example Ref. [36]), a numerical implementation needs more consideration. In Ref. [24] we developed numerical implementation of dimensional regularization. To explain the method, we note that any N -particle one-loop scattering amplitude $A_N^{[1]}$ can be written as a linear combination of the so-called master integrals. The coefficients of such an expansion depend on D ; this dependence can be made explicit by choosing the appropriate basis of master integrals. After dimensional continuation, the final expression in the four-dimensional helicity (FDH) scheme [37, 38] is given by [24]

$$\begin{aligned}
A_N^{[1]} &= \sum_{[i_1|i_5]} \epsilon \times e_{i_1 i_2 i_3 i_4 i_5} I_{i_1 i_2 i_3 i_4 i_5}^{(D+2)} \\
&+ \sum_{[i_1|i_4]} \left(d_{i_1 i_2 i_3 i_4} I_{i_1 i_2 i_3 i_4}^{(D)} + \epsilon \times \hat{d}_{i_1 i_2 i_3 i_4} I_{i_1 i_2 i_3 i_4}^{(D+2)} - \epsilon(1 - \epsilon) \times \hat{\hat{d}}_{i_1 i_2 i_3 i_4} I_{i_1 i_2 i_3 i_4}^{(D+4)} \right) \\
&+ \sum_{[i_1|i_3]} \left(c_{i_1 i_2 i_3} I_{i_1 i_2 i_3}^{(D)} + \epsilon \times \hat{c}_{i_1 i_2 i_3} I_{i_1 i_2 i_3}^{(D+2)} \right) + \sum_{[i_1|i_2]} \left(b_{i_1 i_2} I_{i_1 i_2}^{(D)} + \epsilon \times \hat{b}_{i_1 i_2} I_{i_1 i_2}^{(D+2)} \right) \\
&+ \sum_{[i_1|i_1]} a_{i_1} I_{i_1}^{(D)} \tag{1}
\end{aligned}$$

where we introduced the short-hand notation $[i_1|i_n] = 1 \leq i_1 < i_2 < \dots < i_n \leq N$. The master integrals in Eq. (1) are defined as

$$I_{i_1 \dots i_m}^{(D)} = \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{d_{i_1} \dots d_{i_m}}, \tag{2}$$

with $d_i = d_i(l) = (l + p_1 + \dots + p_i)^2 - m_i^2$. The coefficients $b_{i_1 i_2}$, $\hat{b}_{i_1 i_2}$, $c_{i_1 i_2 i_3}$, $\hat{c}_{i_1 i_2 i_3}$, $d_{i_1 i_2 i_3 i_4}$, $\hat{d}_{i_1 i_2 i_3 i_4}$, $\hat{\hat{d}}_{i_1 i_2 i_3 i_4}$, and $e_{i_1 i_2 i_3 i_4 i_5}$ are independent of the dimensionality.

We can compute these dimension-independent coefficients numerically, within the method of D -dimensional generalized unitarity. To accomplish this, a parametric integration method [23, 24], based on the ideas developed in Ref. [29], is employed. The key point is to extend the dimensionality of the loop momentum to an *integer* D -dimensional value. For one-loop calculations, an extension to five dimensions is sufficient [24, 33]. However, care has to be taken with the dimensional dependence of the spins of the internal particles. The dimensional regularization scheme allows us to choose the dimensionality for internal

degrees of freedom of virtual particles D_s to be equal or larger than the embedded loop-momentum dimensionality. By choosing the parametric form of the integrand in integer (D_s, D) dimensions we can determine the dimension-independent coefficients through partial fractioning of the integrand. The partial fractioning factorizes the calculation of the one-loop amplitude into tree amplitudes [24]. This factorization property is equivalent to the factorization obtained in generalized unitarity methods. The four-dimensional helicity scheme defines the parametric continuation as $D_s \rightarrow 4$, $D \rightarrow 4 - 2\epsilon$ with the constraint $D_s \geq D$, giving the final result of Eq.(1).

We consider now the one-loop scattering amplitude involving a massive quark pair in addition to the gluons: $0 \mapsto t\bar{t} + N$ gluons. The D_s -dependence of the amplitude is linear. This means that we need to compute the integrand for two different values of D_s so that we can separate the D_s -dependent and D_s -independent parts [24]. Because we need well-defined states for fermions when taking the internal fermion propagator on-shell, we must choose the space-time dimensionality to be even, i.e. $D_s = 4$, $D_s = 6$ and $D_s = 8$.

The on-shell internal gluonic polarization states in six and eight dimensions with the momentum vector in five dimensions are straightforward generalizations of the choices made in Refs. [24, 27]. The construction of D_s -dimensional on-shell fermionic lines requires an extension of the four-dimensional Clifford algebra. We need to explicitly construct the D_s Dirac matrices Γ^μ and the $2^{D_s/2-1}$ spin polarization states $u_j^{(s)}(l, m)$ that satisfy the Dirac equation

$$\sum_{j=1}^{2^{D_s/2}} \left(\sum_{\mu=0}^{D-1} l_\mu \Gamma_{ij}^\mu - m \times \delta_{ij} \right) u_j^{(s)}(l, m) = 0, \quad (3)$$

and the completeness relation

$$\sum_{s=1}^{2^{(D_s/2-1)}} u_i^{(s)}(l, m) \bar{u}_j^{(s)}(l, m) = \sum_{\mu=0}^{D-1} l_\mu \Gamma_{ij}^\mu + m \times \delta_{ij}, \quad (4)$$

where the on-shell condition for a fermion with the mass m and momentum l reads $l^2 = m^2$. To construct the explicit higher dimensional Dirac matrices we follow the recursive definition given in Ref. [36]. The 8×8 six-dimensional Dirac matrices are defined in terms of the 4×4 four-dimensional Dirac matrices $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5\}$

$$\Gamma^0 = \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix}, \quad \Gamma^{i=1,2,3} = \begin{pmatrix} \gamma^i & 0 \\ 0 & \gamma^i \end{pmatrix}, \quad \Gamma^4 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix}. \quad (5)$$

It is readily checked that these matrices satisfy the standard anti-commutation relation

$$\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu = 0, \dots, 5. \quad (6)$$

The 16×16 eight-dimensional Dirac matrices are constructed in a similar manner from the six-dimensional Dirac matrices. The D_s -dimensional Dirac matrices are given for a particular representation of the Dirac algebra. Other representations can be obtained by unitary transformations. To construct a set of $2^{D_s/2-1}$ spinors satisfying the Dirac equation we generalize the procedures used in the four-dimensional case. We define the spinors

$$u^{(s)}(l, m) = \frac{(l_\mu \Gamma^\mu + m)}{\sqrt{l_0 + m}} \eta_{D_s}^{(s)}, \quad s = 1, \dots, 2^{D_s/2-1}. \quad (7)$$

For $D_s = 4$ we choose

$$\eta_4^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_4^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (8)$$

and construct recursively the $D_s = 6$ eight-component basis spinors

$$\eta_6^{(1)} = \begin{pmatrix} \eta_4^{(1)} \\ 0 \end{pmatrix}, \quad \eta_6^{(2)} = \begin{pmatrix} \eta_4^{(2)} \\ 0 \end{pmatrix}, \quad \eta_6^{(3)} = \begin{pmatrix} 0 \\ \eta_4^{(1)} \end{pmatrix}, \quad \eta_6^{(4)} = \begin{pmatrix} 0 \\ \eta_4^{(2)} \end{pmatrix}. \quad (9)$$

The eight spinors for $D_s = 8$ are obtained using the obvious generalization. It is easy to see that the spinors constructed in this way do indeed satisfy the Dirac equation.

To check the completeness relation, we need the Dirac-conjugate spinor \bar{u} . One subtlety associated with the fact that we have to deal with complex, rather than real, on-shell momenta is that in order to satisfy the completeness relation Eq. (4), we have to *define* the conjugate spinor as

$$\bar{u}^{(s)}(l, m) = \bar{\eta}_{D_s}^{(s)} \frac{(l_\mu \Gamma^\mu + m)}{\sqrt{l_0 + m}}. \quad (10)$$

Note that the loop momentum is *not* complex conjugated. It is then straightforward to check that the completeness relation Eq.(4) is satisfied.

III. MASSIVE PARTICLES AND THE UNITARITY CUTS

To determine the dimension-independent master integral coefficients in Eq. (1) we use the D -dimensional generalized unitarity method of Ref. [24]. To this end, we parameterize

the integrand of the one-loop amplitude

$$\mathcal{A}_N^{[1]}(l) = \sum_{[i_1|i_5]} \frac{\bar{e}_{i_1 i_2 i_3 i_4 i_5}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4} d_{i_5}} + \sum_{[i_1|i_4]} \frac{\bar{d}_{i_1 i_2 i_3 i_4}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{[i_1|i_3]} \frac{\bar{c}_{i_1 i_2 i_3}^{(D_s)}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{[i_1|i_2]} \frac{\bar{b}_{i_1 i_2}^{(D_s)}(l)}{d_{i_1} d_{i_2}} + \sum_{[i_1|i_1]} \frac{\bar{a}_{i_1}^{(D_s)}(l)}{d_{i_1}}. \quad (11)$$

The left hand side of the equation is completely specified by the Feynman rules. The parametric form on the right hand side of the equation depends on a set of coefficients. To determine the coefficients for a given phase space point we use partial fractioning. This isolates the individual pole structures, thereby dividing the sets of linear equations to be solved into smaller subsets. More importantly, the partial fractioning sets groups of internal lines on-shell. This organizes the left hand side of the equation into products of gauge invariant tree amplitudes, thereby removing the necessity to compute individual Feynman diagrams to evaluate $\mathcal{A}_N^{[1]}(l)$ for a given loop momentum.

This procedure can readily be applied in a situation when massive particles are involved in the scattering process. The presence of massive particles creates more types of master integrals or, equivalently, more different denominator structures in Eq. (11). Furthermore, the single-cut (or tadpole) contributions to one-loop amplitudes have to be calculated so that the tadpole coefficient in Eq. (1) can be determined. These issues complicate the bookkeeping, but do not add conceptual difficulties.

However, a new conceptual issue does appear when dealing with the double cuts shown in Fig. 1. Note that such cuts need only be considered for external massive states, since, if the external on-shell line carries a light-like momentum, the cut in Fig. 1 is set to zero in dimensional regularization. For massive particles these cuts do give non-vanishing contributions. The subtlety arising when such cuts are considered is related to a conflict between generalized unitarity and self-energy insertions on the external lines ².

To see this, we study the contribution from a particular two-particle cut shown in Fig. 1. The only outgoing external line to the left of the cut is the top quark and all other external particles are to the right of the cut. The residue of the one-loop amplitude for such a cut can be schematically written as

$$\text{Res} [\mathcal{A}^{[1]}(t, g_1, \dots, g_n, \bar{t})] \sim \sum_{\text{states}} \mathcal{A}^{[0]}(t, g^*, \bar{t}^*) \times \mathcal{A}^{[0]}(t^*, g^*, g_1, \dots, g_n, \bar{t}), \quad (12)$$

² While we discuss the one-loop amplitude $t\bar{t} + N$ gluons, other processes with massive external lines can be treated in the same way.

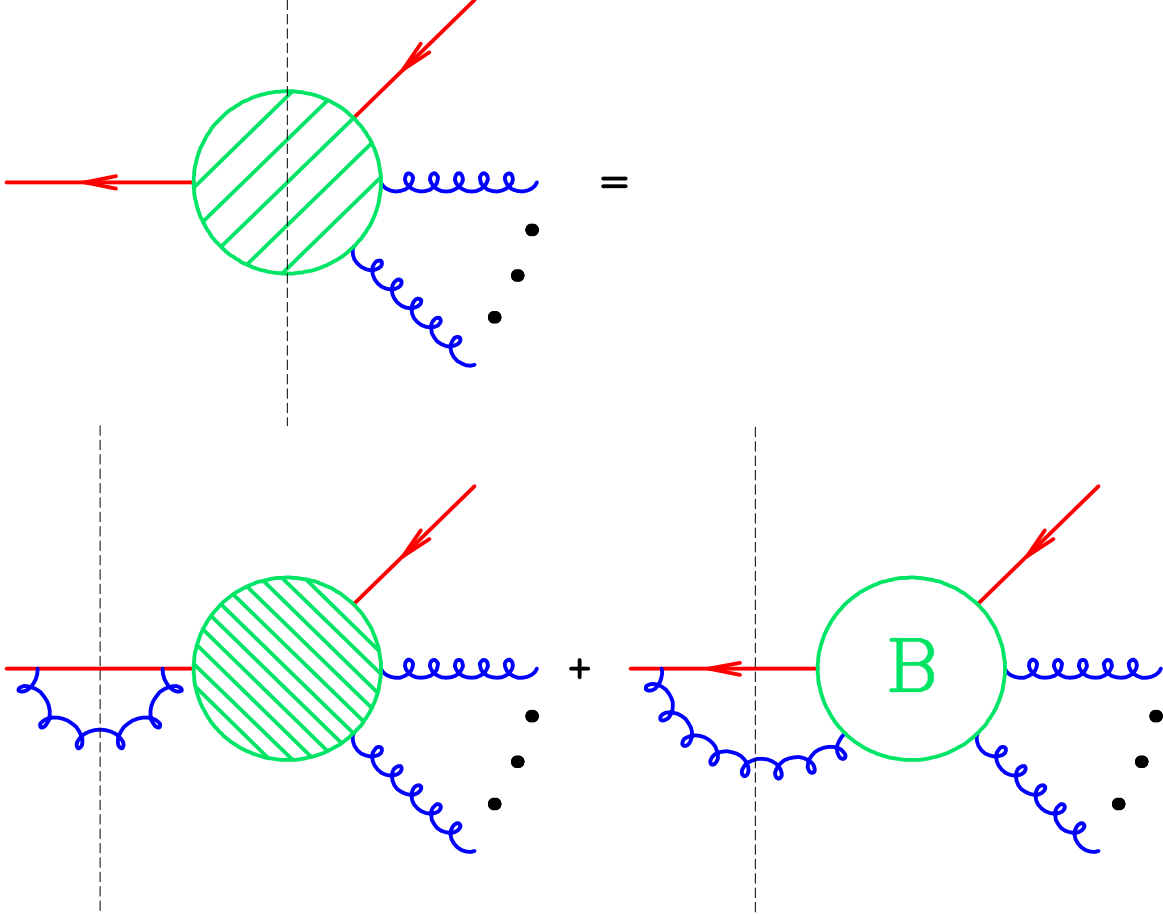


FIG. 1: The external self-energy cut of a general $t\bar{t} + N$ gluon loop amplitude splits in the external self-energy contribution and the remaining higher point contributions. The different shadings of the blobs represent different content.

where t^* and g^* denote the top quark and gluon cut lines respectively and the sum is over the intermediate states of the on-shell top quark and gluon particles of the two cut lines. The factorized on-shell tree amplitudes are given by $\mathcal{A}^{[0]}(t, g^*, \bar{t}^*)$ and $\mathcal{A}^{[0]}(t^*, g^*, g_1, \dots, g_n, \bar{t})$. However, the latter amplitude is not defined. Separating the cut self-energy contribution as indicated in Fig. 1 gives for the tree amplitude

$$\mathcal{A}^{[0]}(t^*, g^*, g_1, \dots, g_n, \bar{t}) = \frac{R(t^*, g^*, g_1, \dots, g_n, \bar{t})}{(p_{t^*} + p_{g^*})^2 - m_t^2} + B(t^*, g^*, g_1, \dots, g_n, \bar{t}). \quad (13)$$

Momentum conservation forces the invariant mass of $t^* + g^*$ to be equal to the top quark mass squared, $(p_{g^*} + p_{t^*})^2 = m_t^2$ making the one-quark reducible part of the amplitude singular.

The singular contribution corresponds to the self-energy correction to the external top quark line. When one-loop scattering amplitudes are calculated using conventional Feyn-

man diagrams, these type of one-particle reducible diagrams are discarded; their effects on the scattering process are accommodated later through the external particle wave function renormalization constants. We would like to follow this approach in conjunction with the generalized unitarity technique, but then care has to be taken with the gauge invariance.

Suppose we subtract the first term in Eq. (13) from the tree amplitude; in recursive calculations this can be done by truncating the recursive steps. It is then easy to see that the remaining part of the amplitude B , the second term in Eq. (13), is no longer gauge invariant. Indeed, the discarded part of the amplitude is related to the self-energy correction on the external top quark line; such self-energy corrections produce on-shell mass and wave-function renormalization factors. While the mass renormalization constant, Z_m , is independent of the gauge-fixing parameter, the on-shell wave-function renormalization factor, Z_2 , is *not*. For this reason we have to ensure that the gauge used in calculating the second term in Eq. (13) and the gauge used in the calculation of the wave-function renormalization factor Z_2 are the same. Since the wave-function renormalization factors are most easily computed in the Feynman gauge, we use this gauge to calculate the residue in Eq. (12). This means that the sum over gluon particle states for the cut in Fig. 1 includes non-physical states e_s^μ such that

$$\sum_{s=1}^{D_s} e_s^\mu e_s^\nu = -g^{\mu\nu} . \quad (14)$$

Note that since the offending cuts never involve gluon self-couplings, ghosts do not need to be considered.

Finally, we note that, for the most part, the coefficients in Eq. (11) are computed using the standard sums over physical states of the on-shell particles associated with cut lines. However, for a limited set of pole terms which contain the external self-energy contributions we need the procedure described in this Section. We emphasize that the conflict between unitarity and self-energy corrections to external massive lines is generic; it appears in any calculation of one-loop scattering amplitudes provided that massive internal or external particles are present.

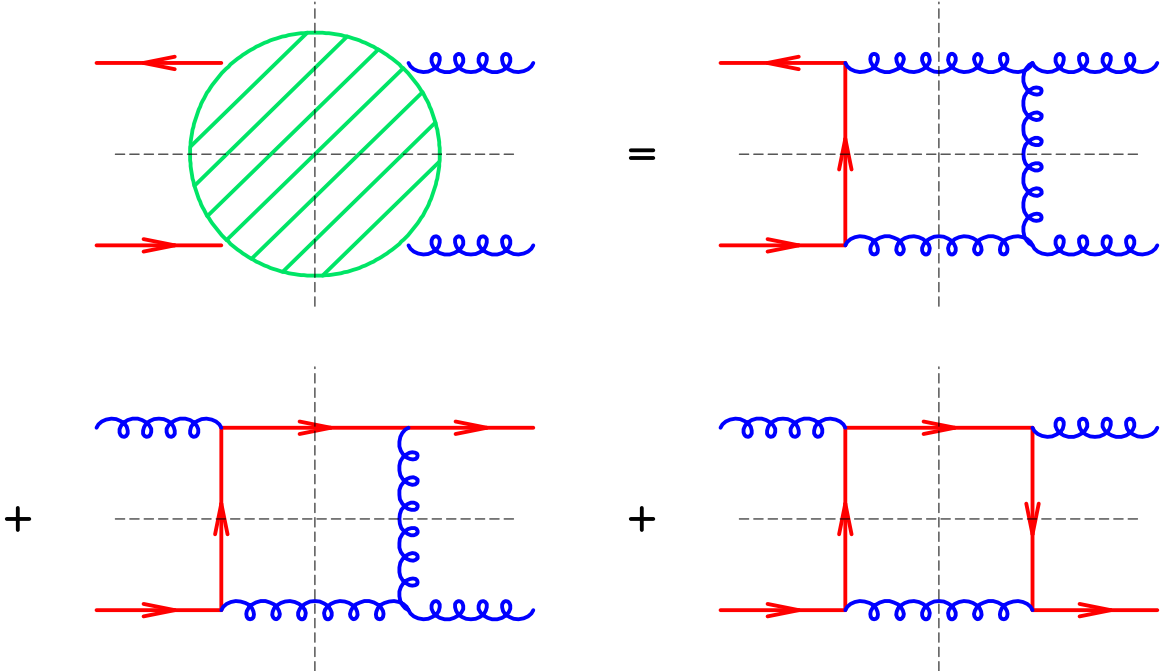


FIG. 2: The quadruple cut of the $t\bar{t} + 2$ gluon amplitude decomposes into 3 gauge invariant contributions, each with its own 4-point master integral. The first box integral contributes to the primitive amplitude $A_L(1_{\bar{t}}, 2_t, 3, 4)$, the second to $A_L(1_{\bar{t}}, 3, 2_t, 4)$ and the third to $A_L(1_{\bar{t}}, 3, 4, 2_t)$.

IV. SCATTERING AMPLITUDES AT ONE-LOOP

To implement the generalized unitarity method in a numerical algorithm, we decompose the $0 \mapsto t\bar{t} + 2, 3$ gluon amplitude into so-called primitive amplitudes [39]³. Within the context of D -dimensional unitarity the primitive amplitudes play a special role. Each primitive amplitude has unique unitarity cuts, i.e. the flavor of the cut lines is uniquely defined. This is shown in Fig. 2 for the example of the quadruple cuts applied to the $0 \mapsto t\bar{t} + 2$ gluon amplitude. This quadruple cut decomposes into three distinct gauge invariant cuts, each with its own master integral. Each of the three individual cuts contributes to one of the three primitive amplitudes $A_L(1_{\bar{t}}, 2_t, g_1, g_2)$, $A_L(1_{\bar{t}}, g_1, 2_t, g_2)$ and $A_L(1_{\bar{t}}, g_1, g_2, 2_t)$.

The method described in this paper is amenable to straightforward numerical implementation. To evaluate a primitive amplitude we consider all pole terms in the partial fractioning of Eq. (11). Double pole terms that correspond to massless two-point functions for light-like incoming momenta and single pole terms that correspond to massless tadpoles are discarded

³ We adopt the conventions and normalizations of Ref. [39] to define the primitive amplitudes.

since the corresponding master integrals vanish in dimensional regularization. The tree amplitudes for each cut are computed using Berends-Giele recurrence relations [40]. Because single particle cuts contribute, we need to evaluate the high multiplicity tree amplitudes $\bar{t}t + \bar{t}t + 2, 3$ gluons

Before discussing numerical results for one-loop $0 \mapsto \bar{t}t + 2, 3$ gluon amplitudes, we remind the reader that, when massive particles are involved, additional renormalization constants are required to arrive at physical predictions. In particular, for massive quarks, on-shell mass and wave function renormalization constants are necessary⁴. For consistency, we need those constants in FDH scheme. As described above, the wave function renormalization constant needs to be computed in the Feynman gauge. The bare quark mass m_0 and the bare quark field ψ_0 are renormalized multiplicatively

$$m_0 = Z_m m, \quad \psi_0 = \sqrt{Z_2} \psi. \quad (15)$$

We find ($D = 4 - 2\epsilon$)

$$\begin{aligned} Z_m = Z_2 &= 1 - C_F g_s^2 c_\Gamma \left(\frac{\mu^2}{m^2} \right)^\epsilon \left(\frac{D_s + 2}{2\epsilon} + \frac{D_s + 6}{2} \right) + O(g_s^4, \epsilon) \\ &\rightarrow 1 - C_F g_s^2 c_\Gamma \left(\frac{\mu^2}{m^2} \right)^\epsilon \left(\frac{3}{\epsilon} + 5 \right) + O(g_s^4, \epsilon), \end{aligned} \quad (16)$$

where g_s is the bare strong coupling constant, c_Γ is the normalization factor,

$$c_\Gamma = \frac{\Gamma(1 + \epsilon)\Gamma(1 - \epsilon)^2}{(4\pi)^{2-\epsilon}\Gamma(1 - 2\epsilon)}, \quad (17)$$

$C_F = (N_c^2 - 1)/(2N_c)$ is the color factor and μ is the scale which is introduced in dimensional regularization to maintain proper dimensionality of the coupling constant. Note that in the last step in Eq. (16) we used $D_s = 4$, as is required in the FDH scheme.

We now present the results of the numerical evaluation of one-loop $0 \mapsto \bar{t}t + 2, 3$ gluon scattering amplitudes in QCD. We do not include diagrams with closed fermion loops. In addition, external wave function renormalization constants and the coupling constant renormalization factors are not included. However, we do include the mass counter-term diagrams which are necessary to obtain a result which is invariant under gauge transformations of the

⁴ Note that the *on-shell* wave-function renormalization constant contains both ultraviolet and infrared divergences. Both show up as poles in ϵ .

external gluons. For presentation purposes, it is convenient to normalize one-loop primitive amplitudes to tree-graph primitive amplitudes

$$\begin{aligned}\mathcal{A}_L^{[1]}(1_{\bar{t}}, 3, \dots, j-1, 2_t, j, \dots, n) &= c_\Gamma \left(\frac{a_L^{(j)}}{\epsilon^2} + \frac{b_L^{(j)}}{\epsilon} + c_L^{(j)} \right) \mathcal{A}_L^{[0]}(1_{\bar{t}}, 3, \dots, j-1, 2_t, j, \dots, n) \\ \mathcal{A}_L^{[1]}(1_{\bar{t}}, 3, \dots, n, 2_t) &= c_\Gamma \left(\frac{a_L^{(n)}}{\epsilon^2} + \frac{b_L^{(n)}}{\epsilon} + c_L^{(n)} \right) \mathcal{A}_L^{[0]}(1_{\bar{t}}, 3, \dots, n, 2_t) .\end{aligned}\quad (18)$$

The coefficients $a_L^{(j)}$ and $b_L^{(j)}$ parameterize divergences of the one-loop scattering amplitude. They can be extracted from Ref. [41]

$$\begin{aligned}\frac{a_L^{(j)}}{\epsilon^2} + \frac{b_L^{(j)}}{\epsilon} &= \frac{1}{2\epsilon} - S_{\bar{t}g}(p_2, p_j, \mu) - \sum_{i=j}^{n-1} S_{gg}(p_i, p_{i+1}, \mu) - S_{gt}(p_n, p_1, \mu) \\ \frac{a_L^{(n)}}{\epsilon^2} + \frac{b_L^{(n)}}{\epsilon} &= \frac{1}{2\epsilon} - S_{\bar{t}t}(p_2, p_1, \mu) .\end{aligned}\quad (19)$$

The functions $S_{f_i, f_{i+1}} = S_{f_{i+1}, f_i}$ depend on the flavor of particles f_i , their momenta p_i and the scale μ . They read

$$S_{\bar{t}\bar{t}} = \frac{1}{\epsilon\beta} \left(\frac{1}{2} \ln \left(\frac{1-\beta}{1+\beta} \right) + i\pi\Theta(d_{\bar{t}\bar{t}}) \right), \quad (20)$$

$$S_{tg} = S_{\bar{t}g} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1}{2} \ln \left(\frac{m_t^2 \mu^2}{d_{tg}^2} \right) + i\pi\Theta(d_{tg}) \right), \quad (21)$$

$$S_{g_1 g_2} = \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\ln \left(\frac{\mu^2}{|d_{g_1 g_2}|} \right) + i\pi\Theta(d_{g_1 g_2}) \right), \quad (22)$$

where $d_{i,j} = 2p_i \cdot p_j$ and $\beta = \sqrt{1 - \frac{4m_t^4}{d_{\bar{t}\bar{t}}^2}}$.

Finally, we need to define the spin states of the gluons and top-quarks. For the gluons we use the conventional definition of the helicity vectors

$$\begin{aligned}p_\mu &= E(1, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \\ \varepsilon_\mu^\pm(p) &= \frac{1}{\sqrt{2}}(0, \cos\theta \cos\phi \mp i \sin\phi, \cos\theta \sin\phi \pm i \cos\phi, -\sin\theta) .\end{aligned}\quad (23)$$

For the massive on-shell quarks ($p = (E, p_x, p_y, p_z)$, $p^2 = m^2$) we use the spinors

$$u_+(p) = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}, \quad u_-(p) = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad (24)$$

$$v_+(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \\ 1 \\ 0 \end{pmatrix}, \quad v_-(p) = \sqrt{E+m} \begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}. \quad (25)$$

The numerical results reported below are obtained in conventional double precision using a FORTRAN 77 program. The evaluation time does not depend on the helicities of the external particles but it does depend on the specific primitive amplitude. It takes less time to evaluate primitive amplitudes where quarks are adjacent, than to evaluate primitive amplitudes where quarks are separated by gluons. The reason for this is that it is computationally more expensive to have more quarks involved in the evaluation of the primitive tree amplitudes. That is, the more quark propagators there are, the longer the evaluation time.

For evaluating the master integrals we use the `QCDLoop` program developed in Ref. [42]. We have verified that our calculations correctly reproduce the divergent parts of primitive amplitudes, given in Eq. (19). For all primitive amplitudes we have checked the gauge invariance by substituting a polarization vector of one of external gluons by its momentum. In addition we performed a Feynman diagram-by-diagram check on the results of the calculation.

A. Scattering amplitudes with two quarks and two gluons

In Table I we present the results for the finite parts of some of the primitive amplitudes $A_L(1_{\bar{t}}, 2_t, g_3, g_4)$, $A_L(1_{\bar{t}}, g_3, 2_t, g_4)$ and $A_L(1_{\bar{t}}, g_3, g_4, 2_t)$ in the FDH scheme. The numerical results are obtained for the scale choice $\mu = E$. We take the mass of the top quark to be $m_t = 1.75$ and choose the following kinematic point ($p = (E, p_x, p_y, p_z)$)

$$\begin{aligned} p_1 &= E(1, 0, 0, \beta), & p_2 &= E(1, 0, 0, -\beta), \\ p_3 &= E(-1, \sin \theta, 0, \cos \theta), & p_4 &= E(-1, -\sin \theta, 0, -\cos \theta). \end{aligned} \quad (26)$$

with $E = 10$, $\beta = \sqrt{1 - m_t^2/E^2}$ and $\theta = \pi/3$. Note that all the external momenta are taken to be outgoing.

On a standard Pentium 2.33 GHz processor, it takes about 7.5, 11 and 12 ms to evaluate the primitive amplitudes $A_L(1_{\bar{t}}, 2_t, 3, 4)$, $A_L(1_{\bar{t}}, 3, 2_t, 4)$ and $A_L(1_{\bar{t}}, 3, 4, 2_t)$ respectively.

Amplitude	tree	c^{cut}	c
$+\bar{t}, +t, +3, +4$	0.0026595 i	5.859738 +11.04762 i	43.74436+11.04762 i
$+\bar{t}, +t, -3, +4$	-0.127261 i	18.49057-2.63910 i	18.49058 -2.63910 i
$+\bar{t}, -t, +3, -4$	1.259555 i	20.69972-0.144581 i	20.52783 -0.14458 i
$+\bar{t}, -t, -3, +4$	-0.4198517	22.16788 -3.40322 i	22.68356 -3.40322 i
$+\bar{t}, +3, +t, +4$	-0.0035643 i	-0.26303343	-0.26303305
$+\bar{t}, -3, +t, +4$	0.170558 i	15.2990066	15.2990071
$+\bar{t}, +3, -t, -4$	-1.688090 i	20.8261462	20.8261462
$+\bar{t}, -3, -t, +4$	0.56269666 i	22.0890527	22.0890523
$+\bar{t}, +3, +4, +t$	0.000905 i	-26.24047 + 40.67377 i	-123.4438 + 40.67377 i
$+\bar{t}, -3, +4, +t$	-0.043298 i	20.00357 -1.69128 i	20.00357 -1.69128 i
$+\bar{t}, +3, -4, -t$	0.4285350 i	21.83688-4.01097 i	21.33165 -4.01097 i
$+\bar{t}, -3, +4, -t$	-0.142845 i	19.48471 -1.85887 i	21.00038 -1.85887 i

TABLE I: The $0 \mapsto \bar{t}t + 2$ gluons results for the primitive tree amplitude and finite parts of the three one-loop primitive amplitudes for various helicities of gluons and top-quarks. Both the cut-constructible and total finite one-loop terms are given.

Approximately half of that time is spent on the calculation of rational parts. We note that the calculation of the cut-constructible part of color-ordered four-gluon amplitude [23] takes about 1 ms. The difference in CPU time between $\bar{t}t + 2$ gluons and the four gluon amplitude is not dramatic. The time difference is the result of several factors. First, a larger number of cuts has to be calculated. Second, in addition to the cut-constructible part we calculate also the rational part. Last, the evaluation of tree level amplitudes with (massive) quarks takes more computational effort.

B. Scattering amplitudes with two quarks and three gluons

In Table II the results for the finite parts of the four primitive amplitudes $A_L(1_{\bar{t}}, 2_t, g_3, g_4, g_5)$, $A^L(1_{\bar{t}}, g_3, 2_t, g_4, g_5)$, $A^L(1_{\bar{t}}, g_3, g_4, 2_t, g_5)$ and $A^L(1_{\bar{t}}, g_3, g_4, g_5, 2_t)$ are presented in the FDH scheme. We take the mass of the top quark to be $m_t = 1.75$, the scale

Amplitude	tree	c^{cut}	c
$+\bar{t}, +t, +3, +4, +5$	-0.000533-0.000137 i	9.584144+6.530925 i	51.8809+6.543042 i
$+\bar{t}, -t, +3, -4, +5$	-0.004540 + 0.018665 i	19.65913-11.77003 i	23.00306-9.699584 i
$+\bar{t}, +t, -3, +4, -5$	-0.004726+ 0.014201 i	33.15950-1.832717 i	33.71943 -3.142751 i
$+\bar{t}, -t, -3, +4, +5$	0.045786 + 0.010661 i	22.84043-6.540697 i	23.03114-7.313041 i
$+\bar{t}, +3, +t, +4, +5$	0.000182 + 0.001369 i	6.517366-1.277070 i	19.37656+7.563101 i
$+\bar{t}, +3, -t, -4, +5$	0.0467366-0.006020 i	19.440997-7.639466 i	20.93024-9.936409 i
$+\bar{t}, -3, +t, +4, -5$	0.019275 -0.0732138 i	15.31910 -3.9278496 i	15.176306-4.102803 i
$+\bar{t}, -3, -t, +4, +5$	-0.018203-0.111312 i	24.13158+1.431596 i	24.70002+1.018096 i
$+\bar{t}, +3, +4, +t, +5$	0.00060-0.001377 i	13.13854+6.157043 i	10.13113+13.83997 i
$+\bar{t}, +3, -4, -t, +5$	-0.047199-0.021516 i	23.90539 -2.168867 i	22.905695-4.284617 i
$+\bar{t}, -3, +4, +t, -5$	-0.015110+0.063118 i	13.54258-7.800591 i	13.50273-8.018127 i
$+\bar{t}, -3, +4, -t, +5$	-0.048800+ 0.112645 i	21.77602+ 2.078051 i	22.52784+1.424481 i
$+\bar{t}, +3, +4, +5, +t$	-0.000252+0.000144 i	-10.35085+45.26276 i	-98.81384+52.81712 i
$+\bar{t}, +3, -4, +5, -t$	0.0050023+0.008871 i	23.944473+2.862220 i	20.92683-0.968026 i
$+\bar{t}, -3, +4, -5, +t$	0.000561-0.004105 i	-2.987822-42.00048 i	-3.834451-43.67103 i
$+\bar{t}, -3, +4, +5, -t$	0.021216-0.011994 i	19.72995-2.120128 i	20.94996-1.684734 i

TABLE II: The $0 \mapsto \bar{t}t + 3$ gluons results for the primitive tree amplitude and finite parts of the four one-loop primitive amplitudes for various helicities of gluons and top-quarks. Both the cut-constructible and total finite one-loop terms are given.

$\mu = E$ and choose the kinematic point

$$\begin{aligned}
p_1 &= E(1, 0, 0, \beta), & p_2 &= E(1, 0, 0, -\beta), \\
p_3 &= E\xi(-1, 1, 0, 0), & p_4 &= E\xi(-\sqrt{2}, 0, 1, 1), \\
p_5 &= -p_1 - p_2 - p_3 - p_4,
\end{aligned} \tag{27}$$

where $E = 10$, $\beta = \sqrt{1 - m_t^2/E^2}$ and $\xi = 2/(1 + \sqrt{2} + \sqrt{3}) = 0.4823619098$.

On a standard Pentium 2.33 GHz processor it takes about 27, 35, 45 and 50 ms respectively to evaluate the primitive amplitudes $A_L(1_{\bar{t}}, 2_t, 3, 4, 5)$, $A_L(1_{\bar{t}}, 3, 2_t, 4, 5)$, $A_L(1_{\bar{t}}, 3, 4, 2_t, 5)$ and $A_L(1_{\bar{t}}, 3, 4, 5, 2_t)$. Comparing these evaluation times

to the $t\bar{t} + 2$ gluon evaluation time, we see that the scaling is similar to the time scaling of the four and five gluon evaluation time in Ref. [23]. Similar to $t\bar{t} + 2$ gluons case, approximately half of the time is spent on the evaluation of the rational part.

V. CONCLUSIONS

In this paper we extended the method of generalized D -dimensional unitarity by computing one-loop scattering amplitudes for processes with massive quarks. We have proposed a solution to the subtleties associated with external self-energies and renormalization. We validated the method by computing the one-loop amplitudes for $0 \mapsto t\bar{t} + 2$ gluons and $0 \mapsto t\bar{t} + 3$ gluons. We have shown that the method is amenable to efficient numerical implementation. The results of this paper show that the generalized D -dimensional unitarity is a robust computational method. It will allow us to carry out NLO calculations for a large number of high multiplicity processes with massive particles, relevant for LHC phenomenology.

Acknowledgments

K.M. is supported in part by the DOE grant DE-FG03-94ER-40833.

-
- [1] Z. Bern *et al.*, “The NLO Multileg Working Group, Summary Report, 5th Les Houches Workshop on Physics at TeV Colliders, Les Houches, France, 11-29 Jun 2007”, arXiv:0803.0494 [hep-ph].
 - [2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217 (1994) [hep-ph/9403226].
 - [3] Z. Bern and A. G. Morgan, Nucl. Phys. B **467**, 479 (1996) [hep-ph/9511336].
 - [4] Z. Bern, L. J. Dixon and D. A. Kosower, Ann. Rev. Nucl. Part. Sci. **46**, 109 (1996) [hep-ph/9602280].
 - [5] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Phys. Lett. B **394**, 105 (1997) [hep-th/9611127].
 - [6] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275 (2005) [hep-th/0412103].

- [7] R. Britto, F. Cachazo and B. Feng, Phys. Rev. D **71**, 025012 (2005) [hep-th/0410179].
- [8] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **71**, 105013 (2005) [hep-th/0501240];
- [9] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **72**, 125003 (2005) [hep-ph/0505055].
- [10] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D **73**, 065013 (2006) [hep-ph/0507005].
- [11] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, Phys. Rev. D **74**, 036009 (2006) [hep-ph/0604195].
- [12] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D **72**, 065012 (2005) [hep-ph/0503132].
- [13] R. Britto, B. Feng and P. Mastrolia, Phys. Rev. D **73**, 105004 (2006) [hep-ph/0602178].
- [14] P. Mastrolia, Phys. Lett. B **644**, 272 (2007) [hep-th/0611091].
- [15] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B **645**, 213 (2007) [hep-ph/0609191];
- [16] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, JHEP **0703**, 111 (2007) [hep-ph/0612277].
- [17] R. Britto and B. Feng, Phys. Rev. D **75**, 105006 (2007) [hep-ph/0612089].
- [18] R. Britto and B. Feng, JHEP **0802**, 095 (2008) [0711.4284 [hep-ph]].
- [19] R. Britto, B. Feng and P. Mastrolia, 0803.1989 [hep-ph].
- [20] R. Britto, B. Feng and G. Yang, 0803.3147 [hep-ph].
- [21] D. Forde, Phys. Rev. D **75**, 125019 (2007) [0704.1835 [hep-ph]].
- [22] W. B. Kilgore, arXiv:0711.5015 [hep-ph].
- [23] R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803** (2008) 003 [arXiv:0708.2398 [hep-ph]].
- [24] W. T. Giele, Z. Kunszt and K. Melnikov, JHEP **0804** (2008) 049 [arXiv:0801.2237 [hep-ph]].
- [25] R. K. Ellis, W. T. Giele and Z. Kunszt, pg. 31, contribution to the NLO multileg working group for the Workshop "Physics at TeV Colliders", Les Houches, France, 11-29 June, 2007, Z. Bern *et al.*, arXiv:0803.0494 [hep-ph].
- [26] C. F. Berger *et al.*, arXiv:0803.4180 [hep-ph].
- [27] W. T. Giele and G. Zanderighi, arXiv:0805.2152 [hep-ph].
- [28] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435** (1995) 59 [arXiv:hep-ph/9409265].
- [29] G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763** (2007) 147 [arXiv:hep-ph/0609007].

- [30] P. Nason, S. Dawson and R. K. Ellis, Nucl. Phys. B **303** (1988) 607.
- [31] W. Beenakker, H. Kuijf, W. L. van Neerven and J. Smith, Phys. Rev. D **40** (1989) 54.
- [32] S. Dittmaier, P. Uwer and S. Weinzierl, arXiv:0804.4389 [hep-ph].
- [33] G. 't Hooft, Nucl. Phys. B **33** (1971) 173.
- [34] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972) 189.
- [35] W. T. Giele and E. W. N. Glover, Phys. Rev. D **46** (1992) 1980.
- [36] J. Collins, *Renormalization*, Cambridge University Press, 1984.
- [37] Z. Bern and D. A. Kosower, Nucl. Phys. B **379** (1992) 451.
- [38] Z. Bern, A. De Freitas, L. J. Dixon and H. L. Wong, Phys. Rev. D **66** (2002) 085002 [arXiv:hep-ph/0202271].
- [39] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **437** (1995) 259 [arXiv:hep-ph/9409393].
- [40] F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.
- [41] S. Catani, S. Dittmaier and Z. Trocsanyi, Phys. Lett. B **500** (2001) 149 [arXiv:hep-ph/0011222].
- [42] R. K. Ellis and G. Zanderighi, JHEP **0802** (2008) 002 [arXiv:0712.1851 [hep-ph]].