

# On cosmic acceleration without dark energy

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## Abstract

We elaborate on the proposal that the observed acceleration of the Universe is the result of the backreaction of cosmological perturbations, rather than the effect of a negative-pressure dark-energy fluid or a modification of general relativity. Through the effective Friedmann equations describing an inhomogeneous Universe after smoothing, we demonstrate that acceleration in our local Hubble patch is possible even if fluid elements do not individually undergo accelerated expansion. This invalidates the no-go theorem that there can be no acceleration in our local Hubble patch if the Universe only contains irrotational dust. We then study perturbatively the time behavior of general-relativistic cosmological perturbations, applying, where possible, the renormalization group to regularize the dynamics. We show that an instability occurs in the perturbative expansion involving sub-Hubble modes, which indicates that acceleration in our Hubble patch may originate from the backreaction of cosmological perturbations on observable scales.

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## I. INTRODUCTION

Recent observations of the expansion history of the Universe indicate that the Universe is presently undergoing a phase of accelerated expansion [1, 2]. The accelerated expansion is usually interpreted as evidence either for a “dark energy” (DE) component to the mass-energy density of the Universe, or for a modification of gravity at large distances. In this paper we explore another possibility, namely that the accelerated expansion is due to the presence of inhomogeneities in the Universe.

In the homogeneous, isotropic, Friedmann-Robertson-Walker (FRW) cosmology, the acceleration (or deceleration) of the expansion may be expressed in terms of a dimensionless parameter  $q$ , proportional to the second time derivative of the cosmic scale factor  $a$ . It is uniquely determined in terms of the relative densities and the equations of state of the various fluids by (overdots denote time derivatives),

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1}{2}\Omega_{\text{TOTAL}} + \frac{3}{2}\sum_i w_i \Omega_i, \quad (1)$$

where  $\Omega_{\text{TOTAL}}$  is the total density parameter and the factors  $\Omega_i$  are the relative contributions of the various components of the energy density with equation of state  $w_i = P_i/\rho_i$  ( $P_i$  and  $\rho_i$  being the pressure and energy density of  $i$ -th fluid). The expansion accelerates if  $q < 0$ . Observations seem to require DE with present values  $w_{DE} \sim -1$  and  $\Omega_{DE} \sim 0.7$  [3]. The negative value of  $w_{DE}$ , indicating a violation of the energy condition  $w > -1/3$  [4], is usually interpreted as the effect of a mysterious dark energy fluid of unknown nature or a cosmological constant of surprisingly small magnitude.

The existence of a negative-pressure fluid or a cosmological constant would have profound implications for physics as well as cosmology. While the observational evidence for the acceleration of the Universe is now compelling, it is important to keep in mind that the evidence for dark energy is indirect; it is *inferred* from the observed time evolution of the expansion rate of the Universe. What is known is that the expansion history of the Universe is not described by the expansion history of an Einstein–de Sitter Universe (a spatially flat, matter-dominated FRW model). While such a departure may be caused by dark energy, there are other possibilities. One possibility is that general relativity is not a good description of gravity on large distance scales. Another possibility is that the Universe is matter-dominated and described by general relativity, and the departure of the expansion rate from

the Einstein–de Sitter model is the result of back reactions of cosmological perturbations. This explanation is the most conservative, since it assumes neither a cosmological constant, a negative-pressure fluid, nor a modification of general relativity.

In this paper we explore the possibility that backreactions of cosmological perturbations is the source of the accelerated expansion [5, 6, 7, 8, 9, 10, 11, 12, 13]. The idea is as follows. We know there exist cosmological perturbations; after all, the Universe is inhomogeneous. To describe the time evolution of a patch of the Universe as large as our local Hubble radius one has to construct the effective dynamics from which observable average properties can be inferred. Of course, this implies a scale-dependent description of inhomogeneities. Suppose further that our Universe is filled with pressureless matter and no DE. If inhomogeneities evolve with time, a local observer would infer that our Universe is not expanding as a homogeneous and isotropic FRW matter-dominated Universe with Hubble rate  $H(t) \propto t^{2/3}$ , where  $t$  is cosmic time. On the contrary, the Universe would appear to have an expansion rate with a time evolution that depends on the nature and time evolution of these perturbations. Potentially, this could lead to an accelerated expansion.

Our idea is actually intimately connected with the general problem of how the nonlinear dynamics of cosmological perturbations on small scales may affect the large-scale “background” geometry. Let us start by discussing this issue in some generality.

The standard approach to cosmology is based both on observational facts, such as the near-perfect isotropy of the Cosmic Microwave Background (CMB) radiation, and on an *a priori* philosophical assumption, the so-called Cosmological or Copernican Principle. According to the Cosmological Principle all comoving cosmic observers at a given cosmic time should see identical properties around them (isotropy around all cosmic observers implies homogeneity, hence the FRW line element). The Cosmological Principle allows one to circumvent our inability to obtain information about the Universe outside our past light-cone by assuming that a symmetry principle is valid *everywhere*. By using the Cosmological Principle, we assume that we are able to determine conditions many Hubble radii away from us by using observational data within our past light-cone, whose region of influence is, by definition, limited to one Hubble radius [14].

An alternative procedure, dubbed *Observational Cosmology*, has also been proposed. It aims at constructing a cosmological model solely in terms of observational facts, thereby avoiding any *a priori* assumptions of global symmetry. It dates back to the works by Kristian

and Sachs in 1966 [15] and Ellis in 1984 [14]. A remarkable feature of this approach is that, by using Einstein's equations, the dataset observable within our past light-cone is precisely sufficient to determine the space-time and its matter content within the same light-cone (see Ref. [14] and references therein).

A crucial ingredient of the Observational Cosmology approach, which is shared by any realistic cosmological model-fitting procedure, is *smoothing*. Observations tell us that the Universe is far from homogeneous and isotropic on small scales. Observationally, we know that homogeneity, *e.g.*, in the galaxy distribution, is only achieved over some large smoothing scale (see *e.g.*, Ref. [16]). When we refer to homogeneity and isotropy of the Universe we tacitly assume that spatial smoothing over some suitably large filtering scale has been applied so that fine-grained details can be ignored (see in this respect the discussion in Refs. [17, 18]). In other words, by the mere assumption that the same background model can be used to describe the properties of nearby and very distant objects in the Universe, the smoothing process is implicit in the way we fit a FRW model to observations. Cosmological parameters like the Hubble expansion rate or the energy density of the various cosmic components are to be considered as volume averaged quantities: only these can be compared with observations [19].

There is, however, a technical difficulty inherent in any smoothing procedure. While matter smoothing is somewhat straightforward (*e.g.*, in the fluid description), smoothing of the space-time metric is more complex and immediately leads to an important and unexpected feature, pointed out by Ellis [14]. Let us assume that Einstein's equations hold on some suitably small scale where the Universe is highly inhomogeneous and anisotropic. Next, suppose we smooth over some larger scale. Einstein's equations are nonlinear: smoothing and evolution (*i.e.*, going to the field equations) will not commute. Hence, the Einstein tensor computed from the smoothed metric would generally differ from that computed from the smoothed stress-energy tensor. The difference is a tensor appearing on the right-hand side (RHS) of Einstein's equations that leads to an extra term in the effective Friedmann equations describing the dynamics of the smoothing domain.

How can this fact be related to the acceleration of the Universe? The answer is that this extra source term need not satisfy the usual energy conditions (according to which our Universe can only decelerate) even if the original matter stress-energy tensor does. The fact that the effective stress-energy tensor emerging after smoothing could lead to a violation

of the energy conditions was originally recognized by Ellis [14].<sup>1</sup> As we will discuss in Sec. II, explicit calculations of the effective Friedmann equations [13, 21, 22] confirm that acceleration is indeed possible even if our Universe is filled solely with matter.

A closely related question is what is the appropriate scale for which the smoothing procedure can fit the standard picture of a homogeneous and isotropic Universe on large scales? Our choice will be that of smoothing over a volume of size comparable with present-day Hubble volume. The precise size of the averaging volume does not matter, provided it is large enough that the fair sample hypothesis applies, *i.e.* that volume averages yield an accurate approximation of statistical ensemble averages. We will nonetheless refer to scales within (outside) the averaging domain as “sub-Hubble” (“super-Hubble”) perturbations. In doing this we are however promoting our super-Hubble, or “zero”-mode, to the role of FRW-like background.

The next question is what are the scales that determine the dynamics of our local background. To answer this question we have to recall what happens in the standard FRW models. The evolution of the global scale factor  $a(t)$ , the zero-mode of FRW models, is fully determined by the matter content of the Universe through the value of  $\Omega_{\text{TOTAL}}$  and  $\Omega_i$ , and via the equation of state  $w_i$  of its components. That is where microphysics enters the game. In other words, in the standard FRW picture the evolution of the Universe as a whole is determined by the dynamics of matter on sub-Hubble scales. Similarly, in the Observational Cosmology approach the dynamics of our local background must be determined by the observed behavior of matter inhomogeneities within our past-light-cone. That is where the backreaction of sub-Hubble inhomogeneities enters the game. This picture will become clear in Sec. II, where we will introduce two scalars, the so-called kinematical backreaction  $Q_D$  and the mean spatial curvature  $\langle R \rangle_D$ , that enter the expression for the energy density and pressure in the effective Friedmann equations governing the mean evolution of our local domain  $D$ . The crucial point is that in the fully general relativistic framework these two scalars are linked together by an integrability condition (which has no analogue in the Newtonian context), whose solution provides the effective equation of state of the backreaction. In order to solve this equation and establish the typical size of these terms one needs a non-perturbative and non-Newtonian approach to the evolution of cosmological

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<sup>1</sup> A closely related discussion can also be found in Ref. [20].

irregularities, as pointed out in Refs. [23, 24].

The fact that the average dynamics naturally leads to new terms in the source implies that it is legitimate to use an effective Friedmann description, provided one takes into account that the effective sources of these equations contain the back-reaction terms.

What will result from our analysis is that the evolution of sub-Hubble perturbations leads to an instability of the perturbative expansion due to the presence of large contributions which depend on a combination of Newtonian and post-Newtonian terms. This instability indicates that the effective scale factor describing the dynamics of our local Hubble patch is fed by the evolution of inhomogeneities within the Hubble radius. This cross-talk between the small-scale dynamics and the effective average dynamics described by super-Hubble, or “zero”-mode, playing the role of FRW-like background might be the crucial ingredient of backreaction that can lead to cosmic acceleration without dark energy. In Ref. [5] a deviation from the pure matter-dominated evolution was obtained by a combination of sub- and super-Hubble modes generated by inflation, the latter being improperly used to amplify the backreaction. In this paper, we will show that the deviation from a matter-dominated background is entirely due to the nonlinear evolution of sub-Hubble modes which may cause a large backreaction (technically due to the disappearance of the filter modeling the volume average), while the super-Hubble modes play no dynamical role.

Our paper is organized as follows. In Sec. II we summarize the effective Friedmann description of an inhomogeneous Universe after smoothing. In Sec. III we discuss how acceleration in our Hubble patch can result from the backreaction of perturbations. Conclusions are drawn in Sec. IV. The Appendix presents the main results of a fourth-order gradient-expansion technique.

## II. EFFECTIVE FRIEDMANN EQUATIONS IN AN INHOMOGENEOUS UNIVERSE

The goal of this section is to compute the time dependence of the local expansion rate of the Universe. For a generic fluid we may take the four-velocity to be  $u^\mu = (1, \vec{0})$ , which amounts to saying that a local observer is comoving with the energy flow of the fluid. For the case of irrotational dust considered in this paper we have the freedom to work in the

synchronous and comoving gauge with line element

$$ds^2 = -dt^2 + h_{ij}(\mathbf{x}, t)dx^i dx^j, \quad (2)$$

where  $t$  is cosmic time.

A fundamental quantity in our analysis is the velocity gradient tensor, which is purely spatial and symmetric because of irrotationality. It is defined as

$$\Theta^i_j = u^i_{;j} = \frac{1}{2}h^{ik}\dot{h}_{kj}. \quad (3)$$

Here dots denote derivatives with respect to cosmic time. The tensor  $\Theta^i_j$ , represents the *extrinsic curvature* of the spatial hypersurfaces orthogonal to the fluid flow. It may be written as

$$\Theta^i_j = \Theta \delta^i_j + \sigma^i_j. \quad (4)$$

Here  $\Theta$  is called the *volume-expansion scalar*, reducing to  $3H$  ( $H$  is the usual Hubble rate) in the homogeneous and isotropic FRW case. The traceless tensor  $\sigma^i_j$  is called the *shear*.

The evolution equations for the expansion and the shear come from the space-space components of Einstein's equations (see *e.g.*, Ref. [25]). They read, respectively, ( $\rho$  is the mass density,  $R$  is the Ricci scalar and  $R^\mu_\nu$  is the Ricci tensor)

$$\dot{\Theta} + \Theta^2 + R = 12\pi G \rho, \quad (5)$$

$$\dot{\sigma}^i_j + \Theta \sigma^i_j + R^i_j - \frac{1}{3}R \delta^i_j = 0. \quad (6)$$

The 0 – 0 component of Einstein's equations is the *energy constraint*

$$\frac{2}{3}\Theta^2 - 2\sigma^2 + R = 16\pi G \rho, \quad (7)$$

where  $\sigma^2 \equiv \frac{1}{2}\sigma^i_j \sigma^j_i$ . The 0 –  $i$  components yield the *momentum constraint*

$$\sigma^i_{j|i} - \frac{2}{3}\Theta_{,j} = 0, \quad (8)$$

where the vertical bar denotes covariant differentiation in the three-space with metric  $h_{ij}$ .

The mass density, in turn, can be obtained from the continuity equation

$$\dot{\rho} = -\Theta \rho, \quad (9)$$

whose solution reads

$$\rho = \rho_0 (h/h_0)^{-1/2}, \quad (10)$$

where  $h \equiv \det h_{ij}$  and the initial conditions have been arbitrarily set at the present time  $t_0$ . Combining the expansion evolution equation with the energy constraint gives the *Raychaudhuri equation*,

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 + 4\pi G\rho = 0. \quad (11)$$

From the latter equation it is straightforward to verify that irrotational pressure fluid elements cannot locally undergo accelerated expansion. (This point was emphasized by Hirata and Seljak [26].) Indeed, defining a local deceleration parameter and using the Raychaudhuri equation, one finds

$$q \equiv -\left(3\dot{\Theta} + \Theta^2\right)/\Theta^2 = 6(\sigma^2 + 2\pi G\rho)/\Theta^2 \geq 0. \quad (12)$$

While it is true that locally the expansion does not accelerate, it is incorrect to assume that acceleration can not occur when the fluid is coarse-grained over a finite domain. The reason is trivial: the time derivative of the average of  $\Theta$  and the average of the time derivative of  $\Theta$  are not the same because of the time dependence of the coarse-graining volume.

Let us denote the coarse-grained value of a quantity  $\mathcal{F}$  by its average over a spatial domain  $D$ :<sup>2</sup>

$$\langle \mathcal{F} \rangle_D = \frac{\int_D \sqrt{h} \mathcal{F} d^3x}{\int_D \sqrt{h} d^3x}. \quad (13)$$

We will take the domain to be comparable with the size of the present Hubble volume<sup>3</sup>.

A first important property follows directly from the smoothing procedure itself: for a generic function  $\mathcal{F}$  one has [21, 22]

$$\langle \mathcal{F} \rangle_D \dot{\phantom{\mathcal{F}}} - \dot{\langle \mathcal{F} \rangle}_D = \langle \mathcal{F}\Theta \rangle_D - \langle \Theta \rangle_D \langle \mathcal{F} \rangle_D. \quad (14)$$

In particular, for the local expansion rate one finds

$$\langle \Theta \rangle_D \dot{\phantom{\Theta}} = \langle \dot{\Theta} \rangle_D + \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \geq \langle \dot{\Theta} \rangle_D. \quad (15)$$

Although  $\langle \dot{\Theta} \rangle_D \leq -\frac{1}{3}\langle \Theta^2 \rangle_D \leq 0$ , the coarse-grained deceleration parameter  $q_D \equiv -3\langle \Theta \rangle_D \dot{\phantom{\Theta}} / \langle \Theta^2 \rangle_D - 1$  is related to  $\langle \Theta \rangle_D \dot{\phantom{\Theta}}$ , which is not the same as  $\langle \dot{\Theta} \rangle_D$ . It is precisely

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<sup>2</sup> Notice that one is not allowed to define the mean cosmological parameters only through an average over directions [26, 27] as cosmological observables, such as the Hubble rate, receive unacceptably large corrections from the same Newtonian terms which become harmless surface terms when averaging over a large volume [6]. We acknowledge discussions with U. Seljak about this issue.

<sup>3</sup> The correct definition of our comoving Hubble radius is  $R_H(t_0) = e^{-\Psi_{t_0}} \int^{t_0} dt e^{\Psi_\ell(t)}$ .



this commutation rule that allows for the possibility of acceleration in our local patch in spite of the fact that fluid elements cannot individually undergo accelerated expansion. This simple argument circumvents the *no-go theorem* adopted in Refs. [26, 27] (and later in [28]), according to which there can be no acceleration in our local Hubble patch if the Universe only contains irrotational dust.

Indeed, let us follow the work of Buchert [21, 22] and define a dimensionless scale factor

$$a_D(t) \equiv \left( \frac{V_D}{V_{D_0}} \right)^{1/3}; \quad V_D = \int_D \sqrt{h} d^3x, \quad (16)$$

where  $V_D$  is the volume of our coarse-graining domain (the subscript “0” denotes the present time). As an averaging volume we may take a large comoving domain, so that  $D$  is constant in time. Alternative choices are however possible (see, *e.g.*, Ref. [29] for a discussion of different averaging procedures).<sup>4</sup>

The coarse-grained Hubble rate  $H_D$  will be

$$H_D = \frac{\dot{a}_D}{a_D} = \frac{1}{3} \langle \Theta \rangle_D. \quad (17)$$

(Notice that with such a coarse-graining,  $H_D$  coincides with the quantity  $\bar{H}$  defined in Ref. [5]). By properly smoothing Einstein’s equations over the volume  $V_D$ , one obtains [21, 22]

$$\frac{\ddot{a}_D}{a_D} = -\frac{4\pi G}{3} (\rho_{\text{eff}} + 3P_{\text{eff}}), \quad (18)$$

$$\left( \frac{\dot{a}_D}{a_D} \right)^2 = \frac{8\pi G}{3} \rho_{\text{eff}}, \quad (19)$$

where we have defined effective energy density and pressure terms

$$\rho_{\text{eff}} = \langle \rho \rangle_D - \frac{Q_D}{16\pi G} - \frac{\langle R \rangle_D}{16\pi G} \quad (20)$$

$$P_{\text{eff}} = -\frac{Q_D}{16\pi G} + \frac{\langle R \rangle_D}{48\pi G}, \quad (21)$$

and we have introduced the *kinematical backreaction*

$$Q_D = \frac{2}{3} \left( \langle \Theta^2 \rangle_D - \langle \Theta \rangle_D^2 \right) - 2 \langle \sigma^2 \rangle_D. \quad (22)$$

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<sup>4</sup> One could alternatively average over a volume of size of the order of the instantaneous particle horizon. The effective dynamics in this case will be accompanied by a stochastic source originated by the statistical nature of the perturbations.

From the effective Friedmann equations, Eqs. (18) and (19), obtained by Buchert in Ref. [21], one immediately obtains the continuity equation for our effective fluid

$$\dot{\rho}_{\text{eff}} = -3H_D(\rho_{\text{eff}} + P_{\text{eff}}). \quad (23)$$

Note that the smoothed continuity equation differs from the local continuity equation. On the other hand, owing to the fact that our coarse-graining volume is comoving with the mass flow, mass conservation is preserved by the smoothing procedure, implying

$$\langle \rho \rangle_D \dot{\phantom{\rho}} = -3H_D \langle \rho \rangle_D. \quad (24)$$

The two quantities  $Q_D$  and  $\langle R \rangle_D$  are not independent. This can be seen by taking the derivative of Eq. (19) and using Eq. (24). The consistency of the system of Eqs. (18), (19), and (24) requires that  $Q_D$  and  $\langle R \rangle_D$  satisfy the *integrability condition* [21]

$$\left( a_D^6 Q_D \right) \dot{\phantom{Q_D}} + a_D^4 \left( a_D^2 \langle R \rangle_D \right) \dot{\phantom{\langle R \rangle_D}} = 0. \quad (25)$$

One should stress that the latter equation, *i.e.*, the link between kinematical backreaction  $Q_D$  and mean curvature  $\langle R \rangle_D$ , is a genuine General Relativistic (GR) effect, having no analogue in Newtonian theory, as the curvature  $R$  of comoving hypersurfaces vanishes identically in the Newtonian limit [21, 25, 30], implying that there exist globally flat Eulerian coordinates  $X^i$ . Indeed, in the Newtonian case, it is immediate to verify that  $Q_D$  is *exactly* (*i.e.*, at any order in perturbation theory) given by the volume integral of a total-derivative term in Eulerian coordinates [23],

$$Q_D^{\text{Newtonian}} = \langle \nabla \cdot [\mathbf{u}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \mathbf{u}] \rangle_D, \quad (26)$$

where  $\mathbf{u}$  is the peculiar velocity, so that by the Gauss theorem it can be transformed into a pure boundary term. It is precisely by this reason that any analysis of backreaction based on the Newtonian approximation, such as that recently performed in v1 of Ref. [31], is not relevant: it will invariably lead to a tiny effect, and to the absence of any acceleration. Indeed, if inhomogeneities only exist on scales much smaller than our Hubble radius and if peculiar velocities are small on the boundary of our Hubble patch then, *within the Newtonian approximation*, the standard FRW matter-dominated model can be applied without any substantial correction from the backreaction [23]. Such a drawback of the Newtonian approximation was also noticed in Refs. [6, 7]. This exact result demonstrates that in order

to deal with the backreaction, going beyond the Newtonian approximation is mandatory, as also stressed in Ref. [13]. Studies of the average dynamics including the lowest post-Newtonian gradient terms in the weak field limit were considered in Refs. [32] and v2 of [31]. However, further and more sizeable terms are expected to contribute to the backreaction once the effective dynamics of the system (including the kinematical backreaction) is considered. We will come back to this issue in subsection IIIC.

The GR integrability condition makes it clear how acceleration in our local Hubble patch is possible. Indeed, it is immediate to realize that the general condition for acceleration in a domain with mean density  $\langle \rho \rangle_D$  is

$$Q_D > 4\pi G \langle \rho \rangle_D. \quad (27)$$

Moreover, a particular solution of the integrability condition for constant  $Q_D$  and  $\langle R \rangle_D$  is

$$Q_D = -\frac{1}{3} \langle R \rangle_D = \text{const.}, \quad (28)$$

which, for negative mean curvature mimics a cosmological constant,  $\Lambda_{\text{eff}} = Q_D$ . More in general, if  $Q_D$  is positive, it may mimic a dynamical dark energy or quintessence.

So far the considerations have been rather general. Now we write the spatial metric in the general form

$$h_{ij} \equiv a^2(t) e^{-2\Psi(\mathbf{x}, t)} [\delta_{ij} + \chi_{ij}(\mathbf{x}, t)], \quad (29)$$

where  $a(t) \propto t^{2/3}$  is the usual FRW scale-factor for a flat, matter-dominated Universe and the traceless tensor  $\chi_{ij}$  contains the remaining modes of the metric, namely one more scalar, as well as vector and tensor modes.<sup>5</sup> Next, when we consider the expansion in some domain  $D$ , we can split the gravitational potential  $\Psi$  into two parts:  $\Psi = \Psi_\ell + \Psi_s$ , where  $\Psi_\ell$  is the long-wavelength mode and  $\Psi_s$  is a collection of short-wavelength modes. Of course “long” and “short” describe wavelengths compared to the size of the domain  $D$ . We can easily take into account the effect of  $\Psi_\ell$  by noting that within  $D$  the factor  $e^{-2\Psi_\ell}$  is a space-independent conformal rescaling of the spatial metric. Let us then write

$$h_{ij} = a^2(t) e^{-2\Psi_\ell(t)} \tilde{h}_{ij}(\mathbf{x}, t), \quad (30)$$

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<sup>5</sup> Indices of  $\chi_{ij}$  will be raised by the Kronecker symbol:  $\chi^i_j \equiv \delta^{ik} \chi_{kj}$  and  $\chi^{ij} \equiv \delta^{ik} \delta^{jl} \chi_{kl}$ .

with  $\tilde{h}_{ij} = e^{-2\Psi_s(\mathbf{x},t)} [\delta_{ij} + \chi_{ij}(\mathbf{x}, t)]$ . The expansion scalar and shear then become

$$\begin{aligned}\Theta &= 3\frac{\dot{a}}{a} + \tilde{\Theta} - 3\dot{\Psi}_\ell, \\ \sigma^i_j &= \tilde{\sigma}^i_j,\end{aligned}\tag{31}$$

where  $\tilde{\Theta}$  and  $\tilde{\sigma}^i_j$  are calculated with  $\tilde{h}_{ij}$ . Note that  $\tilde{\Theta}$  and  $\tilde{\sigma}^i_j$  do not depend *explicitly* on  $\Psi_\ell$ . It should be kept in mind that the local expansion rate is  $\Theta$ , not  $\tilde{\Theta}$ .

The Ricci tensor of comoving space-like hypersurfaces is given by

$$R^i_j = a^{-2}e^{2\Psi_\ell} \left[ \tilde{R}^i_j + \tilde{\nabla}^i \tilde{\nabla}_j \Psi_\ell + \tilde{\nabla}^2 \Psi_\ell \delta^i_j + \tilde{\nabla}^i \Psi_\ell \tilde{\nabla}_j \Psi_\ell - \tilde{\nabla}^k \Psi_\ell \tilde{\nabla}_k \Psi_\ell \delta^i_j \right],\tag{32}$$

where  $\tilde{R}^i_j$  is the Ricci scalar of the metric  $\tilde{h}_{ij}$  and the symbol  $\tilde{\nabla}_i$  denotes the covariant derivative in the 3-space with metric  $\tilde{h}_{ij}$ . For the Ricci scalar we find

$$R = a^{-2}e^{2\Psi_\ell} \left[ \tilde{R} + 4\tilde{\nabla}^2 \Psi_\ell - 2\tilde{\nabla}^k \Psi_\ell \tilde{\nabla}_k \Psi_\ell \right]\tag{33}$$

$$\langle R \rangle_D = a^{-2}e^{2\Psi_\ell} \left\langle \tilde{R} + 4\tilde{\nabla}^2 \Psi_\ell - 2\tilde{\nabla}^k \Psi_\ell \tilde{\nabla}_k \Psi_\ell \right\rangle_D.\tag{34}$$

Notice that  $a_D$  coincides with the scale factor  $\bar{a}$  adopted in Ref. [5], provided we take

$$a_D(t) = a(t) e^{-\Psi_\ell(t) + \Psi_{\ell 0}}\tag{35}$$

with

$$\Psi_\ell(\mathbf{x}_{\text{obs}}, t) \equiv \ln a - \frac{1}{3} \ln \left( \int_D \sqrt{h} d^3x \right) + \text{const.},\tag{36}$$

where the residual dependence of  $\Psi_\ell$  on the spatial coordinate  $\mathbf{x}_{\text{obs}}$  labels the individual comoving volume patch, *i.e.*, the specific *cosmic observer* we are considering. Using Eq. (10) we can rewrite Eq. (36) in the form

$$\Psi_\ell(t) = -\frac{1}{3} \ln \left\langle (1 + \delta_{\text{FRW}})^{-1} \right\rangle_{D_{\text{in}}} + \text{const.},\tag{37}$$

where  $\delta_{\text{FRW}}$  is the *density contrast* with respect to the mean density of a flat, matter-dominated FRW (Einstein-de Sitter) model, defined through  $\rho = (1 + \delta_{\text{FRW}}) / (6\pi G t^2)$ , and “in” denotes the initial time, which for simplicity we have taken to coincide with the end of inflation. For any quantity  $\mathcal{F}$

$$\langle \mathcal{F} \rangle_{D_{\text{in}}} = \frac{\int_D \sqrt{h_{\text{in}}} \mathcal{F} d^3x}{\int_D \sqrt{h_{\text{in}}} d^3x}.\tag{38}$$

By inspecting Eq. (37) one immediately realizes that acceleration may be achieved in those Hubble patches where the mean *rarefaction factor*  $\langle (1 + \delta_{\text{FRW}})^{-1} \rangle_{D_{\text{in}}}$  grows fast enough to

compensate for the Einstein-de Sitter expansion rate  $H = 2/3t$ . Note also that the integral defining  $\Psi_\ell$  is dominated by the dynamics of the most underdense fluid elements, not by the densest ones, so the complex dynamics of highly nonlinear mass concentrations never enters the calculation; by the same reasoning, any intrinsic limitation related to caustic formation would not affect the validity of our backreaction treatment.

The kinematical backreaction  $Q_D$  is non-vanishing and gets contributions only from  $\tilde{\Theta}$  and from the shear  $\tilde{\sigma}^i_j$ :

$$Q_D = \frac{2}{3}\langle\tilde{\Theta}^2\rangle_D - 2\langle\tilde{\sigma}^2\rangle_D, \quad (39)$$

where we used the fact that  $\langle\tilde{\Theta}\rangle_D = 0$  by construction.

In order to have a qualitative understanding of why acceleration can be the natural outcome of the backreaction, let us rewrite the mean expansion rate in terms of the peculiar expansion rate  $\theta$ , defined by  $\Theta = 3H + \theta$ :

$$H_D = \frac{2}{3t} + \frac{1}{3} \frac{\langle(1 + \delta_{\text{FRW}})^{-1}\theta\rangle_{D_{\text{in}}}}{\langle(1 + \delta_{\text{FRW}})^{-1}\rangle_{D_{\text{in}}}}, \quad (40)$$

which shows that the *mean* expansion rate receives a correction with respect to the FRW background value by the peculiar expansion rate of mostly underdense regions (where  $\theta$  is largest), which give the largest contribution to the average. However, as we already noticed, acceleration may be achieved when  $Q_D$  is positive and large enough. This requires a large variance of the volume expansion rate within the averaging domain. At early times, when perturbations are linear,  $\Theta$  is narrowly peaked around its mean background value  $3H$ . When non-linearities set in, the variance increases because of the *simultaneous* presence of largely under- and over-dense regions (in fact, counting only under-dense regions would reduce the variance leading to an under-estimate of  $Q_D$ ) [33].

### III. THE APPEARANCE OF INSTABILITIES

In order to discuss the dynamics of our local Hubble patch, one may proceed along two complementary directions. Either one tries to encode the effect of perturbations into the local scale factor  $a_D$  as done in Ref. [5], or one may try to construct an effective equation of state by computing the backreaction terms  $Q_D$  and  $\langle R \rangle_D$ . We will try to follow a combination of them to see under which circumstances acceleration in our Hubble patch can be achieved.

### A. The effect of super-Hubble modes

The mean curvature generally depends on both sub- and super-Hubble modes. Since  $Q_D$  does not depend explicitly on perturbations with wavelengths larger than the Hubble radius, one immediately concludes that if one considers super-Hubble modes only,  $Q_D$  vanishes and from the integrability condition  $\langle R \rangle_D$  scales like  $a_D^{-2}$ . Therefore the effect of pure super-Hubble perturbations is limited to generating a true local curvature term which may be important but can not accelerate the expansion of the Universe.

As we already anticipated in the Introduction, the effective scale factor describing the dynamics of our local Hubble patch is fed by the evolution of inhomogeneities within the Hubble radius. This cross-talk between the small-scale dynamics and the effective average dynamics is a crucial ingredient, as also pointed out in Ref. [5]. Before coming to this crucial point, let us pause for a moment and show that there is another more technical way to achieve the same conclusion about the role played by super-Hubble modes starting from the spatial-gradient expansion of Einstein equations.

The spatial-gradient expansion is a nonlinear approximation method suitable to describe the long-wavelength part of inhomogeneities in the Universe. This scheme is based on the assumption that observables like the local curvature can be expanded in powers of gradients of the perturbations. To account for the effect of super-Hubble modes at late times we may adopt the so-called renormalization group method applied to the gradient expansion of Einstein equations [34]. This will result in the renormalized long-wavelength solution to Einstein equations, valid also at late times until the long-wavelength perturbations enter the horizon.

Here we sketch a non-perturbative technique to solve Einstein's equations in an inhomogeneous Universe. A more detailed presentation of the method will be presented elsewhere [35].

Our approach makes use of a gradient-expansion approximation (see Refs. [36, 37, 38, 39, 40, 41, 42]). The idea is to describe the dynamics of irregularities in a Universe which contains inhomogeneities on scales larger than the Hubble radius. Working in the synchronous gauge one expands Einstein's equations starting from a space-dependent "seed" metric. The lowest order solution corresponds to the so-called long-wavelength approximation, while adding higher-order gradients leads to a more accurate solution, which hopefully

converges toward the exact one.

The gradient-expansion technique amounts to keeping a finite number of spatial derivatives. This approximation technique is non-perturbative in the sense that by solving for the metric coefficients  $\Psi$  and  $\chi_{ij}$  up to  $2n$  spatial gradients one obtains terms of any order in the conventional perturbative expansion containing up to  $2n$  gradients. For the purpose of this paper, working to four derivatives in  $\Psi$  and  $\chi_{ij}$  will suffice. Note that because the scalar  $\Psi$  appears in the argument of an exponential in the way we write the spatial metric, our gradient-expansion sensibly differs from that used in Refs. [36, 37, 38, 39, 40, 41, 42]. Even if  $\Psi$  is obtained up to a finite number of spatial gradients,  $h_{ij}$  will necessarily contain gradient terms of any order.

Since the cosmological perturbations are generated during inflation, it is natural to set initial conditions for the gravitational perturbations  $\Psi$  and  $\chi_{ij}$  at the end of inflation (effectively coinciding with  $t = t_{\text{in}} = 0$ ). If so, the spatial metric in the super-horizon regime is given by  $h_{ij} = a^2 e^{-2\zeta} \delta_{ij}$  where  $\zeta$  is the curvature perturbation [43]. It is related to the so-called peculiar gravitational potential  $\varphi(\mathbf{x})$  by  $\zeta = 5\varphi/3$  in single-field models of inflation. Therefore, the initial conditions at  $t = 0$  are  $\Psi_{\text{in}} \equiv \Psi(t = 0) = 5\varphi/3$  and  $\chi_{ij}(t = 0) = 0$ . Notice that since cosmological perturbations generated during single-field models of inflation are very nearly Gaussian with a nearly flat spectrum [43], we infer that  $\varphi$  should be regarded as a nearly scale-invariant, quasi-Gaussian random field.

From this analysis we see that  $\Psi$  contains at least a zero-derivative term: this will be our seed metric perturbation, which is the necessary ingredient for gravitational instability to develop. The traceless tensor  $\chi_{ij}$  has at least two spatial gradients. The only exception might come from linear or higher-order tensor modes appearing at the end of inflation. Nonetheless, accounting for these contributions does not quantitatively affect our results.

Let us adopt the gradient-expansion in order to obtain  $\Psi$  and  $\chi_i^j$  including up to four gradients. Under this assumption, the inverse spatial metric can be written as

$$h^{ij} = a^{-2} e^{2\Psi} \left( \delta^{ij} - \chi^{ij} + \chi^i_k \chi^{kj} \right). \quad (41)$$

Writing  $\Theta \equiv 3H + \theta$  where  $H = \dot{a}/a$ , one finds (up to higher-derivative terms)

$$\theta = -3\dot{\Psi} - \frac{1}{2} \chi^{kl} \dot{\chi}_{kl} \quad (42)$$

$$\sigma^i_j = \frac{1}{2} \dot{\chi}^i_j - \frac{1}{2} \chi^{ik} \dot{\chi}_{kj} + \frac{1}{6} \chi^{lk} \dot{\chi}_{lk} \delta^i_j. \quad (43)$$

To solve for these quantities one also needs the 3D Ricci tensor and Ricci scalar of the constant-time hypersurfaces. We start by calculating the 3D Christoffel symbols, which read (up to higher-derivative terms)

$$\Gamma_{jk}^i = -\Psi_{,k} \delta_j^i - \Psi_{,j} \delta_k^i + \Psi^{,i} \delta_{jk} + \frac{1}{2} \left( \chi_{j,k}^i + \chi_{k,j}^i - \chi_{jk}^{,i} \right) + \Psi^{,i} \chi_{jk} - \Psi_{,l} \chi^{il} \delta_{jk}. \quad (44)$$

The Ricci tensor and Ricci scalar read, respectively,

$$\begin{aligned} R^i_j &\equiv \frac{\mathcal{R}^i_j}{a^2} = \frac{e^{2\Psi}}{a^2} \left[ \Psi^{,i} \delta_j^i + \nabla^2 \Psi \delta_j^i + \Psi^{,i} \Psi_{,j} - (\nabla \Psi)^2 \delta_j^i - \chi^{ik} \Psi_{,kj} - \chi^{ik} \Psi^{,k} \Psi_{,j} \right. \\ &\quad + \frac{1}{2} \left( \chi_{,kj}^{ik} + \chi_{j,k}^{,i} - \nabla^2 \chi_j^i \right) - \Psi^{,kl} \chi_{kl} \delta_j^i - \Psi_{,k} \chi^{kl} \delta_j^i \\ &\quad \left. + \frac{1}{2} \Psi^{,k} \left( -\chi_{k,j}^i - \chi_{kj}^{,i} + \chi_{j,k}^i \right) + \Psi_{,k} \Psi_{,l} \chi^{kl} \delta_j^i \right] \end{aligned} \quad (45)$$

$$R \equiv \frac{\mathcal{R}}{a^2} = \frac{e^{2\Psi}}{a^2} \left[ 4\nabla^2 \Psi - 2(\nabla \Psi)^2 + \chi^{ij}_{,ij} - 4\chi^{ij} \Psi_{,ij} - 4\chi^{ij}_{,i} \Psi_{,j} + 2\chi^{ij} \Psi_{,i} \Psi_{,j} \right]. \quad (46)$$

The evolution equations for the peculiar volume expansion scalar and for the shear immediately follow from Eqs. (5), (6) and (7)

$$\begin{aligned} \dot{\theta} + 3H\theta + \frac{1}{2}\theta^2 + \frac{3}{2}\sigma^2 &= -\frac{1}{4a^2} \mathcal{R}, \\ \dot{\sigma}^i_j + 3H\sigma^i_j + \theta\sigma^i_j &= -\frac{1}{a^2} \left( \mathcal{R}^i_j - \frac{1}{3} \mathcal{R} \delta^i_j \right). \end{aligned} \quad (47)$$

Replacing our expressions for  $\theta$ ,  $\chi^i_j$ ,  $\mathcal{R}^i_j$  and  $\mathcal{R}$  in terms of the metric coefficients and retaining only terms containing up to four spatial derivatives, one obtains differential equations for  $\Psi$  and  $\chi_{ij}$ , namely

$$\ddot{\Psi} + 3H\dot{\Psi} = \frac{3}{2}\dot{\Psi}^2 - \frac{1}{2}H\chi^{kl}\dot{\chi}_{kl} - \frac{5}{48}\dot{\chi}^{kl}\dot{\chi}_{kl} - \frac{1}{6}\chi^{kl}\ddot{\chi}_{kl} + \frac{1}{12a^2}\mathcal{R}, \quad (48)$$

$$\begin{aligned} \ddot{\chi}^i_j + 3H\dot{\chi}^i_j &= 3H \left( \chi^{ik}\dot{\chi}_{kj} - \frac{1}{3}\chi^{kl}\dot{\chi}_{kl}\delta^i_j \right) + \left( \dot{\chi}^{ik}\dot{\chi}_{kj} - \frac{1}{3}\dot{\chi}^{kl}\dot{\chi}_{kl}\delta^i_j \right) \\ &\quad + \left( \chi^{ik}\ddot{\chi}_{kj} - \frac{1}{3}\chi^{kl}\ddot{\chi}_{kl}\delta^i_j \right) + 3\dot{\Psi}\dot{\chi}^i_j - \frac{2}{a^2} \left( \mathcal{R}^i_j - \frac{1}{3}\mathcal{R}\delta^i_j \right). \end{aligned} \quad (49)$$

These equations can be solved iteratively. With two gradients only, the conformal Ricci tensor  $\mathcal{R}^i_j$  and scalar  $\mathcal{R}$  coincide with their initial values, *i.e.*, with the curvature of the seed conformal metric  $e^{-2\Psi_{\text{in}}}\delta_{ij}$ . Up to two gradients we obtain

$$\Psi = \frac{5}{3}\varphi + \frac{1}{18} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left[ \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right], \quad (50)$$

$$\chi^i_j = -\frac{1}{3} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left[ D^i_j \varphi + \frac{5}{3} \left( \varphi^{,i} \varphi_{,j} - \frac{1}{3} (\nabla \varphi)^2 \delta^i_j \right) \right], \quad (51)$$



where  $D^i{}_j \equiv \partial^i \partial_j - \frac{1}{3} \nabla^2 \delta^i{}_j$ .

From Eq. (50) we obtain the volume expansion scalar and the shear tensor:

$$\theta = -\frac{1}{3} \left( \frac{a}{a_0} \right)^{-1/2} \left( \frac{2}{H_0} \right) e^{10\varphi/3} \left[ \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right] \quad (52)$$

$$\sigma^i{}_j = -\frac{1}{3} \left( \frac{a}{a_0} \right)^{-1/2} \left( \frac{2}{H_0} \right) e^{10\varphi/3} \left[ D^i{}_j \varphi + \frac{5}{3} \left( \varphi^i \varphi_{,j} - \frac{1}{3} (\nabla \varphi)^2 \delta^i{}_j \right) \right]. \quad (53)$$

Let us first explain how the renormalization group method works at the level of two gradients starting from the solution of Eq. (50) written in the form

$$\Psi = \Psi_{\text{in}} + e^{2\Psi_{\text{in}}} \epsilon (a - a_{\text{in}}), \quad \epsilon \equiv \frac{1}{18} \left( \frac{1}{a_0} \right) \left( \frac{2}{H_0} \right)^2 \left[ \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right], \quad (54)$$

where  $\epsilon \ll 1$ . By taking the limit  $\epsilon \ll 1$  we can isolate the long-wavelength part of  $\Psi$ , in other words,  $\Psi_\ell$ .

The constant  $\Psi_{\text{in}}$  represents the value of the gravitational potential at some initial instant of time when the scale factor is  $a_{\text{in}}$ . We regularize the  $\mathcal{O}(\epsilon)$  secular term by introducing an arbitrary ‘‘scale factor’’  $\mu$  and a renormalized constant  $\Psi_{\text{in}} = \Psi_R(\mu) + \epsilon \delta\Psi(\mu, a_{\text{in}})$ . If we split the term  $(a - a_{\text{in}})$  into  $(a - \mu + \mu - a_{\text{in}})$ , then to first order in  $\epsilon$

$$\Psi = \Psi_R(\mu) + \epsilon \delta\Psi(\mu, a_{\text{in}}) + \epsilon e^{2\Psi_R(\mu)} (a - \mu + \mu - a_{\text{in}}). \quad (55)$$

The counterterm  $\delta\Psi$  is determined in such a way to absorb the  $(\mu - a_{\text{in}})$ -dependent term in the gravitational potential  $\Psi$ :

$$\delta\Psi(\mu, a_{\text{in}}) + e^{2\Psi_R(\mu)} (\mu - a_{\text{in}}) = 0. \quad (56)$$

This defines the renormalization-group transformation

$$\Psi_R(\mu) = \Psi_{\text{in}} + \epsilon e^{2\Psi_R(\mu)} (\mu - a_{\text{in}}), \quad (57)$$

and the renormalization group equation

$$\frac{\partial \Psi_R(\mu)}{\partial \mu} = \epsilon e^{2\Psi_R(\mu)}. \quad (58)$$

The solution of Eq. (58) is

$$\Psi_R(\mu) = -\frac{1}{2} \ln(c_2 - 2\epsilon\mu), \quad (59)$$

where  $c_2 = e^{-10\varphi/3}$  is the constant of integration. Equating  $\mu$  to the generic scale factor we find that the renormalized improved solution for the gravitational potential at the level of two gradients is given by

$$\Psi = \Psi_R(\mu = a) = \frac{5}{3} \varphi - \frac{1}{2} \ln \left[ 1 - \frac{1}{9} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left( \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right) \right]. \quad (60)$$

If expanded up to two gradients, this solution coincides with Eq. (50). Since by construction one should take the long-wavelength part of the argument of the logarithm in the solution of Eq. (60), it is easy to see that the latter matches Eq. (36) expanded up to two gradients. Indeed, write Eq. (36) as

$$\Psi_\ell(\mathbf{x}, t) - \Psi_\ell(\mathbf{x}, t_{\text{in}}) = -\frac{1}{3} \ln \left( \frac{\int_D e^{-3\Psi} d^3x}{\int_D e^{-3\Psi_{\text{in}}} d^3x} \right). \quad (61)$$

Inserting Eq. (50) into Eq. (61) and expanding up to two gradients using the fact that  $\Psi_\ell(\mathbf{x}, t_{\text{in}}) = 5\varphi/3$ , we obtain

$$\begin{aligned} \Psi_\ell(\mathbf{x}, t) - \Psi_\ell(\mathbf{x}, t_{\text{in}}) &\simeq -\frac{1}{3} \ln \left( \frac{\int_D e^{-3\Psi_{\text{in}}} (1 - 3e^{10\varphi/3} \epsilon a) d^3x}{\int_D e^{-3\Psi_{\text{in}}} d^3x} \right) \\ &\simeq -\frac{1}{3} \ln \left( 1 - 3 \frac{\int_D e^{-3\Psi_{\text{in}}} e^{10\varphi/3} \epsilon a d^3x}{\int_D e^{-3\Psi_{\text{in}}} d^3x} \right) \\ &\simeq -\frac{1}{2} \ln \left( 1 - \left\langle \frac{1}{9} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left[ \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right] \right\rangle_{D_{\text{in}}} \right), \end{aligned} \quad (62)$$

which coincides with Eq. (60). Notice, in particular, that Eq. (60) differs from the toy gravitational potential adopted by Hirata and Seljak [26].

Let us compute the corresponding deceleration parameter

$$q = -\frac{\dot{H}_D}{H_D^2} - 1 = -\frac{\dot{H} - \ddot{\Psi}_\ell}{(H - \dot{\Psi}_\ell)^2}, \quad (63)$$

where we have used the fact that  $H_D = H - \dot{\Psi}_\ell$ . Inserting Eq. (60) into Eq. (63), we find that at late times

$$q \sim \frac{3}{2} \cdot \frac{2}{3} - 1 = 0, \quad (64)$$

*i.e.*, the deceleration parameter tends to zero. This result confirms our expectation that at the (resummed) lowest order in the gradient expansion, the Universe turns out to be curvature-dominated at late times, which is equivalent to a Universe with effective equation of state  $w = -1/3$ . What about the resummation of the long-wavelength perturbations at higher order in gradient terms? The curvature term is a series of gradients, and can be written in the form

$$R = \sum_{n \geq 1} e^{2n\Psi_{\text{in}}} c_n a^{n-2}, \quad (65)$$

where  $c_n = \mathcal{O}(\partial^{2n})$  is a coefficient containing  $2n$  gradients. Repeating the resummation procedure outlined for the case of two gradients, one can easily show that at any given order

$n$  the renormalized solution reads

$$\Psi_R(a) \sim -\frac{1}{2n} \ln(1 - 2n c_n a^n). \quad (66)$$

Since at very late times  $\Psi_R \sim -\frac{1}{2} \ln a$ , the corresponding deceleration parameter at late times goes like in Eq. (64). We conclude that at any order in the gradients, at late times the effect of the resummation of the long-wavelength perturbations is simply to generate a curvature term. This conclusion may be obtained also by inspecting Eq. (65) after the constant of integration  $\Psi_{\text{in}}$  has been promoted to the renormalized quantity  $\Psi_R$ . Each term in the series gives a contribution to  $R$  which scales as  $a^{-3} \sim t^{-2}$ . If only long-wavelength perturbations were present, the true scale factor  $a_D$  would scale like

$$a_D = a e^{\frac{1}{2} \ln a} = a^{3/2}, \quad (67)$$

and  $\langle R \rangle_D$  would scale like  $a_D^{-2}$ . Therefore, if *only* long-wavelength perturbations were present, at large times the line-element would take the form of a curvature dominated Universe, with  $h_{ij} \sim t^2 C_{ij}(\mathbf{x})$ , where  $C_{ij}(\mathbf{x})$  is a function of spatial coordinates only.

## B. The effect of sub-Hubble modes

Dealing with the backreaction of sub-Hubble perturbations, and therefore attacking the issue of the cross-talk between the sub-Hubble modes and the homogeneous mode, is more difficult than dealing with the super-Hubble modes because, as we shall see, the gradient expansion displays an instability of the perturbative series.

In the effective Friedmann description of the inhomogeneous Universe one wishes to compute the typical value of the local observables averaged over the comoving volume  $D$ . By that we mean the ensemble average of such a volume average. The cosmological perturbations are treated as variables that take random values over different realizations of volumes  $D$ . In other words, we calculate the typical value of a quantity for a region of given size as the statistical mean over many different similar regions. This typical value is generically accompanied by a variance. If the size of the comoving volume  $D$  is much smaller than the global inflationary volume, then we can imagine placing this volume in random locations within a region whose size is much bigger than the size of  $D$ . By the ergodic property, this is equivalent to taking random samples of the ensemble for a fixed location of the box.

In other words, one can replace the expected value of a given quantity averaged over a given comoving volume  $D$  with the ensemble average of the volume average, denoted by  $\overline{\langle \dots \rangle}_D$ . Since we are interested in the role of sub-Hubble perturbations which cause a tiny variation of the value of the gravitational potential from one Hubble patch to another, the variance of the local mean observables is small. Under these circumstances, we can safely replace the spatial average with the ensemble average. This automatically implies that the perturbations which contribute to the effective dynamics are no longer restricted to receive contributions peaked at modes comparable to the Hubble-size (technically, this means that the window filter function defining the size of the comoving volume  $D$  plays no role) and therefore can be much bigger than of order  $10^{-5}$  (or powers of it)<sup>6</sup>.

Let us start by considering the lowest order in a gradient expansion, *i.e.*, keeping only two spatial derivatives. The mean local curvature will be non-vanishing, but  $Q_D$  will be zero at this order, as  $\dot{\Psi}$  contains at least two spatial derivatives. In such a case, the integrability relation Eq. (25) immediately shows that the only consistent solution is

$$\langle R \rangle_D \propto a_D^{-2}, \quad (68)$$

*i.e.*, the effect of sub-Hubble perturbations at this order is to generate a standard curvature-like term in the effective Friedmann equations, scaling as the inverse square of the scale factor. This simple result holds at any order in perturbation theory (provided that one keeps only two spatial derivatives) and represents a straightforward extension of what found in Ref. [44] where it was shown that to second order in spatial gradients and in the gravitational potential, cosmological perturbations amount only to a renormalization of the local spatial curvature (this result valid up to two gradients was though improperly applied to the findings of Ref. [5], where more than two spatial derivatives were included, for instance through the physical redshift; this point was also noticed in Ref. [8]). The result of Eq. (68) is reminiscent of the so-called *vacuole* model (see, *e.g.*, Ref. [45]).<sup>7</sup> Consider indeed a spherical region of a perfectly uniform Universe. Suppose that the matter inside that spherical region is squeezed

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<sup>6</sup> It is important to stress that although the evolution of the kinematical backreaction and the mean curvature are obtained for averaged fields restricted to the domain  $D$ , the solutions to the averaged equations are actually influenced by inhomogeneities outside the domain  $D$  too, since the initial data are to be constructed non-locally and so take the fields on the whole Cauchy hypersurface into account. We thank T. Buchert for discussions on this issue.

<sup>7</sup> We thank S. Carroll for correspondence on this issue.

into a smaller uniform spherical distribution with higher density. By mass conservation there will be a region in between the overdense sphere and the external Universe that is completely empty. Einstein's equations are exactly solvable for this situation in terms of the Tolman-Bondi metric. The outside FRW Universe is totally unaffected by such a rearrangement. By Birkhoff's theorem, the empty shell will be described by the Schwarzschild metric. Finally, the interior region will behave like a homogeneous and isotropic FRW Universe, but with different values of the cosmological parameters. These parameters will exactly obey the conventional Friedmann equations, and someone who lived inside there would have no way of telling that those parameters did not describe the entire Universe.

However, the situation changes if we consistently account for higher-order derivative terms both in  $\langle R \rangle_D$  and in  $Q_D$ . The lowest non-zero contribution to  $Q_D$  contains four spatial gradients and goes like  $a^2 H^2 \propto a^{-1}$ ; this corresponds to a similar term with four gradients in the mean spatial curvature.

Let us further elaborate on these findings. In all generality one can write

$$\begin{aligned} Q_D &= \sum_{n=2}^{\infty} q_n a^{n-3} \\ \langle R \rangle_D &= \sum_{n=1}^{\infty} r_n a^{n-3}, \end{aligned} \tag{69}$$

where  $q_n$  and  $r_n$  are expansion coefficients containing  $2n$  spatial gradients. (Note that  $q_1 = 0$ , as  $Q_D$  starts from 4 gradients.) One may wonder about the actual range of validity of the gradient-expansion technique. At first sight it might appear to be valid only to describe inhomogeneities on super-Hubble scales, *i.e.*, for comoving wave-numbers  $k \lesssim aH$ . However, this is not really the case! As one can easily check, terms of order  $n$  in the expansion, *i.e.*, terms with  $2n$  gradients, contain the peculiar gravitational potential  $\varphi$  to power  $m$  with  $2n \geq m \geq n$ . The dominant contribution at each order  $n$  (*i.e.*, with  $2n$  gradients) is Newtonian, *i.e.*, coming from terms of the type  $(\partial^2 \varphi)^n$ . However, these terms both in  $Q_D$  and  $\langle R \rangle_D$  sum up to produce negligible surface terms when averaged over a large volume, so that the leading terms become the first post-Newtonian ones, *i.e.*, those proportional to  $(\partial^2 \varphi)^{n-1} (\partial \varphi)^2$ . In other words the expansion is shielded from the effect of the Newtonian terms, which could in principle be almost arbitrarily large, by the volume averaging. The same *protection* mechanism, however, does not apply to the non-Newtonian terms in the expansion, simply because they cannot be recast as surface terms. This simple reasoning

immediately leads to the conclusion that the actual limit of validity of our expansion at order  $n$ , is set by  $(k/aH)^{2n}\varphi^{n+1} \lesssim 1$ . Because of the nearly-Gaussian nature of our inflationary seed  $\varphi$ , it is clear that the lowest-order term able to produce a big contribution to  $Q_D$  and  $\langle R \rangle_D$  appears for  $n = 3$ , *i.e.*, a term with six gradients. The importance of the six-derivative post-Newtonian terms has indeed been stressed also by Notari [12]. It is a disconnected fourth-order moment of  $\varphi$  of the type

$$\langle (\nabla^2\varphi)^2/H_0^4 \rangle \langle (\nabla\varphi)^2/H_0^2 \rangle, \quad (70)$$

having assumed that the spatial average coincides with the ensemble average. At this level an *instability* of the perturbative expansion is produced by the combination of the small post-Newtonian term  $\langle (\nabla\varphi)^2/H_0^2 \rangle$  (of order  $10^{-5}$ ) with the Newtonian term  $\langle (\nabla^2\varphi)^2/H_0^4 \rangle$ , which can be almost arbitrarily large [6], due to the logarithmic dependence on the ultraviolet cut-off (for a scale-invariant spectrum and cold dark matter transfer function).<sup>8</sup>

It is important to stress that the six derivative terms give a contribution to  $Q_D$  which scales like  $a^3H^2 = \text{constant}$ ; similarly the six-gradients contribution to the smoothed curvature scales like  $a^2/a^2 = \text{constant}$ . So these terms give rise to a sizeable *effective cosmological constant-like term* in our local Friedmann equations. In order to estimate correctly the six-gradients terms one needs the metric coefficients  $\Psi$  and  $\chi^i_j$  up to four gradients (whose explicit expressions are given in the Appendix).

The existence of a large contribution at six gradients, however, suggests that higher-order gradient terms will similarly lead to large corrections to the FRW background expansion rate. This is indeed the case. In the large- $n$  limit there will be large contributions coming from perturbations in the quasi-linear regime ( $|\delta_{FRW}| \gtrsim 1$ ). These generic conclusions, however, also tell us that stopping the expansion at six gradients would be completely arbitrary and that, in any case, the perturbative approach cries for a more refined treatment than simply counting powers of the scale factor as done in Ref. [12]. The existence of large corrections to the background should be taken strictly as evidence for an instability of the FRW background caused by nonlinear structure formation in the Universe. The actual quantitative evaluation of their effect on the expansion rate of the Universe would however require a truly non-

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<sup>8</sup> Unlike the standard perturbative approach, we need not require small matter density fluctuations  $\nabla^2\varphi/H_0^2 \ll 1$ , in our approach. This is because in the evaluation of mean observables, powers of  $\nabla^2\varphi/H_0^2$  either give rise to tiny surface terms or get multiplied by the small post-Newtonian term  $(\nabla\varphi)^2/H_0^2$ .

perturbative approach, which is clearly beyond the aim of this paper.

Connected to this fact is a technical obstacle in extending the validity of the gradient expansion to late times and/or to the nonlinear regime. This comes from the fact that the metric determinant may become negative, indicating an internal inconsistency of the approximation. In Ref. [42] the problem is solved by using an “improved” approximation scheme which expresses the metric as a “square;” this choice guarantees non-negativity and leads to a GR extension of the classical Zel’dovich approximation of Newtonian theory. It is then shown that with suitable choice of the initial seed, such an improved approximation provides an excellent match to an exact inhomogeneous solution of Einstein’s field equations, the so called *Szekeres metric* [46], which describes locally axisymmetric (pancake) collapse of irrotational dust. Alternatively, exploiting the non-perturbative continuity equation,  $(1 + \delta_{\text{FRW}})^{-1} = \int_{t_{\text{in}}}^t dt \theta$ , one can easily convince her/himself that the determinant of the metric is always well-defined.

In order to take one step forward in the gradient-expansion approach, we will use the same renormalization group technique previously applied to deal with the backreaction of super-Hubble modes. Let us start by dealing with the case of two gradients. Can we apply the renormalization technique to the case of sub-Hubble perturbations up to two gradients? The answer is yes, since the spatial averages of objects like  $\nabla^2\varphi/H_0^2$  and  $(\nabla\varphi)^2/H_0^2$  can be replaced by the corresponding ensemble averages and therefore are tiny (of the order of  $10^{-5}$ ). The renormalized growing solutions at two gradients reads therefore<sup>9</sup>

$$\begin{aligned} \Psi &= \frac{5}{3}\varphi - \frac{1}{2} \ln \left[ 1 - \frac{1}{9} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left( \nabla^2\varphi - \frac{5}{6} (\nabla\varphi)^2 \right) \right], \\ \chi^i_j &= -\frac{1}{2} \ln \left\{ 1 - \frac{2}{3} \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 e^{10\varphi/3} \left[ D^i_j\varphi + \frac{5}{3} \left( \varphi^i\varphi_{,j} - \frac{1}{3} (\nabla\varphi)^2 \delta^i_j \right) \right] \right\}. \end{aligned} \quad (71)$$

The next step consists in solving for the cosmological perturbations at four gradients. The equations of motion for the gravitational potential  $\Psi$  and for  $\chi^i_j$  at four gradients are given by Eqs. (48) and (49) where the sources in the right-hand-side are computed inserting the

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<sup>9</sup> The long-wavelength part of the factor  $e^{5\varphi(\mathbf{x})/3}$ , associated with each spatial gradient, can be re-absorbed by a redefinition of the spatial coordinates, as noticed in Ref. [26], and does not play any role when evaluating ensemble averages as well as in the backreaction problem (we thank M. Porrati for correspondence on this issue); the small-wavelength part, on the other hand, can be expanded as  $(1 + 5\varphi_s/3) \sim 1$  because  $\varphi_s \sim 10^{-5}$ . Nonetheless, we prefer to show them explicitly because they provide the initial condition  $C_{\text{in}}$  for the renormalization approach.

solutions of Eq. (71).

Upon defining the coefficient

$$\mathcal{E} = \frac{1}{9}e^{10\varphi/3} \left( \nabla^2 \varphi - \frac{5}{6} (\nabla \varphi)^2 \right), \quad (72)$$

the matrix

$$\mathcal{F}^i_j = \left( \partial^i \partial_j \varphi - \frac{5}{6} \partial^i \varphi \partial_j \varphi \right), \quad (73)$$

and the traceless matrix

$$\mathcal{E}^i_j = \frac{2}{3}e^{10\varphi/3} \left[ D^i_j \varphi + \frac{5}{3} \left( \varphi^{,i} \varphi_{,j} - \frac{1}{3} (\nabla \varphi)^2 \delta^i_j \right) \right], \quad (74)$$

the growing solution at four gradients for the gravitational potential assumes the form

$$\begin{aligned} \Psi \simeq & -\frac{1}{24} \text{Tr} \left[ \ln \left( 1 - \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 \mathcal{E} \right) \ln \left( 1 - \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 \mathcal{E} \right) \right] \\ & + \frac{5}{36\mathcal{E}} \text{Tr} \left[ \ln \left( 1 - \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 \mathcal{E} \right) \ln \left( 1 - \left( \frac{a}{a_0} \right) \left( \frac{2}{H_0} \right)^2 \mathcal{E} \right) \mathcal{F} \right]. \end{aligned} \quad (75)$$

A similar solution can be obtained for  $\chi^i_j$  at four gradients. At late times the solution for the gravitational potential grows like  $(\ln a)^2$ . A renormalization procedure can be applied to the solutions at four gradients because there are still “small” perturbative terms at hand, for instance terms like  $(\nabla \varphi)^4$  whose spatial average is small. This amounts to saying that one has to take the solution of Eq. (75), expand the arguments of the logarithms and apply the renormalization procedure described previously at second order in the “perturbative parameter”  $\mathcal{E}$ . The renormalized solution for the gravitational potential will grow like  $(\ln a)$  at large times.

The lesson to learn from this computation is that, if we proceed further and go to six gradients, the unrenormalized gravitational potential, as well as  $\chi^i_j$ , will grow like  $(\ln a)^3$ . This is surely a step forward compared to the simple counting of powers of the scale factor which predicts that, at six gradients, the gravitational potential should grow like  $a^3$ . However, at this stage the renormalization procedure fails because it involves terms like the one in Eq. (70), which may be easily of order unity. Even the resummed perturbative expansion shows an instability produced by the combination of post-Newtonian and Newtonian terms; solutions with  $2n$  gradients are expected to behave like  $(\ln a)^n$ . If taken at face value, such a time-behavior of the gravitational potential would lead to acceleration of our local Hubble patch. To put this indication on firmer grounds (or to disprove it), however, one should go



beyond the perturbative approach adopted in this paper. Our result may spur the efforts toward the search for a nonperturbative description of the dynamics of the system which would account for combinations of large Newtonian and small post-Newtonian terms.

### C. The backreaction in the weak-field approximation

So far, in evaluating the effect of backreaction we have been making use of a perturbative approach in which non-linear dynamical quantities are explicitly expressed in terms of the inflationary seed perturbation  $\varphi$  and its spatial derivatives. This is the reason why higher and higher gradients of  $\varphi$  appear in our results. The same conclusion would hold also in different gauges as well as by using different perturbative schemes.

One might however wonder whether the back-reaction problem can be approached *directly* in terms of non-linearly evolved variables. Related to this issue is the gauge choice. Non-perturbative approaches are indeed possible both in the comoving gauge adopted so far (see Ref. [25]), and in the more commonly used Poisson gauge [47]. Working out the effects of back-reaction in a non-comoving gauge is indeed fully legitimate, provided a well-defined space-time splitting is performed, *e.g.*, by means of the ADM approach [22]. We will here only sketch how back-reaction effects can be evaluated in the Poisson gauge, leaving to a subsequent paper a more detailed and quantitative analysis of the problem.

The line-element of the Poisson gauge reads (see, *e.g.*, Ref. [48])

$$ds^2 = a^2(\tau) \left\{ - (1 + 2\phi_P) d\tau^2 - 2V_i d\tau dx^i + \left[ (1 - 2\Psi_P) \delta_{ij} + h_{ij}^{(T)} \right] dx^i dx^j \right\}. \quad (76)$$

where  $\tau$  is the conformal time and  $a(\tau) \propto \tau^2$  is the FRW background scale-factor for our irrotational dust source. It is important to stress that this line-element is meant to include perturbative terms of any order around the FRW background. The quantities  $V_i$  are pure vectors, *i.e.*, they are divergenceless,  $\partial^i V_i = 0$ , while  $h_{ij}^{(T)}$  represent traceless and transverse (*i.e.*, pure tensor) modes,  $h_i^{(T)i} = \partial^i h_{ij}^{(T)} = 0$  (spatial indices are raised by the Kronecker symbol). Vector and tensor metric modes are, respectively, of  $\mathcal{O}(1/c^3)$  and  $\mathcal{O}(1/c^4)$ . To leading order in powers of  $1/c$  the above line-element is known to take the well-known *weak-field* form

$$ds^2 = a^2(\tau) \left[ - (1 + 2\phi_P) d\tau^2 + (1 - 2\psi_P) \delta_{ij} dx^i dx^j \right], \quad (77)$$

where the scalars  $\phi_P$  and  $\psi_P$  are both  $\mathcal{O}(1/c^2)$  and  $\phi_P = \psi_P = \Phi_N/c^2$ ; the Newtonian

gravitational potential  $\Phi_N$  is related to density fluctuations  $\delta\rho$  by the cosmological Poisson equation  $\nabla^2\Phi_N = 4\pi G a^2 \delta\rho$ . It is easy to realize that this form is accurate enough to describe structure formation within the Hubble radius as long as the considered wavelengths are much larger than the Schwarzschild radius of collapsing bodies [49].

The crucial point is that the kinematical back-reaction will contain the relevant term [22]  $\langle N^2\Theta^2 \rangle_D$ , where  $\Theta = u^\mu{}_{;\mu}$  ( $u^\mu$  being the fluid four-velocity). Here  $N$  is the inhomogeneous lapse function needed to express the Poisson-gauge coordinate time  $t_P = \int d\tau a(\tau)$  as a function of the proper time  $t$  of comoving observers. This issue was already pointed out in Ref. [7] where an approximate explicit expression for  $N$  was given (a second-order perturbative expression can be found in Ref. [6]); a term of the type  $(\nabla\Phi_N)^2$  appears explicitly in  $N$ . What is important for us here is that  $Q_D$  will clearly display the same type of post-Newtonian (hence non-total derivative) terms which were found in the comoving gauge using our gradient expansion, namely terms of the type

$$\left\langle (\nabla^2\Phi_v)^2 (\nabla\Phi_N)^2 \right\rangle, \quad (78)$$

where  $\Phi_v$  is the velocity potential, which coincides (up to a sign) with the gravitational potential  $\Phi_N$  on linear scales; more generally  $\Phi_v$  and  $\Phi_N$  are connected by a cosmological Bernoulli equation (see, *e.g.*, Ref. [25]). Similar terms appear in the mean curvature when projecting onto the comoving observer frame. We stress again that the terms of the type (78) appear only when considering the correct effective description of the average dynamics which has to include the kinematical backreaction term. Notice that this does not amount to saying that post-Newtonian effects are relevant in the dynamical evolution of the gravitational and velocity potentials themselves. Indeed the expression (78) requires evaluation of the generally non-linear potentials  $\Phi_v$  and  $\Phi_N$  which may be readily obtained through the use of standard  $N$ -body simulations. Owing to the non-linear (hence non-Gaussian) nature of the potentials  $\Phi_v$  and  $\Phi_N$ , the average (78) contains both a disconnected term, as in our previous treatment, and a non-zero connected four-point moment which is dominated by mildly non-linear scales, of order a few Mpc. Contrary to what happens in the synchronous gauge when the result is expressed in terms of the initial seed  $\varphi$ , in the weak-field approximation the number of gradients is expected to be finite. However, the time-dependence is generally more involved than in the perturbative approach. It is interesting to note that the combination in Eq. (78) provides a contribution to  $Q_D$  which is of the order of  $H^2$  and, using the linear dependence,

nearly constant in time.

#### IV. CONCLUSIONS

The most astonishing recent observational result in cosmology is the indication that our Universe is presently undergoing a phase of accelerated expansion. One possible explanation of the observations is that the Universe is homogeneously filled with a fluid with negative pressure that counteracts the attractive gravitational force of matter fields. Another possible explanation is a modification of GR on large distance scales.

In this paper we have elaborated on the alternative idea that the backreaction of cosmological perturbations may cause the cosmic acceleration [5, 6, 7, 8, 9, 10, 11, 12, 13]. Following Buchert [21, 22], we have provided the effective Friedmann equations describing an inhomogeneous Universe after smoothing. The effective dynamics is governed by two scalars: the so-called kinematical backreaction  $Q_D$  and the mean spatial curvature  $\langle R \rangle_D$ . They enter in the expression for the effective energy density and pressure in the Friedmann equations governing the mean evolution of a local domain  $D$ . For positive  $Q_D$ , acceleration in our local Hubble patch may be attained despite the fact that fluid elements cannot individually undergo accelerated expansion. Indeed, the very fact that the smoothing process does not commute with the time evolution invalidates the no-go theorem, which states that there can be no acceleration in our local Hubble patch if the Universe only contains irrotational dust.

Through the renormalization group technique, we have then shown that super-Hubble modes can be resummed at any order in perturbation theory yielding a local curvature term  $\sim a_D^{-2}$  at large times. We then turned our attention to the backreaction originating from modes within our Hubble radius, studying perturbatively their time-behavior. In this case our findings indicate that an instability occurs in the perturbative expansion, which may be not taken care of by the renormalization group procedure since terms of the form  $H^2 \langle \delta_{\text{FRW}}^2 (v/c)^2 \rangle \sim H^2$  (where  $v$  is the peculiar velocity) start appearing both in  $Q_D$  and in the mean spatial curvature. Such terms are not as small as order  $10^{-5} H^2$ ; on the contrary the averaging procedure allows the combination of these post-Newtonian and Newtonian terms to acquire values of order  $H^2$ . Since the perturbation approach breaks down, we may not predict on firm grounds that backreaction is responsible for the present-day acceleration

of the Universe. However, it is intriguing that such an instability shows up only recently in the evolution of the Universe and that this picture is further supported by a very general result: even a tiny back-reaction term can drive the cosmological parameters on the averaging domain far away from their global values of the standard FRW model [24]. Other aspects of the scenario discussed in this paper, such as the dynamics of perturbations on observable scales, will be the subject of a forthcoming publication.

## APPENDIX: FOURTH-ORDER GRADIENT-EXPANSION APPROXIMATION TO THE SOLUTION OF EINSTEIN'S FIELD EQUATIONS

For completeness, we give here the explicit expression for  $\Psi$  and  $\chi^i_j$  up to four gradients (we refer the reader to Ref. [35] for the detailed derivation of these results). We have

$$\begin{aligned}
\Psi &= \frac{5}{3}\varphi + \frac{1}{18} \left(\frac{a}{a_0}\right) \left(\frac{2}{H_0}\right)^2 e^{10\varphi/3} \left[ \nabla^2\varphi - \frac{5}{6} (\nabla\varphi)^2 \right] + \frac{1}{504} \left(\frac{a}{a_0}\right)^2 \left(\frac{2}{H_0}\right)^4 e^{20\varphi/3} \\
&\quad \times \left[ \frac{23}{9} (\nabla^2\varphi)^2 - \frac{10}{3} \varphi^{,ij} \varphi_{,ij} - \frac{100}{9} \varphi_{,i} \varphi_{,j} \varphi^{,ij} + \frac{35}{27} \nabla^2\varphi (\nabla\varphi)^2 - \frac{1675}{324} (\nabla\varphi)^2 (\nabla\varphi)^2 \right], \\
\chi^i_j &= -\frac{1}{3} \left(\frac{a}{a_0}\right) \left(\frac{2}{H_0}\right)^2 e^{10\varphi/3} \left[ D^i_j \varphi + \frac{5}{3} \left( \varphi^i \varphi_{,j} - \frac{1}{3} (\nabla\varphi)^2 \delta^i_j \right) \right] \\
&\quad + \frac{1}{504} \left(\frac{a}{a_0}\right)^2 \left(\frac{2}{H_0}\right)^4 e^{20\varphi/3} \left\{ 38 \left( \varphi^{,ki} \varphi_{,kj} - \frac{1}{3} \varphi_{,kl} \varphi^{,kl} \delta^i_j \right) \right. \\
&\quad - \frac{128}{3} \left[ (\nabla^2\varphi) \varphi^i_{,j} - \frac{1}{3} (\nabla^2\varphi)^2 \delta^i_j \right] + \frac{890}{27} (\nabla^2\varphi) (\nabla\varphi)^2 \delta^i_j - \frac{250}{9} (\nabla\varphi)^2 \varphi^i_{,j} \\
&\quad - \frac{640}{9} (\nabla^2\varphi) \varphi^i \varphi_{,j} - \frac{380}{9} \varphi_{,k} \varphi_{,l} \varphi^{,kl} \delta^i_j + \frac{190}{3} \left( \varphi^{,ki} \varphi_{,k} \varphi_{,j} + \varphi^i \varphi_{,kj} \varphi^{,k} \right) \\
&\quad \left. + \frac{1600}{27} (\nabla\varphi)^2 \left( \varphi^i \varphi_{,j} - \frac{1}{3} (\nabla\varphi)^2 \delta^i_j \right) \right\}. \tag{A.2}
\end{aligned}$$

One can verify that these solutions satisfy the energy constraint and the momentum constraint up to the relevant number of gradients. It is also important to stress that these expressions reproduce the perturbative second-order metric (see, *e.g.*, Ref. [6]) when only terms up to second order in  $\varphi$  are kept.

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