

Bounding Gauged Skyrmion Masses

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Normally, standard (ungauged) skyrmion masses are proportional to the coupling of the Skyrme term needed for stability, and so can grow to infinite magnitude with increasing coupling. In striking contrast, when skyrmions are gauged, their masses are bounded above for any Skyrme coupling, and, instead, are of the order of monopole masses, $O(v/g)$, so that the coupling of the Skyrme term is not very important. This boundedness phenomenon and its implications are investigated.

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I. INTRODUCTION

A remarkable feature of theories based upon $SU(N)_L \times SU(N)_R$ global chiral symmetries is the existence of topologically stable field configurations known as skyrmions. The skyrmion carries a topological charge representing the nontrivial homotopy group, $\Pi_3(SU(N))$, the mapping of the gauge group $SU(N)$ onto a time slice into the three dimensions of space. This charge emulates baryon number, and thus skyrmions provide an effective model of the baryons of QCD, and their matrix elements [1, 2, 3].

It is of general interest to consider the skyrmion in the presence of gauge interactions. There are perturbative gauge interactions in nature which the skyrmion-as-baryon must experience, i.e., QED, and the electroweak interactions. The skyrmion further experiences the ρ -meson [4], which has an effective description as a gauge field of isospin. Indeed, gauging chiral Lagrangians promotes the Wess-Zumino term, which generates the global topological current structure, to the Wess-Zumino-Witten term, which not only generates currents but their anomaly structure as well, and is seen to be faithful to an underlying theory of quarks and gluons. Moreover, if one considers pure Yang-Mills theories in higher dimensions that undergo compactification to $D = 4$, one generally finds that the $D = 4$ effective description of KK-modes is a gauged chiral Lagrangian. The skyrmion then matches higher dimensional topological objects [5, 6, 7]. There have also been interesting applications of gauged skyrmions in the context of technibaryon decay [8].

In the present paper, we examine the impact of the diagonal gauging (i.e., the promotion of the diagonal subgroup $SU(2)$ (isospin) of the chiral group $SU(2)_L \times SU(2)_R$ to a gauge group). In particular, we observe novel behavior for the masses of gauge skyrmions which significantly departs from the global case.

The conventional (ungauged) skyrmion is a solution to the chiral model equations of motion [3, 9] supplemented with a “Skyrme term.” The Skyrme term is required to stabilize the core structure, but one finds a sensitivity to the strength of this term in the mass: the mass of the skyrmion is essentially proportional to the square-root of the coupling coefficient of the Skyrme term. This term has to be input, by hand, or to be somehow justified as emerging from a long-distance effective Lagrangian description of a more complicated system, e.g., from some shorter distance scale physics in QCD.

The skyrmion solutions of particular gauged chiral models, however, exhibit unexpected behavior, even though tantalizing hints of it could be gleaned from pioneering numerical studies of gauged skyrmions [10, 11, 12, 13]. Specifically, the mass of the skyrmion increases monotonically from zero with the Skyrme coupling, but does not go to infinity as it would for ungauged skyrmions. Instead, the mass stabilizes to an upper bound, whose scale, $O(v/g)$, is “monopolic”, i.e., it is set by the gauge coupling and the characteristic spontaneous symmetry breaking scale. This

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limit conforms to the masses of magnetic monopoles, which likewise do not vary much above the minimal BPS values [14].

For simplicity, we focus on plain Skyrme-Wu-Yang spherically symmetric hedgehogs. We discuss the simplest system [7, 8], $SU(2)_L \times SU(2)_R$ with gauged diagonal $SU(2)_V$,

$$4\pi E(g, v, \kappa) \equiv \frac{1}{2} \int d^3x \text{Tr} F_{ij} F_{ij} + \frac{v^2}{2} \int d^3x \left(\text{Tr} [D_j, U^\dagger] [D_j, U] + \kappa^2 \text{Tr} ([D_j, U^\dagger], [D_i, U])^2 \right), \quad (1)$$

The spherical hedgehog Skyrme-Wu-Yang Ansatz[3, 15] (i.e. of unit winding/“baryon” number) is

$$A_i = \frac{a(r) - 1}{gr} \epsilon_{ijk} \frac{\tau^j}{2} \hat{x}^k, \quad U = \exp\left(i f(r) \hat{\mathbf{x}} \cdot \boldsymbol{\tau} \right) = \cos f(r) + i \hat{\mathbf{x}} \cdot \boldsymbol{\tau} \sin f(r). \quad (2)$$

(The exponent $f(r) \hat{\mathbf{x}} \cdot \boldsymbol{\tau} \sim \int d^4x A_4$, amounts to the deconstruction Wilson line/link [7].)

This two-scale problem yields an energy $E(g, v, \kappa)$ which has a lower, topological bound [12]. Moreover, it is manifestly *monotonic* in the Skyrme coupling strength κ , because the $\partial/\partial\kappa$ derivative is positive semidefinite, while all implicit dependence of the fields on κ vanishes on-shell (by use of the eqns of motion), and is thus irrelevant, as in the case of the monopole mass varying as a function of the Higgs mass [14].

To familiarize the reader with the bounding arguments, we first summarize the standard results on the simplest, $B = 1$, ungauged skyrmion [3] in Section II. We then define and examine the simplest, $B = 1$, gauged skyrmion; we review lower bounds, $E_{\text{topological}}$ [10, 11, 12, 13]; and, finally, we derive upper bounds for its mass, on the basis of numerical investigation for asymptotically large couplings, in Section III. Asymptotically, we find this actual (bounding) mass of the $B = 1$ simple skyrmion to be merely 2.06 $E_{\text{topological}}$. Our results are qualitatively unchanged upon further introduction of a pion mass [13]—even though, as expected, the corresponding upper bound increases with the mass of the pion. In the last Section, IV, we conclude with discussion and interpretation of the phenomenon.

II. REVIEW OF LOWER BOUNDS OF UNGAUGED SKYRMIONS

In the limit of decoupling of the gauge fields, $g = 0$, or equivalently, $a = 1$, (1) yields the standard single skyrmion reviewed here. It is evident from scaling, below, that all activity occurs at scales of $r = O(\kappa)$, so that $E = O(\kappa v^2)$:

$$\begin{aligned} E(0, v, \kappa) &= \frac{v^2}{2} \int_0^\infty dr \left(r^2 f'^2 + 2 \sin^2 f + \kappa^2 \frac{\sin^4 f}{r^2} + 2\kappa^2 f'^2 \sin^2 f \right) \\ &= \frac{v^2}{2} \int_0^\infty dr \left((r f' - \kappa \frac{\sin^2 f}{r})^2 + 2 \sin^2 f (1 - \kappa f')^2 + 6\kappa f' \sin^2 f \right). \end{aligned} \quad (3)$$

The first two terms in the integrand are positive semi-definite. The last one is a total divergence, $12\kappa\pi^2 r^2 \times$ the topological (Chern-Simons) baryon density of the conventional skyrmion [1],

$$\frac{3\kappa}{2} \partial_r (2f - \sin 2f) = \frac{\kappa r^2}{2} \epsilon^{ijk} \text{Tr} (U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U). \quad (4)$$

Thus, its contribution to the energy is

$$E_{\text{topological}} = \frac{3v^2\kappa}{4} \left(2f(\infty) - 2f(0) - \sin 2f(\infty) + \sin 2f(0) \right) = \frac{3\pi}{2} \kappa v^2. \quad (5)$$

Note that one could choose either sign in completing the above squares.

This is a Bogomol’ny *topological lower bound*. However, it cannot be saturated, as it does in the BPS monopole case (There are no self-dual chiral fields [9]). Saturation would require both squares to vanish, which is impossible: both $f' = 1/\kappa$, and $\kappa \sin f = r$.

This lower bound melts away for vanishing Skyrme term $\kappa = 0$, and blows up for $\kappa \rightarrow \infty$. Thus, there can be no $\kappa \rightarrow \infty$ upper bound for the ungauged skyrmion.

III. GAUGED SKYRMIONS AND THEIR UPPER AND LOWER BOUNDS

By contrast, when gauge fields are introduced, $g \neq 0$, κ -dependence drops out of the lower bound as $\kappa \rightarrow \infty$; moreover, as we show below, *there is an upper bound for the gauged skyrmion* above, remarkably close to the highest lower bound.

The full equation (1) then for the $B = 1$ Ansatz, with r and κ rescaled in units of gv , amounts to

$$E(g, v, \kappa) = \frac{v}{g} \int_0^\infty dr \left(4a'^2 + \frac{2(a^2 - 1)^2}{r^2} + \frac{r^2 f'^2}{2} + a^2 \sin^2 f + \kappa^2 a^4 \frac{\sin^4 f}{2r^2} + \kappa^2 a^2 f'^2 \sin^2 f \right), \quad (6)$$

with boundary conditions $a(0) = 1$, $a(\infty) = 0$; $f(0) = 0$, $f(\infty) = \pi$.

The topological lower bound [12] is

$$E_{\text{topological}} > \frac{2\pi v}{\sqrt{g^2 + \left(\frac{4}{3\kappa v}\right)^2}}, \quad (7)$$

which yields, in the limit $\kappa \rightarrow \infty$, a lower bound of $O(v/g)$, i.e., of the order of the BPS monopole mass. Once gauging is switched on, $1/g^2$ and κ^2 behave analogously to resistors in parallel: as κ blows up, it becomes irrelevant, leaving the scale to be set by g .

The gauged skyrmion has been well studied numerically in refs [10, 11, 12, 13], which detail a remarkable branch structure of solutions. For increasing Skyrme coupling κ , the energy of the simple ($B = 1$) skyrmion is seen to increase, starting from 0. (Specifically, Fig 1 of ref [12], for increasing κgv , the gauged skyrmion energy E curves over, in contrast to that of the ungauged skyrmion on the same graph.) In fact, we show that it flattens out asymptotically, yielding an upper bound, as is the case for the 'tHooft-Polyakov monopole as a function of the Higgs mass [14]. This bound is very close to the highest lower bound.

The full lower bound for the gauged case results from rewriting the above energy as

$$\begin{aligned} E(g, v, \kappa) &= \frac{v}{g} \int_0^\infty dr \left(\left(4a'^2 + \frac{(3\kappa/4)^2}{1 + (3\kappa/4)^2} \frac{a^2 \sin^2(2f)}{4} \right) + \frac{(3\kappa/4)^2}{1 + (3\kappa/4)^2} a^2 \sin^4 f + a^2 \sin^2 f \left(\frac{1}{1 + (3\kappa/4)^2} + \kappa^2 f'^2 \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{r^2 f'^2}{1 + (3\kappa/4)^2} + \kappa^2 a^4 \frac{\sin^4 f}{r^2} \right) + \frac{1}{2} \left(\frac{(3\kappa/4)^2}{1 + (3\kappa/4)^2} r^2 f'^2 + \frac{4(a^2 - 1)^2}{r^2} \right) \right) \\ &= \frac{v}{g} \int_0^\infty dr \left(\left(2a' + \frac{(3\kappa/4)}{\sqrt{1 + (3\kappa/4)^2}} \frac{a \sin(2f)}{2} \right)^2 + \frac{(3\kappa/4)^2}{1 + (3\kappa/4)^2} a^2 \sin^4 f \right. \\ &\quad \left. + a^2 \sin^2 f \left(\frac{1}{\sqrt{1 + (3\kappa/4)^2}} - \kappa f' \right)^2 + \frac{1}{2} \left(\frac{r f'}{\sqrt{1 + (3\kappa/4)^2}} - \kappa a^2 \frac{\sin^2 f}{r} \right)^2 \right. \\ &\quad \left. + \frac{1}{2} \left(r f' \frac{(3\kappa/4)}{\sqrt{1 + (3\kappa/4)^2}} + \frac{2(a^2 - 1)}{r} \right)^2 \right) + \frac{3\kappa v}{4g \sqrt{1 + (3\kappa/4)^2}} \int_0^\infty dr \partial_r (2f - a^2 \sin 2f) \\ &> \frac{2\pi v}{g \sqrt{1 + (4/3\kappa)^2}}, \end{aligned} \quad (8)$$

since each term but the last (surface term) is positive semi-definite. As before, all these terms cannot be nullified simultaneously, and thus the topological bound is not saturated, except in the degenerate case $\kappa = 0$, cf. [12]. The highest value for this lower bound, $2\pi v/g$, holds for $\kappa \rightarrow \infty$; it will be seen that the actual energy is roughly twice this, in that limit.

The Euler-Lagrange equations are

$$a'' + \frac{a(1 - a^2)}{r^2} - \frac{a}{4} \sin^2 f - \frac{\kappa^2 a}{4} f'^2 \sin^2 f - \frac{\kappa^2 a^3}{4r^2} \sin^4 f = 0, \quad (9)$$

$$(r^2 + 2\kappa^2 a^2 \sin^2 f) f'' + 2r f' + 4\kappa^2 a a' f' \sin^2 f + \kappa^2 a^2 f'^2 \sin(2f) - a^2 \sin(2f) - \frac{2\kappa^2 a^4}{r^2} \sin^3 f \cos f = 0, \quad (10)$$

with BCs:

$$a(0) = 1, \quad a(\infty) = 0; \quad f(0) = 0, \quad f(\infty) = \pi. \quad (11)$$

Study of actual numerical solutions for increasing κ^2 , (cf., e.g., branch B of ref [12]), reveals that the scales of the two variables a and f tend to resolve as in Born-Oppenheimer problems, and the monopole in the large Higgs mass limit [14]. Specifically, a asymptotes faster than f , which continues to evolve slowly after a has decayed to 0 by some value $r = R$. Numerically, $R \sim 1.34$, very weakly dependent on κ , for large $\kappa^2 \sim 1000$ (cf. Fig. 1).

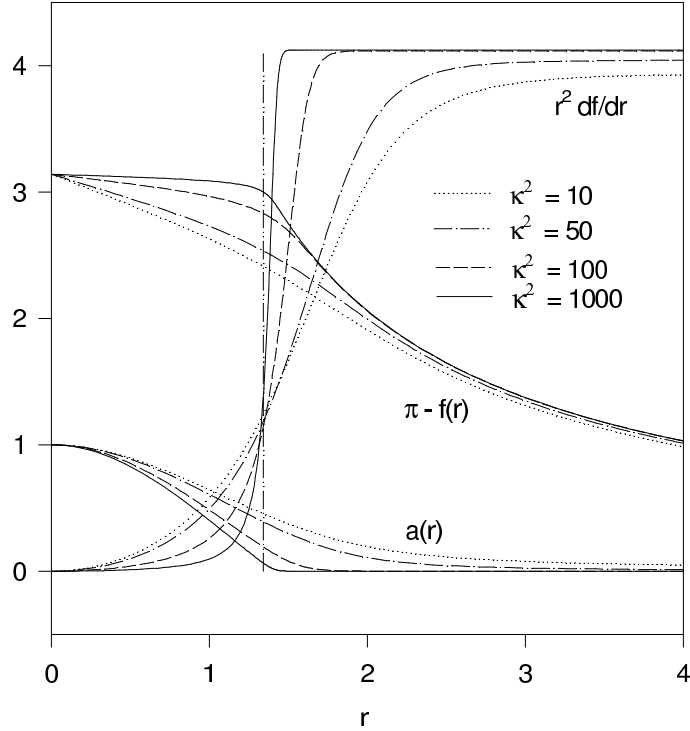


FIG. 1: Numerical study of a ; $\pi - f(r)$; and $r^2 f'$, the corresponding argument of eqn (14), for increasing values of κ^2 . It is evident that the two variables tend to decouple, for sufficiently large κ^2 : the scales of a and f separate at $R \sim 1.34$, demarcated by the vertical dash-double-dot line. Note the abrupt transition of f' from null to $\pi R/r^2$ behavior.

Specifically, for the proximate interval, $[0, R]$,

$$f(r) \sim 0, \quad (12)$$

so that eqn (10) collapses, while (9) effectively reduces to

$$a'' + \frac{a(1-a^2)}{r^2} \sim 0, \quad (13)$$

the celebrated Wu-Yang equation[15] for pure Yang-Mills theory. (Note, however, that this equation, by itself, is scale

invariant: the actual scale R is set through interaction with f . The range of a would spread out, left to itself, but the Skyrme term disfavors overlap of a with f . In the limit, it forces a to attenuate inside the core, before f builds up at R .) Hence $a(r)$ is an attenuating function which reaches $a(R) \sim 0$.

For the distant interval $[R, \infty)$, $a \sim 0$, so that (9) collapses, while (10) reduces to

$$\partial_r(r^2 f') \sim 0, \quad (14)$$

and hence $f \sim \pi(1 - \frac{R}{r})$.

Thus, for $\kappa \rightarrow \infty$, dependence on κ dies out. For solutions (on-shell), the coefficient of κ^2 in the energy collapses,

$$\frac{dE}{d(\kappa^2)} = \frac{\partial E}{\partial(\kappa^2)} = \int_0^\infty dr \left(a^2 f'^2 \sin^2 f + \frac{a^4 \sin^4 f}{2r^2} \right) \sim 1.62 \kappa^{-5/2} \rightarrow 0. \quad (15)$$

In the most important region, the neighborhood of R , activity is apparently dominated by the scale $r \sim \sqrt{\kappa}$; this is quite unlike the characteristic scale of the ungauged skyrmion activity, $r \sim \kappa$.

Since E is monotonic in κ , an upper bound results for (6) in this limit,

$$E(g, v, \kappa) \leq E(g, v, \infty) \sim 12.95 \frac{v}{g}. \quad (16)$$

This upper bound is merely $\frac{g}{v} E(g, v, \infty) \sim 2.06 \times 2\pi$, where 2π is the above-mentioned highest lower bound. In effect, the mass of the gauged skyrmion varies from 0 to $12.95 v/g$, as the Skyrme term ranges from zero to infinite strength. Near zero, the Skyrme coupling κ sets the mass scale, but for large couplings the scale is set by the ‘‘monopole mass’’ scale v/g .

In more numerical detail, for $\frac{g}{v} E(g, v, \infty) \sim 12.95$, the subleading behavior is

$$E(g, v, \kappa) = E(g, v, \infty) - 6.68 \frac{v}{g} \kappa^{-1/2} + O(\kappa^{-1}). \quad (17)$$

In Fig. 2, beyond E , E_{sk} is also plotted. It represents the ‘‘Skyrme term’’, i.e., the last two terms in eqn (6). The decay of E_{sk} goes like $E_{sk} \sim 1.62\kappa^{-1/2}$, as indicated, so this component is subdominant to the contributions of the Wu-Yang and the conventional chiral terms, the leading four terms in eqn (6). Since

$$E_{sk} = \kappa^2 \frac{dE}{d(\kappa^2)} = -\frac{1}{4} \kappa^{-1/2} \frac{dE}{d(\kappa^{-1/2})}, \quad (18)$$

this is seen to be numerically consistent with the above expansion in $\kappa^{-1/2}$ around $\kappa^{-1/2} = 0$.

The boundary condition for $f(r)$ above may be effectively regarded as a ‘‘unit baryon charge constraint’’ [13]; to enforce it more naturally, it is customary to add a pion mass term, $\lambda r^2(1 - \cos f)$ to the integrand of (6), arising out of a term $\text{Tr}(U + U^\dagger) - 4$ in the chiral Lagrangian, where $\lambda = 2(m_\pi/gv)^2$. However, addition of such a term does not alter the qualitative conclusions above.

For example, for $\lambda = 1$, the upper bound is only somewhat higher,

$$\frac{g}{v} E(g, v, \kappa) = 17.5 - 6.0 \kappa^{-1/2} + O(\kappa^{-1}), \quad (19)$$

as expected: since this mass term is positive-semidefinite, the skyrmion mass is a monotonically increasing function of λ .

IV. DISCUSSION

Our present analysis probed the effects on skyrmionic masses of the presence of the diagonal gauge group. The situation is reminiscent of the 't Hooft-Polyakov monopole, for which a similar bound exists owing to gauge fields [14]. While the monopole is a tangle of gauge, Goldstone, and Higgs fields; as the mass of the Higgs field is taken to infinity with fixed VEV, the Higgs serves only to enforce boundary conditions. The monopole in this limit ends up made purely of gauge fields (a Higgsless monopole, as in ref [16]), with a mass of $O(M_W/\alpha)$. Analogously, the gauged skyrmion in (6) consists of gauge fields and Higgs field skyrmions. But, in the interaction with very heavy would-be skyrmions, the last two terms in that system, the ‘‘Skyrme term’’ (viz., E_{sk} of Fig 2), become decreasingly relevant in the energy. Thus, what would have been the infinitely massive skyrmion largely enforces boundary conditions at R . The leading two terms in the energy (the gauge, or Wu-Yang, part) scale as $1/s$ with $r \rightarrow sr$, and, left to

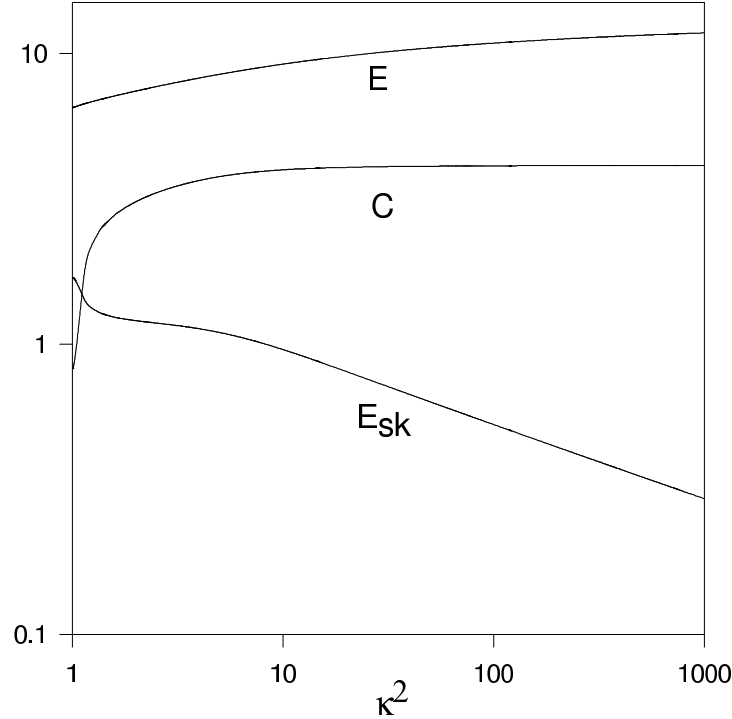


FIG. 2: Numerical study of the energy E , in units of v/g , for increasing κ^2 . The upper bound is at $E(g, v, \infty) \sim 12.95$. Also plotted is E_{sk} , representing the last two terms in eqn (6), the Skyrme term. Further plotted is the coefficient C of the leading tail of $f = \pi(1 - C/r + O(1/r^2))$ for large r , tending to πR in the limit $\kappa \rightarrow \infty$. It may be instructive to note the contrast to the $\pi - 2.16/r^2$ asymptoting of the conventional ungauged skyrmion.

themselves, favor a spread-out integrand to maximize s . The next two terms (the chiral action terms) scale as s , and favor core-shrinking, but the last two terms (the Skyrme terms) oppose this, and stabilize the core to $\sim R$, thereby constraining the gauge field within this range. The mass of the skyrmion ends up of the order characteristic of monopole configurations, $O(v/g)$, superficially oblivious of the Skyrme coupling.

We have thus found the effects of gauging to be significant in the limit of large Skyrme term coefficient. We note, however, that one could [7] (and should [17]) include the effects of additional operators that explicitly involve the Yang-Mills field strength, and are of the same dimension as the gauged Skyrme term utilized here, such as

$\text{Tr}F_{ij}[D^j, U^\dagger][D^j, U] + \text{h.c.}$ There are potentially interesting effects of these, to be considered elsewhere [17].

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