The $B^+ - B^0_d$ Lifetime Difference
Beyond Leading Logarithms

Martin Beneke$^1$, Gerhard Buchalla$^2$, Christoph Greub$^3$, Alexander Lenz$^4$ and Ulrich Nierste$^{2,5}$

1 Institut für Theoretische Physik E, RWTH Aachen, Sommerfeldstraße 28, D-52074 Aachen, Germany.
2 Theory Division, CERN, CH-1211 Geneva 23, Switzerland.
3 Institut für Theoretische Physik, Universität Bern, Sidlerstrasse 5, CH-3012 Berne, Switzerland.
4 Fakultät für Physik, Universität Regensburg, D-93040 Regensburg, Germany.
5 Fermi National Accelerator Laboratory, Batavia, IL 60510-500, USA.

Abstract

We compute perturbative QCD corrections to the lifetime splitting between the charged and neutral $B$ meson in the framework of the heavy quark expansion. These next-to-leading logarithmic corrections are necessary for a meaningful use of hadronic matrix elements of local operators from lattice gauge theory. We find the uncertainties associated with the choices of renormalization scale and scheme significantly reduced compared to the leading-order result. We include the full dependence on the charm-quark mass $m_c$ without any approximations. Using hadronic matrix elements estimated in the literature with lattice QCD we obtain $\tau(B^+)/\tau(B^0_d) = 1.053 \pm 0.016 \pm 0.017$, where the effects of unquenching and $1/m_b$ corrections are not yet included. The lifetime difference of heavy baryons $\Xi_b^0$ and $\Xi_b^-$ is also briefly discussed.

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1 Preliminaries

The Heavy Quark Expansion (HQE) technique provides a well-defined QCD-based framework for the calculation of total decay rates of $b$-flavoured hadrons [1]. The HQE yields an expansion of the decay rate $\Gamma(H_b)$ in terms of $\Lambda_{QCD}/m_b$, where $H_b$ represents any hadron containing a single $b$-quark and any of the light $u,d,s$ (anti-)quarks as valence quarks. $m_b$ is the $b$-quark mass and $\Lambda_{QCD}$ is the fundamental scale of QCD, which determines the size of hadronic effects. In the leading order of $\Lambda_{QCD}/m_b$ the decay rate of $H_b$ equals the decay rate of a free $b$-quark, which is unaffected by the light degrees of freedom of $H_b$. Consequently, the lifetimes of all $b$-flavoured hadrons are the same at this order. The first corrections to the free quark decay appear at order $(\Lambda_{QCD}/m_b)^2$ and are caused by the Fermi motion of the $b$-quark in $H_b$ and the chromomagnetic interaction of the final state quarks with the hadronic cloud surrounding the heavy $b$-quark. These mechanisms have a negligible effect on the lifetime difference between the $B^+$ and $B^0$ mesons, because the strong interaction excellently respects isospin symmetry. At order $(\Lambda_{QCD}/m_b)^3$, however, one encounters weak interaction effects between the $b$-quark and the light valence quark. These effects, known as weak annihilation (WA) and Pauli interference (PI) [1], are depicted in Fig. 1. They are phase-space enhanced with respect to the leading free-quark decay and induce corrections to $\Gamma(H_b)$ of order $16\pi^2(\Lambda_{QCD}/m_b)^3 = \mathcal{O}(5-10\%)$. The measurement of lifetime differences among different $b$-flavoured hadrons therefore tests the HQE formalism at the third order in the expansion parameter.

The calculation of $\Gamma(H_b)$ consists of three steps: the first step is an operator product expansion (OPE) integrating out the heavy $W$ boson, which mediates the weak $b$ decay. This results in an effective $|\Delta B| = 1$ Hamiltonian describing the flavour-changing weak interaction of the Standard Model up to corrections of order $m_b^2/M_W^2$, where $\Delta B$ denotes the change in bottom-quark number:

$$H = \frac{G_F}{\sqrt{2}} V_{cb}^* \sum_{d'=d,s} V_{u'd'} \left[ C_1(\mu_1) Q_1^{u'd'}(\mu_1) + C_2(\mu_1) Q_2^{u'd'}(\mu_1) \right] + \text{h.c.} \quad (1)$$

Here $G_F$ is the Fermi constant and the $V_{ij}$’s are elements of the Cabibbo-Kobayashi-Maskawa
(CKM) matrix. The Wilson coefficients $C_i(\mu_1)$ contain the short-distance physics associated with scales above the renormalization scale $\mu_1$. The weak interaction is encoded in the four-quark operators

$$Q_1^{d'd'} = \bar{b}_j \gamma_\mu (1 - \gamma_5) c_j \bar{\pi}_j \gamma^\mu(1 - \gamma_5) d'_i, \quad Q_2^{d'd'} = \bar{b}_j \gamma_\mu (1 - \gamma_5) c_i \bar{\pi}_j \gamma^\mu(1 - \gamma_5) d'_j,$$

with summation over the colour indices $i$ and $j$. We have omitted penguin operators and doubly Cabibbo-suppressed terms in (1), which have a negligible effect on the $B^+-B_d^0$ lifetime difference. Next the total decay rate $\Gamma(H_b)$ is related to $\bar{H}$ by the optical theorem:

$$\Gamma(H_b) = \frac{1}{2M_{H_b}} \langle H_b | T | H_b \rangle. \quad (3)$$

Here we have adopted the conventional relativistic normalization $\langle H_b | H_b \rangle = 2EV$ and introduced the transition operator:

$$T = \text{Im} i \int d^4 x \, T[H(x)H(0)]. \quad (4)$$

The second step is the HQE, which exploits the hierarchy $m_b \gg \Lambda_{QCD}$ to expand the RHS of (3) in terms of $\Lambda_{QCD}/m_b$. To this end an OPE is applied to $T$ which effectively integrates out the hard loop momenta (corresponding to the momenta of the final state quarks). We decompose the result as

$$T = [T_0 + T_2 + T_3] \left[ 1 + O(1/m_b^4) \right]$$

$$T_3 = T^u + T^d + T_{\text{sing}} \quad (5)$$

Here $T_n$ denotes the portion of $T$ which is suppressed by a factor of $1/m_b^n$ with respect to $T_0$ describing the free quark decay. The contributions to $T_3$ from weak spectator interactions read

$$T^u = \frac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \left[ |V_{ud}|^2 \left( F^u Q^d + F^u S^d Q_S^d + G^u T^d + G_S^u T_S^d \right) \right.$$

$$+ |V_{cd}|^2 \left( F^c Q^d + F^c S^d Q_S^d + G^c T^d + G_S^c T_S^d \right) \left. \right] + (d \rightarrow s)$$

$$T^d = \frac{G_F^2 m_b^2 |V_{cb}|^2}{6\pi} \left[ F^d Q^u + F^d S^u Q_S^u + G^d T^u + G_S^d T_S^u \right]. \quad (6)$$

The superscript $q$ of the coefficients $F^q, F_S^q, G^q, G_S^q$ refers to the $cq$ intermediate state (see Fig. 1). We include singly Cabibbo-suppressed contributions. In writing $T^d$ we have used $|V_{ud}|^2 + |V_{us}|^2 \approx 1$ and $m_s \approx m_u \approx 0$, so that $F^d = F^s$, etc.. Here we encounter the local dimension-6, $\Delta B = 0$ operators

$$Q^q = \bar{b} \gamma_\mu (1 - \gamma_5) q \bar{q} \gamma^\mu(1 - \gamma_5) b, \quad Q_S^q = \bar{b} (1 - \gamma_5) q \bar{q} (1 + \gamma_5) b,$$

$$T^q = \bar{b} \gamma_\mu (1 - \gamma_5) T^q q \bar{q} \gamma^\mu(1 - \gamma_5) T^q b, \quad T_S^q = \bar{b} (1 - \gamma_5) T^q q \bar{q} (1 + \gamma_5) T^q b, \quad (7)$$
where $T^a$ is the generator of colour SU(3). We define the $\Delta B = 0$ operators at the renormalization scale $\mu_0$, which is of order $m_b$. The Wilson coefficients $F^u \ldots G^d_S$ are computed in perturbation theory. When applied to mesons, $T^u$ and $T^d$ correspond to the WA and PI mechanisms of Fig. 1, respectively. In the case of baryons their role is interchanged: $T^u$ encodes the PI effect and $T^d$ describes the weak scattering of the $b$-quark with the valence quark (see Fig. 5). The coefficients in (6) depend on $\mu_0$. Since the hard loops involve the charm quark, they also depend on the ratio $z = m_c^2/m_b^2$. The truncation of the perturbation series makes $F^u \ldots G^d_S$ also dependent on $\mu_1 = O(m_b)$. This dependence diminishes in increasing orders of $\alpha_s$. To the considered order, the dependence on $\mu_0$ cancels between the coefficients and the matrix elements of operators in (6), so that observables are independent of $\mu_0$. The remainder $T_{sing}$ in (5) involves additional dimension-6 operators, which describe power-suppressed contributions to the free quark decay from strong interactions with the spectator quark. The operators in $T_{sing}$ are isospin singlets and do not contribute to the $B^+ - B^0_d$ lifetime difference. The formalism of (5)–(7) applies to weakly decaying hadrons containing a single bottom quark and no charm quarks. Decays of hadrons like the $B_s$ meson with more than one heavy quark have a different power counting than in (5) [2]. In the third step one computes the hadronic matrix elements of the operators in (7). They enter our calculation in isospin-breaking combinations and are conventionally parametrized as [3, 8]

$$
\langle B^+ | (Q^a - Q^d) (\mu_0) | B^+ \rangle = f_B^2 M_B^2 B_1 (\mu_0), \quad \langle B^+ | (Q^a_s - Q^d_s) (\mu_0) | B^+ \rangle = f_B^2 M_B^2 B_2 (\mu_0),
$$

$$
\langle B^+ | (T^u - T^d) (\mu_0) | B^+ \rangle = f_B^2 M_B^2 \epsilon_1 (\mu_0), \quad \langle B^+ | (T^u_s - T^d_s) (\mu_0) | B^+ \rangle = f_B^2 M_B^2 \epsilon_2 (\mu_0).
$$

(8)

Here $f_B$ is the $B$ meson decay constant. In the vacuum saturation approximation (VSA) one has $B_1 (\mu_0) = 1$, $B_2 (\mu_0) = 1 + O(\alpha_s (m_b), \Lambda_{QCD}/m_b)$ and $\epsilon_{1,2} (\mu_0) = 0$. Corrections to the VSA results are of order $1/N_c$, where $N_c = 3$ is the number of colours.

Using the isospin relation $\langle B^0_d | Q^d | B^0_d \rangle = \langle B^+ | Q^d | B^+ \rangle$ we now find from (3) and (6):

$$
\Gamma (B^0_d) - \Gamma (B^+) = \frac{G^2_F m_b^2 V_{ub}^2}{12 \pi} f_B^2 M_B \left( |V_{ud}|^2 \bar{F}^u + |V_{cd}|^2 \bar{F}^c - \bar{F}^d \right) \bar{B}.
$$

(9)

Here we have introduced the shorthand notation

$$
\bar{F}^q (z, \mu_0) = \begin{pmatrix} F^q (z, \mu_0) \\ F^q_s (z, \mu_0) \\ G^q (z, \mu_0) \\ G^q_s (z, \mu_0) \end{pmatrix}, \quad \bar{B} (\mu_0) = \begin{pmatrix} B_1 (\mu_0) \\ B_2 (\mu_0) \\ \epsilon_1 (\mu_0) \\ \epsilon_2 (\mu_0) \end{pmatrix}
$$

(10)

The strong interaction affects all three steps of the calculation. The minimal way to include QCD effects is the leading logarithmic approximation, which includes corrections of order $\alpha_s^n \ln^n (\mu_1 / M_W)$, $n = 0, 1, \ldots$ in the coefficients $C_{1,2} (\mu_1)$ in (1). The corresponding leading order (LO) calculation of the width difference in (9) involves the diagrams in Fig. 1 [1, 3]. Yet LO results are too crude for a precise calculation of lifetime differences. The heavy-quark masses in (9) cannot be defined in a proper way and one faces a large dependence on the renormalization scale $\mu_1$. Furthermore, results for $B_{1,2}$ and $\epsilon_{1,2}$ from lattice gauge theory cannot be matched to the continuum theory in a meaningful way at LO. Finally, as pointed out in [3], at LO the coefficients $F$, $F_s$ in (9) are anomalously small. They multiply the large matrix elements parametrized by
We decompose the Wilson coefficients in (6) as

\[ F^u(z, \mu_0) = C_2^u(\mu_1) F_{11}^{u}(z, x_{\mu_1}, x_{\mu_0}) + C_1(\mu_1) C_2(\mu_1) F_{12}^{u}(z, x_{\mu_1}, x_{\mu_0}) + C_2^2(\mu_1) F_{22}^{u}(z, x_{\mu_1}, x_{\mu_0}) \]
with $x_\mu = \mu/m_b$ and an analogous notation for the remaining Wilson coefficients in (6). The LO coefficients are obtained from the diagrams in Fig. 1. The non-vanishing coefficients read [3]

$$
\begin{align*}
\frac{1}{3} F_{11}^{u,0}(z) &= \frac{1}{2} F_{12}^{u,0}(z) = 3 F_{22}^{u,0}(z) = \frac{1}{2} G_{22}^{u,0}(z) = -(1 - z)^2 \left( 1 + \frac{z}{2} \right), \\
\frac{1}{3} F_{S,11}^{u,0}(z) &= \frac{1}{2} F_{S,12}^{u,0}(z) = 3 F_{S,22}^{u,0}(z) = \frac{1}{2} G_{S,22}^{u,0}(z) = (1 - z)^2 (1 + 2z), \\
\frac{1}{3} F_{E,11}^{c,0}(z) &= \frac{1}{2} F_{E,12}^{c,0}(z) = 3 F_{E,22}^{c,0}(z) = \frac{1}{2} G_{E,22}^{c,0}(z) = - \sqrt{1 - 4z} (1 - z), \\
\frac{1}{3} F_{S,11}^{c,0}(z) &= \frac{1}{2} F_{S,12}^{c,0}(z) = 3 F_{S,22}^{c,0}(z) = \frac{1}{2} G_{S,22}^{c,0}(z) = \sqrt{1 - 4z} (1 + 2z), \\
6 F_{11}^{d,0}(z) &= F_{12}^{d,0}(z) = 6 F_{22}^{d,0}(z) = G_{11}^{d,0}(z) = G_{22}^{d,0}(z) = 6 (1 - z)^2,
\end{align*}
$$

while

$$
\begin{align*}
G_{11}^{u,0} &= G_{12}^{u,0} = G_{S,11}^{u,0} = G_{S,12}^{u,0} = G_{11}^{c,0} = G_{12}^{c,0} = G_{S,11}^{c,0} = G_{S,12}^{c,0} = G_{12}^{d,0} = 0, \\
F_{S,ij} &= G_{S,ij}^{d,0} = 0.
\end{align*}
$$

To obtain the NLO corrections $F_{ij}^{u,1} \ldots G_{S,ij}^{d,1}$ we have calculated the diagrams $E_i$ and the imaginary parts of $D_i$ in Fig. 2. At NLO one becomes sensitive to the renormalization scheme. First, this affects the quantities $m_b, z$ and $\alpha_s$ entering our calculation. The NLO coefficients given below correspond to the use of the pole-mass definition for $m_b$ and the definition of $\alpha_s$ in the $\overline{\text{MS}}$ scheme [11]. $z$ can be either calculated from the pole masses or from the $\overline{\text{MS}}$ masses, because $z = m_0^2/m_b^2 = \overline{m}_0^2(m_c)/\overline{m}_0^2(m_b) + \mathcal{O} (\alpha_s^2)$. Second, the choice of the renormalization scheme is also an issue for the effective four-quark operators appearing at the various stages of our calculation. In the prediction of physical quantities this scheme dependence cancels to the calculated order, nevertheless it must be taken care of when assembling pieces from different theoretical sources. The Wilson coefficients $C_{1,2}$ of $H$ in (1) and $F_{ij}^{u,1} \ldots G_{S,ij}^{d,1}$ depend on the scheme used to renormalize the $\Delta B = 1$ operators in (2), but this dependence cancels in $F_{ij}^{u,1} \ldots G_{S}^{d,1}$. Our results below correspond to the definition of $C_{1,2}$ in [5]. $F_{ij}^{u,1} \ldots G_{S}^{d,1}$ also depend on the renormalization scheme of the $\Delta B = 0$ operators in (7). This dependence cancels only when these coefficients are combined with the hadronic parameters $B_{1,2}$ and $\epsilon_{1,2}$ calculated from lattice QCD. It is therefore important that our scheme is used in the lattice-continuum matching of these quantities. We use the $\overline{\text{MS}}$ scheme with the NDR prescription for $\gamma_5$ [5]. To specify the scheme completely, it is further necessary to state the definition of the evanescent operators appearing in the calculation [12]. We use

$$
\begin{align*}
E[Q] &= \overline{\psi}_\gamma \gamma_\mu \gamma_\rho \gamma_\nu (1 - \gamma_5) q \overline{\psi} \gamma^\nu \gamma^\rho \gamma_\mu (1 - \gamma_5) b - (4 - 8 \varepsilon) Q \\
E[Q_S] &= \overline{\psi}_\gamma \gamma_\mu (1 - \gamma_5) q \overline{\psi} \gamma^\nu \gamma^\mu (1 + \gamma_5) b - (4 - 8 \varepsilon) Q_S
\end{align*}
$$

and analogous definitions of $E[T]$ and $E[T_S]$. When the diagrams $E_1 \ldots E_4$ for e.g. $Q_S$ are calculated in $D = 4 - 2\varepsilon$ dimensions, the result can be expressed as a linear combination of
Figure 2: WA contributions in the next-to-leading order of QCD. The PI diagrams are obtained by interchanging $u$ and $d$ and reversing the fermion flow of the $u$ and $d$ lines. The first line shows the radiative corrections to $\Delta B = 0$ operators, which are necessary for the proper infrared factorization. Not displayed are the diagrams $E_1', E_4'$ and $D_{3-8}$ which are obtained from the corresponding unprimed diagrams by left-right reflection and the reverse of the fermion flow.

$Q_S$ and $E[Q_S]$. Effectively, (14) defines how Dirac strings with two or three Dirac matrices are reduced. (Note that (14) also implies the replacement rules $\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma (1 - \gamma_5) \otimes \gamma^\nu \gamma^\rho \gamma^\sigma (1 - \gamma_5) \rightarrow (16 - 4\varepsilon) \gamma_\mu (1 - \gamma_5) \otimes \gamma^\mu (1 - \gamma_5)$ and $\gamma_\mu \gamma_\nu (1 - \gamma_5) \otimes \gamma^\mu \gamma^\nu (1 + \gamma_5) \rightarrow 4(1 + \varepsilon) (1 - \gamma_5) \otimes (1 + \gamma_5)$.)

The particular choice of the $-8\varepsilon$ terms in (14) is motivated by Fierz invariance: the one-loop matrix elements of e.g. $Q_S$ and its Fierz transform $Q_F^S = -1/2 b_i \gamma_\nu (1 + \gamma_5) b_j \overline{\gamma}_j \gamma^\nu (1 - \gamma_5) q_i$ are in general different. This feature is an artifact of dimensional regularization. With (14) and a corresponding definition of $E[Q_F^S]$, however, Fierz invariance is maintained at the one-loop level. This choice, which has also been made in [5] for the $\Delta B = 1$ operators, has the practical advantage that one can freely use the Fierz transformation at any step of the calculation. In other words: “Fierz-evanescent” operators like $Q_S - Q_F^S$ can be identified with 0.

In the procedure of matching the full theory (eq. (4)) to the effective $\Delta B = 0$ theory, infrared singularities are encountered at $O(\alpha_s)$ both in the full-theory diagrams and in the matrix elements of operators in the effective theory. The diagrams relevant for this issue are $D_1 - D_4$ and $E_1 - E_4$. The singularities cancel in the Wilson coefficients $F$ and $G$, but need to be regularized at intermediate steps of the calculation. We take the $b$-quark on-shell, assign zero 4-momentum to the external light quarks and use dimensional regularization for the infrared (as well as the ultraviolet-
let) divergences. In this case, care has to be taken to treat the Dirac algebra in a consistent way.

In computing the matching condition between $D_1 - D_4$ and $E_1 - E_4$ we have used two different methods, which lead to the same result. In both methods ultraviolet divergences appearing in $E_1 - E_4$ and $D_3$ are subtracted, respectively, by $\Delta B = 0$ and $\Delta B = 1$ counterterms, in the usual way.

In the first method, we distinguish IR singularities arising in loop integrals from UV singularities, and treat the Dirac algebra in strictly four dimensions in the IR-divergent parts. In the second method, IR and UV divergences are not distinguished and $d$-dimensional Dirac algebra is used throughout. In this case evanescent operators $E$, as those given in (14), give a non-vanishing contribution in the matching procedure. This is a subtlety of the IR regulator used in method 2 [13]. If a different IR regulator, such as a gluon mass or method 1, is used, the non-vanishing bare one-loop matrix element of $E$ is cancelled by a finite counterterm, so that $E$ disappears from the NLO matching calculation [5, 12].

We would also like to mention that the Fierz ordering of $\Delta B = 1$ operators is immaterial because Fierz symmetry is respected by the standard NDR renormalization scheme employed by us. This has been checked by using the Fierz form leading to Dirac strings with flavour structure $\overline{b}b \otimes \overline{u}u$ in method 1, and $\overline{b}u \otimes \overline{u}b$ in method 2, and similarly for the contribution with $u \rightarrow d$. (The Fierz form used in method 2 for $\overline{b}d \otimes \overline{d}b$ is such that a closed fermion loop is generated in $D_1 - D_4$.)

In the NLO corrections to (9) we set $|V_{ud}| = 1$ and $V_{cd} = 0$. This introduces an error of order $|V_{cd}|^2 \alpha_s(m_b) z \ln z$, which is well below 1% of $\frac{\tau(B^+)/\tau(B^0_d) - 1}{\tau(B^+)/\tau(B^0_d) - 1}$. Hence (9) only involves the differences $F^{a, (1)}_{ij} - F^{d, (1)}_{ij} \ldots G^{a, (1)}_{S,ij} - G^{d, (1)}_{S,ij}$. Our results for these coefficients read:

$$
F^{u, (1)}_{11}(z, x_{\mu_1}, x_{\mu_0}) - F^{d, (1)}_{11}(z, x_{\mu_1}, x_{\mu_0}) =
\left[ \frac{16}{3} \frac{(1 - z)}{(1 - z) - 4 - 3 z + 3 z^2} \right] [\text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2}] + 
\left[ \frac{4}{3} (1 - z)^2 \frac{(16 + 19 z)}{16 + 19 z} \right] \ln(1 - z) + 
\left[ \frac{4 z}{9} (93 + 40 z - 57 z^2) \right] \ln(z) + 
\left[ \frac{32}{9} (1 - z)^2 \right] \ln(x_{\mu_1}) + \left[ -16 (1 - z)^2 \right] \ln(x_{\mu_0}) +
$$
\[
\left[ \frac{32 (1 - z)}{9} \right] \pi^2 + \frac{2 (1 - z) (152 + 149 z + 155 z^2)}{27}
\]

\[
F_{12}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - F_{12}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\left[ \frac{32 (1 - z) (-4 - 6 z + z^2)}{3} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] +
\left[ \frac{8 (1 - z)^2 (2 + 13 z + 3 z^2)}{3 z} \right] \ln(1 - z) + \left[ \frac{8 z (37 - 6 z - 6 z^2)}{3} \right] \ln(z) +
\left[ \frac{16 (1 - z)^2 (2 + z)}{3} \right] \ln(x_{\mu_1}) + \left[ \frac{16 (1 - z) (6 + 2 z + z^2)}{9} \right] \pi^2 +
\frac{4 (1 - z) (30 + 33 z - 13 z^2)}{3}
\]

\[
F_{22}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - F_{22}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\left[ \frac{16 (19 - z) (-1 + z) z}{9} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] +
\left[ \frac{16 (1 - z)^2 (1 + 2 z)^2}{9 z} \right] \ln(1 - z) + \left[ \frac{4 z (135 + 30 z - 68 z^2)}{27} \right] \ln(z) +
\left[ \frac{16 (1 - z)^2 (8 + z)}{3} \right] \ln(x_{\mu_1}) + \left[ \frac{-8 (1 - z)^2 (8 + z)}{3} \right] \ln(x_{\mu_0}) +
\left[ \frac{16 (1 - z) (6 + 2 z + z^2)}{27} \right] \pi^2 + \frac{4 (1 - z) (544 - 185 z - 68 z^2)}{81}
\]

\[
F_{S,11}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - F_{S,11}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\left[ \frac{32 (1 - z)^2 (1 + 2 z)}{3} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] +
\left[ -8 (1 - z)^2 (2 + 10 z - 3 z^2) \right] \ln(1 - z) +
\left[ \frac{8 z (18 - 155 z + 144 z^2 - 27 z^3)}{9} \right] \ln(z) +
\left[ -48 (1 - z)^2 (1 + 2 z) \right] \ln(x_{\mu_0}) + \frac{-4 (1 - z) (133 - 53 z + 40 z^2)}{27}
\]
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\[ F_{S,12}^{\mu}(z, x_{\mu_1}, x_{\mu_0}) - F_{S,12}^{d}(z, x_{\mu_1}, x_{\mu_0}) = \]
\[ \left[ \frac{64}{3} (1 - z) \frac{(2 - z) (1 + 2 z)}{3} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] + \]
\[ \left[ -\frac{16}{3} (1 - z)^2 \frac{(1 + 2 z + 6 z^2 - 3 z^3)}{3} \right] \ln(1 - z) + \]
\[ \left[ \frac{16}{3} z (4 - 24 z + 18 z^2 - 3 z^3) \right] \ln(z) + \]
\[ \left[ -\frac{32}{3} (1 - z)^2 (1 + 2 z) \right] \ln(x_{\mu_1}) + \left[ -\frac{32}{3} (1 - z)^2 (1 + 2 z) \right] \ln(x_{\mu_0}) + \]
\[ \left[ -\frac{32}{9} (1 - z) z (1 + 2 z) \right] \pi^2 + \frac{8}{3} (1 - z) (-17 - 29 z + 36 z^2) \]

\[ F_{S,22}^{\mu}(z, x_{\mu_1}, x_{\mu_0}) - F_{S,22}^{d}(z, x_{\mu_1}, x_{\mu_0}) = \]
\[ \left[ \frac{32}{9} (1 - z) \frac{(3 - z) (1 + 2 z)}{9} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] + \]
\[ \left[ -\frac{8}{9} (1 - z)^2 \frac{(2 + 5 z + 8 z^2 - 3 z^3)}{9 z} \right] \ln(1 - z) + \]
\[ \left[ -\frac{8}{27} z (18 - 123 z + 82 z^2 - 9 z^3) \right] \ln(z) + \]
\[ \left[ -\frac{32}{3} (1 - z)^2 (1 + 2 z) \right] \ln(x_{\mu_1}) + \left[ -\frac{16}{3} (1 - z)^2 (1 + 2 z) \right] \ln(x_{\mu_0}) + \]
\[ \left[ -\frac{32}{27} (1 - z) z (1 + 2 z) \right] \pi^2 + \frac{4}{81} (1 - z) (-259 - 421 z + 488 z^2) \]

\[ G_{11}^{\mu}(z, x_{\mu_1}, x_{\mu_0}) - G_{11}^{d}(z, x_{\mu_1}, x_{\mu_0}) = \]
\[ \left[ 16 (4 - 3 z) (1 - z) \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] + \]
\[ \left[ (1 - z)^2 (122 + 5 z) \right] \ln(1 - z) + \left[ \frac{z (384 - 256 z - 21 z^2)}{3} \right] \ln(z) + \]
\[ \left[ -24 (1 - z)^2 \right] \ln(x_{\mu_1}) + \left[ -6 (1 - z)^2 (4 + 3 z) \right] \ln(x_{\mu_0}) + \]
\[
\frac{4 \left( 7 - 9 z \right) (1 - z)}{3} \pi^2 + \frac{(1 - z) (-2450 + 2575 z + 517 z^2)}{18}
\]

\[
G_{12}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - G_{12}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\frac{8 \left( 4 - 13 z \right) (1 - z)}{12} \left[ \mathrm{Li}_2(z) + \ln(1 - z) \ln(z) \right] + \frac{2 (1 - z)^2 \left( 2 + 3 z + 13 z^2 \right)}{z} \ln(1 - z) + \frac{4 z \left( 12 + 24 z - 25 z^2 \right)}{3} \ln(z) + \frac{12 (1 - z)^2 \left( 14 + z \right)}{\ln(z)} \ln(1 - z) + \frac{[-12 (1 - z)^2 (8 + z)] \ln(x_{\mu_0}) +}{\left[ \frac{4 (1 - z) (6 + 2 z + z^2)}{3} \right] \pi^2 + \frac{(1 - z) (818 - 667 z - 19 z^2)}{9}}
\]

\[
G_{22}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - G_{22}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\frac{-8 \left( 1 - z \right) \left( 36 + 31 z + 5 z^2 \right)}{3} \left[ \mathrm{Li}_2(z) + \ln(1 - z) \ln(z) \right] + \frac{4 (1 - z)^2 \left( -1 + 68 z + 5 z^2 \right)}{3 z} \ln(1 - z) + \frac{4 z \left( 162 - 102 z - z^2 \right)}{9} \ln(z) + \frac{-4 (1 - z)^2 (8 + z)}{\ln(z)} \ln(x_{\mu_1}) + \frac{2 (1 - z)^2 (8 + z)}{\ln(z)} \ln(x_{\mu_0}) + \frac{2 (1 - z) \left( 60 + 77 z + 7 z^2 \right)}{9} \pi^2 + \frac{(1 - z) \left( -2803 + 2786 z + 725 z^2 \right)}{27}
\]

\[
G_{S,11}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - G_{S,11}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\frac{[-18 (1 - z)^2 (1 + 2 z)] \ln(1 - z) + \left[ -44 \left( 4 - 3 z \right) z^2 \right] \ln(z) +}{\frac{4 (1 - z) (28 + 103 z - 164 z^2)}{9}}
\]

\[
G_{S,12}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - G_{S,12}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) =
\frac{[16 (1 - z) (1 + 2 z)] \left[ \mathrm{Li}_2(z) + \ln(1 - z) \ln(z) \right] +}{\frac{16 (1 - z) (1 + 2 z)}{16}}
\]
\[
\left[ -\frac{4 (1 - z)^2 (1 + z) (1 + 2 z)}{z} \right] \ln(1 - z) + \left[ \frac{4 z (6 - 51 z + 28 z^2)}{3} \right] \ln(z) + \\
\left[ -\frac{24 (1 - z)^2 (1 + 2 z)}{z} \right] \ln(x_{\mu_1}) + \left[ \frac{-8 (1 - z) z (1 + 2 z)}{3} \right] \pi^2 + \\
\frac{4 (1 - z) (-53 - 80 z + 82 z^2)}{9}
\]

\[
G_{S,22}^{u,(1)}(z, x_{\mu_1}, x_{\mu_0}) - G_{S,22}^{d,(1)}(z, x_{\mu_1}, x_{\mu_0}) = \\
\left[ \frac{16 (1 - z) (1 + 2 z) (3 + 5 z)}{3} \right] \left[ \text{Li}_2(z) + \frac{\ln(1 - z) \ln(z)}{2} \right] + \\
\left[ \frac{-2 (1 - z)^2 (-2 + 31 z + 64 z^2 + 3 z^3)}{3 z} \right] \ln(1 - z) + \\
\left[ \frac{2 z (36 - 336 z + 62 z^2 + 9 z^3)}{9} \right] \ln(z) + \\
\left[ \frac{8 (1 - z)^2 (1 + 2 z)}{9} \right] \ln(x_{\mu_1}) + \left[ \frac{4 (1 - z)^2 (1 + 2 z)}{9} \right] \ln(x_{\mu_0}) + \\
\left[ \frac{-4 (1 - z) (1 + 2 z) (9 + 7 z)}{9} \right] \pi^2 + \frac{(1 - z) (385 + 1519 z - 3278 z^2)}{27}
\]  

(15)

Here \(\text{Li}_2(z) = - \int_0^z dt \ln(1 - t)/t\) is the dilogarithm function. Any dependence on infrared regulators has cancelled from the coefficients in (15) showing that infrared effects properly factorize. As another check we have verified that the dependence on \(\mu_1\) cancels analytically to the calculated order.

For our numerical studies we choose the following range for the input parameters:

\[
\alpha_s(M_Z) = 0.118 \pm 0.003, \quad m_b = 4.8 \pm 0.1 \text{ GeV}, \quad z = 0.085 \pm 0.015.
\]

(16)

Throughout this paper we always remove \(\mathcal{O}(\alpha_s^2)\) terms from the calculated coefficients. (For instance, at NLO we write a product such as \(C_{1,\text{LO}}^2 F_u^\mu + 2 C_{1,\text{LO}} dC_1 F_{\text{LO}}^\mu\), where \(C_{1,\text{NLO}} = C_{1,\text{LO}} + dC_1\) denotes the NLO Wilson coefficient.) In all terms we use the two-loop expression for the running coupling \(\alpha_s\) in QCD with five flavours. Numerical values for the calculated coefficients can be found in Tab. 1. The two contributions from \((F_u - F_d)B_1 + (G_u - G_d)\epsilon_1\) and from \((F_S^u - F_S^d)B_2 + (G_S^u - G_S^d)\epsilon_2\) to \(\Gamma(B_0^d) - \Gamma(B^+)\) are separately scheme-independent. Tab. 1 reveals that the former part is expected to give the dominant contribution to the desired width difference. The results also show a substantial improvement of the \(\mu_1\)-dependence in the NLO compared to LO. This dependence is plotted in Fig. 3 for the two Wilson coefficients of the important vector operators. The approximation employed in [8] setting \(z = 0\) in the NLO correction is also plotted. Expectedly, the accuracy of this approximation decreases.
Table 1: Numerical values for the coefficients in (9) for \( \alpha_s(M_Z) = 0.118 \) and \( \mu_0 = m_b = 4.8 \) GeV.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \mu_1 )</th>
<th>( m_b/2 )</th>
<th>( m_b )</th>
<th>( 2m_b )</th>
<th>( 3m_b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_{u,LO} - F_{d,LO} )</td>
<td>0.865</td>
<td>0.270</td>
<td>-0.176</td>
<td>0.280</td>
<td>0.261</td>
</tr>
<tr>
<td>( F_{u,NLO} - F_{d,NLO} )</td>
<td>0.396</td>
<td>0.460</td>
<td>0.386</td>
<td>0.469</td>
<td>0.452</td>
</tr>
<tr>
<td>( F_{S,LO} - F_{S,NLO} )</td>
<td>0.002</td>
<td>0.042</td>
<td>0.105</td>
<td>0.043</td>
<td>0.042</td>
</tr>
<tr>
<td>( F_{S,NLO} - F_{S,NLO} )</td>
<td>0.035</td>
<td>0.033</td>
<td>0.026</td>
<td>0.031</td>
<td>0.035</td>
</tr>
<tr>
<td>( G_{u,NLO} - G_{d,NLO} )</td>
<td>-8.665</td>
<td>-8.501</td>
<td>-8.154</td>
<td>-8.718</td>
<td>-8.280</td>
</tr>
<tr>
<td>( G_{S,LO} - G_{S,NLO} )</td>
<td>2.679</td>
<td>2.404</td>
<td>2.231</td>
<td>2.420</td>
<td>2.385</td>
</tr>
<tr>
<td>( G_{S,NLO} - G_{S,NLO} )</td>
<td>1.668</td>
<td>1.850</td>
<td>1.902</td>
<td>1.854</td>
<td>1.843</td>
</tr>
</tbody>
</table>

Figure 3: Dependence of \( F^u - F^d \) and \( G^u - G^d \) on \( \mu_1/m_b \) for the input parameters in (16) and \( \mu_0 = m_b \). The solid (short-dashed) line shows the NLO (LO) result. The long-dashed line shows the NLO result in the approximation of [8], i.e. \( z \) is set to zero in the NLO corrections.

For small \( \mu_1 \), because the difference to the exact NLO result is of order \( \alpha_s(\mu_1) z \ln z \). For the final result of our coefficients we estimate the \( \mu_1 \)-dependence in a more conservative way: we vary \( \mu_1 \) in \( F^u \ldots G^u \) and \( F^d \ldots G^d \) independently. Further the variation with \( z \) and \( \alpha_s(M_Z) \) in the ranges of (16) is calculated and all these sources of theoretical uncertainty are symmetrized.
individually and added in quadrature. The dependence on $z$ is only an issue for $G^u - G^d$. We find:

$$
\begin{array}{|c|c|c|c|}
\hline
 & \text{NLO} & \text{LO} & \text{app} \\
\hline
F^u - F^d & 0.460 \pm 0.101 & 0.270 \pm 0.480 & 0.440 \pm 0.119 \\
F^u_S - F^d_S & 0.033 \pm 0.046 & 0.042 \pm 0.052 & 0.025 \pm 0.045 \\
G^u - G^d & -8.50 \pm 0.40 & -8.62 \pm 0.90 & -8.00 \pm 0.32 \\
G^u_S - G^d_S & 1.85 \pm 0.08 & 2.40 \pm 0.23 & 1.80 \pm 0.10 \\
\hline
\end{array}
$$

(17)

The quoted central values correspond to the choice $\mu_1 = m_b$ and the central values in (16). The third column in (17) shows the result for the approximation of [8], setting $z = 0$ in the NLO corrections. For $\mu_1 = m_b$, this approximation reproduces the size of the NLO corrections to $F^u - F^d$ and $G^u_S - G^d_S$ to better than 15% . The small NLO correction to $G^u - G^d$ is, however, overestimated. The NLO result for this coefficient, which is largest in magnitude, is better reproduced by the LO result than by the approximation of [8].

The origin of the $\alpha_s(\mu_1) z \ln z$ terms, which are the main cause of the discrepancy between the first and third column in (17), can be traced back to diagram $D_{11}$ of Fig. 2. This diagram defines the scheme of the charm-quark mass. One can absorb the $\alpha_s(\mu_1) z \ln z$ terms into the LO by replacing $z$ with $\bar{z} = \frac{m_c^2}{m_c^2(\mu)}$, which implies the replacement

$$
F^{u,(1)} \rightarrow F^{u,(1)} - \frac{\alpha_s}{4\pi} \frac{\partial F^{u,(0)}}{\partial z} \gamma_m^{(0)} z \ln z
$$

(18)

in the NLO corrections to $F^u$ and similarly in the other Wilson coefficients. Here $\gamma_m^{(0)} = 8$ is the LO anomalous dimension of the quark mass. This procedure sums the terms of order $\alpha_s^2(\mu_1) z \ln^n z$ with $n = 0, 1, \ldots$ to all orders in perturbation theory. This can be seen by performing an OPE of the transition operator $T$ which treats $m_c$ as a light mass scale: then increasing powers of $m_c$ correspond to $\Delta B = 0$ operators of increasing dimension and $m_c$ and $m_b$ enter the result at the same scale $\mu_1$ at which the OPE is performed. In every order of the perturbation series $\ln z$ is split into $\ln(\mu_1^2/m_b^2)$ contained in the Wilson coefficients and $\ln(\mu_1^2/m_b^2)$ residing in the matrix elements. Since there are no dimension-8 operators with charm-quark fields contributing to $\Gamma(B^0_d) - \Gamma(B^+)$, no terms of order $m_c^2 \ln(m_c^2/m_b^2)$ can occur. From our NLO results we can indeed verify that the procedure in (18) removes the $\alpha_s(\mu_1) z \ln z$ terms, while e.g. terms of order $\alpha_s(\mu_1) \ln z$ persist as expected, because there are dimension-10 operators with charm-quark fields of the type $m_c(bq)(\bar{q}b)(\bar{c}c)$. Using $\bar{z} = 0.055$ rather than $z = 0.085$ in the coefficients tabulated in the third column of (17) indeed removes the disturbing discrepancy with the NLO result for $G^u - G^d$. Also the central values of $F^u - F^d$ and $G^u_S - G^d_S$ move closer to the NLO result, while no significant improvement occurs for $F^u_S - F^d_S$.

The width difference in (9) involves the product $\bar{F}^{\ell_1 T} B$, which is independent of the renormalization scheme and scales. In order to compare the scheme dependent coefficients $\bar{F}^{\ell_1}$ with the calculation in [8] for $z = 0$, we need to take into account that the coefficients in [8] are defined for matrix elements in HQET rather than in full QCD. The matching relation connecting
HQET and full-QCD matrix elements of the four operators $\bar{O}$ used in [8] has the form
\[
\langle \bar{O} \rangle_{\text{QCD}}(m_b) = \left( 1 + \frac{\alpha_s(m_b)}{4\pi} \hat{C}_{1}^{\overline{\text{MS}}} \right) \langle \bar{O} \rangle_{\text{HQET}}(m_b),
\]
where the $4 \times 4$ matrix $\hat{C}_{1}^{\overline{\text{MS}}}$ can be found in eq. (36) of [8]. The renormalization scheme of operator matrix elements in full QCD is identical in our paper and in [8,9]. The only further difference is that the operators $\bar{O}$ are linear combinations, $\bar{O} = S \bar{Q}$, of our basis $\bar{Q} = (Q, Q_s, T, T_s)^T$ with
\[
S = \begin{pmatrix}
\frac{1}{3} & 0 & 2 & 0 \\
0 & -\frac{2}{3} & 0 & -4 \\
\frac{4}{9} & 0 & -\frac{1}{3} & 0 \\
0 & -\frac{8}{9} & 0 & \frac{2}{3}
\end{pmatrix}.
\]
(This simple relation holds beyond tree level because the renormalization schemes are identical. The preservation of Fierz-symmetry by the choice of evanescent operators in (14) is important for this property.) It follows that our coefficients $\bar{F}$ are related to the corresponding coefficients $\bar{A} + \frac{\alpha_s}{4\pi} \bar{B}$ in [8] at scale $\mu = m_b$ through
\[
\frac{1}{3} \left( \bar{F}^{(0)} + \frac{\alpha_s}{4\pi} \bar{F}^{(1)} \right)^T = \bar{A}^T S + \frac{\alpha_s}{4\pi} \left( \bar{B}^T S - \bar{A}^T \hat{C}_{1}^{\overline{\text{MS}}} S \right).
\]
Here we have suppressed flavour labels $q = u, d$ and the double indices $ij = 11, 12, 22$ referring to the $\Delta B = 1$ coefficients $C_i C_j$ (see (11)). Note that in the notation of [8] labels $u, d$ are interchanged with respect to our convention and that the coefficients with label 12 are defined with a relative factor of two. Using (21) we have verified that the results of [8] obtained for $z = 0$ are in agreement with ours in this limit.

3 Phenomenology

3.1 $\tau(B^+)/\tau(B^0_d)$

One can directly use (9) to predict the desired lifetime ratio:
\[
\frac{\tau(B^+)}{\tau(B^0_d)} - 1 = \tau(B^+) \left[ \Gamma(B^0_d) - \Gamma(B^+) \right]
\]
\[
= 0.0325 \left( \frac{|V_{cb}|}{0.04} \right)^2 \left( \frac{m_b}{4.8 \text{GeV}} \right)^2 \left( \frac{f_B}{200 \text{MeV}} \right)^2 \times 
\]
\[
\left[ (1.0 \pm 0.2) B_1 + (0.1 \pm 0.1) B_2 - (18.4 \pm 0.9) \epsilon_1 + (4.0 \pm 0.2) \epsilon_2 \right].
\]
Here $\tau(B^+) = 1.653$ ps has been used in the overall factor and the hadronic parameters $B_1 \ldots \epsilon_2$ are normalized at $\mu_0 = m_b$ throughout this section.

In [3] it has been noticed that without a detailed study of the hadronic parameters one expects $\tau(B^+)/\tau(B^0_d)$ to deviate from 1 by up to $\pm 20\%$. This feature originates from the large coefficient
of $\epsilon_1$ and persists in our NLO prediction in (22), because the NLO corrections to $G_u - G_d$ are small. Confronting (22) with the recent measurements [14, 15],

$$\frac{\tau(B^+)}{\tau(B^0_d)} = \begin{cases} 1.082 \pm 0.026 \pm 0.012 & \text{(BABAR)} \\ 1.091 \pm 0.023 \pm 0.014 & \text{(BELLE)} \end{cases}$$

one expects $|\epsilon_1|$ to be significantly smaller than $1/N_c = 1/3$, i.e. nonfactorizable contributions appear to be small. This result is confirmed by the existing computations of the $\epsilon_1$'s in quenched lattice QCD [9, 10]. However, due to its large coefficient sophisticated non-perturbative methods are definitely necessary to compute $\epsilon_1$ sufficiently accurately. The other important term in (22) is the first one: the NLO enhancement of $F_u - F_d$ in (17) has altered the coefficient of $B_1$ in (22) from $1.06 \pm 0.10$ in the LO to $1.0 \pm 0.2$. While from the LO result not even the sign of this contribution was known, the NLO result now clearly establishes a positive contribution of order 3% to $\tau(B^+)/\tau(B^0_d)$ from the term involving $B_1$.

The hadronic parameters have been computed in [9] using the same renormalization scheme as in the present paper. They read

$$(B_1, B_2, \epsilon_1, \epsilon_2) = (1.10 \pm 0.20, 0.79 \pm 0.10, -0.02 \pm 0.02, 0.03 \pm 0.01).$$

Using $|V_{cb}| = 0.040 \pm 0.0016$ from a CLEO analysis of inclusive semileptonic $B$ decays [16], the world average $f_B = (200 \pm 30)$ MeV from lattice calculations [17] and $m_b = 4.8 \pm 0.1$ GeV in (22), we find

$$\frac{\tau(B^+)}{\tau(B^0_d)} = 1.053 \pm 0.016 \pm 0.017, \quad \left[ \frac{\tau(B^+)}{\tau(B^0_d)} \right]_{LO} = 1.041 \pm 0.040 \pm 0.013,$$

where the first error is due to the errors on the NLO coefficients as given in (22) and the hadronic parameters (24), and the second error is the overall normalization uncertainty due to $m_b$, $|V_{cb}|$ and $f_B$ in (22). The first error reduces to 0.008 in NLO and 0.038 in LO, if the errors on the hadronic parameters are neglected, demonstrating the substantial reduction of scale dependence at NLO in comparison with the LO. This result is gratifying as the strong scale dependence observed at LO had been a major motivation for a NLO analysis. This is also seen in Fig. 4, where we show the lifetime ratio as a function of the renormalization scale $\mu_1$. We should, however, emphasize that the result and error given in (25) do not include the effects of $1/m_b$ corrections and unquenching, which could well be on the order of 0.05. The NLO result slightly exceeds the central value of the LO result and improves the agreement with the experimental value in (23).

### 3.2 $\tau(\Xi^0_b)/\tau(\Xi^-_b)$

The SU(3)$_F$ anti-triplet ($\Lambda_b \sim bud$, $\Xi^0_b \sim bus$, $\Xi^-_b \sim bds$) comprises the $b$-flavoured baryons whose light degrees of freedom are in a 0$^+$ state. These baryons decay weakly. Baryon lifetimes have attracted a lot of theoretical attention: the measured $\Lambda_b$ lifetime falls short of $\tau(B^0_d)$ by roughly 20% [18], which has raised concerns about the applicability of the HQE to baryons. Unfortunately this interesting topic cannot yet be addressed at the NLO level, because $\tau(\Lambda_b)/\tau(B^0_d)$...
receives contributions from the SU(3)$_F$-singlet portion $T_{sing}$ of the transition operator in (5) and NLO corrections to $T_{sing}$ are unknown at present. Further the hadronic matrix elements entering $\tau(\Lambda_b)/\tau(B^0_d)$ involve penguin contractions of the operators in (7), which are difficult to compute. It is, however, possible to predict the lifetime splitting within the iso-doublet $(\Xi^0_b, \Xi^-_b)$ with NLO precision. The corresponding LO diagrams are shown in Fig. 5. For $\Xi_b$’s the weak decay of the valence $s$-quark could be relevant: the decays $\Xi^-_b \to \Lambda_b\pi^-$, $\Xi^-_b \to \Lambda_b e^-\nu_e$ and $\Xi^0_b \to \Lambda_b\pi^0$ are triggered by $s \to u$ transitions and could affect the total rates at the $\mathcal{O}(1\%)$ level [19]. Once the lifetime measurements reach this accuracy, one should correct for this effect. To this end we define

$$\Gamma(\Xi_b) \equiv \Gamma(\Xi_b) - \Gamma(\Xi_b \to \Lambda_b X) = \frac{1 - B(\Xi_b \to \Lambda_b X)}{\tau(\Xi_b)} \equiv \frac{1}{\tau(\Xi_b)}$$

for $\Xi_b = \Xi^0_b, \Xi^-_b$, (26)

where $B(\Xi_b \to \Lambda_b X)$ is the branching ratio of the above-mentioned decay modes. Thus $\Gamma(\Xi_b)$ is the contribution from $b \to c$ transitions to the total decay rate. In analogy to (9) one finds

$$\Gamma(\Xi^-_b) - \Gamma(\Xi^0_b) = \frac{G^2_F m_b^2 |V_{cb}|^2}{12\pi} f_B M_B \left(|V_{ud}|^2 \bar{F}^u + |V_{cd}|^2 \bar{F}^c - \bar{F}^d\right) \cdot \vec{B}\Xi_b. \quad (27)$$
Here $\tilde{B}^{\Xi_b} = (L_{11}^{\Xi_b}(\mu_0), L_{12}^{\Xi_b}(\mu_0), L_{22}^{\Xi_b}(\mu_0), L_{22}^{\Xi_b}(\mu_0))^{T}$ comprises the hadronic parameters defined as

$$
\langle \Xi_b^0 | (Q^u - Q^d)(\mu_0) | \Xi_b^0 \rangle = f_B^2 M_B M_{\Xi_b} L_{11}^{\Xi_b}(\mu_0),
$$

$$
\langle \Xi_b^0 | (Q^S - Q^T_S)(\mu_0) | \Xi_b^0 \rangle = f_B^2 M_B M_{\Xi_b} L_{12}^{\Xi_b}(\mu_0),
$$

$$
\langle \Xi_b^0 | (T^u - T^d)(\mu_0) | \Xi_b^0 \rangle = f_B^2 M_B M_{\Xi_b} L_{22}^{\Xi_b}(\mu_0),
$$

$$
\langle \Xi_b^0 | (T^S - T^T_S)(\mu_0) | \Xi_b^0 \rangle = f_B^2 M_B M_{\Xi_b} L_{22}^{\Xi_b}(\mu_0).
$$

(28)

In contrast to the $B$ meson system, the four matrix elements in (28) are not independent at the considered order in $\Lambda_{QCD}/m_b$. Since the light degrees of freedom are in a spin-0 state, the matrix elements $\langle \Xi_b | 2Q^u_S + Q^T \Xi_b \rangle$ and $\langle \Xi_b | 2T^u_S + T^T \Xi_b \rangle$ are power-suppressed compared to those in (28) (see e.g. [1,3]). This, however, is not true in all renormalization schemes, in the $\overline{MS}$ scheme used by us $2Q^u_S + Q^T$ and $2T^u_S + T^T$ receive short-distance corrections, because hard gluons can resolve the heavy $b$-quark mass. This feature is discussed in [7]. These short-distance corrections are calculated from the diagrams $E_1 \ldots E_4$ in Fig. 2. For our scheme we find

$$
\left( \begin{array}{c}
L_{11}^{\Xi_b}(m_b) \\
L_{22}^{\Xi_b}(m_b)
\end{array} \right) = \left[ -\frac{1}{2} + \frac{\alpha_s(m_b)}{4\pi} \left( \begin{array}{cc}
-28/3 & -7 \\
-14/9 & 7/2
\end{array} \right) \right] \left( \begin{array}{c}
L_{11}^{\Xi_b}(m_b) \\
L_{22}^{\Xi_b}(m_b)
\end{array} \right) + O(\frac{\Lambda_{QCD}}{m_b}).
$$

(29)

As an important check we find that the dependence on the infrared regulator drops out in (29). With (29) we can express the width difference in (27) in terms of just the two hadronic parameters $L_{11}^{\Xi_b}$ and $L_{22}^{\Xi_b}$. We find

$$
\frac{\tau(\Xi_b)}{\tau(\Xi_b^0)} - 1 = \frac{\tau(\Xi_b^0)}{\tau(\Xi_b)} \left[ \Gamma(\Xi_b) - \Gamma(\Xi_b^0) \right]
$$

$$
= 0.59 \left( \frac{|V_{cb}|}{0.04} \right)^2 \left( \frac{m_b}{4.8 \text{ GeV}} \right)^2 \left( \frac{f_B}{200 \text{ MeV}} \right)^2 \frac{\tau(\Xi_b^0)}{1.5 \text{ ps}} \times \left[ (0.04 \pm 0.01) L_1 - (1.00 \pm 0.04) L_2 \right],
$$

(30)

with $L_i = L_i^{\Xi_b}(\mu_0 = m_b)$. For the baryon case there is no reason to expect the color-octet matrix element to be much smaller than the color-singlet ones, so that the term with $L_2$ will dominate.
the result. The hadronic parameters $L_{1,2}$ have been analysed in an exploratory study of lattice HQET [20] for $\Lambda_b$ baryons. Up to SU(3)$_F$ corrections, which are irrelevant in view of the other uncertainties, $L_i^{\Xi_b}$ and $L_i^{\Lambda_b}$ are equal.

4 Conclusions

We have computed the Wilson coefficients in the heavy quark expansion to order $(\Lambda_{QCD}/m_b)^3$ for the $B^+ - B^0_d$ lifetime difference at next-to-leading order in perturbative QCD. These coefficients depend on the scheme and scale $\mu_0$ used to define the matrix elements of the $\Delta B = 0$ operators in the effective theory. Our scheme is specified by the NDR prescription for $\gamma_5$, $\overline{\text{MS}}$ subtraction and the definition of evanescent operators given in (14). The $\mathcal{O}(\alpha_s)$ accuracy is crucial for a satisfactory matching of the Wilson coefficients to the matrix elements determined with lattice QCD. Current lattice calculations, which are still in a relatively early stage in this case, yield, when combined with our calculations, $\tau(B^+)/\tau(B^0_d) = 1.053 \pm 0.016 \pm 0.017$ [see (25)].

The effects of unquenching and $1/m_b$ corrections are not yet included, but could well be on the order of 0.05. Next-to-leading order corrections to $\tau(B^+)/\tau(B^0_d)$ were recently computed in the approximation $m_c = 0$ [8]. Taking the limit $m_c \to 0$ of our results we find agreement with this calculation.

A substantial improvement of the NLO calculation is the large reduction of perturbative uncertainty reflected in the scale dependence of $\Delta B = 1$ Wilson coefficients from the standard weak Hamiltonian. This scale dependence had been found to be very large at leading order, preventing even an unambiguous prediction of the sign of $\tau(B^+)/\tau(B^0_d) - 1$ up to now [3]. With this major source of uncertainty removed by the NLO calculation, further progress will depend on continuing advances in the evaluation of the nonperturbative hadronic matrix elements and the computation of $1/m_b$-suppressed effects.

References


