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Pattern Formation in the Early Universe

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The formation of regular patterns is a well-known phenomenon in condensed matter physics. Systems that exhibit pattern formation are typically driven and dissipative with pattern formation occurring in the weakly non-linear regime and sometimes even in more strongly non-linear regions of parameter space. In the early universe, parametric resonance can drive explosive particle production called preheating. The fields that are populated then decay quantum mechanically if their particles are unstable. Thus, during preheating, a driven-dissipative system exists. We have shown previously that pattern formation can occur in two dimensions in a self-coupled inflaton system undergoing parametric resonance. In this paper, we provide evidence of pattern formation with more realistic initial conditions in both two- and three-dimensions. We show that the patterns are spatio-temporal, leading to a distinctive, but probably low-amplitude peak in the gravitational wave spectrum. We also discuss putting power from resonance into patterns on cosmological scales in order to explain the observed excess power at $128h^{-1}Mpc$, and why this seems unlikely.

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I. INTRODUCTION

Much recent work has been done on the topic of preheating in inflationary cosmology. Preheating is a stage of explosive particle production which results from the resonant driving of particle modes by an inflaton oscillating in its potential at the end of inflation [1–3].

In regions of parameter space where parametric resonance is effective, much of the energy of the inflaton is transferred to bands of resonant wave modes. This energy transfer is non-thermal and can lead to interesting non-equilibrium behavior. Two examples of the non-equilibrium effects that can be produced are non-thermal phase transitions [4–7] and baryogenesis [8,9]. The non-thermal phase transitions induced during preheating can sometimes lead to topological defect formation [10–12], even at energies above the eventual final thermal temperature. Furthermore, non-linear evolution of the field when quantum decay of the resonantly produced particles is small leads to a chaotic power-law spectrum of density fluctuations [13,14].

In a previous letter [16], we presented evidence for a new phenomenon that can arise from preheating: pattern formation. It has long been known that many condensed matter systems exhibit pattern formation*. Examples of pattern forming systems which have been studied are ripples on sand dunes, cloud streets and a variety of other convective systems, chemical reaction-diffusion systems, stellar atmospheres and vibrated granular materials. All of these physical systems have two features in common. They are all driven in some manner, i.e. energy is input to the system, and they are all dissipative, usually being governed by diffusive equations of motion. Typically, patterns are formed in these systems in the weakly non-linear regime before the energy introduced into the system overwhelms the dissipative mechanism. Sometimes, patterns persist beyond the weakly non-linear regime as well.

At the end of inflation, the inflaton ϕ is homogeneous with small perturbations $\delta\phi$ imprinted on it due to quantum fluctuations. The inflaton then oscillates about the minimum of its potential, giving an effectively time dependent mass to fields with which it is coupled. The time dependent mass drives exponential growth in the population of bands of particle wave modes. Many of the fields into which the inflaton can decay resonantly are also unstable to quantum decay. For these reasons, at the end of inflation, we are considering fields which are driven, due to resonant particle creation, and also dissipative, due to quantum decay. In [16], we were able to show that pattern formation occurs in a chaotic inflationary model with a self-coupled inflaton in the weakly non-linear regime.

We considered a $\lambda\phi^4$ theory with the addition of a phenomenological decay term to mimic the inflaton's quantum decay. This model without the decay term has been studied extensively in the literature [2,3,13,15,14,17,20],

*For an extensive review of pattern formation in condensed matter systems, see [18,19] and a similar model including the decay term has also been studied [21].

In [16], we used restricted initial conditions. We only seeded the resonant band with small fluctuations of order $\sim 10^{-3}$ of $\langle\phi\rangle$, then simulated the field's evolution. We found that the resonant modes interacted and formed patterns. Here, we use initial conditions appropriate to the vacuum at the end of chaotic inflation. We also extend our study to a 3-dimensional volume. In [16], we did not point out the spatio-temporal nature of the patterns, which, at the time was not evident due to an unfortunate coincidence in the pattern which presented itself at the timesteps at which we viewed the data. Here, we note the spatio-temporal behavior. We also discuss resonance giving rise to patterns on cosmological scales, when it might occur and why it seems unlikely.

The paper is ordered as follows: In section II, we present the model we are investigating. In section III, we discuss the initial conditions appropriate for the end of inflation. In section IV, we present results of our simulations in two dimensions. In section V, we present results of our three-dimensional simulations, then we discuss possible implications of our results and conclude the paper.

II. THE $\lambda\phi^4$ MODEL WITH PHENOMENOLOGICAL DAMPING

Our field equation in comoving coordinates is

$$\ddot{\phi} + 3H\dot{\phi} + \gamma\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + \lambda\phi^3 = 0 \tag{1}$$

where γ is a decay constant and λ is the self-coupling of the field. Converting to conformal time $t = a(\tau)\tau$ and rescaling the field $\varphi = a\phi$, then further rescaling: $t \to t/\sqrt{\lambda}\varphi_0$, $x \to x/\sqrt{\lambda}\varphi_0$ and $\varphi \to \varphi\varphi_0$, where φ_0 is the value of the inflaton at the end of inflation. This gives us a new equation

$$\ddot{\varphi} + a\Gamma\dot{\varphi} - \nabla^2\varphi - (\dot{a}\Gamma + \frac{\ddot{a}}{a})\,\varphi + \varphi^3 = 0, \tag{2}$$

where $\Gamma = \gamma/\sqrt{\lambda}\varphi_0$. In the $\lambda\phi^4$ theory, the averaged equation of state during preheating is that of radiation, therefore, $\frac{\ddot{a}}{a} = 0$.

It should be noted that pattern formation in the inflaton system is conceptually distinct from condensed matter systems for at least two reasons. First, the equations we study are wave equations with damping, not diffusive equations. Secondly, we expect wave patterns to be formed while the homogeneous mode decays, therefore pattern formation will be a temporary phenomenon, at least in the model above in which gravity is neglected. The driving in $\lambda\phi^4$ preheating comes from the large initial value of the inflaton at the end of inflation, causing the field to roll and oscillate in its potential. This should be considered in contrast to the typical condensed matter system, in which energy is introduced via boundary conditions (in a convective system) or by a vibrating bed (in a granular material system), and the energy input is essentially constant.

For $\Gamma = 0$, the resonant modes lie in the interval

$$\frac{3}{2} < k^2 < \sqrt{3}. \tag{3}$$

For $\Gamma \neq 0$, neglecting expansion we can introduce $\psi = \varphi\, e^{\frac{\Gamma}{2}t}$ giving

$$\ddot{\psi} - (\nabla^2 + \frac{\Gamma^2}{4})\psi + e^{-\Gamma t}\psi^3 = 0. \tag{4}$$

Therefore, for small Γ, we expect the resonance bands to be slightly shifted, because the ∇^2 term is shifted by $\Gamma^2/4$, and we expect the resonance to diminish over time, due to the exponential damping of the potential with time. It should also be noted that the resonance structure of the equation can be quite sensitive to changes in the potential [20], although in our case the resonance structure remains intact.

As the effect that we are trying to isolate is non-linear, we resort to numerical simulation of the field equation.

III. INITIAL CONDITIONS

The inflaton is homogeneous after inflation except for sub-Hubble quantum fluctuations and super-Hubble classical fluctuations that *were* quantum fluctuations at the start of inflation. In the $\lambda\phi^4$-theory of preheating, it is only a small band of the sub-Hubble modes that can be amplified.

For early times we may expand ϕ about a homogeneous piece and then linearise Eq. (2). Let $\phi(t,\mathbf{x}) = \Phi(t) + \psi(t,\mathbf{x})$, where $\langle\psi\rangle_{spatial} \equiv (1/V)\int \psi d^3x = 0$. We obtain:

$$\ddot{\Phi} + \lambda\Phi^3 = 0 \tag{5}$$

$$\ddot{\psi}_{\mathbf{k}} + \left(k^2 + 3\lambda\Phi^2(t)\right)\psi_{\mathbf{k}} = 0 \tag{6}$$

where we have taken the Fourier transform in the latter equation.

The initial conditions for Φ are those of the end of slow-roll, which is normally supposed to be when $\ddot{a} = 0$. Let $\Phi(0) = \phi_0$ and $\dot{\Phi}(0) = \dot{\phi}_0$.

To find the initial conditions for the $\psi_{\mathbf{k}}$ we quantise ψ at time $t = 0$:

$$\hat{\psi} = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_k}}\left(\hat{a}^\dagger_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{7}$$

and

$$\hat{\dot{\psi}} = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} -i\sqrt{\frac{\omega_k}{2}}\left(\hat{a}^\dagger_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} - \hat{a}_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{8}$$

where $\omega_k^2 = k^2 + 3\lambda\phi_0^2$.

Then it is easy to see that $\langle\hat{\psi}\rangle_{quantal} \equiv \langle 0|\hat{\psi}|0\rangle = 0$ and $\langle\hat{\psi}^2\rangle_{quantal} = (1/V)\sum_{\mathbf{k}}(1/2\omega_k)$.

For initial conditions for the classical field ψ, we then take $\langle\ldots\rangle_q = \langle\ldots\rangle_s$. If we define the Fourier transform so that $\psi = (1/\sqrt{V})\sum_{\mathbf{k}}\psi_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}$, then $\langle\psi^2\rangle_s = (1/V)\sum_{\mathbf{k}}|\psi_{\mathbf{k}}|^2$. To obtain $\langle\hat{\psi}^2\rangle_q = \langle\psi^2\rangle_s$ we require that *on average* $|\psi_{\mathbf{k}}|^2 = (1/2\omega_k)$.

This may be achieved by choosing:

$$\psi_{\mathbf{k}} = \frac{1}{\sqrt{2\omega_k}}|n|e^{2i\pi r} \tag{9}$$

where n is a number randomly taken from a Gaussian distribution with mean = 0 and variance = 1, and r is a random number taken from the interval $[0,1]$. One must remember that $\psi_{-\mathbf{k}} = \psi^*_{\mathbf{k}}$.

Requiring that $\langle\hat{\dot{\psi}}^2\rangle_q = \langle\dot{\psi}^2\rangle_s$ we similarly obtain the momentum initial conditions:

$$\dot{\psi}_{\mathbf{k}} = -i\omega_k\psi_{\mathbf{k}} \tag{10}$$

with $\dot{\psi}_{-\mathbf{k}} = -\dot{\psi}^*_{\mathbf{k}}$.

IV. SIMULATION RESULTS IN TWO DIMENSIONS

To simulate the evolution of the field, we discretise the spatial derivatives to fourth-order in Δx, and we use a leapfrog integrator which is accurate to second order in Δt.

Setting the box size to 256 gridpoints per dimension and such that the resonant wave number in the box is 16 is enough to give many different resonant modes in the box, but still have good resolution of the wave, so this is the box size we used. We use periodic boundary conditions.

We set out to identify the weakly non-linear regime. In [13] it is shown that, without the decay terms, the self-coupled inflaton system's non-linear time evolution proceeds as follows: First, the resonant band amplitude grows. Next, when the amplitude in the resonant band is high enough for non-linear effects to become important, period doubling occurs and subsidiary peaks develop in the power spectrum. Further peaks then develop and the spectrum broadens and approaches an exponential spectrum. We tuned Γ such that the amplitude of the resonant band grew, but little period doubling occurred. In this regime, only resonant mode wavelengths exist in the box and they interact with each other non-linearly.

In the expanding case, the increase of the effective damping coefficient $\gamma_{eff} = a\Gamma$ with increasing scale factor a makes it easier to keep the system in the weakly nonlinear regime relative to the non-expanding case, since damping grows with time.

We found the smallest value for Γ such that the system remained weakly non-linear was of order 10^{-5}. This is two orders of magnitude smaller than that in the non-expanding case.

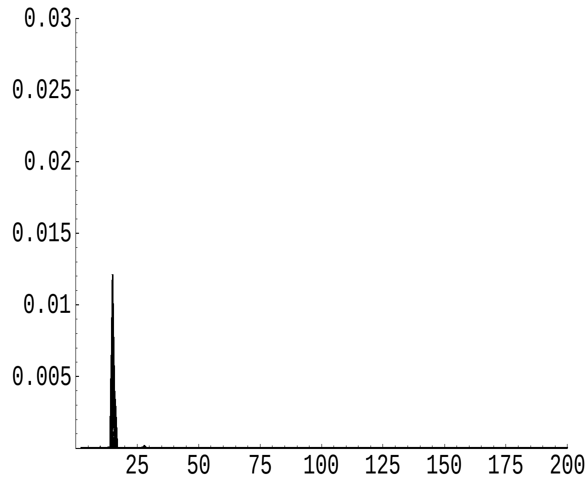

FIG. 1. A superposition of power spectra taken from a simulation with $\Gamma = 0.00005$. We plot the amplitude of the power spectra vs. wave number. Note that the period doubling modes are only weakly populated, indicating that the simulation is in the weakly non-linear regime.

In Figure 1 we plot a superposition of the power spectrum at various times during a simulation with $\Gamma = 0.00005$. It is possible to see that the system stays in the weakly non-linear regime for the entire simulation.

When wave patterns form, the specific pattern which

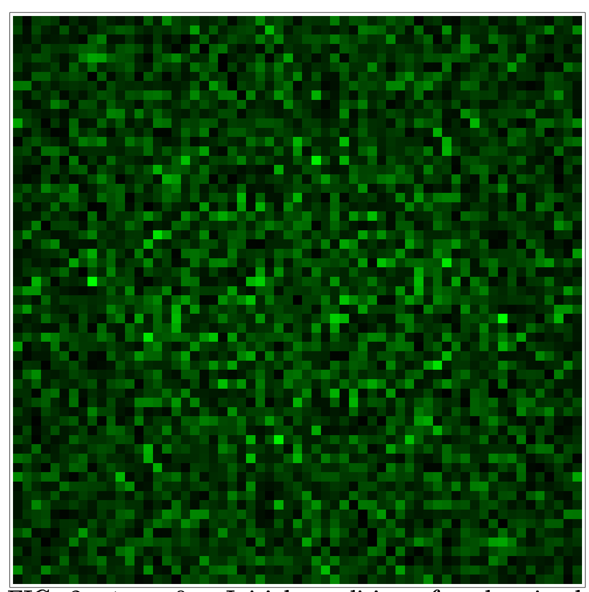

FIG. 2. $t = 0$. Initial conditions for the simulation ($\Gamma = 0.00005$). $\tilde{\phi}(k_x, k_y)$, the Fourier transform of the inflaton is plotted. The zero mode has been deleted for plotting purposes, and the surrounding modes populated with vacuum amplitudes from Eq. 9. Only the region of interest is plotted.

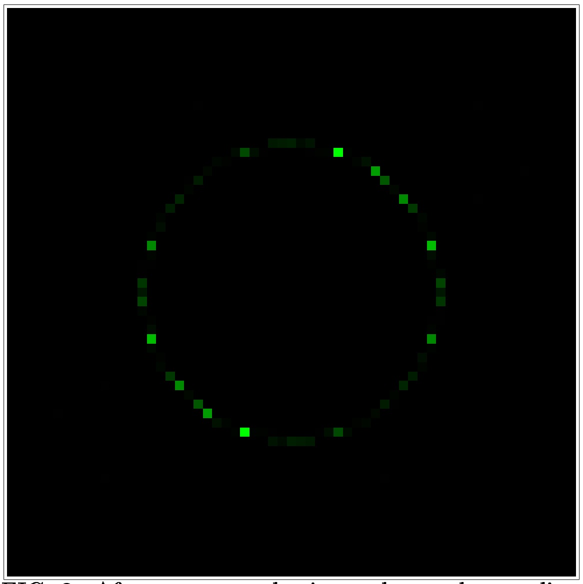

FIG. 3. After resonance begins to boost the amplitude of the resonant mode. Notice the brightening (increasing amplitude) of the resonant modes. $t = 840$.

arises is due to the non-linear interaction of the wave modes. The amplitude of wave modes separated by different angles grows at different rates. Modes separated by angles with the fastest growing amplitudes dominate the solution and form the wave pattern.

It can also occur that patterns vary temporally. This is the situation we find in the $\lambda\phi^4$ model. What we see in both the expanding and non-expanding cases is that a pattern emerges from the fluctuation background, then the peaks and valleys begin to move relative to each other.

The temporal dependence is almost periodic. Peaks and troughs in the field energy align along one direction

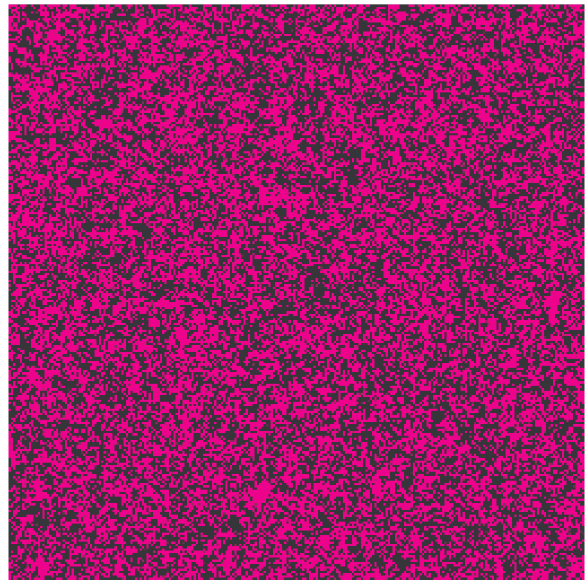

FIG. 5. $t = 0$. The initial conditions in configuration space.

in a ripple-like pattern, then the pattern flips to align in a direction orthogonal to the original direction. We say the dependence is 'almost' periodic because the field flips back and forth, but the timing of the flips varies as the field evolves. This leads us to believe that, for instance, if there were a background driving field that gave constant energy input (as opposed to the decaying background in chaotic inflation) the flipping would be truly periodic.

In Figures 2, 3 and 4 we present snapshots of the evolution of the Fourier transform of the inflaton in two dimensions in an expanding universe. And in plots 5, 6 and 7 we present snapshots of the evolution of $\phi(x, y)$ in configuration space in two dimensions in an expanding universe. Notice the change in direction between the

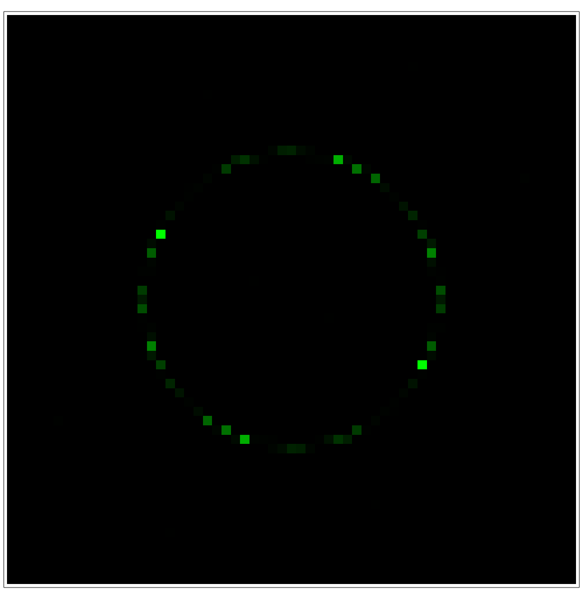

FIG. 4. The final wave pattern. $t = 1020$.

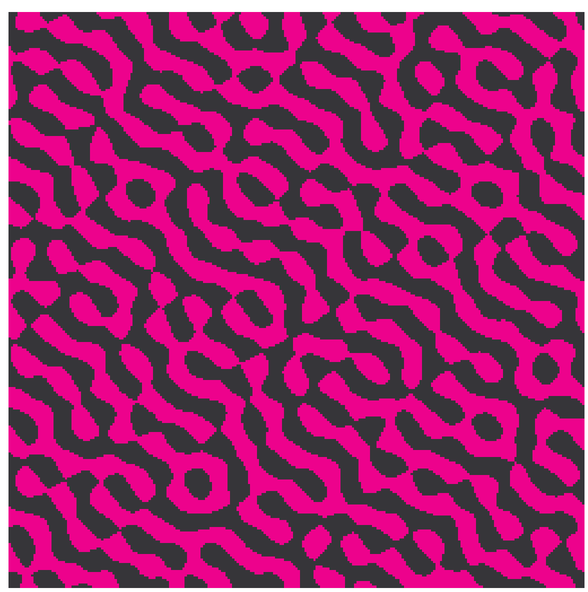

FIG. 6. Wave pattern at intermediate time. $t = 840$.

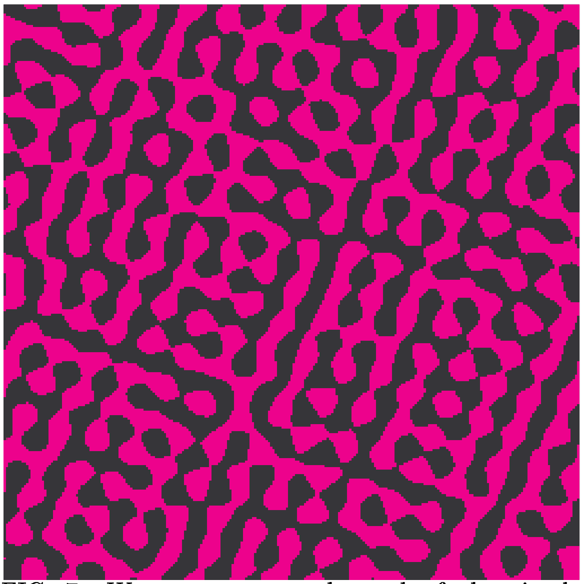

FIG. 7. Wave pattern at the end of the simulation. $t = 1020$.

ripples in Figure 6 and 7.

In the expanding case (w.r.t. the non-expanding case), the patterns are more clearly delineated and look less noisy. This is due to the fact that, in the expanding universe case, the field equations (in conformal time) have an effective symmetry breaking potential, with barrier height $(a'\gamma)^2/4\lambda$, which is constant in time in a radiation dominated universe (the case we are considering). Therefore, the pattern is actually a network of domain walls separated by the characteristic wavelength of the resonance.

V. PATTERNS IN THREE DIMENSIONS

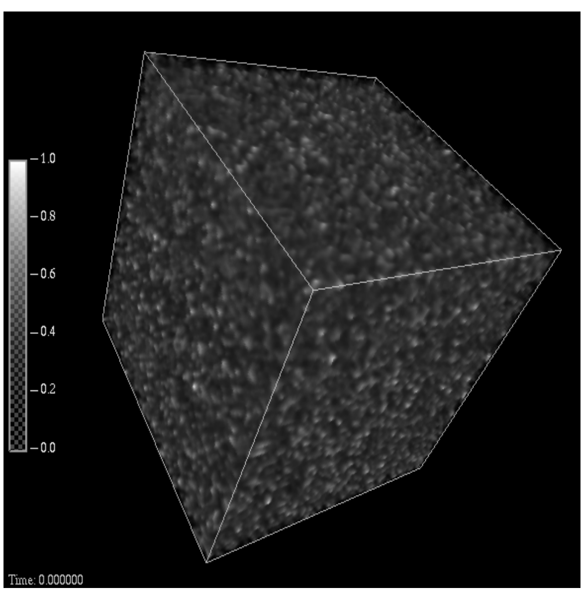

FIG. 8. Initial conditions for 3-D simulation. $t = 0$.

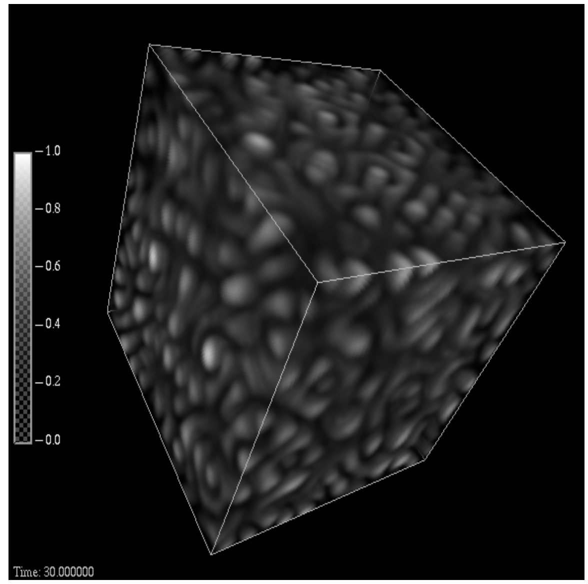

FIG. 9. Wave pattern at intermediate time. $t = 100$.

Using the same simulation techniques in three dimensions, we also find spatio-temporal patterns. We plot the field in configuration space in Figures 8, 9 and 10. The behavior of these patterns is similar to those found in two dimensions: the pattern forms at the resonant wavelength, then the peaks and valleys in the energy density begin to move with respect to each other. In these simulations, we set the physical box size such that there were eight resonant wavelengths per dimension in the box.

VI. DISCUSSION

Since the patterns that we see vary in time, we expect

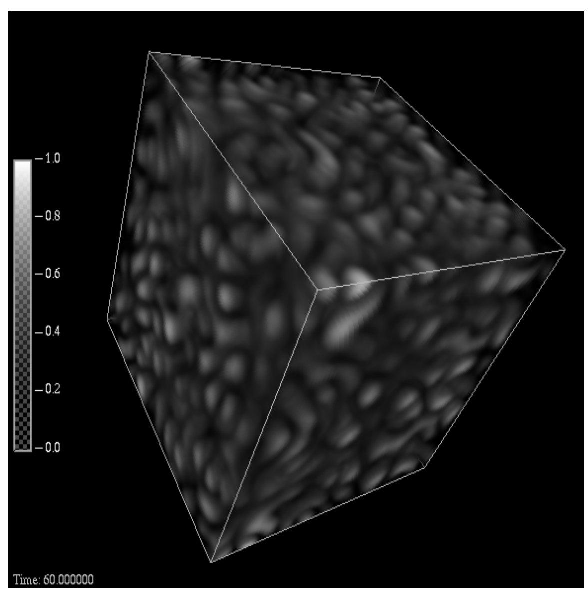

FIG. 10. Wave pattern at end of simulation. $t = 300$.

there to be a peak in the gravitational wave spectrum at the resonant frequency for preheating in the weakly non-linear regime. Similar peaks in the gravitational wave spectrum have been seen [22] in simulations of undamped preheating. These peaks correspond to the resonant and period doubled frequencies. In the pattern forming regime, we expect to see similar peaks of lower amplitude, since the dissipative terms keep the resonant peak in our simulation at lower amplitude than the undamped system. Since the peak in the gravitational wave spectrum from pattern formation will be of smaller amplitude than that of the fully chaotic system, it will be very difficult to detect. However, since the pattern has directionality, this might help in extracting a signal.

We would also like to discuss the possibility of patterns occurring on cosmological scales, since, if we could put resonant power at $128h^{-1}Mpc$, we would have an explanation for the observed excess power found at these scales. Recently, Bassett [23] and a number of other authors [24], [25], [26] have investigated the question of whether resonance can amplify modes with wavelength larger than the Hubble radius at the end of inflation. It turns out that this is possible due to the large-scale (many Hubble volumes) coherence of the inflaton at the end of inflation. Due to the large scale coherence, super-Hubble modes can be amplified due to what can be thought of as down-scattering from the oscillating zero mode. It turns out, though [26], that in practice it is difficult to find models in which the amplification changes the spectrum significantly. This is due to the fact that during inflation, long wave $m^2 \ll H^2$ fluctuations $\delta\chi$ in fields that have effective masses are damped exponentially.

$$\delta\chi = \left[\frac{H^4}{8\pi^2 m^2}\left(1 - \exp\left(-\frac{2m^2}{3H}t\right)\right)\right]^{\frac{1}{2}} \tag{11}$$

Therefore, during preheating, although super-Hubble modes are amplified, they cannot be amplified enough to be significant.

Let us suppose, however, that it is possible to construct a model in which the field χ which develops the resonance (the field coupled to the inflaton) at the end of inflation is massless during the inflationary epoch, and therefore has fluctuations of order the fluctuations in the inflaton $\sim 10^{-6}$. Thus, super-Hubble fluctuations in χ will not be damped by an exponential factor. Given this argument, it may seem possible to introduce fluctuations on large scales. The difficulty comes when one tries to introduce a bump at a particular scale. Essentially, for small k, resonance bands always vary slowly, since they are just small perturbations of the resonant amplitude of the zero mode, and it thus becomes difficult to put a bump at a particular scale without also putting power at all scales nearby, therefore, at best, a bump in the spectrum would be very broad.

A possible workaround for this problem would be to construct a theory where a resonance occurs during the inflationary epoch. The scale of the perturbation would then grow with the rest of the inflationary perturbations.

As a final comment, we would like to point out that the system of equations which we have investigated is a low-energy pion model. This suggests that patterns might turn up in driven pion systems.

VII. CONCLUSIONS

In this paper, we have presented new evidence that there is a pattern forming regime in a $\lambda\phi^4$ theory during preheating at the end of inflation. We have used vacuum initial conditions appropriate to sub-Hubble modes at the end of inflation. We have also shown that patterns arise in both two- and three-dimensions. Since the patterns vary spatio-temporally, gravitational waves will be produced, however, relative to the gravitational waves produced in an undamped model, their amplitude will be small, and therefore, unless the directionality of the pattern can aid in detection, extremely difficult to detect.

We have speculated on the possibility of putting a resonant band at cosmological scales, and conclude that, due to the broadening of the band at small k, this is not feasible.

But, we point out that another physical system which could have similar pattern forming behavior is the pion system.

Acknowledgements

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