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***B* decays in the *upsilon* expansion**

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Abstract

Theoretical predictions for B decay rates are rewritten in terms of the $\Upsilon(1S)$ meson mass instead of the b quark mass, using a modified perturbation expansion. The theoretical consistency of this expansion is shown both at low and high orders. Our method improves the behavior of the perturbation series for semileptonic and nonleptonic inclusive decay modes, as well as for exclusive decay form factors. The results are applied to the determination of the semileptonic B branching ratio, charm counting, the ratio of $B \rightarrow X\tau\bar{\nu}$ and $B \rightarrow Xe\bar{\nu}$ decay rates, and form factor ratios in $B \rightarrow D^*e\bar{\nu}$ decay. We also comment on why it is not possible to separate perturbative and nonperturbative effects in QCD.

I. INTRODUCTION

Heavy hadron decays can be computed using a systematic expansion in powers of $\alpha_s(m_Q)$ and Λ_{QCD}/m_Q , where m_Q is the mass of the heavy quark and Λ_{QCD} is the nonperturbative scale parameter of the strong interactions. In the $m_Q \rightarrow \infty$ limit, inclusive decay rates are given by free quark decay and the order Λ_{QCD}/m_Q corrections vanish [1,2]. The leading nonperturbative corrections of order $\Lambda_{\text{QCD}}^2/m_Q^2$ are parameterized by two hadronic matrix elements [2–6] λ_1 and λ_2 . These results are now used to determine the CKM matrix elements $|V_{cb}|$ and $|V_{ub}|$, using experimental data on inclusive semileptonic B meson decays. For exclusive decays at lowest order in $1/m_Q$, one can compute form factors in terms of a perturbative series in α_s times the Isgur-Wise function [7]. Nonperturbative corrections are parameterized by hadronic matrix elements, and are of order $(\Lambda_{\text{QCD}}/m_Q)^n$. In some cases at zero recoil, the order $1/m_Q$ correction vanishes [8].

At present, the largest theoretical uncertainties in the computation of inclusive $B \rightarrow X_c e \bar{\nu}$ and $B \rightarrow X_u e \bar{\nu}$ decay rates arise from poor knowledge of quark masses. The pole mass of a heavy quark is an infrared (IR) sensitive quantity, and is not well defined beyond perturbation theory [9,10]. This is related to the bad behavior of perturbative corrections to the inclusive decay rate when it is written in terms of the pole mass [11,12]. A simple method of avoiding problems with the quark mass is to use instead the B mesons mass. Unfortunately, the B meson and b quark masses differ by order Λ_{QCD} , and so this reintroduces a Λ_{QCD}/m_b correction to the inclusive decay rate. The decay rate has also been rewritten in terms of the infrared safe $\overline{\text{MS}}$ mass of the b quark, but the uncertainties still remain sizable because the value of the $\overline{\text{MS}}$ mass is not known very accurately, and because the first few terms in the perturbation series still remain large.

In this paper, the theoretical predictions for semileptonic B decays are rewritten in terms of the $\Upsilon(1S)$ meson mass rather than the b quark mass. At the same time a modified perturbation expansion (referred to as the *upsilon expansion*) must be used, which is discussed in detail in Sec. II. This procedure eliminates the uncertainty due to the m_b^5 factor in the decay rates, and at the same time improves the behavior of the perturbation series. Our formulae relate measurable quantities to one another and the resulting perturbation series is free of renormalon ambiguities. We will also see numerically that the perturbative corrections are small when the B decay rate is written in terms of the Υ mass. We have discussed the procedure in an earlier publication [13], where it was applied to inclusive semileptonic B decays. In this article, we describe the method in more detail and also explain why it leads to an improvement in the perturbation series expansion. The method is applied to inclusive nonleptonic decays as well as to exclusive decays. The perturbation series is better behaved for all the processes we have examined. The semileptonic and nonleptonic decay computations can be combined to obtain new values for the semileptonic branching ratio and the average charm multiplicity in B meson decay.

The outline of this paper is as follows. In Section II we define the *upsilon expansion*. In Sec. III we show that it is the theoretically consistent way of eliminating the b quark mass from the combination of the perturbation theory results for B decay rates and the Υ mass. We also comment on the role of nonperturbative effects in the theoretical expression for the Υ mass. Exclusive decays are discussed in IV. In Sec. V we give the results for inclusive semileptonic and nonleptonic decay rates in the *upsilon expansion*, the ratio of decay rates

$R_\tau = \mathcal{B}(B \rightarrow X_c \tau \bar{\nu}) / \mathcal{B}(B \rightarrow X_c e \bar{\nu})$, the semileptonic B branching ratio, and charm counting in B decay. Sec. VI contains our conclusions and comments on further applications of our results.

II. THE UPSILON EXPANSION¹

The results for the OPE calculation of inclusive B decay rates can be schematically written as

$$\Gamma(B \rightarrow X_{ijk}) = \frac{G_F^2 |V_{CKM}|^2}{192\pi^3} m_b^5 (PS) \left[1 + \frac{\alpha_s}{\pi} \epsilon + \frac{\alpha_s^2}{\pi^2} (\beta_0 + 1) \epsilon^2 + \frac{\alpha_s^3}{\pi^3} (\beta_0^2 + \dots) \epsilon^3 + \dots \right. \\ \left. + \frac{1}{m_b^2} (\lambda_1 + \lambda_2) + \dots \right], \quad (1)$$

where the precise coefficients are not shown. ijk denotes an arbitrary final state created by the $b \rightarrow ijk$ transition at short distance (e.g., $ijk = c\ell\bar{\nu}$, $c\bar{c}s$, etc.), V_{CKM} contains CKM matrix elements, PS is the b quark decay phase space at tree level including color factors (e.g., $ijk = u\bar{u}d$ is a factor 3 enhanced compared to $ijk = ue\bar{\nu}$), m_b is the b quark pole mass, $\beta_0 = 11 - 2n_f/3$ is the first coefficient of the QCD β -function, and α_s is the running coupling constant in the \overline{MS} scheme at the scale μ . It is convenient to expand the coefficient of a given α_s order in powers of β_0 which is related to the Brodsky–Lepage–Mackenzie (BLM) prescription [14].² For nonleptonic and rare B decays there is another large parameter in Eq. (1), $\ln(m_W/m_b) \sim 2.8$, which will be discussed in Sec. IV. These large logarithms can be summed using the renormalization group equations. The reason for introducing the variable $\epsilon = 1$ is explained in the next paragraph.

Schematically, the perturbative expansion of the Υ mass in terms of m_b is

$$\frac{m_\Upsilon}{2m_b} \sim 1 - \frac{(\alpha_s C_F)^2}{8} \left\{ 1\epsilon + \frac{\alpha_s}{\pi} [(\ell + 1)\beta_0 + 1] \epsilon^2 \right. \\ \left. + \frac{\alpha_s^2}{\pi^2} [(\ell^2 + \ell + 1)\beta_0^2 + \dots] \epsilon^3 + \dots \right\}, \quad (2)$$

where $\ell = \ln[\mu/(m_b \alpha_s C_F)]$, $C_F = 4/3$, and the precise coefficients are again not shown. Note that this series is of the form $\{\alpha_s^2, \alpha_s^3 \beta_0, \alpha_s^4 \beta_0^2, \dots\}$, whereas the corrections in Eq. (1) are of order $\{\alpha_s, \alpha_s^2 \beta_0, \alpha_s^3 \beta_0^2, \dots\}$. We will show in Sec. III that to combine these two equations

¹ B decays written in terms of the Υ mass should be referred to as *upsilon-expanded*, even though the expansion parameter will be denoted by ϵ . Blame this on Donald Knuth (or on the Greeks?), as v ($\text{\$}\epsilon\text{\$}$) is similar to v ($\text{\$}v\text{\$}$) in \TeX .

²The subscript BLM will be used to denote terms of order $\epsilon^n \beta_0^{n-1}$ in the perturbation expansion. In some cases, the entire ϵ^2 contribution has not been computed, and only the BLM piece is known. In cases where the entire ϵ^2 term is known, the BLM contribution is about ten times larger than the non-BLM part.

in a theoretically consistent manner, terms of order α_s^n in Eq. (2) should be viewed as if they were only of order α_s^{n-1} . For this reason, the power of ϵ in Eq. (2) is one less than the power of α_s . It is also convenient to choose the same renormalization scale, μ , when Eqs. (1) and (2) are combined. The prescription of counting $[\alpha_s(m_b)]^n$ in the B decay rate as order ϵ^n , and $[\alpha_s(m_b)]^n$ in the Υ mass as order ϵ^{n-1} will be called the epsilon expansion. It is expected that the infrared sensitivity present separately in Eqs. (1) and (2) will cancel to all orders in perturbation theory in ϵ . Some arguments that support this will be given in the next section. In the epsilon expansion Eq. (1) takes the form

$$\Gamma(B \rightarrow X_{ijk}) = \frac{G_F^2 |V_{CKM}|^2}{192\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 (PS') \left[1 + \epsilon + (\beta_0 + 1)\epsilon^2 + \dots\right], \quad (3)$$

where PS' means that the b decay phase space is also expanded in ϵ . We emphasize that the coefficients in the series in ϵ contain different orders in α_s . We show in Sec. III that this is theoretically consistent. Our results in Sec. IV and V demonstrate that this expansion simultaneously eliminates the uncertainty in the theoretical predictions related to the dependence on the poorly known quark masses and also improves the behavior of the perturbation series.

The precise form of Eq. (1) depends on the decay channel under consideration. The expression for the Υ mass in terms of m_b is given by [15,16],

$$\begin{aligned} \frac{m_\Upsilon}{2m_b} = 1 - \frac{(\alpha_s C_F)^2}{8} \left\{ 1\epsilon + \frac{\alpha_s}{\pi} \left[\left(\ell + \frac{11}{6} \right) \beta_0 - 4 \right] \epsilon^2 \right. \\ \left. + \left(\frac{\alpha_s \beta_0}{2\pi} \right)^2 \left(3\ell^2 + 9\ell + 2\zeta(3) + \frac{\pi^2}{6} + \frac{77}{12} \right) \epsilon^3 + \dots \right\}. \end{aligned} \quad (4)$$

The ellipsis denote terms of order α_s^5 , and non-BLM terms of order α_s^4 . For μ of order m_b , Eq. (4) shows no sign of convergence; for $\mu = m_b$ it yields $m_\Upsilon = 2m_b(1 - 0.011\epsilon - 0.016\epsilon^2 - 0.024_{\text{BLM}}\epsilon^3 - \dots)$, using $\alpha_s(m_b) = 0.22$. (This value of $\alpha_s(m_b)$ will be used whenever we present numerical results.) The bad behavior of this series is unimportant (it would be somewhat better in terms of the $\overline{\text{MS}}$ mass at a low scale). The only physical question is what happens when we use Eq. (4) to predict B decay rates in terms of m_Υ .

Finally, decays with a charm quark in the final state depend both on m_b and m_c . It is convenient to express these decay rates in terms of m_Υ and λ_1 instead of m_b and m_c , using Eq. (4) and

$$m_b - m_c = \overline{m}_B - \overline{m}_D + \left(\frac{\lambda_1}{2\overline{m}_B} - \frac{\lambda_1}{2\overline{m}_D} \right) + \dots, \quad (5)$$

where $\overline{m}_B = (3m_{B^*} + m_B)/4 = 5.313 \text{ GeV}$ and $\overline{m}_D = (3m_{D^*} + m_D)/4 = 1.973 \text{ GeV}$. One could imagine replacing m_c by the J/ψ mass using a procedure similar to that for the b quark. However, the nonperturbative errors in this case are not under control.

III. THEORETICAL CONSISTENCY

In Ref. [13] we showed that our method of combining the expansion of inclusive B decay rates with the expansion of the Υ mass to eliminate the dependence on the b quark mass

is consistent at large orders in perturbation theory in the BLM approximation. In this section we briefly repeat this argument (Sec. A) and show that the epsilon expansion is also consistent at first and second order in ϵ , including non-Abelian contributions (Sec. B). Our arguments prove that, as far as the cancellation of infrared sensitive contributions in the perturbative expansion of the B decay is concerned, the epsilon expansion is equivalent to the use of a short-distance mass like the $\overline{\text{MS}}$ mass. They also explain why the epsilon expansion requires powers of α_s to be counted differently in m_Υ and B decay rates. We also comment on the relation between perturbative and nonperturbative effects in the Υ mass (Sec. C).

A. Renormalons in the large β_0 approximation

The pole mass m_b is an infrared sensitive quantity, which is not well defined beyond perturbation theory. It can be related to an infrared safe mass, such as the $\overline{\text{MS}}$ mass \overline{m}_b via (for $n_f = 4$) [17]

$$\frac{m_b}{\overline{m}_b(m_b)} = 1 + \frac{4\alpha_s}{3\pi} \epsilon + (1.56\beta_0 - 1.07) \frac{\alpha_s^2}{\pi^2} \epsilon^2 + \dots \quad (6)$$

This relation has terms of the form $\alpha_s^n \beta_0^{n-1} n!$ at high orders, leading to a renormalon ambiguity. The Borel transform of the perturbation series relating the pole mass to the $\overline{\text{MS}}$ mass is [9,10]

$$\widetilde{m}_b(u) = \overline{m}_b \left\{ \delta(u) + \frac{C_F}{\beta_0} \left[6e^{-Cu} \left(\frac{\mu}{m_b} \right)^{2u} (1-u) \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} - \frac{3}{u} + \dots \right] \right\}, \quad (7)$$

where C is a scheme dependent constant ($C = -5/3$ in $\overline{\text{MS}}$). The leading renormalon ambiguity is given by the $u = 1/2$ pole, yielding

$$\Delta m_b = -\frac{2C_F}{\beta_0} e^{-C/2} \Lambda_{\text{QCD}}. \quad (8)$$

The α_s perturbation series for the inclusive decay rate written in terms of the pole mass also has a renormalon singularity. As a result, the inclusive decay perturbation series is badly behaved at high orders. Renormalons cancel when the inclusive decay rate is written in terms of the $\overline{\text{MS}}$ mass [18–21], rather than the pole mass. While this cancellation is present at high orders, the perturbation series written in terms of the $\overline{\text{MS}}$ mass at a renormalization scale μ of order m_b still contains large corrections at low orders (see, Ref. [11,12] and some examples in Sec. V). On the other hand, the perturbation series written in terms of the $\overline{\text{MS}}$ mass at a low scale has a better behavior. But such a choice still does not remove the quark mass uncertainty in the decay rates.

The expansion of the Υ mass also becomes better behaved when written in terms of the $\overline{\text{MS}}$ mass at a low scale (of order the Bohr radius, $m\alpha_s$), rather than in terms of the pole mass. This fact is important for extracting the short distance masses of the bottom and top quarks from the Υ system and from the location of the peak of the $1S$ $t\bar{t}$ resonance in the total cross section for $t\bar{t}$ production in e^+e^- annihilation close to threshold. One might think that

the numerical results obtained in the upilon expansion are equivalent to the result obtained by extracting a short-distance b quark mass from the Υ mass and then using it to predict B decay rates. Due to the fact that only the first few terms in the perturbative expansions are known, this is not the case because in this procedure the information on the correlations between the short-distance mass, the strong coupling constant and the renormalization scale is lost. The upilon expansion, which completely eliminates the intermediate step of using any quark mass takes full advantage of these correlations and leads to smaller uncertainties.

The potential between two static quarks in a color singlet state, $V(r) = -C_F g_s^2/r$, also has renormalon ambiguities. The Borel transform is given by [22]

$$\tilde{V}(r, u) = -\frac{4C_F e^{-Cu}}{\beta_0} \frac{1}{r} (\mu r)^{2u} \frac{\Gamma(1/2 + u) \Gamma(1/2 - u)}{\Gamma(2u + 1)}. \quad (9)$$

The leading renormalon ambiguity of V also comes from the $u = 1/2$ pole,

$$\Delta V = \frac{4C_F}{\beta_0} e^{-C/2} \Lambda_{\text{QCD}}. \quad (10)$$

It follows from Eqs. (7) and (9) that the order Λ_{QCD} ambiguity cancels in $2m_b + V$ [23,24].

Equation (9) also explains the mismatch between the power of α_s and ϵ in Eq. (2), which may seem somewhat unnatural at first sight. The coefficient of $\alpha_s^{n+2} \beta_0^n$ in Eq. (2) is given by the Υ matrix element of the coefficient of u^n in the expansion of $\tilde{V}(u)$ in Eq. (9) about $u = 0$, multiplied by $n!$. For large enough n , up to corrections suppressed by at least one power of n , the terms in this expansion of $\tilde{V}(r, u)$ are proportional to

$$\frac{(2u)^n}{r} \sum_{p=0}^n \frac{[\ln(\mu r)]^p}{p!} \simeq \mu (2u)^n. \quad (11)$$

This explains why in the BLM approximation and in higher orders in ϵ , the terms in Eq. (2) are generically of the form³

$$n! \alpha_s^{n+2} \beta_0^n \left(\frac{\ell^n}{n!} + \frac{\ell^{n-1}}{(n-1)!} + \dots + 1 \right), \quad (12)$$

since the Υ matrix element of $\ln(\mu r) \sim \ell$. For large n ,

$$\left(\frac{\ell^n}{n!} + \frac{\ell^{n-1}}{(n-1)!} + \dots + 1 \right) \rightarrow \exp(\ell) = \frac{\mu}{m_b \alpha_s C_F}, \quad (13)$$

reducing the difference between the powers of α_s and β_0 to one, thus eliminating the mismatch in the large order behaviors of Eqs. (1) and (2). This has to happen since m_Υ is a

³The actual coefficients of the logarithms in Eq. (2) differ from those displayed in Eq. (12) because of multiple insertions of the corrections to the pure Coulomb potential, $-C_F \alpha_s/r$. The contributions arising from multiple insertions, however, only constitute corrections suppressed by at least one power of n .

physical quantity, so the renormalon ambiguities must cancel in Eq. (2) between $2m_b$ and the potential plus kinetic energies [23,24].

If NRQCD exists as a well defined low energy effective field theory, then the $\Lambda_{\text{QCD}}^3 r^2$ renormalon in the potential corresponding to the $u = 3/2$ pole in Eq. (9) must cancel against the leading renormalon in the kinetic energy contribution to the Υ mass. While this has not been shown explicitly, it is quite conceivable that this happens. Studying higher order renormalons is less interesting as there are certainly nonperturbative contributions of order $\Lambda_{\text{QCD}}^4 / (m\alpha_s)^3$ to m_Υ (e.g., from the vacuum expectation value of $G_{\mu\nu}^2$ [25–27]), and so higher order renormalon effects are of the same order as other nonperturbative effects which are known to exist.

B. Infrared sensitivity at low orders in ϵ

One can study the infrared sensitivity of Feynman diagrams by introducing a fictitious infrared cutoff λ of order Λ_{QCD} . The infrared sensitive terms are nonanalytic in λ^2 , such as $(\lambda^2)^{n/2}$ or $\lambda^{2n} \ln \lambda^2$. These terms arise from the low-momentum part of Feynman diagrams. This is similar to what occurs in chiral perturbation theory, where the nonanalytic terms in the quark mass (or equivalently, the pion mass-squared) come from the low-momentum behavior of Feynman diagrams. Diagrams which are more infrared sensitive, i.e., have contributions $(\lambda^2)^{n/2}$ or $\lambda^{2n} \ln \lambda^2$ for small values of n , should have larger nonperturbative contributions. In particular, linear infrared sensitivity, i.e., terms of order λ , are a signal of Λ_{QCD} effects, quadratic sensitivity, i.e., terms of order $\lambda^2 \ln \lambda^2$ are a signal of Λ_{QCD}^2 effects, etc.

In the following we show that the linear infrared sensitivity present separately in the expansion of the B decay rate and in the expansion of the Υ mass in terms of the b quark mass cancel at least to order ϵ^2 when they are combined using the epsilon expansion. (At order ϵ this is equivalent to the cancellation of the leading renormalon ambiguity in the BLM approximation, discussed previously.) The argument is essentially a combination of those in recent papers by Beneke [24], and by Sinkovics, Akhoury, and Zakharov [21]. As far as the cancellation of infrared sensitive contributions in the perturbative expansion of B decay is concerned, the epsilon expansion is therefore equivalent to a scheme where a short-distance mass like the $\overline{\text{MS}}$ mass is used.

The proof proceeds in two steps. First one notes that the linear IR sensitivity cancels in inclusive decay widths of a heavy quark when it is expressed in terms of an infrared safe short distance mass (such as the $\overline{\text{MS}}$ mass). To leading order in α_s this was shown in Refs. [18–20,9]. To order α_s^2 , where non-Abelian contributions first occur, the cancellation of the linear IR sensitivity was shown only recently in Ref. [21]. This cancellation of the linear IR sensitivity is expected to hold at higher orders in α_s as well, but the demonstration of this appears highly non-trivial. The second step is to show the cancellation of the linear IR sensitivity when the Υ mass is expressed in terms of an infrared safe short distance mass. This was done by Beneke [24], and we will repeat some of his arguments below. The reason is to explain why it is imperative to assign one less power of ϵ to each term in Eq. (4) than the power of α_s even at low orders in ϵ .

Let us first consider the leading contribution to the heavy quark self energy, shown in



FIG. 1. One-loop correction to the heavy quark self energy, and the tree-level contribution to the $Q\bar{Q}$ potential.

Fig. 1.⁴ Since we are only interested in the infrared behavior, we may calculate the self energy diagram in terms of the bare mass, m_0 , which differs from the $\overline{\text{MS}}$ mass only by ultraviolet subtractions. The self energy is given by

$$-i\Sigma = -C_F g_s^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\mu (\not{p} + \not{q} + m_0) \gamma^\mu}{[(p+q)^2 - m_0^2 + i\varepsilon](q^2 + i\varepsilon)}. \quad (14)$$

To obtain the leading infrared contribution to the mass shift coming from the region of small loop momentum, we can set $p = m_0 v$, where v is the four-velocity of the heavy quark ($v^2 = 1$). This gives

$$\delta m = -iC_F g_s^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(v \cdot q + i\varepsilon)(q^2 + i\varepsilon)}. \quad (15)$$

The integration over q^0 is simplest by closing the contour in the upper half plane, yielding

$$\delta m = \frac{1}{2} C_F g_s^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\mathbf{q}^2}. \quad (16)$$

Of course, this formula is only valid in the infrared. Restricting the range of integration to $|\mathbf{q}| < \lambda$, where λ is of order Λ_{QCD} , it gives $\delta m \sim \alpha_s \lambda$.

Let us next consider the heavy quark potential. It was shown in Ref. [24] that the potential in momentum space is free of linear IR sensitivity. The linear IR sensitivity of the contribution of the potential to m_Υ arises in a momentum space calculation from taking its matrix element. This occurs because the momentum space wave function $\sim 1/(1+a^2q^2)^2$ does not suppress the low momentum contribution of $V(q)$. In coordinate space, the wave function $\sim \exp(-r/a)$ is exponentially small at long distances, and the IR sensitivity of the contribution of the potential to m_Υ arises from Fourier transforming $V(q)$ into coordinate space to obtain $V(r)$. Therefore, the linear infrared sensitivity of the potential energy can be isolated by restricting the range of integration in the Fourier transform [24]

$$\delta V = \int_{|\mathbf{q}| < \lambda} \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{q}). \quad (17)$$

Choosing λ of order Λ_{QCD} , much smaller than $1/a$, the $e^{i\mathbf{q}\cdot\mathbf{r}}$ factor can be replaced by 1, and the IR sensitive term is a constant times $\alpha_s \lambda$.⁵ Since this contribution to the potential is independent of r , its Υ matrix element is simply itself,

⁴The argument in this and the next paragraph are directly from Beneke [24].

⁵The assumption $\Lambda_{\text{QCD}} \ll 1/a$ is essential for the argument to hold. For the $\Upsilon(1S)$ ground state this assumption is valid.

$$\langle \Upsilon | \alpha_s \lambda | \Upsilon \rangle \sim \alpha_s \lambda. \quad (18)$$

From Eq. (18) it is evident that the infrared sensitive contribution in the Υ matrix element of the heavy quark potential is of one lower order in α_s than the Υ matrix element of the heavy quark potential itself in the limit $\lambda \rightarrow 0$,

$$\langle \Upsilon | \frac{\alpha_s}{r} | \Upsilon \rangle \sim m_b \alpha_s^2. \quad (19)$$

This mismatch arises because the Υ matrix element of $1/r$ is of order the inverse Bohr radius $1/a \sim m_b \alpha_s$. Consequently, it is the infrared sensitive contribution in the order α_s^2 term in the perturbative series for m_Υ in Eq. (4) which cancels the infrared sensitive contribution of the order α_s term in the perturbative series for inclusive B decay rates in Eq. (1) (or equivalently for the b quark pole mass). It was shown in Ref. [24] that this cancellation of the linear IR sensitivity occurs similarly to all orders in α_s .⁶

Thus it is shown that using the epsilon expansion for B decay rates eliminates all linear infrared sensitivity at least to order ϵ^2 . This is the order to which most B decay rates have been computed, and the order to which we will present results in Secs. IV and V. While we cannot prove that this cancellation is present at all orders in perturbation theory, this is likely in view of the fact that it holds both at large orders in the bubble summation approximation and in the first two orders including non-Abelian contributions. Consequently, the epsilon expansion is insensitive to any order Λ_{QCD} ambiguity in the b quark mass which is present in perturbation theory.

C. Nonperturbative corrections to the Υ mass

An important theoretical uncertainty in applying the epsilon expansion is the size of nonperturbative corrections to the Υ mass in Eq. (4). The dynamics of the Υ system can be described using NRQCD [32]. The leading nonperturbative corrections to m_Υ arise from matrix elements in the Υ of H_{light} , the Hamiltonian of the light degrees of freedom. In B mesons, the leading nonperturbative correction to the B meson mass is due to the matrix element of H_{light} , which is the $\bar{\Lambda}$ parameter of order Λ_{QCD} . The Λ_{QCD} dependence is different for the Υ . H_{light} is the integral of a local Hamiltonian density,

$$H_{\text{light}} = \int d^3x \mathcal{H}_{\text{light}}(x). \quad (20)$$

The radius of the Υ is $a \sim 1/(m_b \alpha_s)$, so the matrix element of H_{light} is of order $a^3 \Lambda_{\text{QCD}}^4$, by dimensional analysis. (Note that the matrix element of H_{light} is of order Λ_{QCD}^4 , not m_b^4 . Terms that grow with m_b can be treated using NRQCD perturbation theory.) Using $1/a \sim 1 \text{ GeV}$,

⁶For Beneke's all order argument to hold one has to assume that the heavy quark potential does not contain any infrared divergences (or equivalently infrared sensitivity to scales below the inverse Bohr radius) to all orders in α_s . It was argued in Ref. [28] that this assumption may not hold beyond order α_s^3 . To order α_s^3 , i.e. to order ϵ^2 , this assumption is proven by explicit calculations [29–31].

and $\Lambda_{\text{QCD}} \sim 350 \text{ MeV}$, of order a constituent quark mass, gives a nonperturbative correction of 15 MeV. Using instead $\Lambda_{\text{QCD}} \sim 500 \text{ MeV}$ gives a correction of 60 MeV. We will use 100 MeV as a conservative estimate of the nonperturbative contribution to m_Υ .

For large enough quark mass, the dominant nonperturbative corrections to the quarkonia masses are believed to come from the gluon condensate, $\langle \alpha_s G_{\mu\nu}^2 \rangle$. Its contribution to the mass of the n^{th} S -wave state as calculated in Refs. [26,27] is

$$\delta m^{(n)} = m_b \frac{n^6 e_n \pi \langle \alpha_s G_{\mu\nu}^2 \rangle}{(m_b C_F \tilde{\alpha}_s)^4}, \quad (21)$$

where α_s in the denominator should be evaluated at the scale of the order of the Bohr radius of the n^{th} state, $\tilde{\alpha}_s \sim \alpha_s(a^{-1}/n^2)$, and $e_1 = 1.41$, $e_2 = 1.59$, etc., are numerical constants. Values of the gluon condensate in the literature vary in the range $\langle \alpha_s G_{\mu\nu}^2 \rangle \sim (0.05 \pm 0.03) \text{ GeV}^4$, and so the uncertainties in Eq. (21) are sizable. In Ref. [15], Eq. (21) is estimated to give a 60 MeV contribution to the $\Upsilon(1S)$ mass. Equation (21) seems to be enhanced by $1/\alpha_s$ compared to our naive estimate of $a^3 \Lambda_{\text{QCD}}^4$ above. This is due to the fact that $\langle \alpha_s G_{\mu\nu}^2 \rangle$ and not $\langle G_{\mu\nu}^2 \rangle$ is renormalization group invariant. In our estimate the operator $\mathcal{H}_{\text{light}}$ is renormalized at the scale $\mu = a^{-1}$. The renormalization group scaling from $\mu = a^{-1}$ to a low scale $\mu \sim 1 \text{ GeV}$ gives an additional factor $\alpha_s(1 \text{ GeV})/\alpha_s(a^{-1})$, which is parameterically of order $1/\alpha_s(a^{-1})$. Since $a^{-1} \sim 1 \text{ GeV}$, we have not included this factor in our estimate.

Our equations depend on the Υ mass, and make no reference to the wave function of the Υ . In fact, there are much larger nonperturbative contributions to the $\Upsilon(1S)$ wave function, which nevertheless, do not affect the energy of the state [27]. The reason is that the energy can be obtained in terms of the matrix element of the NRQCD effective Hamiltonian, whereas the wave function has no simple operator expansion. The situation is similar to other well-studied processes, such as the cross-section for $e^+e^- \rightarrow \text{hadrons}$, where the total cross-section has an OPE description and much smaller nonperturbative corrections than other non-OPE observables such as event shapes [33].

From the results presented in Ref. [13], one can see three indications that the nonperturbative corrections to Eq. (4) may be smaller than typical expectations based on Eq. (21). First, the sign of $\langle \alpha_s G_{\mu\nu}^2 \rangle$ is positive, and a large value for it increases $|V_{cb}|$ as extracted from the inclusive semileptonic B width. Thus, for a very large nonperturbative correction to m_Υ (say, above 500 MeV or so), the consistency of the inclusive and exclusive $|V_{cb}|$ determinations is significantly worse. Second, in Ref. [13] we determined $|V_{cb}|$ using both the mass of the $\Upsilon(1S)$ and $\Upsilon(2S)$. Eq. (21) shows that nonperturbative corrections to the n^{th} S -wave state scale as n^6 . If we assume that Eq. (21) holds at least approximately also for the size of nonperturbative effects for higher radial excitations of the Υ , the nonperturbative contributions would be 64 times larger for the $2S$ than for the $1S$. The fact that the resulting values of $|V_{cb}|$ coincide at the 5% level at order ϵ^2 [13] indicates that there cannot be a large nonperturbative correction to the mass of the $\Upsilon(1S)$. Third, the ‘‘most nonperturbative’’ that an Υ can be is for it to look like a $B - \bar{B}$ bound state, which occurs just at threshold for decay into two mesons. The nonperturbative contribution to the $B + \bar{B}$ mass is $2\bar{\Lambda}$, and is of order 1 GeV. Assuming again that this is approximately the nonperturbative contribution for $n = 3$, one finds that the $\Upsilon(1S)$ state has a nonperturbative contribution of order $1 \text{ GeV}/3^6 \sim 1.4 \text{ MeV}$, an absurdly small value.

Finally, note that one cannot separate perturbative and nonperturbative contributions to a physical quantity, since the perturbation series is a divergent series.⁷ One can treat the perturbation series as an asymptotic series, and sum the terms until they no longer decrease in magnitude. The resulting difference from the complete answer (which is not known) can be called the nonperturbative contribution. If the perturbation series is poorly behaved, so that one can keep only a few terms in the asymptotic expansion, then one would expect that the error in the asymptotic expansion is large—i.e., that nonperturbative effects are also large. To make this more precise, consider the e^+e^- total cross-section into hadrons at $s = Q^2$, in the limit of massless quarks. By dimensional analysis, the cross-section must have the form

$$Q^2 \sigma(s = Q^2) = f(\alpha_s(Q)), \quad (22)$$

The dimensionless function $f(\alpha_s(Q))$ includes both perturbative and non-perturbative effects, and has an asymptotic series expansion in α_s . For an asymptotic expansion, the best approximation to f is given by summing the series till the terms start to increase. The error at the minimum term, which is typically of the form

$$e^{-c/\alpha_s(Q)} \sim \left(\frac{\Lambda_{\text{QCD}}}{Q} \right)^{c\beta_0/4\pi} \quad (23)$$

can be considered the nonperturbative contribution to f .

The terms in the α_s expansion for the Υ mass in Eq. (4) start increasing already at order ϵ^2 . Using the order ϵ term as an estimate of the error in the asymptotic series gives 100 MeV, which can be taken as an estimate for the size of nonperturbative contributions. When one combines m_Υ with B decay rates, the perturbation series is much better behaved, and one expects a much smaller nonperturbative contribution to B decay rates expressed in terms of m_Υ . In other words, the dominant part of the large nonperturbative contributions to m_Υ are closely related to the Λ_{QCD} uncertainty in the pole mass. What is relevant for our analysis is the nonperturbative contributions to m_Υ after these pole mass effects have been removed. The residual contributions are expected to be much smaller, of order $\sim \Lambda_{\text{QCD}}^4/(m_b\alpha_s)^3$, and formally of the same order as $1/m_b^3$ corrections which have been neglected in the inclusive decay rates. It is these residual nonperturbative effects for which we have used the estimate of 100 MeV.

To summarize, we think that the size of nonperturbative corrections to m_Υ relevant for our analysis are hardly known at the order of magnitude level (the conventional estimates yield hundreds of MeV, but our arguments above indicate that they could equally be tens of MeV only). The best way of determining them is directly from the experimental data. We would like to find B decay observables sensitive to m_Υ for which parameter-free predictions

⁷This separation is not possible even if an infrared cutoff is introduced, since the perturbation series is still divergent. The $n!$ growth in the coefficients due to infrared renormalons is removed, but there are other sources of factorial growth in the coefficients, such as the number of diagrams at n^{th} order [34].

can be made. The best candidates are shape variables in inclusive decays or form factor ratios in exclusive decays. The reason is that observables like total inclusive rates are only moderately sensitive to m_γ , and a possible discrepancy between experiment and theory could always be interpreted as a different value for a CKM angle, as opposed to constraining nonperturbative corrections to m_γ . The most promising observables are the ones which are the most sensitive to $\bar{\Lambda}$: the mean photon energy in $B \rightarrow X_s \gamma$, and the form factor ratio $R_1(w) = h_V(w)/h_{A_1}(w)$ in exclusive $B \rightarrow D^* e \bar{\nu}$ decay. These are discussed in Secs. V.B and IV, respectively.

IV. EXCLUSIVE DECAYS

Exclusive semileptonic $B \rightarrow D^* e \bar{\nu}$ decays depend on four form factors,

$$\begin{aligned} \frac{\langle D^*(v', \epsilon) | V^\mu | B(v) \rangle}{\sqrt{m_{D^*} m_B}} &= i h_V \varepsilon^{\mu\alpha\beta\gamma} \epsilon_\alpha^* v'_\beta v_\gamma, \\ \frac{\langle D^*(v', \epsilon) | A^\mu | B(v) \rangle}{\sqrt{m_{D^*} m_B}} &= h_{A_1}(w+1) \epsilon^{*\mu} - (h_{A_2} v^\mu + h_{A_3} v'^\mu) (\epsilon^* \cdot v), \end{aligned} \quad (24)$$

where $w = v \cdot v'$. In the infinite mass limit $h_{A_2} = 0$ and the other three are equal to the Isgur-Wise function [7]. One linear combination of the form factors is not measurable when the lepton masses are neglected. It is conventional [35] to define two ratios of the three measurable form factors

$$R_1(w) = \frac{h_V(w)}{h_{A_1}(w)}, \quad R_2(w) = \frac{h_{A_3}(w) + (m_{D^*}/m_B)h_{A_2}(w)}{h_{A_1}(w)}. \quad (25)$$

Here we shall concentrate on the prediction for $R_1(1)$ at order $(\epsilon^2)_{\text{BLM}}$ in the epsilon expansion. Using the notation and results of Ref. [35], the theoretical prediction for R_1 is

$$R_1(1) = \frac{C_1}{C_1^5} + \frac{\bar{\Lambda}}{2m_c} + \frac{\bar{\Lambda}}{2m_b} [1 - 2\eta(1)] + \dots, \quad (26)$$

where the ellipses denote the known order $\alpha_s/m_{c,b}$ and unknown order $1/m_{c,b}^2$ corrections. If the QCD sum rule prediction $\eta(1) = 0.6 \pm 0.2$ [36] is approximately correct, then the $\bar{\Lambda}/m_b$ correction is much smaller than the $\bar{\Lambda}/m_c$ term. One can avoid relying on a theoretical estimate for $\eta(1)$, since the deviation of R_2 from unity measures $\eta(w)$ directly if the χ_2 subleading Isgur-Wise function is neglected. Since χ_2 parameterizes $1/m_{c,b}$ corrections due to insertion of the chromomagnetic operator in the Lagrangian, it is expected to be small and a QCD sum rule calculation supports this [37]. Neglecting χ_2 seems to be less model dependent than relying on the previously mentioned result for $\eta(1)$. The relation between $R_2(1)$ and $\eta(1)$ when χ_2 is neglected is

$$\eta(1) = [1 - R_2(1)] \frac{2m_c}{\bar{\Lambda}(1 + 3z)}, \quad (27)$$

where $z = m_c/m_b$.

The perturbative correction $C_1^5 = \eta_A$ is known to order $\alpha_s^2\beta_0$ [38],

$$C_1^5 = \eta_A = 1 + \frac{\bar{\alpha}_s}{4\pi} C_F[\phi(z) - 2] + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^2 \beta_0 C_F \left[\frac{5}{6}\phi(z) - \frac{7}{3}\right], \quad (28)$$

where $\bar{\alpha}_s = \alpha_s(\sqrt{m_c m_b}) \simeq 0.27$ and

$$\phi(z) = -3 \frac{1+z}{1-z} \ln(z) - 6. \quad (29)$$

From results given in Ref. [20], we can obtain the coefficient C_1 to order $\alpha_s^2\beta_0$

$$C_1 = 1 + \frac{\bar{\alpha}_s}{4\pi} C_F[\phi(z) + 2] + \left(\frac{\bar{\alpha}_s}{4\pi}\right)^2 \beta_0 C_F \left[\frac{13}{6}\phi(z) + \frac{25}{3}\right]. \quad (30)$$

To combine these results with Eq. (4), it seems simplest to reexpress $\bar{\alpha}_s$ in terms of $\alpha_s(m_b)$. This yields

$$\frac{C_1}{C_1^5} = 1 + \frac{\alpha_s}{\pi} C_F + \left(\frac{\alpha_s}{4\pi}\right)^2 \beta_0 C_F 8 \left(\frac{1}{3} - \frac{\ln z}{1-z}\right). \quad (31)$$

Thus we obtain at order $(\epsilon^2)_{\text{BLM}}$ in the epsilon expansion

$$R_1(1) = 1.21 + 0.07\epsilon + 0.01_{\text{BLM}}\epsilon^2 + 0.01\lambda_1/\text{GeV}^2 + [0.31R_2(1) - 0.25]. \quad (32)$$

The quantity λ_1 appears in this result because we used Eq. (5) to eliminate the dependence on the charm quark mass. The last term in square brackets is the $\bar{\Lambda}/m_b$ contribution in Eq. (26), and we have assumed that it is small, so that terms of order ϵ or λ_1 from this contribution have been omitted. A 100 MeV nonperturbative contribution to m_Υ changes the prediction for R_1 by ± 0.021 , which is about twice the order $(\epsilon^2)_{\text{BLM}}$ correction. Probably the largest uncertainty is due to the neglected $1/m_{c,b}^2$ corrections [39], which need to be better understood before Eq. (32) may be trusted at the percent level.

One can compare Eq. (32) with the conventional treatment. The size of the order $\alpha_s^2\beta_0$ terms is usually quoted in terms of the BLM scale. The BLM scale of $C_1^5 = \eta_A$ is $0.51\sqrt{m_c m_b}$ and that of $C_1 + C_2 + C_3 = \eta_V$ is $0.92\sqrt{m_c m_b}$ [38]. Surprisingly, the BLM scale of C_1 is much smaller, $0.16\sqrt{m_c m_b}$, and that of C_1/C_1^5 in Eq. (31) is only slightly larger, $0.13m_b$. These low BLM scales show that the conventional perturbation series for R_1 is poorly behaved. The perturbation series in Eq. (32) cannot be converted into a BLM scale since it mixes different orders in α_s . However, to compare with conventional results one can define an effective BLM scale by using the ratio of the $(\epsilon^2)_{\text{BLM}}$ and ϵ terms, just as the conventional treatment used the ratio of the $(\alpha_s^2)_{\text{BLM}}$ and α_s terms. The effective BLM scale of Eq. (32) is $0.89m_b$, which is large and allows for a reliable expansion.

The present experimental results are [40]

$$R_1 = 1.18 \pm 0.30 \pm 0.12, \quad R_2 = 0.71 \pm 0.22 \pm 0.07. \quad (33)$$

This is consistent with Eq. (32), although the errors are sizable. If $R_{1,2}(1)$ will be measured in the future with one (or at most a few) percent errors, then it may be possible to constrain nonperturbative contributions to m_Υ at the 100 MeV level, which would be very interesting.

To summarize, Eq. (32) predicts the form factor ratio R_1 independent of $\bar{\Lambda}$ and the quark masses. If this is tested and works at the few percent level, then we can make better and still model independent predictions for B decay form factors in general, including the ones to the orbitally excited mesons D_1 and D_2^* [41]. Even if nonperturbative contributions to m_Υ cannot be constrained too well, our results show that expressing B decay rates in terms of m_Υ is a useful and theoretically consistent parameterization.

V. INCLUSIVE B DECAY RATES

In this section, we apply the epsilon expansion to inclusive B decays. The b quark decay rate has long been known for arbitrary masses of the four fermions participating in the weak decay, both at tree level [42] and at order α_s [43]. The order $\Lambda_{\text{QCD}}^2/m_b^2$ nonperturbative corrections will be taken into account for all decay modes, however, α_s corrections to these will be neglected. The numerical results below correspond to the choice $\mu = m_b$ and $\alpha_s(m_b) = 0.22$, and μ -dependence will be discussed as well.

Nonleptonic decays are more complicated than semileptonic decays. It is less clear whether local duality holds in nonleptonic decays, since there is no smearing variable [44,45]. There are also corrections not present in semileptonic decay because the four-quark operators run in the effective theory below the weak scale. While the complete series of leading [46] and subleading [47] logarithms have been summed, we cannot consistently use these results, since that would involve summing series of the form $[\alpha_s \ln(m_W/m_b)]^n$ and $\alpha_s[\alpha_s \ln(m_W/m_b)]^n$ (where $\ln(m_W/m_b) \sim 2.8$) without summing $\alpha_s^{n+1}\beta_0^n$ (where $\beta_0 \sim 9$). Instead, we follow an approach somewhat similar to Ref. [48] based on the observation that the largest corrections arise at low orders in α_s . This is quite reasonable, as the order $[\alpha_s \ln(m_W/m_b)]^2$ term dominates the sum of leading logs (there is no order $\alpha_s \ln(m_W/m_b)$ term due to color conservation), and the order α_s term dominates the sum of next-to-leading logs. For $b \rightarrow c$ nonleptonic decays we include the order α_s correction, and at order α_s^2 the terms containing $[\ln(m_W/m_b)]^2$, $\ln(m_W/m_b)$, and β_0 .⁸ For $b \rightarrow u$ nonleptonic decays we include the order α_s and order $[\alpha_s \ln(m_W/m_b)]^2$ corrections only.

A. Semileptonic B decays

Implications of our method for semileptonic B decays were studied in Ref. [13], and here we repeat the results briefly for completeness. Estimates of the small non-BLM contributions to the decay rates at order α_s^2 , which were taken into account in Ref. [13], will be neglected here. For massless leptons, the α_s correction to free quark decay is known analytically [50], and the order $\alpha_s^2\beta_0$ correction has been determined in Ref. [11].

⁸It was argued that other order α_s^2 terms might also be unexpectedly large due to Coulomb enhancement [49].

1. $B \rightarrow X_u e \bar{\nu}$

The $B \rightarrow X_u e \bar{\nu}$ decay rate at order $(\epsilon^2)_{\text{BLM}}$ in the epsilon expansion is [13]

$$\Gamma(B \rightarrow X_u e \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 \left[1 - 0.115\epsilon - 0.035_{\text{BLM}}\epsilon^2 - \frac{9\lambda_2 - \lambda_1}{2(m_\Upsilon/2)^2} + \dots \right], \quad (34)$$

The perturbation series, $1 - 0.115\epsilon - 0.035_{\text{BLM}}\epsilon^2$, is far better behaved than the series in terms of the pole mass, $1 - 0.17\epsilon - 0.13_{\text{BLM}}\epsilon^2$, or the series expressed in terms of the $\overline{\text{MS}}$ mass at $\mu = m_b$, $1 + 0.30\epsilon + 0.19_{\text{BLM}}\epsilon^2$. The uncertainty in the B decay rate using Eq. (34) is much smaller than that in Eq. (1), both because the perturbation series is better behaved, and because the Υ mass is better known (and better defined) than the b quark mass.

Eq. (34) yields a relation between $|V_{ub}|$ and the total semileptonic $B \rightarrow X_u e \bar{\nu}$ decay rate with very small uncertainty,⁹

$$|V_{ub}| = (3.06 \pm 0.08 \pm 0.08) \times 10^{-3} \times \left(\frac{\mathcal{B}(B \rightarrow X_u e \bar{\nu})}{0.001} \frac{1.6 \text{ ps}}{\tau_B} \right)^{1/2}, \quad (35)$$

where we have used $\lambda_2 = 0.12 \text{ GeV}^2$ and $\lambda_1 = (-0.25 \pm 0.25) \text{ GeV}^2$. The first error is obtained by assigning an uncertainty in Eq. (34) equal to the value of the ϵ^2 term and the second is from assuming a 100 MeV uncertainty in Eq. (4). The scale dependence of $|V_{ub}|$ due to varying μ in the range $m_b/2 < \mu < 2m_b$ is less than 1%. The uncertainty in λ_1 makes a negligible contribution to the total error. It is unlikely that $\mathcal{B}(B \rightarrow X_u e \bar{\nu})$ will be measured without significant experimental cuts, for example, on the hadronic invariant mass [51]. Our method should reduce the uncertainties in such analyses as well.

2. $B \rightarrow X_c e \bar{\nu}$

The $B \rightarrow X_c e \bar{\nu}$ decay rate at order $(\epsilon^2)_{\text{BLM}}$ is [13]

$$\Gamma(B \rightarrow X_c e \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 0.533 \left[1 - 0.096\epsilon - 0.029_{\text{BLM}}\epsilon^2 - (0.28\lambda_2 + 0.12\lambda_1)/\text{GeV}^2 \right], \quad (36)$$

For comparison, the perturbation series in this relation when written in terms of the pole mass is $1 - 0.12\epsilon - 0.07_{\text{BLM}}\epsilon^2$. Equation (36) implies

$$|V_{cb}| = (41.6 \pm 0.8 \pm 0.7 \pm 0.5) \times 10^{-3} \times \eta_{\text{QED}} \left(\frac{\mathcal{B}(B \rightarrow X_c e \bar{\nu})}{0.105} \frac{1.6 \text{ ps}}{\tau_B} \right)^{1/2}, \quad (37)$$

⁹In Eqs. (35) and (37), we have included the non-BLM terms of order ϵ^2 as well (see Ref. [13]).

where $\eta_{\text{QED}} \sim 1.007$ is the electromagnetic radiative correction. The uncertainties come from assuming an error in Eq. (36) equal to the ϵ^2 term, the 0.25 GeV^2 error in λ_1 , and a 100 MeV error in Eq. (4), respectively. The second uncertainty is reduced to ± 0.3 by extracting λ_1 from the electron spectrum in $B \rightarrow X_c e \bar{\nu}$; see Eq. (38). The agreement of $|V_{cb}|$ with other determinations (such as exclusive decays) is a check that nonperturbative corrections to Eq. (4) are indeed small.

In Ref. [52] $\bar{\Lambda}$ and λ_1 were extracted from the lepton spectrum in $B \rightarrow X_c e \bar{\nu}$ decay. With our approach, there is no dependence on $\bar{\Lambda}$, so we can determine λ_1 directly with small uncertainty. Considering the observable $R_1 = \int_{1.5 \text{ GeV}} E_e (d\Gamma/dE_e) dE_e / \int_{1.5 \text{ GeV}} (d\Gamma/dE_e) dE_e$, a fit to the same data yields

$$\lambda_1 = (-0.27 \pm 0.10 \pm 0.04) \text{ GeV}^2. \quad (38)$$

The central value includes corrections of order $\alpha_s^2 \beta_0$ [53]. The first error is dominated by $1/m_b^3$ corrections [54]. We varied the dimension-six matrix elements between $\pm(0.5 \text{ GeV})^3$, and combined their coefficients in quadrature in the error estimate. The second error is from assuming a 100 MeV uncertainty in Eq. (4). The central value of λ_1 at tree level or at order α_s is within 0.03 GeV^2 of the one in Eq. (38). This value of λ_1 can be used consistently in theoretical expressions up to order $\alpha_s^2 \beta_0$ [55].

3. $B \rightarrow X_c \tau \bar{\nu}$, $B \rightarrow X_u \tau \bar{\nu}$ and R_τ

The order $\Lambda_{\text{QCD}}^2/m_b^2$ corrections are taken from [56,57], and the order $\alpha_s^2 \beta_0$ corrections from [48]. The $B \rightarrow X_c \tau \bar{\nu}$ decay rate at order $(\epsilon^2)_{\text{BLM}}$ is

$$\Gamma(B \rightarrow X_c \tau \bar{\nu}) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3} \left(\frac{m_\tau}{2}\right)^5 0.114 \left[1 - 0.070\epsilon - 0.016_{\text{BLM}}\epsilon^2 - (0.59\lambda_2 + 0.25\lambda_1)/\text{GeV}^2 \right], \quad (39)$$

For comparison, the perturbation series in this relation when written in terms of the pole mass is $1 - 0.097\epsilon - 0.064_{\text{BLM}}\epsilon^2$.

For the $B \rightarrow X_u \tau \bar{\nu}$ decay rate we find

$$\Gamma(B \rightarrow X_u \tau \bar{\nu}) = \frac{G_F^2 |V_{ub}|^2}{192\pi^3} \left(\frac{m_\tau}{2}\right)^5 0.361 \left[1 - 0.08\epsilon - (0.34\lambda_2 - 0.02\lambda_1)/\text{GeV}^2 \right], \quad (40)$$

For comparison, the order ϵ correction written in terms of the pole mass is $1 - 0.16\epsilon$.

The ratio $R_\tau = \mathcal{B}(B \rightarrow X_c \tau \bar{\nu})/\mathcal{B}(B \rightarrow X_c e \bar{\nu})$ is interesting because it directly probes lepton mass effects, and it can be predicted precisely in the standard model. The largest uncertainty in the prediction is λ_1 , and varying it in the range given in Eq. (38) yields $0.220 < R_\tau < 0.227$.¹⁰ Thus, in the standard model, measurements of R_τ can be used to constrain λ_1 [56,58,59]. The most recent ALEPH data, $\mathcal{B}(B \rightarrow X_c \tau \bar{\nu}) = (2.72 \pm 0.20 \pm 0.27)\%$ [60],

¹⁰For this calculation we have set $\mathcal{B}(B \rightarrow X_c e \bar{\nu}) = 10.5\%$, and neglected its uncertainty.

implies at the 1σ level $\lambda_1 < -0.36 \text{ GeV}^2$. However, at the 1.5σ level this upper bound already becomes positive, which is not very interesting since λ_1 is expected to be negative. R_τ is also interesting for constraining new physics, e.g., possible (tree level) charged Higgs contributions in the large $\tan\beta$ region [61].

B. $B \rightarrow X_s\gamma$

The mean photon energy over the region $E_0 < E_\gamma < E_\gamma^{\text{max}}$ in inclusive $B \rightarrow X_s\gamma$ decay for sufficiently small E_0 is sensitive to $\bar{\Lambda}$ and independent of $\lambda_{1,2}$ when the decay rate is expressed in terms of m_B and $\bar{\Lambda}$ instead of m_b [62]. We expect that stringent constraints can be obtained on the nonperturbative corrections to m_γ when a measurement of the photon energy spectrum is available with E_0 lowered to around 2 GeV [62,63].

Our method reduces the theoretical uncertainties in the total $B \rightarrow X_s\gamma$ decay rate only moderately. The $B \rightarrow X_s\gamma$ decay rate is usually normalized to the semileptonic $B \rightarrow X_c e \bar{\nu}$ rate to eliminate the m_b^5 factor from the prediction. The two biggest ingredients of the total theoretical error are the scale dependence of the $B \rightarrow X_c e \bar{\nu}$ rate computation itself (without the m_b^5 factor), and to a lesser extent the normalization to the $B \rightarrow X_c e \bar{\nu}$ rate due to the uncertainty of m_c/m_b and the experimental error.

C. Nonleptonic B decays

The order $\Lambda_{\text{QCD}}^2/m_b^2$ corrections are taken from [2,5]. We neglect the strange quark mass, and the deviation of $|V_{cs}|^2 + |V_{cd}|^2$ and $|V_{ud}|^2 + |V_{us}|^2$ from unity.

The most interesting decay mode is $b \rightarrow c\bar{c}(s+d)$.

$$\Gamma(B \rightarrow X_{c\bar{c}(s+d)}) = \frac{G_F^2 |V_{cb}|^2}{64\pi^3} \left(\frac{m_\gamma}{2}\right)^5 0.212 \left[1 + 0.156\epsilon + (0.074_{\text{BLM}} + 0.157L^2 + 0.046L)\epsilon^2 - (0.45\lambda_2 + 0.41\lambda_1)/\text{GeV}^2 \right], \quad (41)$$

The variable $L = 1$ is introduced to distinguish between terms that come from different powers of $\ln(m_W/m_b)$. Equation (41) has been written with the renormalization group scaling factor expanded in powers of $\alpha_s \ln(M_W/m_b)$. One can effectively include the complete leading logarithmic series by replacing the $0.157L^2$ term by $0.113L^2$. This replacement also holds for Eqs. (42), (43) and (44). The epsilon expansion does not affect the order $\epsilon^2 L$ and $\epsilon^2 L^2$ terms, i.e., they are the same as in the conventional α_s expansion. The behavior of the order ϵ and $(\epsilon^2)_{\text{BLM}}$ terms have improved significantly. When written in terms of the pole mass, these terms are $1 + 0.199\epsilon + 0.151_{\text{BLM}}\epsilon^2$ [48]. The cancellation we find is rather nontrivial. The expansion of m_b^5 when reexpressed in terms of m_γ generates positive contributions at order $\epsilon^{1,2}$. This is why there was a significant cancellation in the perturbation series for $b \rightarrow ue\bar{\nu}$ decay, since the conventional perturbation series expansion for this decay had negative $\epsilon^{1,2}$ terms. On the other hand, the conventional perturbation series for nonleptonic $b \rightarrow c\bar{c}(s+d)$ has positive $\epsilon^{1,2}$ terms, and reexpressing the overall m_b^5 factor in terms of m_γ only makes them larger. It is the expansions of the tree level

$b \rightarrow c\bar{c}(s+d)$ phase space, which generates large negative contributions at order $\epsilon^{1,2}$ to make the expansion reasonably well-behaved.

For nonleptonic $b \rightarrow c\bar{u}(s+d)$ decay, the rate is

$$\Gamma(B \rightarrow X_{c\bar{u}(s+d)}) = \frac{G_F^2 |V_{cb}|^2}{64\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 0.533 \left[1 - 0.026\epsilon + (-0.006_{\text{BLM}} + 0.157L^2 + 0.099L)\epsilon^2 - (0.28\lambda_2 + 0.12\lambda_1)/\text{GeV}^2 \right], \quad (42)$$

When written in terms of the pole mass, the order ϵ and $(\epsilon^2)_{\text{BLM}}$ terms are $1 - 0.048\epsilon - 0.045_{\text{BLM}}\epsilon^2$.

The decays mediated by $b \rightarrow u$ transitions are much less important, and therefore we keep fewer terms than before.

$$\Gamma(B \rightarrow X_{u\bar{u}(s+d)}) = \frac{G_F^2 |V_{ub}|^2}{64\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 \left[1 - 0.045\epsilon + 0.157L^2\epsilon^2 \right], \quad (43)$$

For comparison, the order ϵ correction written in terms of the pole mass is $1 - 0.099\epsilon$.

$$\Gamma(B \rightarrow X_{u\bar{c}(s+d)}) = \frac{G_F^2 |V_{ub}|^2}{64\pi^3} \left(\frac{m_\Upsilon}{2}\right)^5 0.533 \left[1 + 0.11\epsilon + 0.157L^2\epsilon^2 \right], \quad (44)$$

This is the only case where the upsilon expansion increases the order ϵ correction compared to when it is written in terms of the pole mass, $1 + 0.09\epsilon$.

D. Semileptonic branching ratio and charm counting

The theoretical prediction for the semileptonic B branching ratio \mathcal{B}_{SL} was discussed in [64]. It has been emphasized repeatedly that the calculation of the $b \rightarrow c\bar{c}(s+d)$ rate is the least reliable. With our expansion in Eq. (41), the ratio of the $(\epsilon^2)_{\text{BLM}}$ correction to the order ϵ term is significantly smaller than that calculated in terms of the pole masses, so the $b \rightarrow c\bar{c}(s+d)$ rate should be under better control. With our central values for the inclusive decay rates given in the previous section we obtain

$$\mathcal{B}_{\text{SL}} = 11.8\%, \quad n_c = 1.22, \quad (45)$$

for the semileptonic branching ratio and the average charm multiplicity, which is consistent with other recent analyses. Here we have used $|V_{ub}/V_{cb}| = 0.1$. The sensitivity of these results to λ_1 , for example, is small when it is varied in the range obtained in Eq. (38). If we had not expanded the series of leading logarithms, n_c would remain unchanged, but \mathcal{B}_{SL} would increase to 12.1%. Including a +100 MeV nonperturbative contribution to the Υ mass changes these values by a negligible amount.

VI. CONCLUSIONS

We have shown that inclusive and exclusive B decay rates can be predicted in terms of the $\Upsilon(1S)$ mass instead of the b quark mass. It is crucial to our analysis to use the upsilon

expansion in ϵ rather than the conventional expansion in powers of α_s . Our formulae relate only physical quantities to one another. They result in smaller theoretical uncertainties than existing numerical predictions, and the behavior of the perturbation series is improved. Moreover, the uncertainties can be estimated without resorting to cumbersome arguments, and they can be checked using the experimental data.

In a previous paper [13], we had shown how our method works for inclusive semileptonic B decays. In this article, we have shown how it applies to exclusive decays and to nonleptonic decays. In the new cases studied, the epsilon expansion also improves the behavior of the perturbation series. The biggest theoretical uncertainty at present is a reliable bound on the effect of nonperturbative contributions to the Υ mass (other than the linearly infrared sensitive piece of the pole mass). The best way of determining this seems to be directly from the experimental data. The comparison of V_{cb} extracted using the epsilon expansion with previous determinations already suggests that these contributions are not large. We are investigating whether other processes can be used to provide an independent test of this result.

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