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Duality and Exact Results in Product Group Theories

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Abstract

We study the non-perturbative behavior of some $N = 1$ supersymmetric product-group gauge theories with the help of duality. As a test case we investigate an $SU(2) \times SU(2)$ theory in detail. Various dual theories are constructed using known simple-group duality for one group or both groups in succession. Several stringent tests show that the low-energy behavior of the dual theories agrees with that of the electric theory. When the theory is in the confining phase we calculate the exact superpotential. Our results strongly suggest that, in general, dual theories for product groups can be constructed in this manner, by using simple-group duality for both groups. Turning to a class of theories with $SU(N) \times SU(M)$ gauge symmetry we study the renormalization group flows in the space of the two gauge couplings and show that they are consistent with the absence of phase transitions. Finally, we show that a subset of these theories, with $SU(N) \times SU(N-1)$ symmetry break supersymmetry dynamically.

1 Introduction and Summary.

1.1 Motivation.

The past two years have seen dramatic progress in the study of the non-perturbative behavior of supersymmetric (SUSY) gauge theories. Duality, which relates the strongly coupled behavior of one gauge theory to the weakly coupled behavior of another, has emerged as a key idea in this study [1], [2], [3], [4]. Most of the work in this context has focused on gauge theories with simple gauge groups. While some work has been done involving theories with non-simple groups [5], one would like to understand them in more detail. There are several reasons for this:

1. Such an investigation will serve as a non-trivial check of simple-group duality. Gauging a global symmetry in two theories related by Seiberg duality is often a relevant perturbation, and the equivalence of the resulting two theories will give further evidence for duality.

2. Product groups often arise in the course of dualizing theories with simple groups, once one goes beyond the simplest matter representations [6], [7], [8], [9].

3. Many phenomenologically interesting chiral gauge theories consist of product gauge groups.

4. Several classes of product group theories exhibit dynamical SUSY breaking.

In this paper we focus on a relatively simple product-group theory: an $SU(2) \times SU(2)$ gauge theory. We construct duals to this theory and study its non-perturbative behavior. We expect that the insights obtained are applicable to more complicated product-group theories. In particular, we extend some of our analysis to $SU(N) \times SU(M)$ theories.

Since we will closely follow Seiberg's original work [2] it is useful to briefly review his main results here. Seiberg studied SUSY $SU(N)$ gauge theories with N_f flavors of fundamental matter fields. He found that when $N_f \leq N_c + 1$ the theory confines and its superpotential is determined by holomorphy and symmetries. When $N_f > N_c+1$, there are points in the moduli space where extra particles become light. In this regime Seiberg constructed an $SU(N_f - N_c)$ theory and gave strong arguments showing that it has the same low-energy behavior as the $SU(N_c)$ theory. Moreover, in a sense, as the original theory becomes more strongly coupled the $SU(N_f - N_c)$ theory becomes more weakly coupled. Thus, one could regard it as being dual to the $SU(N_c)$ theory.

1.2 The SU(2) × SU(2) Theories.

In the first part of this paper, (sections 2 - 4), we extend Seiberg's results to the $SU(2)_1 \times SU(2)_2$ theory. The theory we study has $2n$ $SU(2)_1$ fundamentals, $2m$ $SU(2)_2$ fundamentals, and one field transforming as a fundamental under both groups. We will refer to this theory as the $[n, m]$ model. We will analyze the theory as n, m are varied[1]. As in the case of SUSY QCD, we will find that for small values of n and m, $(n, m \leq 2)$, the theory is confining. For larger values of n and m the theory can be in the non-Abelian Coulomb phase, and we construct dual descriptions for it. The analysis in this case is qualitatively different depending on whether $n, m > 2$, or only one of them is greater than 2. We discuss these different possibilities below.

1.2.1 The Duality Regime.

We begin our study of duality in section 2 by considering the $[n, m]$ models with both $n, m \geq 3$. In this case, each $SU(2)$, considered separately, has $N_f > N_c + 1 = 3$ flavors, and one expects a dual theory to exist. In fact, with some thought, several theories can be constructed which could, potentially, have the same low-energy behavior as the original $[n, m]$ theory (we will sometimes refer to this theory as the electric theory). For example, one can turn off, at first, the gauge coupling of the second gauge group. The resulting $SU(2)$ gauge theory has a well known dual which has a global symmetry corresponding to the second $SU(2)$. It is natural to guess that on gauging this symmetry one gets a theory which agrees with the electric one in the infra-red. One can now carry this process one step further and dualize the second $SU(2)$ symmetry as well, thereby getting another dual theory. Note that by construction these theories have the same global symmetries as the original electric one, and the 't Hooft anomaly matching conditions for these symmetries are satisfied.

Dualizing $SU(2)_1$ first, we construct two dual theories. One with gauge group $SP(2n-4) \times SU(2)_2$, and the other with gauge group $SP(2n-4) \times SP(4n+2m-10)$[2]. Dualizing $SP(2)_2$ first, one would obtain similar duals, with n and m exchanged.

The question we investigate is this: do these dual theories really have the same infra-red physics as the original electric theory? To analyze this, the low energy behavior is probed in two different ways:

First, in section 2.2, mass terms are added for some of the fields in the electric theory. The electric theory then flows to a new low-energy theory. As we show, the dual theories flow to the duals of this low-energy theory, and the relations between the strong coupling scales of the electric and dual theories change consistently in the process.

[1] One simplifying feature of this theory is that it is non-chiral. The ability to add mass terms for all fields provides better control on its infra-red behavior.

[2] We use $SP(N)$ duals rather than $SU(N)$ duals, as the global symmetries are more manifest in them.

Second, in section 2.3, the moduli spaces of the electric and dual theories are shown to agree by comparing various flat directions in them. We find that along these flat directions, the two theories are related by simple-group duality. In particular, we establish that the chiral rings in the two theories are the same.

Taken together, these checks strongly suggest that the dual theories have the same low energy behavior as the electric theory. Thus, while these duals arise naturally when one gauge coupling is much bigger than the other, they are in fact valid for arbitrary ratios of the couplings.

1.2.2 The "Partially Confining" Models.

In section 3 we study the "partially confining" models, in which one of the groups, say $SU(2)_1$, is confining, when the other gauge coupling is turned off. This class includes the $[2,m]$ and $[1,m]$ models. A convenient starting point for studying the electric theory is the limit $\Lambda_1 \gg \Lambda_2$, where Λ_1, Λ_2 are the strong coupling scales of $SU(2)_1$, $SU(2)_2$ respectively. In this limit the first gauge group confines at the scale Λ_1, generating a non-perturbative superpotential. Below this scale one can use an effective theory in terms of the $SU(2)_1$ mesons, some of which transform under $SU(2)_2$. The low energy theory is therefore an $SU(2)$ gauge theory.

One can use SP duality to construct an $SP(2m-2)$ dual of this theory. It is interesting, however, to see whether the same theory is obtained by flowing down from the $[n,m]$ duals we discussed above. We analyze this question in section 3.1. The relevant dual to consider is the $[3,m]$ dual with gauge group $SU(2) \times SP(2m+2)$. On flowing to the $[2,m]$ theory, the $SU(2) \times SP(2m+2)$ dual theory is indeed broken to an $SP(2m-2)$ subgroup. One can show that the non-perturbative superpotential of the electric theory must arise in the dual theory from instanton-like configurations with winding in both the $SU(2)$ and the *partially broken* $SP(2m+2)$ subgroups. This makes this case somewhat different from the non-perturbative effects in simple-group theories which arise only when the dual group is completely broken. We expect the non-perturbative configurations in the present case to include, but not be restricted to instantons lying in the diagonal $SU(2)$ subgroup of $SU(2) \times SP(2m+2)$. With the non-perturbative superpotential in place, this dual theory has the same infra-red behavior as the electric one.

We then turn to the $[1,m]$ models considered in section 3.2. Here the electric theory itself is intriguing. In the limit $\Lambda_1 \gg \Lambda_2$ $SU(2)_1$ has a quantum modified moduli space. Symmetry considerations show that an axion-dilaton field must arise in the low-energy effective theory in order to cancel anomalies via the Green-Schwarz mechanism. However, since this field is generated dynamically and symmetries do not fix it uniquely, it is not straightforward to determine it. Duality provides a convenient way to do so. Starting with the $SP(2m-2)$ dual

theory described above and flowing to the $[1, m]$ case, one finds that the theory is higgsed, as usual, but the scale at which it is broken depends on a modulus. This modulus is the required dilaton in the electric theory. With this identification, the electric theory and the resulting dual once again agree in the infra-red.

There is another dual theory for the $[1, m]$ case obtained by flowing down from the $[3, m]$ dual with gauge group $SP(2m-4) \times SP(4m-4)$. While we have not analyzed it in generality, we show in section 3.3 that for $m = 3$, this dual too reproduces the infra-red behavior of the electric theory. This already is quite remarkable since the electric theory has a quantum modified moduli space.

Note that while we analyze the electric theory in the limit $\Lambda_1 \gg \Lambda_2$, the dual descriptions we use are valid more generally. Their equivalence to the electric theory indicates that the electric description too is valid for all values of Λ_1/Λ_2.

1.2.3 Conclusions From the Study of the Duality Regime.

The central lessons that emerge from this study of duality are as follows:

First, as mentioned in the very beginning, from the point of view of duality in simple-group theory, gauging an additional group provides a highly non-trivial consistency check of duality.

Second, we have established that dual theories can be constructed by alternately dualizing the various factors in a product group. This reduces the problem of constructing dual theories for the product group cases to the simpler problem of constructing duals for each of the individual groups. The infra-red equivalence of the electric and dual theories follows from simple-group duality in the limit when one gauge coupling is much bigger than the other. But to establish this in general requires the detailed tests described above.

Finally, all the evidence obtained is consistent with the behavior of the product-group theory changing smoothly as the ratio of the two gauge couplings is varied. In particular there is no evidence for a phase transition, which would drastically alter the infra-red behavior[3].

While we have only studied in detail the $SU(2) \times SU(2)$ case, we expect most of our results to hold in general for product-group theories.

1.3 The Confining Models.

We end our study of the $SU(2) \times SU(2)$ theories by considering the confining models in section 4. By analyzing the theory in the two limits, $\Lambda_1 \gg \Lambda_2$ and $\Lambda_2 \gg \Lambda_1$ we find the exact superpotentials and the massless particles in these models. We show that the descriptions obtained in these two limits agree and in fact argue that they should be valid more generally,

[3] There is some lore to the effect that such transitions cannot occur in supersymmetric theories [1], [10]. Our results are in accord with this.

for all values of Λ_1/Λ_2. Again, this is in agreement with the picture emerging from studying the dual models, as the confining models can be obtained by flowing down from the dual theories.

We derive the superpotentials of the confining models by adding the contributions generated by the two groups. We expect this to be true quite generally as well.

Finally, we note that our discussion dealt with theories which were driven into the confining regime by adding suitable mass terms. One can also go into this regime by adding Yukawa terms, without accompanying mass terms (we give an example of this in Sect. 2.4). The exploration of such theories, especially from the point of view of supersymmetry breaking, is left for the future.

1.4 The SU(N) $\times$ SU(M) Theories and Supersymmetry Breaking.

We conclude in section 5 by studying some features of the more general $SU(N) \times SU(M)$ theories.

First, in section 5.1 we study the renormalization group flows in the space of the two gauge couplings. We do this in the vicinity of the two fixed points obtained by turning off one or the other gauge coupling. We find a simple criterion to decide when the gauge coupling that is initially turned off, is a relevant perturbation. We then use this criterion to argue that the flows are consistent locally with the absence of a phase transition.

Second, in section 5.2, we extend our results for the $SU(2)_1 \times SU(2)_2$ theories and construct a set of duals for the $SU(N) \times SU(M)$ theories.

Finally, in section 5.3, as an illustration of the richness of this class of theories, we analyze a subset consisting of $SU(N) \times SU(N-1)$ theories and show that they break supersymmetry after adding appropriate matter fields and Yukawa couplings.

The presentation in the paper follows the order outlined in the introduction.

2 The SU(2)$_1$ $\times$ SU(2)$_2$ Models.

In this section we study the non-perturbative dynamics of an $SU(2)_1 \times SU(2)_2$ gauge theory with matter fields in the fundamental representation. We explore the different phases of the theory as its matter content is varied. We note that the $SU(2)_1 \times SU(2)_2$ theory is non-chiral – all its matter fields can be given mass terms. This allows for a variety of probes of the infrared physics.

The theory we consider has one field, $Q_{\alpha\dot{\alpha}}$, transforming as a fundamental under both groups, $2n$ fields transforming as fundamentals of the first group, $L_{\alpha i}$, and $2m$ fields transforming as fundamentals of the second group, $R_{\dot{\alpha} a}$. Here $\alpha = 1, 2$, $\dot{\alpha} = 1, 2$ are the gauge

Table 1: Field Content of Electric Theory

	$SU(2)_1$	$SU(2)_2$	$SU(2n)$	$SU(2m)$	$U(1)$	$U(1)_R$
$Q_{\alpha\dot{\alpha}}$	$\square$	$\square$	1	1	$-mn$	$m-1$
$L_{\alpha i}$	$\square$	1	$\square$	1	m	$1-m/n$
$R_{\dot{\alpha}a}$	1	$\square$	1	$\square$	n	0

indices of $SU(2)_1$, $SU(2)_2$ respectively and $i = 1 \ldots 2n$, $a = 1 \ldots 2m$. We will refer to this theory as the $[n,m]$ theory, or the $[n,m]$ model. The field content of the $[n,m]$ theory is summarized in Table 1.

In analogy to QCD we sometimes refer to SU- and SP-fundamentals as "quarks", and to a pair of fundamentals as one "flavor". The $[n,m]$ theory has the non-anomalous global symmetry group $SU(2n) \times SU(2m) \times U(1) \times U(1)_R$. For $n, m \geq 0$ there is a one-parameter family of $U(1)_R$ symmetries. As a result, $U(1)_R$ charges of fields and their dimensions at superconformal infra-red fixed points are not uniquely determined. The charges of the fields under the nonanomalous global symmetries are also given in Table 1. Our conventions and notations are summarized in Appendix A. We denote the strong coupling scales of the two factors in $SU(2)_1 \times SU(2)_2$ by Λ_1 and Λ_2 respectively. The charges of the fields and the strong coupling scales under the various anomalous symmetries are given in Appendix B.

2.1 The Duals of the $[n,m]$ Models.

In this section we will construct theories dual to the $[n,m]$ models. Our basic building block will be the dual of an $SP(N)$ gauge theory with fundamental matter first constructed in [2] and subsequently studied in [11]. By applying this SP duality to one or both of the $SU(2)$ groups we will construct two kinds of dual theories[4]. The dual theories, by construction, will have the same global symmetries as the original electric theory and the 't Hooft anomalies for these symmetries will match with those in the electric theory. In the following sections we will subject the dual theories to other non-trivial checks of duality. In Section 2.2 we will change the infra-red behavior of the electric theory by adding mass terms and show that the dual theories flow to new ones in a consistent way. In Section 2.3 we will study the consistency of duality with deformations along flat directions. This test is crucial in verifying that the moduli spaces and chiral rings of the electric and magnetic theories are the same. Some relevant information regarding duality in SP groups is summarized in Appendix A.

[4]Other dual theories can also be constructed involving $SU(N)$ groups. We focus on the SP duals here since the global symmetries are more manifest in them.

As mentioned above, two kinds of theories dual to the $[n,m]$ models will be constructed. We discuss them in turn below. In constructing the first dual to the $SU(2)\times SU(2)$ theory it is useful to consider the theory in the limit in which the $SU(2)_2$ strong coupling scale, Λ_2, is negligible compared to the $SU(2)_1$ scale, Λ_1. In this limit one has an $SU(2)_1$ theory with $n+1$ flavors. For $n\geq 3$, it has an equivalent infra-red description in terms of a dual theory with gauge group $SP(2n-4)$ [11]. The dual theory has $n+1$ flavors of "dual quarks", $q^{\dot\alpha}_\lambda$ and l^i_λ, transforming as fundamentals of $SP(2n-4)$, and $\lambda = 1...2n-4$ is the $SP(2n-4)$ gauge index. In addition, the theory contains $SP(2n-4)$ singlet fields which are mapped to the mesons of the original $SU(2)_1$:

$$X \equiv \frac{1}{2}\varepsilon^{\alpha_1\alpha_2}\varepsilon^{\dot\alpha_1\dot\alpha_2}Q_{\alpha_1\dot\alpha_1}Q_{\alpha_2\dot\alpha_2}\ , \qquad \mathcal{L}_{ij}\equiv L_i\cdot L_j\ , \qquad V_{\dot\alpha i}\equiv Q_{\dot\alpha}\cdot L_i\ , \tag{2.1}$$

where the product denotes $SU(2)_1$ contraction (our conventions are also given in Appendix A). The dual theory has a superpotential [11]:

$$W \;=\; \frac{1}{4\mu_1}\,\left(-X\,q^{\dot\alpha_1}\cdot q^{\dot\alpha_2}\varepsilon_{\dot\alpha_1\dot\alpha_2} \;+\; 2\;V_{\dot\alpha i}\,q^{\dot\alpha}\cdot l^i \;+\; \mathcal{L}_{ij}\,l^i\cdot l^j\right)\ . \tag{2.2}$$

The dimension-one parameter μ_1 is introduced in order to match the dimensions of the electric and magnetic mesons in the ultraviolet [2]. The parameter μ_1 and the strong coupling scales Λ_1 of $SU(2)_1$, and Λ_1' of $SP(2n-4)$ satisfy the scale matching relation (A.6) [11]:

$$\Lambda_1^{5-n}\Lambda_1'^{2n-4} \;=\; 16\,(-1)^{n-1}\,\mu_1^{n+1}\ . \tag{2.3}$$

We note in particular, that this $SP(2n-4)$ theory has a global $SU(2)$ symmetry corresponding to $SU(2)_2$ in the electric theory. By gauging it in the $SP(2n-4)$ theory one expects to get a dual to the $SU(2)_1\times SU(2)_2$ theory. The resulting $SP(2n-4)\times SU(2)_{2'}$ theory will be referred to as the **first dual**. Its field content is summarized in Table 2 (in order to avoid confusion we refer to the $SU(2)_2$ symmetry in the dual as $SU(2)_{2'}$).

The $SU(2)_{2'}$ gauge group has $2(2n+m-2)$ fundamentals[5]:

$$\varepsilon_{\dot\alpha\dot\alpha_1}q^{\dot\alpha_1}_\lambda\ ,\frac{1}{\mu_1}V_{\dot\alpha i}\ ,\ R_{\dot\alpha a}\ . \tag{2.4}$$

The Wilsonian gauge coupling (strong coupling scale) of $SU(2)_{2'}$ is uniquely determined, by its charges under the various anomalous symmetries, given in Table 6 of Appendix B, to be:

$$\Lambda_2'^{\,8-2n-m} \;=\; (-1)^n\,2^{n-2}\,\frac{\Lambda_1^{5-n}\Lambda_2^{5-m}}{\mu_1^{2+n}}\ . \tag{2.5}$$

[5] We use the antisymmetric tensor to lower the $SU(2)_{2'}$ index of q, to conform with our definition of SP doublets as having lower gauge indices.

Table 2: Field Content of First Dual

	$SP(2n-4)$	$SU(2)_{2'}$	$SU(2n)$	$SU(2m)$	$U(1)$	$U(1)_R$
$q^{\dot{\alpha}}_{\lambda}$	$\square$	$\square$	1	1	mn	$2-m$
l^{i}_{λ}	$\square$	1	$\overline{\square}$	1	$-m$	m/n
$R_{\dot{\alpha}a}$	1	$\square$	1	$\square$	n	0
$V_{\dot{\alpha}i}$	1	$\square$	$\square$	1	$m-mn$	$m-m/n$
X	1	1	1	1	$-2mn$	$2(m-1)$
$\mathcal{L}_{ij}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	$2m$	$2(1-m/n)$

The numerical coefficient above is derived at the end of this section by requiring consistency of this dual with deformations along the $X \neq 0$ flat direction.

Although the construction of the $SP(2n-4) \times SU(2)_{2'}$ dual above was motivated by considering the electric theory in the limit $\Lambda_1 \gg \Lambda_2$, we will find through several checks in the subsequent sections that the electric and dual theory in fact agree in the infra-red regardless of the value of Λ_1/Λ_2. As a first test we note here that the 't Hooft anomalies in the electric theory and the dual are guaranteed to match by construction (this can also be checked explicitly by using Tables 1 and 2). It is also worth mentioning that one can clearly repeat the above mentioned procedure to dualize $SU(2)_2$ instead of $SU(2)_1$ thereby obtaining an $SU(2)_{1'} \times SP(2m-4)$ dual theory. The analysis of this dual is very analogous to that of the $SP(2n-4) \times SU(2)_{2'}$ theory; consequently we focus on the latter in this paper.

We now extend this process one step further by dualizing $SU(2)_{2'}$ in the first dual thereby giving another dual theory which we will call the **second dual**. The dual of $SU(2)_{2'}$ with $2n+m-2$ flavors (2.4) is an $SP(4n+2m-10)$ gauge theory with $2n+m-2$ flavors $p^{\lambda}_{\dot{\lambda}}$, $v^{i}_{\dot{\lambda}}$ and $r^{a}_{\dot{\lambda}}$, where $\dot{\lambda} = 1 \ldots 4n+2m-10$ is the $SP(4n+2m-10)$ gauge index. In addition, the $SU(2)_{2'}$ mesons constructed from (2.4) appear as basic fields:

$$\begin{array}{lll} A_{\lambda_1\lambda_2} \equiv q_{\lambda_2} \cdot q_{\lambda_1} \ , & D_{\lambda i} \equiv q_{\lambda} \cdot V_i \ , & G_{\lambda a} \equiv q_{\lambda} \cdot R_a \ , \\ W_{ij} \equiv V_i \cdot V_j \ , & Y_{ia} \equiv V_i \cdot R_a \ , & \mathcal{R}_{ab} \equiv R_a \cdot R_b \ , \end{array} \tag{2.6}$$

where for conciseness we have omitted in eq. (2.6) the various powers of μ_1 from (2.4) that are needed to correctly match the dimensions. The full gauge symmetry in the theory is then $SP(2n-4) \times SP(4n+2m-10)$. The theory has the superpotential:

$$\begin{aligned} W &= \frac{1}{4\mu_1}\left(X\ A_{\lambda_1\lambda_2}\, J^{\lambda_1\lambda_2} - 2\ D_{\lambda_1 i}\, l^{i}_{\lambda_2}\ J^{\lambda_1\lambda_2} + \mathcal{L}_{ij}\, l^{i}_{\lambda_1}\ l^{j}_{\lambda 2}\ J^{\lambda_1\lambda_2}\right) \\ &+ \frac{1}{4\mu_2}\left(\mathcal{R}_{ab}\, r^a \cdot r^b + 2\ G_{\lambda a}\, p^{\lambda} \cdot r^a + \frac{2}{\mu_1}\ Y_{ia}\, v^i \cdot r^a \right. \end{aligned} \tag{2.7}$$

$$+\ A_{\lambda_1\lambda_2}\, p^{\lambda_1}\cdot p^{\lambda_2} + \frac{2}{\mu_1}\ D_{\lambda i}\, p^{\lambda}\cdot v^i + \frac{1}{\mu_1^2}\ W_{ij}\, v^i\cdot v^j\Big)\ ,$$

where the first three terms come from the superpotential (2.2) expressed in terms of the fields of the second dual, and μ_2 is a dimension-one parameter needed to relate the ultraviolet dimensions of the mesons (2.6) in the electric and magnetic theories. As in eq. (2.3), the parameter μ_2 and the scales Λ_2' of $SU(2)_{2'}$ and $\bar{\Lambda}_2$ of $SP(4n+2m-10)$ obey the scale matching relation (A.6):

$$\Lambda_2'^{\,8-2n-m}\,\bar{\Lambda}_2^{4n+2m-10} \;=\; 16\,(-1)^m\,\mu_2^{2n+m-2}\ . \tag{2.8}$$

Substituting $\Lambda_2'^{\,8-2n-m}$ from (2.5) we find the scale matching relation for the scale $\bar{\Lambda}_2$ of $SP(4n+2m-10)$:

$$\bar{\Lambda}_2^{4n+2m-10} \;=\; 2^{6-n}\,(-1)^{n+m}\,\frac{\mu_2^{2n+m-2}\mu_1^{2+n}}{\Lambda_1^{5-n}\Lambda_2^{5-m}}\ . \tag{2.9}$$

It follows from the superpotential (2.7) that the fields X, $A_{\lambda_1\lambda_2}J^{\lambda_1\lambda_2}$, $D_{\lambda i}$ and l^i_λ are heavy in the second dual. Their equations of motion are:

$$\begin{aligned} A_{\lambda_1\lambda_2}\ J^{\lambda_2\lambda_1} &= 0\,, \\ X &= \frac{1}{2n-4}\,\frac{\mu_1}{\mu_2}\,J_{\lambda_1\lambda_2}\ p^{\lambda_1}\cdot p^{\lambda_2}\,, \\ l^i_{\lambda_1} &= \frac{1}{\mu_2}\,J_{\lambda_1\lambda_2}\ p^{\lambda_2}\cdot v^i\,, \\ D_{\lambda i} &= -\,\mathcal{L}_{ij}\ l^j_\lambda\ . \end{aligned} \tag{2.10}$$

The first equation in (2.10) sets the trace part of the anti-symmetric field $A_{\lambda_1\lambda_2}$ to zero. The remaining light fields (see Table 3) are therefore the traceless part of the anti-symmetric field, $A'_{\lambda_1\lambda_2}$, which transforms under $SP(2n-4)$ only, p^λ_λ, which is a fundamental under both groups, v^i_λ and r^a_λ, which are $SP(4n+2m-10)$ fundamentals, and $G_{\lambda a}$, which are $SP(2n-4)$ fundamentals. The theory also contains a number of singlets under both groups: $\mathcal{L}_{ij}$, $\mathcal{R}_{ab}$, Y_{ia} and W_{ij}. Together, these fields saturate the 't Hooft anomalies of the original $[n,m]$ theory.

To summarize, the second dual has an $SP(2n-4)\times SP(4n+2m-10)$ gauge group with an antisymmetric tensor and $2n+2m-5$ flavors of $SP(2n-4)$, $2n-2+m$ flavors of $SP(4n+2m-10)$, and has a superpotential:

$$\begin{aligned} W &= -\frac{1}{4\mu_1\mu_2^2}\,\mathcal{L}_{ij}\ v^i\cdot p^{\lambda_1}J_{\lambda_1\lambda_2}v^j\cdot p^{\lambda_2}\ + \\ &+\ \frac{1}{4\mu_2}\,(\mathcal{R}_{ab}\,r^a\cdot r^b + 2G_{\lambda a}\,p^\lambda\cdot r^a + \frac{2}{\mu_1}Y_{ia}\ v^i\cdot r^a + A'_{\lambda_1\lambda_2}\,p^{\lambda_1}\cdot p^{\lambda_2} + \frac{1}{\mu_1^2}W_{ij}\ v^i\cdot v^j)\ . \end{aligned} \tag{2.11}$$

The scale $\bar{\Lambda}_1$ of $SP(2n-4)$ in the second dual can be found by using the various anomalous symmetries of Table 6 (Appendix B)

$$\bar{\Lambda}_1^{5-2m} = c(n,m)\ \frac{\Lambda_2^{5-m}}{\mu_1\ \mu_2^{m-1}}\,, \tag{2.12}$$

Table 3: Field Content of Second Dual

	$SP(2n-4)$	$SP(4n+2m-10)$	$SU(2n)$	$SU(2m)$	$U(1)$	$U(1)_R$
$p^{\lambda}_{\dot{\lambda}}$	$\square$	$\square$	1	1	$-mn$	$m-1$
$v^{i}_{\dot{\lambda}}$	1	$\square$	$\overline{\square}$	1	$mn-m$	$1-m+\frac{m}{n}$
$r^{a}_{\dot{\lambda}}$	1	$\square$	1	$\overline{\square}$	$-n$	1
$A'_{\lambda_1\lambda_2}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	1	1	$2mn$	$2(2-m)$
$G_{\lambda a}$	$\square$	1	1	$\square$	$mn+n$	$2-m$
$\mathcal{L}_{ij}$	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	$2m$	$2(1-\frac{m}{n})$
$\mathcal{R}_{ab}$	1	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$2n$	0
Y_{ia}	1	1	$\square$	$\square$	$m+n-mn$	$m-\frac{m}{n}$
W_{ij}	1	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1	$2m-2mn$	$2m-2\frac{m}{n}$

up to a constant. We will show in Section 2.2.2 that consistency of duality with the mass flows implies recursion relations on $c(m,n)$ as m and n are varied, eqs. (2.24) and (2.29). Furthermore, in Section 3.3, consistency of duality in the $[n,1]$ models will fix $c(3,1)=-2$, eq. (3.26). Together with the recursion relations (2.24) and (2.29) this will allow us to determine the constant in eq. (2.12):

$$c(n,m) = (-)^m\ 2^{m+n-3}\ . \tag{2.13}$$

Symmetry considerations do allow some field dependence in eq. (2.12), since there is one combination of fields and scales that is invariant under all global symmetries. However, such a field dependence cannot occur. For example, it would introduce unphysical singularities in the Wilsonian gauge coupling. Also, it would not be consistent with the mass flows considered in the next section.

We should note that by repeating the procedure described above in the opposite order and dualizing $SU(2)_2$ first followed by $SU(2)_1$ we would obtain another dual theory of the second kind with gauge group $SP(2n+4m-10)\times SP(2m-4)$. Additional duals could in principle be obtained by continuing to alternatingly dualize the two groups. However, $SP(2n-4)$ now contains an antisymmetric tensor, and its dual is still unknown.

We complete the construction of the $[n,m]$ duals by determining the constant in the scale matching relation (2.5) for Λ'_2 – the scale of $SU(2)_{2'}$ in the first dual. As was mentioned in the discussion preceding eq.(2.5) symmetries determine that

$$\Lambda_2'^{\,8-2n-m} \;=\; C\ \frac{\Lambda_1^{5-n}\Lambda_2^{5-m}}{\mu_1^{2+n}}\ . \tag{2.14}$$

We find the constant C below by going along an X flat direction and demanding consistency of this deformation with the first dual. Along the $X \neq 0$ flat direction, the electric theory breaks to the diagonal $SU(2)_D$ with the $2n+2m$ doublets L_i and R_a, with scale $\Lambda_D^{6-n-m} = \Lambda_1^{5-n}\Lambda_2^{5-m}/X^2$. In the first dual (2.2), one flavor of the dual quarks (q) becomes heavy and can be integrated out. The $SP(2n-4)$ theory then has n flavors and confines, thereby generating a nonperturbative superpotential (A.4). Adding this superpotential to (2.2) (with the fields q integrated out and the result rewritten in terms of the $SP(2n-4)$ mesons), it is easy to see that the only fields that remain light are the $SU(2)_{2'}$-quarks $V_{\dot{\alpha}i}/\mu_1$, and that the superpotential for the light fields vanishes. The scale of the $SU(2)_{2'}$ after integrating out the heavy fields will be denoted by Λ'_{2L}. Requiring that this scale coincide with the scale of the diagonal $SU(2)_D$ in the electric theory we find:

$$\Lambda'^{\,6-n-m}_{2L} \;=\; \left(\frac{-X}{2\mu_1}\right)^{n-2} C\;\frac{\Lambda_1^{5-n}\;\Lambda_2^{5-m}}{\mu_1^{n+2}} \;=\; \frac{\Lambda_1^{5-n}\;\Lambda_2^{5-m}}{X^2}\;, \tag{2.15}$$

where the X dependence of the middle term arises since we integrate out the fields q at the mass scale $-X/2\mu_1$, and use the scale matching relation (A.7). To complete the identification of $SU(2)'$ with $SU(2)_D$ we further note that the fields V_i/μ_1 should be identified with L_i. This fixes $\mu_1 = \sqrt{X}$. Solving for C in eq. (2.15) now we find that

$$C \;=\; (-1)^n\, 2^{n-2}\;, \tag{2.16}$$

thereby obtaining the coefficient in eq. (2.5).

2.2 Mass Flows and Scale Matching.

In this section we subject the above proposed duality to a strict consistency check. First we change the infra-red behavior of the electric theory by adding mass terms for some of its fields. We then show that the first and second dual theories described above flow in the infra-red to the appropriate duals of the low-energy electric theory. This constitutes a non-trivial check on the equivalence of the low energy behavior of the electric and dual theories. In the subsequent discussion we refer to the renormalization group flow from the original infra-red theory to the new one, obtained after adding mass terms, as a "mass flow".

There are three basic mass flows to consider. First, upon giving a mass to one flavor of $SU(2)_1$, such as $L_{i=1,2}$, the $[n,m]$ theory flows to the $[n-1,m]$ theory. Similarly, giving a mass to one flavor of $SU(2)_2$, such as $R_{a=1,2}$, one obtains the $[n,m-1]$ model. Finally, one can give a mass to the field Q, which transforms under both groups, and integrate it out. The resulting low energy electric theory then has two decoupled $SU(2)$ gauge groups. In the following we discuss these three different flows, first in the first dual (Section 2.2.1), and then in the second dual (Section 2.2.2).

We will also demonstrate the consistency of the scale matching relations (2.3), (2.5), (2.9) and (2.12) with the mass flows. The meaning of the scale matching relations is that the parameters $\mu_{1,2}$, the scales $\Lambda_{1,2}$ of the electric theory and the scales $\bar{\Lambda}_{1,2}$ of the dual theory have to obey (2.9), (2.12), in order for the correlation functions of the electric and magnetic theories to agree, including normalization. Similarly, in the first dual, Λ_1', Λ_2' and μ_1 have to obey (2.3), (2.5). For any given value of $[n,m]$ the parameters $\Lambda, \bar{\Lambda}$ and μ in the scale matching relations can be absorbed by redefining the various operators. It may thus seem that the scale matching relations are trivial. It is nontrivial, however (see ref. [12]), that the scale matching relations, as we show below, are consistent with the various flows. In Section 3, we provide additional nontrivial checks on the scale matching relations when we consider the properties of the partially confining models.

2.2.1 Mass Flows in the First Dual.

Since the first dual was obtained by dualizing $SU(2)_1$ only, it is not affected by the $[n, m-1]$ flow[6]. We therefore turn to the $[n-1, m]$ flow. Adding a mass term $M\mathcal{L}_{12}$ for one flavor of $SU(2)_1$ in the electric theory, the theory flows to the $[n-1, m]$ model, with the scale of $SU(2)_1$ given by ${\Lambda_{1L}}^{5-(n-1)} = M\,{\Lambda_1}^{5-n}$.

The first dual is higgsed to $SP(2n-6) \times SU(2)_{2'}$ much like in the case of a single SP group. We therefore do not discuss it further except to note the change in the $SU(2)_{2'}$ scale. Adding the term $M\mathcal{L}_{12}$ to the superpotential (2.2), the fields l_1 and l_2 get vevs with $l_1 \cdot l_2 = -2\,M\,\mu_1 \equiv v^2$. Plugging their D-flat vevs

$$l_\lambda^i = \begin{cases} \delta_\lambda^i \, \sqrt{-2\mu_1 M} & \text{if } \lambda = 1,2 \\ 0 & \text{otherwise,} \end{cases} \tag{2.17}$$

into the superpotential (2.2), the $SU(2)_2$ doublets $q_{1\dot{\alpha}}$, $q_{2\dot{\alpha}}$, $V_{\dot{\alpha}1}/\mu_1$ and $V_{\dot{\alpha}2}/\mu_1$ become heavy, with mass $v/2$. $SU(2)_2$ now has $2(n-1)+m-2$ flavors, and its scale is, using (A.7):

$${\Lambda_2'}_L^{\,8-2(n-1)-m} \;=\; \frac{v^2}{4}\,{\Lambda_2'}^{8-2n-m} \;=\; (-1)^{n-1}\,2^{n-3}\,\frac{{\Lambda_{1L}}^{5-(n-1)}\,{\Lambda_2}^{5-m}}{\mu_1^{2+(n-1)}}\;, \tag{2.18}$$

in agreement with equation (2.5) written in terms of $\Lambda_{1_L}, \Lambda'_{2_L}$.

The last mass flow to consider in the first dual is the decoupling of the common field Q. Upon adding a mass term $M\,X$, the electric theory consists of two disjoint $SU(2)$ groups, with n, m flavors respectively, and scales ${\Lambda_{1L}}^{6-n} = M\,{\Lambda_1}^{5-n}$ and ${\Lambda_{2L}}^{6-m} = M\,{\Lambda_1}^{5-m}$. Correspondingly, in the first dual, upon adding the term $M\,X$ to the superpotential (2.2) the fields q acquire vevs:

$$q_\lambda^{\dot{\alpha}} = \begin{cases} \delta_\lambda^{\dot{\alpha}} \, \sqrt{-2\mu_1 M} & \text{if } \lambda = 1,2 \\ 0 & \text{otherwise,} \end{cases} \tag{2.19}$$

[6] Except for the scale Λ_2'; its change is given by $m \to m-1$ in eq. (2.5).

such that $SP(2n-4) \times SU(2)_{2'}$ breaks to $SP(2n-6) \times SU(2)_D$, where $SU(2)_D$ is the diagonal group. Plugging the vevs (2.19) into (2.2), we see that $V_{\dot{\alpha} i}$, l_1^i and l_2^i are massive. Their equations of motion set them to zero and the superpotential reduces to

$$W \;=\; \frac{1}{4\mu_1}\, \mathcal{L}_{ij}\; l^i_{\lambda_1}\; l^j_{\lambda_2}\; J^{\lambda_1 \lambda_2}\;, \tag{2.20}$$

with $\lambda_{1,2} = 3 \ldots 2n-4$. The light fields include the singlets $\mathcal{L}_{ij}$, the $SP(2n-6)$ fundamentals $l^{i=1..2n}$, and the $SU(2)_D$ doublets $R_{a=1..2m}$. Clearly, $SP(2n-6)$ is dual to the electric $SU(2)_1$, as is evident from applying (A.8): ${\Lambda'_1}_L^{2n-6} = \frac{2}{-2M\mu_1} {\Lambda'_1}^{2n-4}$, which shows that Λ'_{1_L} is precisely the scale of the dual of the low-energy electric theory with ${\Lambda_1}_L^{6-n} = M\,\Lambda_1^{5-n}$. The other unbroken group in the low-energy theory, $SU(2)_D$, should be identified with the electric $SU(2)_2$, which was left untouched by the first duality operation. Matching the scale of the diagonal $SU(2)_D$ to the scales of $SP(2n-4)$ and $SU(2)_{2'}$, and using (2.3), (2.5), we obtain $\Lambda_D^{6-m} \sim \langle q^1 \cdot q^2 \rangle {\Lambda'_1}^{2n-4} {\Lambda'_2}^{8-2n-m} \sim M \Lambda_2^{5-m}$ so that the scale of the diagonal $SU(2)_D$ is equal to the scale of the original $SU(2)_2$ as expected[7].

2.2.2 Mass Flows in the Second Dual.

The flow $[n,m] \to [n,m-1]$ is the simplest flow in the second dual, since $SU(2)_2$ was dualized last. Adding a mass M for the first flavor of $SU(2)_2$ in the electric theory corresponds to adding the term $M\mathcal{R}_{12}$ to the superpotential (2.11) in the second dual. The scale of the low-energy electric $SU(2)_2$ is ${\Lambda_2}_L^{5-(m-1)} = M\Lambda_2^{5-m}$ (A.7). In the dual theory, the equation of motion for $\mathcal{R}_{12}$ is

$$r^1 \cdot r^2 = v^2 = -2\,M\mu_2 \tag{2.21}$$

Taking the vevs of r^a along the D-flat directions

$$r^a_{\dot{\lambda}} = \begin{cases} \delta^a_{\dot{\lambda}}\, \sqrt{-2M\mu_2} & \text{if } \dot{\lambda} = 1,2 \\ 0 & \text{otherwise,} \end{cases} \tag{2.22}$$

we find that $SP(4n+2m-10)$ is higgsed down to $SP(4n+2(m-1)-10)$ and that the superpotential (2.11) gives mass to the $\dot{\lambda} = 1,2$ components of all $SP(4n+2m-10)$ fundamentals as well as to the singlets $\mathcal{R}_{12}$, $\mathcal{R}_{1a}$, $\mathcal{R}_{2a}$, $G_{\lambda 1}$, $G_{\lambda 2}$, Y_{i1} and Y_{i2}. The resulting low-energy theory is therefore $SP(2n-4) \times SP(4n+2(m-1)-10)$ and its superpotential is as in (2.11) with a now taking $2(m-1)$ values and $\dot{\lambda} = 3 \ldots 4n+2m-10$, as can be seen by integrating out the heavy fields.

Note also that $SP(2n-4)$ now has two fewer flavors, since the fields $p^\lambda_{\dot{\lambda}}$ with $\dot{\lambda} = 1,2$, and the fields $\frac{1}{\mu_2} G_{\lambda a=1,2}$ become heavy, with mass $v/2$. It therefore has $2n+2(m-1)-5$ flavors

[7] The numerical factor in the relation between Λ_D, Λ'_1 and Λ'_2 arises because we are working in the $\overline{\text{DR}}$ scheme here.

as expected in $[n, m-1]$ case. The scale $\bar{\Lambda}_{1L}$ of the low-energy $SP(2n-4)$ theory can be found by matching at the scale of the mass of the heavy flavors (A.7):

$$\begin{aligned} \bar{\Lambda}_{1L}^{5-2(m-1)} &= -\frac{M\mu_2}{2}\, c(n,m)\, \frac{\Lambda_2^{5-m}}{\mu_1\, \mu_2^{m-1}} = -\frac{1}{2}\, c(n,m)\, \frac{\Lambda_{2L}^{5-(m-1)}}{\mu_1\, \mu_2^{(m-1)-1}} \\ &= c(n,m-1)\, \frac{\Lambda_{2L}^{5-(m-1)}}{\mu_1\, \mu_2^{(m-1)-1}}\ , \end{aligned} \tag{2.23}$$

where the equality on the second line above is (2.12) with $m \to m-1$. Consistency of the $[n,m] \to [n,m-1]$ flow with (2.12) therefore implies a recursion relation for $c(n,m)$:

$$c(n,m) = -2\, c(n,m-1)\ . \tag{2.24}$$

Finally, the scale of $SP(4n+2(m-1)-10)$ in the low-energy magnetic theory is (A.8):

$$\bar{\Lambda}_{2L}^{4n+2(m-1)-10} = \frac{2}{\langle r^1 \cdot r^2 \rangle}\, \bar{\Lambda}_2^{4n+2m-10} = -\frac{\bar{\Lambda}_2^{4n+2m-10}}{M\mu_2}\ . \tag{2.25}$$

This is clearly consistent with (2.9) after taking $m \to m-1$ in it and identifying $\Lambda_{2L}^{5-(m-1)} = \Lambda_2^{5-m} M$.

We now consider the flow $[n,m] \to [n-1,m]$. Adding a mass term $M\mathcal{L}_{12}$ for one flavor of $SU(2)_1$ in the electric theory, the theory flows to the $[n-1,m]$ model, with the scale of $SU(2)_1$ given by $\Lambda_{1L}^{5-(n-1)} = M\, \Lambda_1^{5-n}$.

In the second dual, upon adding the term $M\mathcal{L}_{12}$ to the superpotential (2.11), the $\mathcal{L}_{12}$ equation of motion is:

$$p^{\lambda_1} \cdot v^1\, J_{\lambda_1\lambda_2}\, p^{\lambda_2} \cdot v^2 = 2\, M\, \mu_1\, \mu_2^2\ . \tag{2.26}$$

The following choice of vevs satisfies (2.26) as well as the D-flatness conditions:

$$p_{\lambda=1}^{\lambda=1} = p_{\lambda=3}^{\lambda=2} = v_{\lambda=2}^{i=1} = v_{\lambda=4}^{i=2} = v \equiv \left(-2M\mu_1\mu_2^2\right)^{1/4}\ , \tag{2.27}$$

with all other components vanishing, and breaks the group down to $SP(2(n-1)-4) \times SP(4(n-1)+2m-10)$. The fields $p_{\lambda=1,3}^{\lambda}$, $p_{\dot{\lambda}}^{\lambda=1,2}$ and $v_{\dot{\lambda}}^{i=1,2}$ are eaten, and the superpotential (2.11) gives masses to the broken $SP(4n+2m-10)$ components ($\dot{\lambda} = 1 \ldots 4$) of $r_{\dot{\lambda}}^a$ and $v_{\dot{\lambda}}^{\hat{i}}$, where $\hat{i} = 3 \ldots 2n$, and to the broken $SP(2n-4)$ components ($\lambda = 1,2$) of $G_{\lambda a}$ and $A'_{\lambda_1\lambda_2}$. It also gives rise to masses for the gauge singlets $\mathcal{L}_{ij}$, Y_{ia}, W_{ij} with i or j equal to 1 or 2. After integrating out the heavy fields, the resulting superpotential is of the form (2.11) with $i = 3 \ldots 2n$, $\lambda = 3 \ldots 2n-4$ and $\dot{\lambda} = 5 \ldots 4n+2m-10$, as expected for the second dual of the $[n-1,m]$ model. The only fields which transform under the unbroken groups and become massive from the superpotential are the $SP((2(n-1)-4)$ fundamentals $\frac{1}{\mu_2} A'_{\lambda_1=1,2;\ \lambda_2>2}$, which

mix with the $p^{\lambda>2}_{\lambda=2,4}$ components. These two flavors of the unbroken $SP((2(n-1)-4)$ have equal masses $v/2$.

The scale $\bar{\Lambda}_{1L}$ of $SP(2(n-1)-4)$ is affected by two factors under this flow: the $SP(2n-4)$ in the high-energy theory is broken to $SP(2(n-1)-4)$ by the expectation values (2.27) of the fields $p_{\lambda=1,3}$, while at the same time two flavors – the fields $\frac{1}{\mu_2}A'_{\lambda_1=1,2;\ \lambda_2>2}$ and $p^{\lambda>2}_{\lambda=2,4}$ mentioned above – of the unbroken $SP(2(n-1)-4)$ gain mass $v/2$ from the superpotential (2.11). Using (A.8) and (A.7), we can find the matching condition for $\bar{\Lambda}_1$ for this flow:

$$\begin{aligned}\bar{\Lambda}_{1L}^{5-2m} &= \left(\frac{v}{2}\right)^2 \frac{2}{v^2}\,\bar{\Lambda}_1^{5-2m} = \frac{1}{2}\,c(n,m)\,\frac{\Lambda_2^{5-m}}{\mu_1\,\mu_2^{m-1}} \\ &= c(n-1,m)\,\frac{\Lambda_2^{5-m}}{\mu_1\,\mu_2^{m-1}}\ . \end{aligned} \tag{2.28}$$

Eq. (2.28) implies a recursion relation for $c(n,m)$:

$$c(n,m) = 2\ c(n-1,m)\ , \tag{2.29}$$

which, together with eq. (2.24) and (3.26) implies that $c(m,n) = (-1)^m 2^{m+n-3}$.

The scale $\bar{\Lambda}_{2\,L}$ of the $SP(4(n-1)+2m-10)$ can be found using (A.8):

$$\begin{aligned}\bar{\Lambda}_{2\,L}^{4(n-1)+2m-10} &= 4v^{-4}\,\bar{\Lambda}_2^{4n+2m-10} = -\frac{2\bar{\Lambda}_2^{4n+2m-10}}{M\,\mu_1\,\mu_2^2} \\ &= (-1)^{(n-1)+m}2^{6-(n-1)}\,\frac{\mu_2^{2(n-1)+m-2}\,\mu_1^{2+(n-1)}}{\Lambda_{1\,L}^{5-(n-1)}\,\Lambda_2^{5-m}}\ , \end{aligned} \tag{2.30}$$

which is the correct scale for the second dual of the $[n-1,m]$ model (equation (2.9)).

Finally, we consider adding a mass M for the field X. Recall that upon integrating out the field X in the electric theory, one gets a theory with two separate $SU(2)$ gauge groups; $SU(2)_1$ with n flavors, and $SU(2)_2$ with m flavors.

In the second dual, X is not present since it becomes heavy and gets integrated out. However, the equation of motion (2.10) relates it to the light field p, which transforms under both $SP(2n-4)$ and $SP(4n+2m-10)$. Thus, the appropriate term to add to the superpotential, eq. (2.11), is given by:

$$\delta W = \frac{1}{2n-4}\,\frac{\mu_1}{\mu_2}\,M\,J_{\lambda_1\lambda_2}p^{\lambda_1}\cdot p^{\lambda_2} \tag{2.31}$$

Once it is added the field p becomes heavy and can be integrated out, so that the $SP(2n-4)$ and $SP(4n+2m-10)$ gauge theories are now decoupled (except for the superpotential). Now however, $SP(4n+2m-10)$ has $N_f = n+m$ and confines for $n \geq 3$[8]. One is then left with an $SP(2n-4)$ gauge group with an antisymmetric, $2m$ fundamentals and a number of

[8]For $n \leq 2$ the discussion in this section is not valid. See following sections.

singlets (including the mesons of the confining $SP(2n-4)$). The superpotential of this theory is partly given by (2.11), after substituting for the field p, from its equations of motion, in terms of the fields $\mathcal{L}_{ij}$, v^i, $G_{a\lambda}$ and $A'_{\lambda_1\lambda_2}$, and going over to the $SP(4n+2m-10)$ mesons. In addition, there is a non-perturbative contribution to the superpotential generated by the confining $SP(4n+2m-10)$.

The dynamics of this $SP(2n-4)$ gauge theory is thus very involved. Although we cannot rigorously argue that this is indeed the case, we conjecture that in the infra-red, the theory gives two decoupled sectors – one corresponding to the electric $SU(2)_2$ with its m flavors, and the other corresponding to an $SP(2n-6)$ gauge theory that is the dual of the electric $SU(2)_1$.

This picture can be substantiated by considering the special case $n=3$, for which $SP(2n-4)=SU(2)$ and $SP(4n+2m-10)=SP(2m+2)$. Once the field p is integrated out, $SU(2)$ is left with the $2m$ doublets $G_{\lambda a}$ (notice there is no antisymmetric in this case). $SP(2m+2)$, on the other hand, has $N_f=N_c+2$ and confines, generating the superpotential $\mathrm{Pf}\tilde{M}$ (A.4), where $\tilde{M}$ denote collectively the $SP(2m+2)$ mesons $M^{ij}=v^i\cdot v^j$, $M^{ab}=r^a\cdot r^b$ and $M^{ia}=v^i\cdot r^a$. These mesons become heavy due to the superpotential (2.11); similarly, the fields Y_{ia}, W_{ij} and $\mathcal{R}_{ab}$ become heavy. Substituting for the fields p as well as for the other heavy fields, we find a vanishing superpotential. The second dual therefore gives an $SU(2)$ gauge theory with m flavors and no superpotential, corresponding to the the electric $SU(2)_2$ (the theories are not identical, there is no simple field redefinition that relates their fundamentals). As for the electric $SU(2)_1$, this theory confines for the case $n=3$ we are considering. The non-perturbative superpotential that this theory generates probably arises in the dual through some complicated dynamics. Aside from this superpotential however, the magnetic and electric theory are clearly identical in the infra-red.

2.3 Deformations Along Flat Directions.

As a further check of the duals we constructed, we now analyze their moduli spaces. To do this, various fields in the electric and magnetic theories are given expectation values along D- and F-flat directions. We then verify that the resulting theories are equivalent in the infra-red. In the process we also show that the chiral rings of the electric and magnetic theories are identical. As is common in the study of duality, classical restrictions on the chiral ring of the electric theory will sometimes arise through complicated, nonperturbative effects in the dual description.

We limit our discussion to the second dual of section 2.1, since, for the purpose of this analysis, the first dual is essentially the same as simple SP duals.

We begin with the flat direction parametrized by the expectation value of the field X. In the electric theory the X expectation value breaks the gauge symmetry to the diago-

nal $SU(2)_D$. The diagonal gauge group has now $2n+2m$ doublets and a scale $\Lambda_D^{6-n-m} = \Lambda_1^{5-n}\,\Lambda_2^{5-m}/X^2$. Its dual is an $SP(2n+2m-6)$ gauge theory with $2n+2m$ fundamentals, and a scale obeying:

$$\bar{\Lambda}_D^{2n+2m-6} = \frac{16(-)^{n+m}\,\mu_D^{n+m}\,X^2}{\Lambda_1^{5-n}\,\Lambda_2^{5-m}}\ . \tag{2.32}$$

We will show below that the $SP(2n-4)\times SP(4n+2m-10)$ dual theory flows to this dual after deforming it along the $X\neq 0$ flat direction, thereby establishing its equivalence with the electric theory.

By the equation of motion for the field X, eq. (2.10), giving an expectation value to X in the electric theory is equivalent to giving an expectation value to $p\cdot p$ in the magnetic theory. The D- and F-flat conditions that follow from (2.11) determine the expectation values of the field p:

$$p_{\dot\lambda}^{\lambda} = \begin{cases} \sqrt{-\frac{\mu_2\,X}{\mu_1}}\,\delta_{\dot\lambda}^{\lambda} & \text{if } \dot\lambda = 1,...,2n-4 \\ 0 & \text{otherwise}\ . \end{cases} \tag{2.33}$$

The p expectation value (2.33) breaks the $SP(2n-4)\times SP(4n+2m-10)$ gauge symmetry down to $SP(2n-4)_D\times SP(2n+2m-6)$. All fundamentals of $SP(4n+2m-10)$ decompose into fundamentals of the unbroken $SP(2n+2m-6)$, denoted in the following by hatted fields, and fundamentals of the diagonal unbroken $SP(2n-4)_D$, denoted in the following by barred fields (e.g. the fields v^i decompose into $\hat{v}^i$ which are fundamentals of the unbroken $SP(2n+2m-6)$, and $\bar{v}^i$ which are fundamentals of $SP(2n-4)_D$). The field p decomposes under $SP(2n-4)_D\times SP(2n+2m-6)$ as follows: $p=(\mathbf{1},\mathbf{1})+(\square\!\square,\mathbf{1})+(\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1})+(\square,\square)$, where $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ is traceless. The $(\square\!\square,\mathbf{1})$ and $(\square,\square)$ are eaten by the Higgs mechanism. The $(\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1})$ part of p pairs with the field A' and becomes massive due to the superpotential. Thus only the singlet part of p remains massless, and corresponds to the field X. Furthermore, by inspecting the superpotential (2.11) after substituting the vevs (2.33), we observe that the fields G_a and $\bar{r}^a$ become massive and can be integrated out. After integrating out the heavy fields the resulting superpotential is:

$$\begin{aligned} W &= \frac{1}{4\,\mu_1^2\,\mu_2}\,(W_{ij}-X\,\mathcal{L}_{ij})\,\bar{v}^i\cdot\bar{v}^j + \frac{1}{4\,\mu_2}\,\mathcal{R}_{ab}\,\hat{r}^a\cdot\hat{r}^b \\ &+ \frac{1}{2\,\mu_1\,\mu_2}\,Y_{ia}\,\hat{v}^i\cdot\hat{r}^a + \frac{1}{4\,\mu_1^2\,\mu_2}\,W_{ij}\,\hat{v}^i\cdot\hat{v}^j\ . \end{aligned} \tag{2.34}$$

The diagonal $SP(2n-4)_D$ now has only $2n$ fundamentals, the fields $\bar{v}^i$, and is therefore confining, generating a nonperturbative superpotential (A.4) $W_{n.p.}\sim \mathrm{Pf}M$, with $M^{ij}\equiv\bar{v}^i\cdot\bar{v}^j$ being the confined degrees of freedom. Adding $W_{n.p.}$ to the superpotential (2.34) we obtain:

$$\begin{aligned} W &= \frac{1}{4\,\mu_1^2\,\mu_2}\,(W_{ij}-X\,\mathcal{L}_{ij})\,M^{ij} + \frac{1}{4\,\mu_2}\,\mathcal{R}_{ab}\,\hat{r}^a\cdot\hat{r}^b \\ &+ \frac{1}{2\,\mu_1\,\mu_2}\,Y_{ia}\,\hat{v}^i\cdot\hat{r}^a + \frac{1}{4\,\mu_1^2\,\mu_2}\,W_{ij}\,\hat{v}^i\cdot\hat{v}^j - \frac{\mathrm{Pf}M}{2^{n-3}\,\bar{\Lambda}_D^{2n-3}}\ , \end{aligned} \tag{2.35}$$

where $\bar{\Lambda}_D^{2n-3}$ is the scale of the diagonal $SP(2n-4)_D$. One linear combination of W and $\mathcal{L}$ obtains mass together with the meson M and can be integrated out. The equations of motion for the heavy fields imply $M^{ij}=0$ and $W_{ij}=X\mathcal{L}_{ij}$. Note that the second equality reproduces a classical constraint in the electric theory. Integrating the heavy fields out, the superpotential becomes:

$$W=\frac{1}{4\ \mu_2}\ \mathcal{R}_{ab}\ \hat{r}^a\cdot\hat{r}^b+\frac{1}{2\ \mu_1\ \mu_2}\ Y_{ia}\ \hat{v}^i\cdot\hat{r}^a+\frac{1}{4\ \mu_1^2\ \mu_2}\ X\ \mathcal{L}_{ij}\ \hat{v}^i\cdot\hat{v}^j\ . \tag{2.36}$$

Eq.(2.36) can be identified with the superpotential of the $SP(2n+2m-6)$ dual of the electric $SU(2)_D$ theory mentioned above. To see this note that once X acquires a vev the fields transforming under $SU(2)_D$ can be taken to be R_a and $\frac{Q\cdot L_i}{\sqrt{X}}$. On dualizing $SU(2)_D$ with this matter content one gets a superpotential given precisely by eq.(2.36), with μ_1 set equal[9] to $\sqrt{X}$. Thus, the electric and the dual theory match along the $X\neq 0$ flat direction.

Next we consider the flat direction $\mathcal{R}_{ab}\neq 0$. We begin by giving $\mathcal{R}_{ab}$ a rank 2 expectation value. For the sake of definiteness we take $\mathcal{R}_{12}\neq 0$. This expectation value completely higgses $SU(2)_2$. The low-energy electric theory is then the $SU(2)_1$ gauge theory with $2n+2$ doublets $(Q_{\dot\alpha},L_i)$, and the massless singlets $\mathcal{R}_{a\hat{a}}$ – the moduli describing the flat direction and the components of the $SU(2)_2$ matter fields that are not eaten. In the following, the indices $\hat{a},\hat{b}=1,2$. By the usual rules of SP duality, this low-energy $SU(2)_1$ has a dual description in terms of an $SP(2n-4)$ theory. We will see below that the magnetic theory correctly reduces to this $SP(2n-4)$ theory along the $R_{12}\neq 0$ direction.

In the dual theory, the expectation value of $\mathcal{R}_{12}$ gives mass to 2 fundamentals of $SP(4n+2m-10)$. The low-energy $SP(4n+2m-10)$ theory (whose scale will be denoted by $\bar{\Lambda}_{2_L}$) now has $4n+2m-6$ fundamentals. It is therefore confining, and generates a nonperturbative superpotential (A.4), $W_{n.p.}\sim \mathrm{Pf}\mathcal{M}$. Here $\mathcal{M}$ denote the mesons of the confining $SP(4n+2m-10)$: $M^{i\lambda}=v^i\cdot p^\lambda$, $M^{ij}=v^i\cdot v^j$, $M^{\lambda\lambda'}=p^\lambda\cdot p^{\lambda'}$, $M^{a'b'}=r^{a'}\cdot r^{b'}$ (with $a',b'=3,...,2m$ only), $M^{a'i}=r^{a'}\cdot v^i$ and $M^{a'\lambda}=r^{a'}\cdot p^\lambda$. We now integrate out the heavy flavor $r^{\hat{a}}$ from (2.11), add the nonperturbatively generated superpotential along the $\mathcal{R}_{12}\neq 0$ flat direction, and rewrite the resulting superpotential in terms of the mesons defined above, much like we did when considering the $X\neq 0$ flat direction:

$$\begin{aligned}W &= -\frac{\mathrm{Pf}\mathcal{M}}{2^{2n+m-6}\ \bar{\Lambda}_{2_L}^{4n+2m-9}}-\frac{1}{4\mu_1\ \mu_2^2}\ \mathcal{L}_{ij}\ M^{i\lambda}\ M^j_\lambda\\ &+ \frac{1}{4\ \mu_2\ \mathcal{R}_{12}}\left(-\mathcal{R}_{\hat{a}a'}\ M^{a'b'}\ \mathcal{R}^{\hat{a}}_{b'}+2\ \mathcal{R}_{\hat{a}a'}\ M^{a'\lambda}\ G^{\hat{a}}_\lambda+\frac{2}{\mu_1}\ \mathcal{R}_{\hat{a}a'}\ M^{a'i}\ Y_i^{\hat{a}}\right.\end{aligned}$$

[9] Equivalently we can perform the X- and μ_1-dependent field redefinition $Y\to Y\sqrt{X}$, $\hat{v}\to\hat{v}\mu_1/\sqrt{X}$, so that the fields have the same symmetries as those in the dual of $SU(2)_D$. As a result μ_1 disappears from the superpotential and the scale matching relation.

$$-G_{\lambda\hat{a}}\ M^{\lambda\nu}\ G_{\nu}^{\hat{a}} - \frac{2}{\mu_1}\ G_{\lambda\hat{a}}\ M^{\lambda i}\ Y_i^{\hat{a}} - \frac{1}{\mu_1^2}\ Y_{i\hat{a}}\ M^{ij}\ Y_j^{\hat{a}}\Big) \tag{2.37}$$

$$+\ \ \frac{1}{4\ \mu_2}\ \Big(2\ G_{\lambda a'}\ M^{\lambda a'} + \frac{2}{\mu_1}\ Y_{ia'}\ M^{ia'} + \mathcal{R}_{a'b'}\ M^{a'b'} + A'_{\lambda_1\lambda_2}\ M^{\lambda_1\lambda_2} + \frac{1}{\mu_1^2}\ W_{ij}\ M^{ij}\Big)\ .$$

From the superpotential eq.(2.37) we see that the only fields that remain massless are the $SP(2n-4)$ fundamentals $G_{\lambda\hat{a}}, M_\lambda^i$ and the gauge singlets $\mathcal{L}_{ij}, Y_{\hat{a}i}, \hat{M} = M^{\lambda\lambda'}J_{\lambda\lambda'}$ and $\mathcal{R}_{a\hat{a}}$. The nonperturbative superpotential $W_{n.p}$ vanishes after imposing the equations of motion for the heavy fields, and the superpotential of the remaining massless fields is:

$$W = -\frac{1}{4\ \mu_1\ \mathcal{R}_{12}}\ X\ \varepsilon_{\hat{a}\hat{b}}\ G_\lambda^{\hat{a}}\ G^{\lambda\hat{b}} + \frac{1}{2\ \mu_1\ \mu_2\ \mathcal{R}_{12}}\ Y_{\hat{a}i}\ G_\lambda^{\hat{a}}\ M^{i\lambda} + \frac{1}{4\ \mu_1\ \mu_2^2}\ \mathcal{L}_{ij}\ M_\lambda^i\ M^{j\lambda}\ , \tag{2.38}$$

where we used the relation $\hat{M} = (2n-4)\mu_2 X/\mu_1$ (2.10).

But this superpotential, as promised, is precisely the superpotential of the dual of the low-energy electric $SU(2)_1$ with $2n+2$ doublets, eq. (2.2). To see this note that once $\mathcal{R}_{12} \neq 0$, the matter fields transforming under $SU(2)_1$ in the electric theory can be taken to be L_i and $\frac{R_a \cdot Q}{\sqrt{\mathcal{R}_{12}}}$. Eq. (2.38) suggests that the field dual to L_i be identified with $l_\lambda^i \equiv \frac{1}{\mu_2} M_\lambda^i$. In addition, on identifying the field dual to $\frac{R_a \cdot Q}{\sqrt{\mathcal{R}_{12}}}$ with $q_\lambda^{\hat{a}} \equiv \frac{G_\lambda^{\hat{a}}}{\sqrt{\mathcal{R}_{12}}}$ and $V_{\hat{a}i} \equiv \frac{Y_{\hat{a}i}}{\sqrt{\mathcal{R}_{12}}}$ we find that eq. (2.38) agrees with the required superpotential of the dual of the low-energy $SU(2)_1$. This agreement shows that the electric and dual theory agree along $\mathcal{R}_{12}$ flat direction too.

Classically, the chiral ring of the electric theory satisfies a number of relations. These ensure, for example, that $\mathcal{R}_{ab}$ cannot have an expectation value of rank greater than 2. In order to see how these relations arise in the dual theory let us return to eq. (2.37). As was mentioned above, several fields get mass due to bilinear couplings in this superpotential. Integrating them out gives rise to equations relating the heavy fields with the light ones and these, in fact, correctly reproduce the relations in the chiral ring of the electric theory. For example, the field $M^{a'b'}$ gets a mass by pairing with $\mathcal{R}_{a'b'}$. On integrating it out we find that

$$\mathcal{R}_{a'b'} = \frac{1}{\mathcal{R}_{12}}\ \mathcal{R}_{\hat{a}a'}\ \mathcal{R}_{b'}^{\hat{a}}\ . \tag{2.39}$$

These are the relations in the chiral ring of the electric theory which result in $\mathcal{R}_{ab}$ having an expectation value of rank ≤ 2. Notice that, whereas in the electric theory these are classical relations, in the dual they arise after including the effects of confinement, as in (2.37).

We can continue turning on further expectation values in eq.(2.38). For example, an $\mathcal{L}_{12}$ expectation value gives mass to two $SP(2n-4)$ fundamentals, so that this group confines (recall that along this flat direction the electric theory is completely higgsed). It is easy to work out the massless spectrum that follows from eq. (2.38), after accounting for the nonperturbative superpotential generated by the confining $SP(2n-4)$, and to see that it

precisely matches that of the electric theory. The superpotential in terms of the massless fields vanishes in both theories too. Thus, along the $\mathcal{L}_{12}, \mathcal{R}_{12} \neq 0$ flat direction the electric and magnetic theories flow to the same (trivial) infra-red theory. Furthermore, when the analysis of the chiral ring is extended to this case, one finds that the magnetic theory reproduces all the constraints in the electric theory, so that the chiral rings in the two cases are the same.

This concludes our discussion of the flat directions in the electric and dual theories. As we have seen, the behavior of these theories agrees along various flat directions. This provides strong additional evidence for their equivalence at low energies.

2.4 Flows by Yukawa Perturbations.

In this section we discuss briefly the flows due to Yukawa perturbations in the second dual. We add the term

$$W = \lambda^{ia}\ Y_{ia} \tag{2.40}$$

to the superpotential (2.11) with a Yukawa-coupling matrix of rank $P \leq \min\{2m, 2n\}$. Upon adding the perturbation (2.40), the dual $SP(2n-4) \times SP(4n+2m-10)$ theory flows to a new fixed point. The fields v^i and r^a get expectation values that obey the F-flatness conditions

$$v^i \cdot r^a = -2\ \mu_1\ \mu_2\ \lambda^{ia}, \quad r^a \cdot r^b = v^i \cdot v^j = 0\ , \tag{2.41}$$

and, along the D-flat directions, can be taken to be:

$$v^i_\lambda = \begin{pmatrix} \sqrt{s_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sqrt{s_2} & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{s_P} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}\ , \tag{2.42}$$

$$r^a_\lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \sqrt{s_1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sqrt{s_2} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \sqrt{s_P} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix}\ , \tag{2.43}$$

where by a field redefinition we took the rank-P Yukawa matrix to be diagonal, with eigenvalues $\lambda_1 \ldots \lambda_P$ and $s_i = -2\mu_1\mu_2\lambda_i$; in eqs. (2.42), (2.43) the $SP(4n+2m-10)$ gauge indices $\dot{\lambda}$ are taken to enumerate the rows.

The expectation values (2.42), (2.43), higgs the dual theory to $SP(2n-4) \times SP(4n+2m-10-2P)$. We will not consider in general the new fixed point, but will only note the interesting case when the Yukawa coupling has rank $P = 2n+m-5$. In this case, the $SP(4n+2m-10)$ group is completely broken, while the matter content of the $SP(2n-4)$ theory consists of the antisymmetric tensor A', the $2m-P$ fundamentals $G_{a\lambda}$ with $a > P$, and $2n+m-5$ of the components of $p_{\dot{\lambda}}$ (half of the components of $p_{\dot{\lambda}}$ and the components of $G_{a\lambda}$ with $a \leq P$ become heavy, as can be seen by substituting the expectation values of v^i and r^a in the superpotential (2.11)). Symmetry considerations show that this theory confines for $m \leq 3$; the $m = 3$ case exhibits confinement without chiral symmetry breaking.

We will not analyze the confining phase of this theory in detail in this paper, our only purpose here is to note that it is possible to flow to the confining phase by perturbing the superpotential with dimension-3 terms, without having to add mass terms for any field. In the more interesting chiral product-group theories, such flows to the confining phase might be interesting from the point of view of supersymmetry breaking. We leave the detailed investigation of this for future work.

3 The "Partially Confining" Models.

3.1 The [2, m] Models.

In this section we study theories in which one of the two electric gauge groups is in the confining regime. By this we mean, more precisely, that one of the two groups, say $SU(2)_1$, has three or fewer flavors and would therefore confine in the absence of $SU(2)_2$. These theories have a rich set of non-perturbative phenomena which duality helps elucidate. This section deals with the $[2,m]$ models, and the following two sections deal with the $[1,m]$ theories.

A convenient starting point for studying the $[2,m]$ models is the limit $\Lambda_1 \gg \Lambda_2$, in which $SU(2)_1$ confines at the scale Λ_1. Below this scale one can use an effective theory in terms of the $SU(2)_1$ mesons, $X \equiv Q^2$, $\mathcal{L}_{ij} \equiv L_i \cdot L_j$ and $V_{\dot{\alpha}i} \equiv Q_{\dot{\alpha}} \cdot L_i$ with $i,j = 1 \ldots 4$. Non-perturbative effects in the confining group give rise to a superpotential (A.4)

$$W = -\frac{1}{\Lambda_1^3}\left(X\ \mathrm{Pf}\mathcal{L} - \frac{1}{4}\,W_{ij}\,\mathcal{L}_{kl}\,\varepsilon^{ijkl}\right)\ , \tag{3.1}$$

with $W_{ij} = V_i \cdot V_j$ as in section 2. $SU(2)_2$ in this effective theory has $2m+4$ doublets, $R_{\dot{\alpha}a}$

and $V_{\dot{\alpha}i}/\Lambda_1$, and its scale is given by[10]:

$$\Lambda_{2'}^{4-m} = \frac{\Lambda_2^{5-m}}{\Lambda_1} \ . \tag{3.2}$$

The dual theories are best understood by starting with the $[3,m]$ duals and flowing to the $[2,m]$ case after adding a mass term for one L flavor. In the previous section we presented two types of $[3,m]$ duals: one where only one $SU(2)$ is dualized, and the other where both $SU(2)$s are dualized. In the former, the $[3,m] \to [2,m]$ flow is very similar to the $N_f = 3 \to N_f = 2$ flow in basic SP duality [11]. We therefore only discuss here duals of the second type, in which both groups are dualized. There are two such theories. Dualizing $SU(2)_1$ followed by $SU(2)_2$, a $[3,m]$ dual with gauge group $SU(2) \times SP(2m+2)$ is obtained. Reversing this order, an $SP(2m-4) \times SP(4m-4)$ theory is obtained instead.

We first discuss the $[3,m] \to [2,m]$ flow in the $SU(2) \times SP(2m+2)$ dual. The analysis of this flow has much in common with that of section 2.2.2. Therefore we only point out the essential differences here.

Adding a mass term $M\mathcal{L}_{12}$ to the superpotential (2.11) of the $[3,m]$ theory, the $SU(2) \times SP(2m+2)$ dual gauge group gets broken to its $SP(2m-2)$ subgroup. Several fields get heavy either through the Higgs mechanism or by pairing with other fields through couplings in the superpotential (2.11). The fields that remain light are r_λ^a, v_λ^i, which transform under the $SP(2m-2)$ gauge symmetry, and $\mathcal{L}_{ij}$, $\mathcal{R}_{ab}$, Y_{ia}, W_{ij} and X $(i,j = 3...6)$, which are gauge singlets[11]. The resulting superpotential is:

$$W = \frac{1}{4\mu_2} \left(\mathcal{R}_{ab}\ r^a \cdot r^b + \frac{1}{\mu_1^2}\ W_{ij}\ v^i \cdot v^j + \frac{2}{\mu_1}\ Y_{ia}\ v^i \cdot r^a \right) \ . \tag{3.3}$$

Since the $SU(2)$ subgroup is completely broken, one might expect the non-perturbatively generated superpotential (3.1) to arise from an instanton in this subgroup. However, careful consideration of the zero-modes involved shows that the contribution from this instanton vanishes. Furthermore, the scale matching relations eqs. (2.12) and (2.9), with $n = 3$, imply the following relation between the scale Λ_1 of the $[2,m]$ electric theory ($\Lambda_1^3 = M\Lambda_{1H}^2$, with Λ_{1H} the corresponding scale in the $[3,m]$ theory) and the scales $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ of the $[3,m]$ magnetic theory:

$$\frac{1}{\Lambda_1^3} \sim \frac{\bar{\Lambda}_1^{5-2m}\ \bar{\Lambda}_2^{2m+2}}{M\ \mu_1^4\ \mu_2^5} \ . \tag{3.4}$$

[10] A constant could in principle appear in this scale matching relation, multiplying the right hand side. However it can be shown to be 1, by first establishing, by considering the $m \to m-1$ flow, that it is m-independent, and then evaluating it in the confining $[2,1]$ model by adding mass terms, calculating the vevs of various fields and demanding consistency with the Konishi anomaly.

[11] The field X arises as follows: unlike the general n case, the $n = 3$ $SU(2) \times SP(2m+2)$ theory does not have the $A'_{\lambda_1\lambda_2}$ field. As a result, one component of p_λ^λ does not get mass and can be identified with X.

Hence, since Λ_1 is the scale appearing in (3.1), this nonperturbative term must arise from instanton-like configurations with one unit of winding in both the $SU(2)$ and the $SP(2m+2)$ subgroups. While we have not actually calculated this contribution, the counting of zero modes suggests that it can arise in this manner. We expect these configurations to include, but not be restricted to, instantons that lie in a diagonal $SU(2)$ subgroup of $SU(2) \times SP(2m+2)$.

This non-perturbative effect is different from those encountered in the simple-group case, in that it arises from configurations with components lying in *partially* broken subgroups. It should be a generic feature of product group theories.

Once we accept that (3.1) does arise, we find that the resulting theory is the expected dual of the electric theory (in the limit $\Lambda_1 \gg \Lambda_2$) with which we started. Below the scale Λ_1, the electric theory was $SU(2)_2$ with $2m+4$ doublets, whose dual should be an $SP(2m-2)$ theory. This is precisely the theory we find by flowing down from the $[3,m]$ dual. The required matter fields and superpotential are also in agreement with those found above in the dual theory, provided we take $\mu_1 = \Lambda_1$. Note that the scale μ_1 in eq. (3.3) arises because Y_{ia}/μ_1 and W_{ij}/μ_1^2 are the canonically normalized fields. In the electric theory, $V_{\dot{\alpha}i}/\Lambda_1$ are the correctly normalized $SU(2)_2$ doublets, consequently, μ_1 is set equal to Λ_1. In fact, in the $SP(2) \times SP(2m+2)$ dual discussed above, the strong coupling scale of $SP(2m+2)$ is given by (2.9), with $n=3$ and $\Lambda_1 \to \Lambda_{1H}$. Flowing down to the $[2,m]$ dual, we find that the $SP(2m-2)$ scale is given by

$$\bar{\Lambda}_2^{2m-2} = 16\ (-1)^m\ \frac{\mu_2^{2+m}\ \mu_1^4}{\Lambda_2^{5-m}\ \Lambda_1^3}\ , \tag{3.5}$$

or, setting $\mu_1 = \Lambda_1$,

$$\bar{\Lambda}_2^{2m-2} = 16\ (-1)^m\ \frac{\mu_2^{2+m}\ \Lambda_1}{\Lambda_2^{5-m}}\ , \tag{3.6}$$

which is in agreement with (3.2) and the SP-duality matching relation (A.6):

$$\Lambda_{2'}^{4-m}\ \bar{\Lambda}_2^{2m-2} = 16\ (-1)^m\ \mu_2^{2+m}\ . \tag{3.7}$$

We end our description of this dual theory with one final comment. From the point of view of the electric theory, we could, strictly speaking, justify the above-mentioned description of low-energy physics only in the limit $\Lambda_1 \gg \Lambda_2$. The dual description on the other hand is valid for all values of the ratio Λ_1/Λ_2. The equivalence between the dual and electric theories implies then that the electric description too must be more generally valid.

As mentioned above, the $[3,m]$ theory also has a dual with gauge group $SP(2m-4) \times SP(4m-4)$. In this case, the flow to the $[2,m]$ theory follows the discussion in Section 2.2.2 very closely. The resulting theory has an $SP(2m-4) \times SP(4m-6)$ symmetry. In particular, none of the groups is completely broken, and no non-perturbatively generated superpotential

is expected to arise. In spite of this, by studying its flat directions as in Section 2.3 and by deforming the theory after adding mass terms, as in Section 2.2 one can show that the infra-red behavior of this theory is identical to that of the electric theory.

3.2 Dynamically-Generated Dilaton: $[2, \mathbf{m}] \to [1, \mathbf{m}]$.

We continue our discussion of the partially confining models by turning next to the $[1, m]$ case. We will see that in these theories the electric theory itself has some interesting features. Specifically, we find that a dilaton is dynamically generated in the low energy electric theory and we will see how it can be understood, in more conventional terms, by flowing from a $[2, m]$ dual theory to a $[1, m]$ dual theory. In the subsequent section we will then study in detail the equivalence between the electric theory and its various duals.

Let us start, as in the previous section, by considering the electric theory in the limit $\Lambda_1 \gg \Lambda_2$. At an energy scale of order Λ_1 one can go over to an effective theory in terms of the mesons of $SU(2)_1$, X, $V_{\dot{\alpha} i}$, and $\mathcal{L}_{ij}$, with a non-perturbative superpotential given by:

$$W = A \ (\ X \ \mathcal{L}_{12} - W_{12} - \Lambda_1^4 \ ). \tag{3.8}$$

We can now consider the effects of the $SU(2)_2$ group in this effective theory. In particular, one would like to know how its strong coupling scale is related to that in the microscopic theory. At first sight it might seem that these two are equal since the group has $2m + 2$ fundamentals both in the ultraviolet theory and in the effective theory, where $V_{\dot{\alpha} i}$ contributes two fundamentals. However, a little thought involving the symmetries in the problem shows that this cannot be the case. In fact the effective theory presents us with a puzzle. There is a non-anomalous symmetry in the high energy theory under which Q has charge 1, each L has charge -1 and each R has charge $-\frac{1}{m}$. But in the effective theory this symmetry is anomalous, since the field $V_{\dot{\alpha} i}$ has charge 0 under this symmetry. How can this be possible? The answer lies in the Green-Schwarz anomaly cancellation mechanism. Let us consider those points in moduli space where $V_{\dot{\alpha} i}$ is zero and the $SU(2)_2$ symmetry is unbroken. We see from eq. (3.8) that at such points the quantum modification of the constraint will force X and $\mathcal{L}_{12}$ to acquire vevs which break the global symmetry described above. The corresponding Goldstone boson then enters the low energy theory as an axion and an appropriate shift in this field, along with the rotations of the R fields, is then a non-anomalous symmetry of the theory. Since this is a supersymmetric theory, the partner of the axion field acts as a dilaton. Because of this axion-dilaton field, the strong coupling scale of $SU(2)_2$ at low-energy is related to its value in the microscopic theory in a field-dependent way. Note that the quantum deformation of the superpotential played a crucial role in the discussion above. As a consequence, from the point

of view of the electric theory, we can regard the origin of the dilaton as a truly dynamical effect.

Now let us be more specific. Symmetry considerations tell us that the strong coupling scale in the low-energy theory is given by:

$$\Lambda_{2L}^{5-m} = \Lambda_2^{5-m} \ \frac{\mathcal{L}_{12}}{\Lambda_1^2} \ f(\frac{X\mathcal{L}_{12}}{\Lambda_1^4}) \ , \tag{3.9}$$

where $f(\frac{X\mathcal{L}_{12}}{\Lambda_1^4})$ above is an arbitrary function. We see below how duality will help determine it completely[12]. In the process we will also find that the dual theory provides a much more straightforward explanation for the field dependence of the coupling: it arises because, as usual, the dual of the $[1,m]$ theory is obtained by higgsing the dual of the $[2,m]$ theory. However, in this case, the scale of the resulting $[1,m]$ theory is not uniquely determined and depends on a modulus. This is the required dilaton in the electric theory.

Let us discuss this in more detail now. We start with the $[2,m]$ model. The dual which is useful to consider has a gauge group $SP(2m-2)$ and was discussed at some length in the previous section. Here we take μ_1 in eq.(3.3) to be Λ_1 and normalize the W_{ij} and Y_{ia} fields accordingly. The superpotential in the dual theory is given by the sum of (3.1) and (3.3) and is:

$$W = \frac{1}{4\mu_2} \ \left(\mathcal{R}_{ab} \ r^a \cdot r^b + \frac{1}{\Lambda_1^2} W_{ij} \ v^i \cdot v^j + \frac{2}{\Lambda_1} \ Y_{ia} \ v^i \cdot r^a \right) - \frac{1}{\Lambda_1^3} \ \left(X \ \mathrm{Pf}\mathcal{L} - \frac{1}{4} \ W_{ij} \ \mathcal{L}_{kl} \ \varepsilon^{ijkl} \right) \ . \tag{3.10}$$

Now we add a mass term for one flavor of the L field:

$$\delta W = m^{34} \ \mathcal{L}_{34} \ . \tag{3.11}$$

The equation of motion for $\mathcal{L}_{34}$ then reproduces the expected quantum modified constraint:

$$X\mathcal{L}_{12} - W_{12} - \Lambda_1^3 m^{34} = 0 \ , \tag{3.12}$$

while the equation of motion for W_{34} is:

$$\frac{\mathcal{L}_{12}}{\Lambda_1} + \frac{1}{2\mu_2} \ v^1 \cdot v^2 = 0 \ . \tag{3.13}$$

The last equation implies that the $SP(2m-2)$ theory is broken to a $SP(2m-4)$ subgroup. We see that the scale $\bar{\Lambda}_{2L}$ of the low-energy $SP(2m-4)$ theory, which is given by (A.8), depends on a modulus, $\mathcal{L}_{12}$, and is:

$$\bar{\Lambda}_{2L}^{2m-4} = \bar{\Lambda}_2^{2m-2} \ \left(-\frac{\Lambda_1}{\mu_2 \mathcal{L}_{12}} \right) . \tag{3.14}$$

[12] This dependence can be fixed in other ways, too. For example one can flow down to the [1,2] model, give masses to the different fields and ensure that the vevs are in accord with the Konishi anomaly.

Substituting for $\bar{\Lambda}_2$ from eq. (3.6) we find that

$$\bar{\Lambda}_{2L}^{2m-4} = 16(-1)^{m+1}\mu_2^{1+m}\frac{\Lambda_1^2}{\Lambda_2^{5-m}\mathcal{L}_{12}} \ . \tag{3.15}$$

This is consistent with the standard scale matching relation (A.6), applied to Λ_{2L} and $\bar{\Lambda}_{2L}$ only if the scale Λ_{2L} of the low-energy $SU(2)_2$ in the electric theory is given by:

$$\Lambda_{2L}^{5-m} = \Lambda_2^{5-m}\ \frac{\mathcal{L}_{12}}{\Lambda_1^2} \ . \tag{3.16}$$

On comparing with eq. (3.9) we see that this determines the function f to be a constant equal to 1.

We end this section with one final comment. As in the $[2,m]$ models, the low-energy description of the electric theory used above could be justified only in the limit when $\Lambda_1 \gg \Lambda_2$. However, the $SP(2m-4)$ theory obtained above is valid for all values of the ratio Λ_1/Λ_2. Duality therefore allows us to conclude that this description of the electric theory must be more generally valid.

3.3 The [1, m] Models.

We will continue our study of the $[1,m]$ models in this section by establishing in some detail the equivalence of the electric and dual theories in the infra-red.

Let us begin by summarizing the important features of the electric theory. As we saw in the previous section the low-energy properties of the electric theory can be described in an effective theory consisting of the mesons[13], X, $\mathcal{L}$, $V_{1\dot{\alpha}}$ and $V_2^{\dot{\alpha}}$, with a superpotential eq. (3.8):

$$W = A\ (X\mathcal{L} - V_{1\dot{\alpha}}V_2^{\dot{\alpha}} - \Lambda_1^4) \ . \tag{3.17}$$

The scale of $SU(2)_2$ in this effective theory, Λ_{2L}, is given by eq. (3.16) (i.e. eq. (3.9) with $f \equiv 1$).

For $m > 2$, $SU(2)_2$ is in the nonabelian Coulomb phase and has a dual description in terms of an $SP(2m-4)$ gauge group. This was the dual theory considered in the previous section. The matter fields in this theory are $2m+2$ dual quarks $v^i, (i=1,2); r^a, (a=1,...,2m)$, and the gauge singlet mesons $\mathcal{R}_{ab}$, $\frac{1}{\Lambda_1}Y_{ia}$ and $\frac{1}{\Lambda_1^2}W_{12}$, with W_{12} corresponding to the electric meson $V_{1\dot{\lambda}}V_2^{\dot{\lambda}}$. The superpotential of the dual theory, after solving the constraint (3.17) for the gauge singlet meson W_{12}, becomes [11]:

$$W = \frac{1}{4\mu_2}\ \mathcal{R}_{ab}\ r^a \cdot r^b + \frac{1}{2\mu_2\Lambda_1}\ Y_{ia}\ v^i \cdot r^a + \frac{X\mathcal{L}-\Lambda_1^4}{4\mu_2\Lambda_1^2}\ v_i \cdot v^i. \tag{3.18}$$

[13]Hereafter, by $\mathcal{L}$ we mean $\mathcal{L}_{12}$

The scale of the dual theory $\bar{\Lambda}_2$ is determined by the matching relation (A.6) for SP-duality:

$$\bar{\Lambda}_2^{2m-4}\Lambda_{2L}^{5-m} = 16\ (-)^{m+1}\mu_2^{m+1}\ . \tag{3.19}$$

We will consider the moduli space of this dual theory in some detail.

Consider first those points where $X\mathcal{L} \neq \Lambda_1^4$. At such points, it follows from eq. (3.18) that two of the dual quarks, v^i, are massive, and $SP(2m-4)$ confines. This is in accord with the electric theory: when $X\mathcal{L} \neq \Lambda_1^4$, eq. (3.17) shows that the fields $V_{i\dot{\lambda}}$ are forced to have nonzero expectation values and break $SU(2)_2$. In the dual theory, we can integrate out the massive quarks. On adding the confining superpotential generated by $SP(2m-4)$ with the $2m$ doublets r^a, it is easy to see that all $SP(2m-4)$ mesons, along with the singlets $\mathcal{R}_{ab}$ are massive. The massless degrees of freedom are the mesons Y_{ia}, X and $\mathcal{L}$. The superpotential for the massless degrees of freedom vanishes. In particular, there is a point in the moduli space where all the light fields Y_{ia}, X, $\mathcal{L}$ have zero expectation values. At that point, all the global symmetries are unbroken and the theory therefore exhibits confinement without chiral symmetry breaking. One can also show, as a consistency check, that the 't Hooft anomalies are saturated by the light mesons at that point.

In contrast, along the flat direction $X\mathcal{L} = \Lambda_1^4$, the theory is in the non-Abelian Coulomb phase.

Finally, we give expectation values to all electric mesons Y_{ia}, X, $\mathcal{L}$ and $\mathcal{R}_{ab}$. This makes all dual quarks massive and one can now integrate them out thereby obtaining a pure $SP(2m-4)$ theory in the infra-red. The scale of this theory is given by (A.7) $\bar{\Lambda}_{2L}^{3m-3} = \mathrm{Pf}\tilde{M}\ \bar{\Lambda}_2^{2m-4}$, where $\tilde{M}$ is the mass matrix of the $SP(2m-4)$ quarks, which can be read off eq.(3.18). Gaugino condensation in the low energy theory generates the superpotential [11] $W = (m-1)\ 2^{\frac{m-3}{m-1}}\ \epsilon_{m-1}\ \bar{\Lambda}_{2L}^3$, where ϵ_k denotes the k-th root of unity. Evaluating the Pfaffian of the mass matrix $\tilde{M}$ we obtain for this superpotential, after using the scale matching conditions (3.16), (3.19):

$$W = (1-m)\ \epsilon_{m-1}\ \left[\frac{\mathrm{Pf}\ \mathcal{R}}{\Lambda_2^{5-m}}\ (X - \frac{\Lambda_1^4}{\mathcal{L}}) - \frac{\mathcal{R}^{m-1}\cdot Y^2}{2^{m+1}\ (m-1)!\ \Lambda_2^{5-m}\ \mathcal{L}}\right]^{\frac{1}{m-1}}\ , \tag{3.20}$$

where all indices are contracted with the appropriate ε-symbols. We can add mass perturbations, $\delta W = \frac{1}{2}m_{\mathcal{R}}\cdot\mathcal{R} + m_{\mathcal{L}}\mathcal{L} + m_X X$ to (3.20) and compute the vevs of the meson fields:

$$\begin{aligned}
\langle X\rangle &= \epsilon_1\sqrt{\frac{\Lambda_1^4\ m_{\mathcal{L}}}{m_X}} + \epsilon_2\sqrt{\frac{\Lambda_2^{5-m}\ \mathrm{Pf}\ m_{\mathcal{R}}}{m_X}} \\
\langle\mathcal{L}\rangle &= \epsilon_1\sqrt{\frac{m_X\ \Lambda_1^4}{m_{\mathcal{L}}}} \\
\langle\mathcal{R}_{ab}\rangle &= (m_{\mathcal{R}}^{-1})_{ab}\ \epsilon_2\ \sqrt{m_X\ \Lambda_2^{5-m}\ \mathrm{Pf}\ m_{\mathcal{R}}}\ ,
\end{aligned} \tag{3.21}$$

where $\epsilon_1, \epsilon_2 = \pm 1$. These vevs coincide with the ones determined by holomorphy and the various limits[14]. Taking the limit of vanishing masses in various orders now allows us to explore the moduli space. One finds from eq. (3.22) that a generic flat direction is given by arbitrary expectation values for X and $\mathcal{L}$, while the rank of $\mathcal{R}_{ab}$ is restricted to be ≤ 2. We saw in our discussion of the $[n,m]$ models, in a different way, how this restriction on the rank of $\mathcal{R}_{ab}$ arose. Here we see, once again, that while in the electric theory this restriction arose classically, in the dual it arises as a consequence of a non-perturbative effect. The description of the flat directions obtained above in the dual theory agrees completely with that of the electric theory.

This brings us to the end of our discussion for the $SP(2m-4)$ dual theory. We turn next to another dual of the $[1,m]$ theory. It can be obtained by first dualizing $SU(2)_2$ and subsequently dualizing the first group. The resulting theory has an $SP(4m-8) \times SP(2m-4)$ gauge symmetry. The analysis is considerably more complicated in this case and we will be able to carry it out only partially for the case $m = 3$. Even so, as we will see, this constitutes a very non-trivial test of duality, especially in view of the quantum deformed moduli space in the electric theory.

The matter content and superpotential of the $SP(4m-8) \times SP(2m-4)$ theory can be deduced from Table 3 and the accompanying discussion, in particular, eq. (2.11), after the following replacements have been made: $n \to m$, $m \to 1$, $\mathcal{L} \to \mathcal{R}$, $\mathcal{R} \to \mathcal{L}$, $a \to i = 1,2$ and $i \to a = 1, ..., 2m$. Note that μ_1, Λ_1 and $\bar{\Lambda}_1$ now refer to the *second* group in Table 3 – $SP(4n+2m-10)$, with $n \to m$ and $m \to 1$ – while μ_2, Λ_2 and $\bar{\Lambda}_2$ refer to the *first* group – $SP(2n-4)$, with $n \to m$ of Table 3. We will denote, as in Table 3, by λ the indices under $SP(2m-4)$, and with $\dot{\lambda}$ – the $SP(4m-8)$ ones.

We will only analyze the dynamics for the simplest case[15], $m = 3$. Then the dual is $SP(4) \times SU(2)$. The $SU(2)$ gauge group has a matter content of six doublets, $\frac{1}{\mu_1} G_{\lambda i}$, $J_{\lambda\lambda_1} p^{\lambda_1}_{\dot{\lambda}}$,

[14]For general n, m, with mass terms added for all fields, the gaugino condensates of the low-energy pure $SU(2)$ theories are determined by the symmetries:

$$\begin{aligned} \langle \lambda_1 \lambda_1 \rangle &= \epsilon_1 \, 32 \, \pi^2 \, \sqrt{m_X \, \Lambda_1^{5-n} \, \mathrm{Pf} \; m_{\mathcal{L}}} \; f_1(t) \\ \langle \lambda_2 \lambda_2 \rangle &= \epsilon_2 \, 32 \, \pi^2 \, \sqrt{m_X \, \Lambda_2^{5-m} \, \mathrm{Pf} \; m_{\mathcal{R}}} \; f_2(t) \; , \end{aligned} \tag{3.22}$$

where $\epsilon_{1,2} = \pm 1$, and $f_{1,2}$ are arbitrary functions of $t = (\Lambda_1^{5-n} \; \mathrm{Pf} \; m_{\mathcal{L}})/(\Lambda_2^{5-m} \; \mathrm{Pf} \; m_{\mathcal{R}})$. As in ref.[13], holomorphy, the large mass or small $\Lambda_{1,2}$ limits, and scale matching, allow us to conclude that $f_{1,2} \equiv 1$. From the Konishi anomaly equations, we can now determine the exact dependence of the vacuum expectation values on the masses and Wilsonian gauge couplings, which, for $n = 1$ coincide with the ones obtained from the superpotential (3.20).

[15]Analyzing e.g. the $m = 4$ case requires understanding the nonperturbative dynamics of $SP(4)$ with a traceless antisymmetric tensor and ten fundamentals and superpotential given in (2.11). For a superpotential of this type, this is an interesting unsolved problem. We note only that the deconfining method of [6] (see also [7], [9]) is not of immediate help in this case – the antisymmetric tensor reappears after dualizing once.

and is therefore in the confining phase. The $SU(2)$ mesons are $\mathcal{M} = \frac{1}{\mu_1^2} G_1 \cdot G_2$, $\mathcal{N}_{i\dot\lambda} = \frac{1}{\mu_1} G_i \cdot p_{\dot\lambda}$ and $\mathcal{K}_{\dot\lambda\dot\lambda'} = -p_{\dot\lambda} \cdot p_{\dot\lambda'}$. Rewriting the superpotential (2.11) in terms of these mesons, and adding the nonperturbative piece generated by the confining $SU(2)$, the superpotential becomes:

$$\begin{aligned} W &= \frac{-1}{4\ \mu_2\ \mu_1^2}\ \mathcal{R}_{ab}\ v^a \cdot \mathcal{K} \cdot v^b + \frac{1}{4\ \mu_1} \mathcal{L}\ r_i \cdot r^i + \frac{1}{2\ \mu_1\ \mu_2}\ Y_{ai}\ v^a \cdot r^i + \frac{1}{2}\ \mathcal{N}_i \cdot r^i \\ &+ \frac{1}{4\ \mu_1\ \mu_2^2}\ W_{ab}\ v^a \cdot v^b - \frac{1}{\bar\Lambda_2^3}\ \left(\mathcal{M}\ \mathrm{Pf}\mathcal{K} - \frac{1}{4}\ \mathcal{N}_i \cdot \mathcal{N}^i \cdot \mathcal{K} \right)\ , \end{aligned} \tag{3.23}$$

where $v^a \cdot \mathcal{K} \cdot v^b = v^a_{\dot\nu}\ J^{\dot\nu\dot\lambda}\ \mathcal{K}_{\dot\lambda\dot\mu}\ J^{\dot\mu\dot\rho}\ v^b_{\dot\rho}$ and $\mathcal{N}_i \cdot \mathcal{N}^i \cdot \mathcal{K} = \mathcal{N}_{i\dot\lambda}\ \mathcal{N}_{j\dot\lambda'}\ \varepsilon^{ij}\ \mathcal{K}_{\dot\nu\dot\nu'}\ \varepsilon^{\dot\lambda\dot\lambda'\dot\nu\dot\nu'}$. The $SU(2)$ mesons $\mathcal{N}^i_{\dot\lambda}$ and the quarks $r^i_{\dot\lambda}$ are massive. To find the masses of the heavy fields, it is convenient to decompose $\mathcal{K}_{\dot\nu_1\dot\nu_2} = J_{\dot\nu_1\dot\nu_2}\mu_1 X/(2\mu_2) + \tilde{\mathcal{K}}_{\dot\nu_1\dot\nu_2}$, using (2.10) with μ_1 and μ_2 interchanged. We denote by $\tilde{\mathcal{K}}$ the traceless part of $\mathcal{K}$. The equation of motion for the meson $\mathcal{M}$ implies

$$0 = \mathrm{Pf}\mathcal{K} = \frac{\mu_1^2 X^2}{4\ \mu_2^2} + \mathrm{Pf}\tilde{\mathcal{K}}\ . \tag{3.24}$$

The mass matrix of the fields $(r^i_{\dot\lambda}, \mathcal{N}^i_{\dot\lambda})$ can be read off eq.(3.23) and its Pfaffian is proportional to:

$$\begin{aligned} &\det \begin{pmatrix} \frac{\mathcal{L}}{\mu_1}\ J^{\dot\lambda_1\dot\lambda_2} & J^{\dot\lambda_1\dot\lambda_2} \\ J^{\dot\lambda_1\dot\lambda_2} & -\frac{\mu_1\ X}{\mu_2\ \bar\Lambda_2^3}\ J^{\dot\lambda_1\dot\lambda_2} - \frac{2}{\bar\Lambda_2^3}\ \tilde{\mathcal{K}}^{\dot\lambda_1\dot\lambda_2} \end{pmatrix} \\ &= \left(4\ \mathrm{Pf}\tilde{\mathcal{K}}\ \left(\frac{\mathcal{L}}{\mu_1\ \bar\Lambda_2^3} \right)^2 + \left(1 + \frac{\mathcal{L}\ X}{\mu_2\ \bar\Lambda_2^3} \right)^2 \right)^2 = \left(1 + \frac{2\ \mathcal{L}\ X}{c(3,1)\ \Lambda_1^4} \right)^2\ . \end{aligned} \tag{3.25}$$

In order to obtain the second equality in (3.25), we used (3.24), and the scale matching relation for the scale $\bar\Lambda_2$, eq.(2.12) with the appropriate replacements discussed earlier (i.e., $\bar\Lambda_2^3 = c(3,1)\Lambda_1^4/\mu_2$; to avoid confusion we have not interchanged m and n in $c(n,m)$). The mass matrix (3.25) is therefore non-degenerate, provided $X\,\mathcal{L} \neq \Lambda_1^4$. The requirement that the dual theory reproduces exactly the modification of the quantum moduli space in the electric theory thus fixes the constant

$$c(3,1) = -2\ , \tag{3.26}$$

which, using the recursion relations (2.24), (2.29) allows us to determine the constant in the scale matching relation (2.12). In this case we can integrate out the fields $\mathcal{N}$, r. We are left with an $SP(4)$ theory with a traceless antisymmetric tensor $\tilde{\mathcal{K}}$ (the traceless part of $\mathcal{K}$) and six fundamentals (the fields v^a) and a complicated superpotential, whose precise form can be obtained but is not essential to the subsequent discussion. It can be shown that this theory is in the confining phase and generates a nonperturbative superpotential:

$$W_{n.p.} \sim v^6\ \tilde{\mathcal{K}}^2\ . \tag{3.27}$$

The subsequent algebra is straightforward but somewhat tedious and we only describe it in words here. Adding (3.27) to (3.23) (after integrating out the heavy fields $\mathcal{N}$ and r), and rewriting it in terms of $SP(4)$ mesons gives us the required superpotential. From it one can see that all mesons gain mass, mixing with the singlets. The equations of motion for the singlets require that the expectation values of the heavy meson fields vanish (hence the precise form of the superpotential (3.27) is not essential, since it vanishes after the heavy mesons are integrated out). The only massless degrees of freedom that remain, finally, are X, Y_{ia} and $\mathcal{L}$ and the superpotential for them vanishes. At the origin – where all these light fields have zero expectation values – one again sees that the global symmetries are unbroken and that 't Hooft's conditions are saturated by these fields, exactly as in the case of the electric theory (and the $SP(2m-4)$ dual discussed above).

On the other hand, along the flat direction X $\mathcal{L} = \Lambda_1^4$, as follows from (3.25), the mass matrix is degenerate, and the massless spectrum of the dual $SP(4)$ theory now has eight fundamentals and a traceless antisymmetric tensor. By using the symmetries, we can deduce that no superpotential (which is nonsingular at the origin) can be written in terms of the mesons and baryons. Hence, new massless degrees of freedom have to descend in the low-energy theory, and the theory is probably in the non-Abelian Coulomb phase. However, as was remarked earlier (see footnote), presently we do not have a sufficient understanding of the non-Abelian Coulomb phase of the SP-theories with antisymmetric tensor and fundamental matter content to carry this analysis further. Even so, the agreement obtained so far is already a non-trivial check of the equivalence between this dual and the electric theory.

4 The Confining Models.

We end our study of the $SU(2) \times SU(2)$ theories by considering the confining models. Some of these models, namely the $[0,0], [1,0]$ and $[2,0]$, were studied in ref. [13]. In this section we study the remaining $[2,1]$ and $[1,1]$ models (we note that the latter were also studied in ref.[14]). We will see how the exact superpotential can be determined in these cases.

We first consider the $[2,1]$ model. It is convenient, in determining the exact superpotential, to consider the theory in the two limits $\Lambda_1 \gg \Lambda_2$ and $\Lambda_1 \ll \Lambda_2$. We start with the limit $\Lambda_1 \gg \Lambda_2$. In this limit $SU(2)_1$ has 6 doublets and its non-perturbative dynamics generates a superpotential:

$$W = -\frac{1}{\Lambda_1^3} \left(X \ \mathrm{Pf}\mathcal{L} - \frac{1}{4} \, V \cdot V \cdot \mathcal{L} \right) , \tag{4.1}$$

with $V \cdot V \cdot \mathcal{L} = \mathcal{L}_{ij} V_{\dot{\alpha} k} V_{\dot{\beta} l} \varepsilon^{ijkl} \varepsilon^{\dot{\alpha}\dot{\beta}}$. The low-energy $SU(2)_2$ now has 6 doublets, R_a and $\frac{1}{\Lambda_1} V_i$, and one expects a nonperturbative superpotential to be generated in this theory as well.

Eq.(4.1) can be viewed as giving rise to Yukawa couplings in this low-energy theory. The simplest guess to make for the $SU(2)_2$ non-perturbative contribution is to assume that it is unaffected by the presence of these Yukawa couplings. The exact superpotential would then be given by:

$$W = -\frac{X\ \mathrm{Pf}\mathcal{L} - \frac{1}{4}\ W\cdot\mathcal{L}}{\Lambda_1^3} - \frac{\mathcal{R}\ \mathrm{Pf}W - \frac{1}{4}\ Y^2\cdot W}{\Lambda_2^4\ \Lambda_1^3}\ , \tag{4.2}$$

where $Y^2\cdot W = Y_{ia}Y_{jb}W_{kl}\varepsilon^{ijkl}\varepsilon^{ab}$ and $W_{ij} = V_i\cdot V_j$.

This simple guess in fact turns out to be correct. There are several ways to see this. First, one can add masses for all the fields and show that the resulting expectation values agree with those determined by the Konishi anomaly. Second, one can add masses for a few fields and flow to the models analyzed in ref. [13]. One finds that eq. (4.2) correctly reduces to the superpotential for these models. Third, there are other terms consistent with the symmetries that can be written besides those above, for example one can take the ratio of any two terms in eq.(4.2) and obtain additional terms. However, such terms always result in singularities at points in field space where there is no physical reason to expect them. This is especially true from the point of view of the dual theory which in this case is a weakly coupled and completely Higgsed theory, so that no such singularities can occur in it. Finally, as we see below, proceeding in a similar way, we obtain exactly the same infra-red physics in the opposite limit, $\Lambda_2 \gg \Lambda_1$. This strongly suggests that no terms of order Λ_2/Λ_1 are being left out in eq.(4.2).

In the $\Lambda_2 \gg \Lambda_1$ limit, $SU(2)_2$ gets strong first and its dynamics generates the constraint (3.17). Below the scale Λ_2, a dilaton is dynamically generated, as discussed in section 3.2, and the Wilsonian coupling of $SU(2)_1$ is field-dependent and is determined by eq.(3.16) (with $m = 1$, and Λ_1 and Λ_2 interchanged). The low-energy $SU(2)_1$ now has 6 doublets. Proceeding as above, by adding the two superpotentials and solving the constraint for the $SU(2)_2$ meson $V_1\cdot V_2$, we find the following superpotential in this limit:

$$W = -\frac{\mathrm{Pf}\mathcal{L}}{\Lambda_1^3}\ \left(X - \frac{\Lambda_2^4}{\mathcal{R}}\right) + \frac{Y^2\cdot\mathcal{L}}{4\ \Lambda_1^3\ \mathcal{R}}\ . \tag{4.3}$$

Although the two superpotentials, eq. (4.3) and (4.2), look different they describe the same infrared physics. For example, (4.3) can be obtained from (4.2) by integrating out the field W which is massive along the flat direction with $\mathcal{R} \neq 0$. It can be shown that along other flat directions as well, (4.3) and (4.2) lead to the same massless spectrum and interactions. By adding mass terms and taking them to zero in various limits one can show that the moduli spaces are also identical, as eq. (4.3) too yields the correct vevs, given by (3.22) and the Konishi anomaly. In particular, the point of vanishing expectation values and unbroken global symmetries, which is obviously part of the moduli space of (4.2) is also part of the

moduli space in the $\Lambda_2 \gg \Lambda_1$ limit, eq. (4.3). At this point the theory exhibits confinement without chiral symmetry breaking and the required 't Hooft anomaly matching conditions are saturated by the fields X, $\mathcal{R}$ and Y_{ia}.

In view of all this evidence we conclude that eq.(4.2) (or equivalently eq.(4.3)) is the exact superpotential for the low-energy $[2,1]$ theory. As we saw above, this superpotential can be obtained by simply summing the contributions of the two groups together. We expect this to be true, generally, for product group theories when they have a quantum moduli space; their exact superpotentials should therefore be straightforward to determine.

It is worth emphasizing that even though the superpotential, eq.(4.2) (or equivalently eq. (4.3)) and the related description in terms of the moduli fields was derived initially in the limit when one of the gauge couplings was much stronger than the other, the evidence discussed above shows that it is valid more generally.

Finally, we discuss the $[1,1]$ model. The exact superpotential in this case can be determined by integrating out one flavor of the L fields. Starting with eq. (4.2) and integrating out in addition the heavy field W_{12} we arrive at the exact superpotential for the $[1,1]$ model (this form was obtained previously by the "integrating-in" procedure in ref.[14]):

$$W = L \ \left(\mathcal{L} \ \mathcal{R} \ X - \mathcal{L} \ \Lambda_2^4 - \mathcal{R} \ \Lambda_1^4 - Y^2 \right) \ . \tag{4.4}$$

where L is the Lagrange multiplier[16]. A few comments are in order about this superpotential:

i. It can also be derived by considering the theory in the two limits $\Lambda_1 \gg \Lambda_2$ and $\Lambda_1 \ll \Lambda_2$ and adding the contributions of the two groups, as was done for the $[2,1]$ models.

ii. As expected, it is symmetric under exchanging the two groups.

iii. As eq.(4.4) shows, the $[1,1]$ model has a quantum-deformed moduli space. It is interesting to note, that although each gauge group individually has a quantum-modified moduli space, and the origin is excluded from its quantum moduli space, in the product group theory it is possible to go to the origin[17]. At this point the global symmetries are unbroken, and 't Hooft's conditions are saturated by X, Y_{ia} and $\mathcal{L}$ (or equivalently, $\mathcal{R}$, since it has the same symmetries).

This concludes our discussion of the confining models.

[16] The same form is also obtained by starting with eq.(4.3) after suitably redefining the Lagrange multiplier.

[17] By the origin we mean the point where all the light fields have zero vevs and the global symmetries are unbroken. A heavy field like W_{12} which is a singlet under all global symmetries has a vacuum expectation value at this point though, due to the quantum-modified constraint.

Table 4: The field content and nonanomalous $U(1)$ symmetries of the $SU(N) \times SU(M)$ theory.

	$SU(N)$	$SU(M)$	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_R$
Q	$\square$	$\square$	0	0	$M - N_f$	0
$L_{i=1,\ldots,N_f-M}$	$\square$	**1**	N_f	0	M	$\frac{2(N_f-N)}{N_f-M}$
$\bar{L}_{I=1,\ldots N_f}$	$\overline{\square}$	**1**	$M - N_f$	0	0	0
$R_{a=1,\ldots,M_f-N}$	**1**	$\square$	0	M_f	$\frac{N(N_f-M)}{M_f-N}$	$\frac{2(M_f-M)}{M_f-N}$
$\bar{R}_{A=1,\ldots,M_f}$	**1**	$\overline{\square}$	0	$N - M_f$	0	0

5 The SU(N) × SU(M) Models.

In this section we consider a generalization of the $SU(2) \times SU(2)$ theory, with an $SU(N) \times SU(M)$ gauge group and the matter content of a single field transforming under both gauge groups, and additional fields transforming under the first or second group alone as fundamentals or antifundamentals. N_F will denote the number of flavors of the $SU(N)$ group, when $SU(M)$ is turned off. Likewise, M_F will denote the total number of flavors under the $SU(M)$ group. The particle content of the model, as well as the charges of the fields with respect to the nonanomalous $U(1)$ symmetries are given in Table 4. Note that in general these theories are chiral.

By varying the four parameters, N, N_f, M and M_f, one can explore the non-perturbative dynamics of these theories. We will not do so here. Rather we will content ourselves with discussing three aspects of these models. First, we will study the renormalization group flows in the space of the two gauge couplings in these theories. These flows can be analyzed in the vicinity of the two fixed points obtained by turning off one or the other gauge coupling. We will find a simple criterion to decide when the gauge coupling, which is initially turned off, is a relevant perturbation. We than use this criterion to show that the flows are locally consistent with the absence of a phase transition. Second, we will construct a dual theory by alternately dualizing the two gauge groups. Finally, we will analyze a subset of these theories with an $SU(N) \times SU(N-1)$ symmetry and show that they dynamically break supersymmetry after adding suitable Yukawa terms.

5.1 Renormalization Group Flows.

In this section we consider the RG flows in the space of the two gauge couplings of the $SU(N) \times SU(M)$ theories introduced above.

In $N = 1$ supersymmetric Yang-Mills theories there exists an exact relation between the

anomalous dimensions of the various fields Φ_i and the beta function [15] :

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \frac{3T(G) - \Sigma_i\ T(R_i)\ (1-\gamma_i)}{1 - T(G)\ \frac{\alpha}{2\pi}}. \tag{5.1}$$

where $\alpha = \frac{g^2}{4\pi}$, $T(R)\ \delta^{ab} = \mathrm{Tr}_R(T^a T^b)$ and $T(G) = T(R = \text{adjoint})$. γ_i is the anomalous dimension of the chiral superfield Φ_i and is related to the full scaling dimension by $\gamma_i + 2 = 2d_i$. Our subsequent discussion will rely mainly on the numerator on the RHS in eq. (5.1), denoted by num β

$$\text{num } \beta = -\frac{\alpha^2}{2\pi} \left[3T(G) - \Sigma_i\ T(R_i)(1-\gamma_i)\right]. \tag{5.2}$$

Note that fixed points can arise when this numerator vanishes[18].

For the $SU(N) \times SU(M)$ models with the field content of Table 4, the numerators of the beta functions for the two gauge groups[19] are given by:

$$\begin{aligned} \text{num } \beta_N &= -\frac{\alpha_N^2}{2\pi}\left[3N - N_f + \frac{M}{2}\ \gamma_Q + \frac{N_f - M}{2}\ \gamma_L + \frac{N_f}{2}\ \gamma_{\bar{L}}\right] \\ \text{num } \beta_M &= -\frac{\alpha_M^2}{2\pi}\left[3M - M_f + \frac{N}{2}\ \gamma_Q + \frac{M_f - N}{2}\ \gamma_R + \frac{M_f}{2}\ \gamma_{\bar{R}}\right] . \end{aligned} \tag{5.3}$$

As we show below, equations (5.3) allow us to deduce some nontrivial facts about the flow diagram in these models. There are three cases to consider:

i. Each gauge group is in the interacting non-Abelian Coulomb phase in the limit when the other gauge interaction is turned off, i.e. the inequalities $3N > N_f > 3N/2$ and $3M > M_f > 3M/2$ hold.

ii. One of the two groups, say $SU(N)$, is in the interacting non-Abelian Coulomb phase, when the $SU(M)$ interaction is turned off. On the other hand, the $SU(M)$ group, is in the magnetically free phase when the $SU(N)$ coupling is turned off. That is, $3N > N_f > 3N/2$ and $M + 1 < M_f < 3M/2$.

iii. Each group is in the magnetic free phase, in the absence of the other gauge coupling, i.e., $N + 1 < N_f < 3N/2$ and $M + 1 < M_f < 3M/2$.

In each case we will analyze the behavior of the RG flows in the neighborhood of the two fixed points obtained by turning one of the gauge couplings off. We will then speculate on the simplest global RG flows consistent with this local behavior. We discuss the first case at some length. The analysis in the other cases is quite analogous and we discuss it somewhat briefly.

[18] The numerator is proportional to the anomaly in the R-symmetry current that determines the scaling dimensions at the fixed point.

[19] Eq. (5.1) is valid for the beta functions of the gauge couplings in a product-group theory as well, as is clear, e.g. from the instanton derivation (first two papers in ref. [15]).

5.1.1 $3N > N_f > 3N/2$ and $3M > M_f > 3M/2$.

We start by setting $g_M = 0$ and considering the fixed point in the resulting $SU(N)$ theory. A great deal of evidence, from the large-N limit [16] and especially now from duality [2], shows that the $SU(N)$ theory has a non-trivial fixed point that is attractive (i.e. along the line $g_M = 0$) in the infra-red. At this fixed point the theory is simply $SU(N)$ SQCD with N_f flavors and the anomalous dimensions of all $SU(N)$-(anti)fundamentals are equal, $\gamma_Q = \gamma_L = \gamma_{\bar{L}} \equiv \gamma^*_{g^*_N,0}$. From the first of eqs.(5.3) we can then calculate the anomalous dimensions at the fixed point $(g^*_N, 0)$:

$$\gamma^*_{g^*_N,0} = -\frac{3N - N_f}{N_f} \ . \tag{5.4}$$

In order to understand the RG flows in the neighbourhood of this fixed point we need to know in addition the beta function of the $SU(M)$ group. In the vicinity of the fixed point $(g^*_N, 0)$ the anomalous dimensions are still given by (5.4), up to small corrections proportional to the deviation from the fixed point. Substituting the anomalous dimension of the field Q in the second of eqs.(5.3), we obtain for the numerator of the beta function of the $SU(M)$ gauge coupling

$$\begin{aligned} \text{num } \beta_M\Big|_{(g_N \simeq g^*_N, g_M \ll 1)} &= -\frac{\alpha_M^2}{2\pi} \left[3M - M_f + \frac{N}{2}\, \gamma^*_{g^*_N,0} + ... \right] \\ &\simeq \frac{\alpha_M}{2\pi} \left[\frac{N}{2\ N_f}\, (3N - N_f) - 3M + M_f \right] \ . \end{aligned} \tag{5.5}$$

It follows that the coupling g_M is irrelevant if

$$3M - M_f < \frac{N}{2\ N_f}\, (3N - N_f) \ , \tag{5.6}$$

and relevant if the inequality is reversed[20]. With this information in hand the RG flows are completely determined in the vicinity of the $(g^*_N, 0)$ fixed point.

We now turn to the $(0, g^*_M)$ fixed point. Since the fixed point for the pure $SU(M)$ theory is attractive in the infra-red we know that the g_M coupling is irrelevant at this fixed point. On considering the effects of the $SU(N)$ group in the neighbourhood of this fixed point we have, from an analysis very similar to that above, that the g_N coupling is irrelevant provided

$$3N - N_f < \frac{M}{2\ M_f}\, (3M - M_f) \ , \tag{5.7}$$

and is relevant if this inequality is reversed.

[20] In the case of an equality in (5.6), we cannot draw any rigorous conclusion about the sign of the beta function, without additional information about the terms denoted by the ellipses in eq. (5.5).

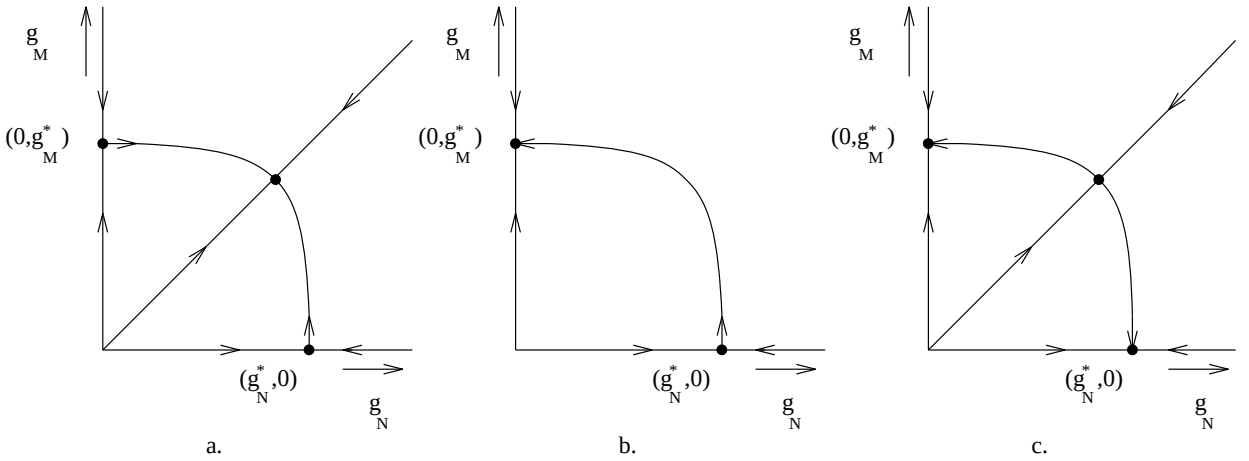

Figure 1: Three different RG flow diagrams in the space of the two gauge couplings. (a) and (b) are the simplest flows consistent with the behavior in the vicinity of the fixed points, $(g_N^*, 0)$ and $(0, g_M^*)$, while (c) is not realized (see text).

We find four possibilities depending on whether the two inequalities, (5.6) and (5.7) are met or not. We discuss these in turn.

First, we note that (5.6) and (5.7) cannot be simultaneously satisfied: iterating them once we obtain the inequality $3M - M_f \leq \frac{N}{2N_f}\frac{M}{2M_f}(3M - M_f)$, which is clearly violated, since $\frac{N}{2N_f}$ and $\frac{M}{2M_f}$ are both smaller than $\frac{1}{3}$ (recall that we are considering the theories in the conformal window). This rules out one possibility. Had it been allowed, there would have to be a phase boundary in the space of the two couplings and associated with it a phase transition as the two couplings are varied. For example, the simplest global flow diagram consistent with this possibility is shown in Fig. 1(c). One sees that for g_N small enough the theory flows to the $g_N = 0$ fixed point, in contrast when g_N is big enough it flows to the $g_M = 0$ fixed point.

There is some lore [1], [10], that phase transitions are not allowed in SUSY theories. As best as we can tell, this lore should apply to couplings in the superpotential. For such couplings holomorphy suggests that a phase boundary would have to be of (real) codimension two. But a surface of codimension two is not a boundary at all, since one can interpolate around it. Thus there are no phase boundaries and therefore no phase transitions can occur. It is not clear to us, if this lore is directly applicable in the present case. We note for example, that the gauge couplings under consideration are *not* the Wilsonian gauge couplings and their renormalizations involve the anomalous dimensions (eq. (5.1)) which are not holomorphic functions of the gauge couplings. Nevertheless, our results in this case are consistent with the absence of a phase transition. In fact, the subsequent discussion in this section will be consistent with the absence of phase transitions too. It is worth noting that this was equally true for our discussion of $SU(2) \times SU(2)$ duality.

We now return to considering the remaining possibilities presented by (5.6) and (5.7). If none of these inequalities hold, then at the fixed point for each gauge group the other gauge coupling is relevant (fig. 1(a)), and we expect the theory to flow to a nontrivial fixed point where both couplings are nonvanishing.

One would like to understand the global nature of these RG flows somewhat better. One limit in which they can be analyzed is the large-N limit first used for this purpose by Banks and Zaks [16]. In this limit, $N, N_f, M, M_f \to \infty$, with Mg_M^2 and Ng_N^2 kept fixed, and $N_f/N = 3 - \epsilon_N$ and $M_f/M = 3 - \epsilon_M$. One finds, in this case, that the flow diagram can be explicitly worked out in the region where $g_N^2 N$ and $g_M^2 M$ are both small. The resulting flow diagram is shown in Fig. 1(a). There is a unique fixed point (away from both the axes) and it is infra-red attractive with respect to both the gauge couplings.

Aside from the large-N limit one cannot in general make any rigorous statement about the global nature of the RG flows. Nevertheless it is worth noting that the simplest RG flow diagram consistent with the local analysis performed above continues to correspond to Fig 1(a). In fact this simple ansatz for the flow diagram is also the only one consistent with the absence of phase transitions. If there were another fixed point there would also necessarily be a phase boundary and therefore a possible phase transition.

Returning to the inequalities (5.6) and (5.7) we find there is one more possibility. Namely that only one of the two, (5.6) or (5.7) holds[21]. In this case again a large-N analysis can be performed. The resulting flow diagram is shown in Fig. 1(b) (for definiteness this diagram is drawn for the case when (5.7) is met but not (5.6)). One sees that in this case the theory flows to the corresponding Seiberg fixed point in the infra-red. For the general case again one cannot make any rigorous statement but Fig. 1(b) continues to be the simplest flow diagram consistent with the local information at hand. It is also the only diagram without any phase transitions.

5.1.2 $\mathbf{3N > N_f > 3N/2}$ and $\mathbf{M + 1 < M_f < 3M/2}$.

In the case when one, or both, theories are in the magnetic free phase, the flows have to be discussed separately. First we consider the case when one of the groups, say $SU(N)$, is in the interacting non-Abelian Coulomb phase (i.e. $N_f > 3N/2$), while the other is in the magnetic free phase ($M + 2 \leq M_f \leq 3M/2$). Turning on weakly the $SU(M)$ gauge coupling at the $SU(N)$ fixed point, we find the same condition (5.6) for g_M to be irrelevant. This condition cannot be satisfied now: note that for $3N/2 < N_f < 3N$, the r.h.s. of the inequality (5.6)

[21] The region in the 4-dimensional parameter space where, eq.(5.6), $N_f \geq M, M_f \geq N$, and $3N > N_f > 3N/2, 3M > M_f > 3M/2$ hold simultaneously, is quite complicated. However, it is easy to show that it is not empty, by finding a particular point, e.g. $N = 6, N_f = 10, M = 8, M_f = 22$, that satisfies all inequalities.

obeys $N(3N-N_f)/(2N_f) < N/2$. On the other hand, for our model $N \leq M_f$ (see Table 4). Therefore, from (5.6) we obtain that $3M - M_f < M_f/2$, or $M_f > 2M$, which contradicts $SU(M)$ being in the magnetic free phase. This implies that g_M is always relevant at the $(g_N^*, 0)$ conformal fixed point. With the analysis of the flows in the vicinity of the $(g_N^*, 0)$ fixed point complete, we now turn to the $(0, g_M^*)$ fixed point. When $g_N = 0$, the theory flows to a magnetic free phase. The gauge group is $SU(M_f - M)$ and the degrees of freedom are the gauge singlet mesons $Q \cdot \bar{R}$ and $R \cdot \bar{R}$ and the dual quarks. Weakly gauging the flavor subgroup $SU(N) \subset SU(M_f)_L$, we find that it has $N_f + M_f - M$ flavors, i.e. $M_f - M$ more flavors than at the ultraviolet fixed point. Its beta function near the infra-red free magnetic fixed point is approximately given by

$$\beta_N \simeq -\frac{\alpha_N^2}{2\pi} \ [\ 3N - (M_f + N_f - M)\ ] \ . \tag{5.8}$$

Therefore, the coupling g_N is irrelevant or relevant at this fixed point depending on whether the following inequality is met or not:

$$3N - N_f < M_f - M \ . \tag{5.9}$$

This presents us with two possibilities, both of which can be realized (in terms of the allowed values of N, N_f, M and M_f). If (5.9) is met, the simplest ansatz for the RG flows would be that the theory flows to the magnetic free phase of $SU(M)$ in the infrared, with the $SU(N)$ gauge coupling becoming irrelevant (i.e. the flow of Fig. 1(b))[22]. If the inequality (5.9) is reversed, then the simplest RG flow diagram would correspond to having one non-trivial fixed point away from both axes, and attractive in the infra-red as in Fig. 1(a).

5.1.3 $\mathbf{N + 2 \leq N_f \leq 3N/2}$ and $\mathbf{M + 2 \leq M_f \leq 3M/2}$.

Finally we can consider the case when both groups are in the magnetic free phase, i.e. $N+2 \leq N_f \leq 3N/2$ and $M+2 \leq M_f \leq 3M/2$ hold. In this case, the beta function for g_N is still given by eq. (5.8), in the vicinity of the $(0, g_M^*)$ fixed point. In an analogous manner analyzing the g_M coupling in the vicinity of the $(g_N^*, 0)$ coupling gives the beta function for g_M as:

$$\beta_M \simeq -\frac{\alpha_M^2}{2\pi} \ [\ 3M - (N_f + M_f - N)\ ] \ . \tag{5.10}$$

Thus g_M is irrelevant if the

$$3M - M_f < N_f - N \tag{5.11}$$

condition is met.

[22] An explicit solution is e.g. $N = 26, N_f = 77, M = 22, M_f = 24$. Note that $SU(N)$ is close to becoming infrared free, $N_f = 3N - 1$; gauging the $SU(M)$ flavor symmetry is enough to drive g_N infrared free.

Table 5: The field content of the dual of $SU(N) \times SU(M)$.

	$SU(N_f - N)$	$SU(M_f + N_f - N - M)$
$\bar{p}$	$\overline{\square}$	$\overline{\square}$
l^i	$\square$	**1**
r^a	**1**	$\square$
v^I	**1**	$\square$
$\bar{r}^A$	**1**	$\overline{\square}$
G_a	$\square$	**1**

However in this case it is easy to show that neither (5.9) nor (5.11) can hold. Eq. (5.9) implies $3N - N_f < M_f - M \leq M/2 \leq N_f/2$, which requires $N_f > 2N$ and contradicts $SU(N)$ being in the magnetic free phase; a similar contradiction follows from (5.11). Therefore in this case, we expect the theory to flow to a non-trivial fixed point. The correspondingly simplest flow diagram in this case is given by Fig. 1(a).

5.2 Duality in SU(N) × SU(M).

In this section we briefly discuss how dual theories for the $SU(N) \times SU(M)$ models can be constructed by alternately dualizing the two groups. The analysis is very close to that of the $SU(2) \times SU(2)$ theories which were discussed at length in Section 2. Consequently we do not describe the construction of the duals in detail and mainly present the resulting final form.

The field content of the $SU(N) \times SU(M)$ theory was described in Table 4. As was discussed in Section 2 two kinds of duals can be constructed: those in which one of the groups is dualized and those in which both groups are dualized. Here we only consider the duals obtained by dualizing both the $SU(N)$ and $SU(M)$ groups. For definiteness we describe the dual obtained by dualizing the $SU(N)$ group followed by the $SU(M)$ group below. This dual theory has an $SU(N_f - N) \times SU(M_f + N_f - N - M)$ gauge symmetry. The matter content consists of fields transforming under the dual gauge group as shown in Table 5 (we explicitly display the nonabelian global symmetries indices; they have the same ranges as in Table 4, and lower (upper) indices denote fundamental (antifundamental) representations).

In addition the dual theory contains the following gauge singlet fields:

$$\begin{aligned} Y_{IA} &= \bar{L}_I \cdot Q \cdot \bar{R}_A \\ \mathcal{R}_{aA} &= R_a \cdot \bar{R}_A \\ \mathcal{L}_{iI} &= L_i \cdot \bar{L}_I \ , \end{aligned} \tag{5.12}$$

whose charges with respect to the nonanomalous global symmetries can be determined from

Table 4 and their definitions (5.12). The dual theory also has a superpotential given by:

$$W = -\frac{1}{\mu_1\ \mu_2}\ \mathcal{L}_{iI}\ l^i \cdot v^I \cdot \bar{p} + \frac{1}{\mu_2}\ G_a \cdot r^a \cdot \bar{p} + \frac{1}{\mu_2}\ \mathcal{R}_{a\,A}\ r^a \cdot \bar{r}^A + \frac{1}{\mu_1\ \mu_2}\ Y_{IA}\ v^I \cdot \bar{r}^A\ . \qquad (5.13)$$

The strong coupling scales of the gauge groups in the dual theories obey matching relations which can be derived similarly to the corresponding relations in the $SU(2) \times SU(2)$ theories.

The detailed analysis of these dual theories is left for future study. We expect that such an analysis will help in understanding the non-perturbative behavior of the $SU(N) \times SU(M)$ theories, much as it did in the $SU(2) \times SU(2)$ case.

We conclude with one comment. The process of alternately dualizing the groups described above can be continued to construct other dual theories. In this case a finite set of duals of the $SU(N) \times SU(M)$ theories is generated. Starting with the second dual – the $SU(N_f - N) \times SU(N_f + M_f - N - M)$ theory constructed above – and dualizing the $SU(N_f - N)$ gauge group we obtain a third dual: an $SU(M_f - M) \times SU(N_f + M_f - N - M)$ gauge group with $N_f + M_f - M - N$ flavors of $SU(M_f - M)$ and $N_f + M_f - M$ flavors of $SU(N_f + M_f - N - M)$. This theory has the same gauge group, field content and superpotential as the second dual of the $SU(N) \times SU(M)$ theory, constructed by dualizing the $SU(M)$ gauge group first. In other words, the chain of duals that can be constructed by alternately dualizing the two gauge groups closes.

5.3 SU(N) × SU(N − 1) and Supersymmetry Breaking.

In this section we consider a subclass[23] of the $SU(N) \times SU(M)$ models: the models with $N_f = N-1$, $M = N-1$, $M_f = N$, which we will refer to as the $SU(N) \times SU(N-1)$ models. We will see that these models dynamically break supersymmetry after suitable Yukawa couplings are added in the superpotential. The $SU(N) \times SU(N-1)$ models are further generalizations of the $SU(3) \times SU(2)$ model [17] of dynamical supersymmetry breaking. Other generalizations have been considered in [18], [19], [20].

We begin by analyzing the classical moduli space of these theories. It is described, [17], [21], by the gauge invariant chiral superfields Y_{IA}, defined in (5.12), the field $\bar{\mathcal{B}} = b^\alpha (\bar{L}^{N-1})_\alpha$, where $b^\alpha = (Q^{N-1})^\alpha$ (α is the $SU(N)$ index), and $\bar{b}^A = (\bar{R}^{N-1})^A$. These fields obey the classical constraints $Y_{IA}\ \bar{b}^A = 0$ and $\bar{\mathcal{B}}\ \bar{b}^A \sim (Y^{N-1})^A$. Adding a renormalizable tree-level superpotential

$$W_{tree} = \lambda^{IA}\ Y_{IA}\ , \qquad (5.14)$$

one finds that for a Yukawa matrix of maximal rank $(N-1)$ the Y_{IA} and $\bar{\mathcal{B}}$ flat directions are lifted. To see this, consider the $\bar{R}_A$ equation of motion. It implies that $\bar{L}_I \cdot Q_{\dot{\alpha}} = 0$ (recall

[23] The models for general M are considered in a forthcoming publication [27].

that rank $\lambda = N-1$), which in turn implies that $Y_{IA} = 0$ and $\bar{\mathcal{B}} \sim \det(\bar{L}_I \cdot Q_{\dot{\alpha}}) = 0$. However, the flat directions corresponding to $\bar{\mathcal{B}} = 0$, $\bar{b}^A \neq 0$ and $Y_{IA} = 0$ are not lifted. While the $SU(N-1)$ group is completely broken along these directions the $SU(N)$ group is completely unbroken. These $\bar{b}^A \neq 0$ flat directions can be lifted in the classical theory if in addition to the Yukawa coupling in eq. (5.14) we add a term $\alpha_A \bar{b}^A$ so that the full tree-level superpotential is:

$$W_{tree} = \lambda^{IA}\ Y_{IA} + \alpha_A \bar{b}^A. \tag{5.15}$$

Performing an analysis similar to the one above, one can show that all classical flat directions are lifted once α_A and λ^{IA} are chosen such that $\lambda^{IA}\alpha_A \neq 0$.

We turn next to the quantum theory and first study it in the limit $\Lambda_1 \ll \Lambda_2$. In this limit, the $SU(N-1)$ group is confining, with N flavors, and nonperturbative effects generate [4] a superpotential

$$W = \frac{b^\alpha\ M_{\alpha A}\ \bar{b}^A - \det\ M}{\Lambda_2^{2N-3}}, \tag{5.16}$$

where the mesons of $SU(N-1)$ are $M_{\alpha A} = Q_\alpha \cdot \bar{R}_A$, and the baryons b^α and $\bar{b}^A$ were defined above. Now we weakly gauge the $SU(N)$ global symmetry. The low-energy $SU(N)$ gauge group has N flavors, with $M_{\alpha A} \sim \square$ and $b^\alpha, \bar{L}_I^\alpha \sim \overline{\square}$. Consequently, the $SU(N)$ gauge group confines as well and dynamically generates a superpotential. The scale of the low-energy $SU(N)$ theory is $\Lambda_{1_L}^{2N} = \Lambda_1^{2N+1}/\Lambda_2$. Its mesons are the fields Y_{IA}, the field $\mathcal{P}_A = b \cdot M_A$ (this field vanishes classically), while the baryons are $\bar{\mathcal{B}}$ and $\mathcal{B} = \det M$ (which vanishes classically as well). The exact superpotential of this model is then given by[24]:

$$W = \frac{\mathcal{P}_A\ \bar{b}^A - \mathcal{B}}{\Lambda_2^{2N-3}} + A\ \left(\mathcal{P} \cdot Y^{N-1} - \bar{\mathcal{B}}\ \mathcal{B} - \Lambda_1^{2N+1}\Lambda_2^{2N-3}\right)\ . \tag{5.17}$$

We now add the tree-level perturbation $W_{tree} = \lambda^{IA} Y_{IA} + \alpha_A \bar{b}^A$. In order to lift all classical flat directions we choose, $\lambda^{IA} = \delta^{IA}\lambda^I$, if $A < N$, $\lambda^{IN} = 0$, and $\alpha_A = \delta_{1A}\alpha$. The F-term equations of motion for the fields $\bar{b}^A$ that follow from this superpotential then imply that :

$$\mathcal{P}_1/\mathcal{P}_N \to \infty\ , \tag{5.18}$$

while the F-term equations for $\mathcal{P}_A$ and Y_{IA} imply that

$$\mathcal{P}_1/\mathcal{P}_N \to 0\ . \tag{5.19}$$

[24]This superpotential can be justified by arguments analogous to those of Section 4. The form of the quantum modification of the constraint in (5.17) arises after taking account of the scale Λ_{1_L} of the low-energy $SU(N)$ theory, as well as the fact that some of its "quarks" are $SU(N-1)$-composites and are normalized with appropriate powers of Λ_2.

In deriving eq. (5.19) it is useful to think in terms of the mesons of $SU(N)$, $\mathcal{P}_A$ and Y_{IA} with masses, $\bar{b}^A/\Lambda_2^{2N-3}$ and λ^{IA} respectively. The vevs of these mesons can be expressed in terms of their masses in the standard way and this leads to eq. (5.19) for $\mathcal{P}_1$ and $\mathcal{P}_N$.

Clearly, eq. (5.18) and eq. (5.19) are not compatible. Thus not all of the F-term conditions can be satisfied and the theory breaks supersymmetry.

One assumption made in the discussion above was that the Kähler potential is nonsingular in terms of the light moduli fields $Y_{IA}, \bar{\mathcal{B}}$ and $\bar{b}^A$. There are two kinds of singularities one might worry about: those that can occur in the finite region of moduli space and those that can occur at the boundaries of moduli space (when some fields go to ∞). We do not expect singularities in the finite region of moduli space in the cases considered here. For example, even though we do not do so, the description used above can be derived from a weakly-coupled, completely Higgsed, dual theory in which no singularities can occur in the finite region of moduli space. The finite region of moduli space can also be studied, ref. [23], [24], by adding one more flavor of the L fields and analyzing the resulting theory which now has a quantum moduli space and showing that the moduli fields saturate the 't Hooft anomalies. Finally, one does not expect singularities at the boundary of moduli space to be relevant to the discussion of SUSY breaking, since all flat directions are lifted classically and we therefore do not expect to be driven to infinite field values in the quantum theory.

It is also worth commenting on the relation between R symmetries and SUSY breaking in these models. First we consider the choice of the tree-level superpotential made in the discussion of SUSY breaking above; $\lambda^{IA} = \delta^{IA}\lambda^I$, if $A < N$, $\lambda^{IN} = 0$, and $\alpha_A = \delta_{1A}\alpha$. In this case one finds that the superpotential eq. (5.15) preserves a flavor-dependent, non-anomalous R symmetry. The charge assignments of the fields under this symmetry are, $\bar{R}_N \sim 2(3N - N^2 - 1)/(N-1)$, $\bar{R}_{A<N} \sim 2N/(N-1)$, $Q \sim -2/(N-1)$ and $\bar{L}_I \sim 0$. Thus, the fact that this model breaks supersymmetry, as we found above, is in accordance with the discussion of ref. [22].

In fact we could have used this R symmetry to argue that SUSY is broken. For this purpose note that all the flat directions are lifted in the classical theory with the superpotential of eq. (5.15). Furthermore, all the gauge invariant moduli, entering the superpotential eq. (5.17), except for $\mathcal{P}_A$ with $A \neq N$, are charged under the R symmetry. Therefore, if all the F-term conditions are met the R symmetry must be broken. Since there are no classical flat directions we conclude that SUSY is broken. The only other alternative is that the F-term constraints are not all met but then again SUSY must be broken.

In contrast consider the case of a tree-level superpotential with $\lambda^{IA} = \delta^{IA}\lambda^I$, if $A < N$, $\lambda^{IN} = 0$, α_1 and $\alpha_N \neq 0$ and all other $\alpha_A = 0$. Since the condition $\lambda^{IA}\alpha_A \neq 0$ is met all the flat directions are lifted classically but in this case one can show that there is no non-anomalous

R symmetry. However, an argument similar to the one given above, eq. (5.18) and eq. (5.19), allows us to conclude that SUSY is broken in this case as well.

We close this section with some comments[25] on the case of vanishing α. As we saw above, in this case, the classical flat direction with $\bar{b}^A \neq 0$ is not lifted. Along this flat direction, the $SU(N-1)$ gauge group is completely broken, while the $SU(N)$ group is unbroken. Thus one might expect quantum effects to play an important role along it. For simplicity, we set $\lambda^{IA} = \delta^{IA}\lambda^I$, if $A < N$ and $\lambda^{IN} = 0$ as above. In this case, there is one solution to the F-term equations obtained from the exact superpotential given by the sum of eq. (5.17) and eq. (5.14). This solution involves $\mathcal{P}_A \to 0$, $\bar{b}^N \to \infty$ and $Y_{II} \to \infty$ for $I = 1 \ldots N-1$. This "runaway" solution is somewhat surprising, since $Y_{IA} \neq 0$ is not a classically flat direction. To understand the origin of this solution consider the flat direction along which the fields $\bar{R}^{\dot{\alpha}}_A$, with $A < N$ have vacuum expectation values which we denote as $\bar{R}^{\dot{\alpha}}_A = \bar{R}\delta^{\dot{\alpha}}_A$. The flat direction then corresponds to varying $\bar{R}$. When $\bar{R} \to \infty$, the Yukawa term in W_{tree} gives a large mass to all $SU(N)$ flavors. The low-energy theory along this classical flat direction is then a pure $SU(N)$ gauge theory. Gaugino condensation in this low-energy theory generates a superpotential $W \sim \Lambda_{1L}^3 = (\Lambda_1^{2N+1}\ \mathrm{det}\lambda\ \bar{R}^{N-1})^{\frac{1}{N}}$. The gradient of this superpotential with respect to $\bar{R}$ is proportional to $\bar{R}^{-\frac{1}{N}}$, and pushes the field $\bar{R}$ to infinity, thus restoring supersymmetry. Strictly speaking, the behavior of the energy along this direction depends on both the superpotential and the Kähler potential. Since, as mentioned above, the infra-red $SU(N)$ theory is strongly coupled along this flat direction, the Kähler potential is difficult to determine exactly. Nevertheless, some preliminary analysis suggests that the corrections to the classical Kähler potential for $\bar{R}$ are small along this direction, leading to the conclusion that SUSY is probably restored.

It is also worth mentioning that the behavior of the superpotential along the baryonic flat direction discussed above can also be recovered from the exact superpotential, eq. (5.17). On adding the tree-level superpotential eq. (5.14), one can solve for all the other fields through their F-term equations in terms of the antibaryon $\bar{b}^N$. This gives a superpotential $W \sim (\Lambda_1^{2N+1}\ \mathrm{det}\lambda\ \bar{b}^N)^{\frac{1}{N}}$, which is identical to the one obtained above once one identifies $\bar{R}^{N-1}$ with $\bar{b}^N$.

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A Notations. Duality and Scale Matching for SP(2N).

We take $SP(2N)$ to denote the SP group whose fundamental representation is of dimension $2N$ so that $SP(2) = SU(2)$. The dimension of the $SP(2N)$ group is $N(2N+1)$. The generators T^a for a representation R obey $\mathrm{Tr}T^aT^b = T(R)\delta^{ab}$, where $T(\square\!\square) = N+1$, $T(\square) = T(\overline{\square}) = 1/2$ and $T(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = N-1$. Here $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ is a traceless antisymmetric tensor, and $\square\!\square$ is the adjoint representation (symmetric tensor).

The one-loop coefficient of the beta function of the $SP(2N)$ theory with $2N_f$ fundamentals $Q_{i\lambda}$, $(i = 1, ..., 2N_F,\ \lambda = 1, ..., 2N)$ is $b_0 = 3T(\square\!\square) - 2N_fT(\square) = 3N + 3 - N_f$. The D-flat directions of the electric theory are described by the gauge invariant chiral superfields (the $SP(2N)$-"mesons")

$$M_{ij} = Q_{i\lambda_1}\ J^{\lambda_1\lambda_2}\ Q_{j\lambda_2} \equiv Q_i \cdot Q_j\ . \tag{A.1}$$

Eq. (A.1) defines our summation convention: we always take the SP-fundamentals to have lower indices and raise (lower) SP indices using the $SP(2N)$ invariant antisymmetric tensors

$$J^{\mu\nu} = \mathrm{diag}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ ...\ , \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \tag{A.2}$$

and

$$J_{\mu\nu} = \mathrm{diag}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ ...\ , \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}\ , \tag{A.3}$$

such that $J^{\mu\nu}J_{\nu\lambda} = \delta^\mu_\lambda$ and $Q^\alpha = J^{\alpha\beta}Q_\beta$. For $SP(2)$ we denote the invariant tensor by $\varepsilon_{\alpha\beta}$ with $\varepsilon^{12} = \varepsilon_{21} = 1$, in accord with (A.2, A.3).

For $N_f \leq N+2$, the theory is confining [11]. The $N_f = N+2$ theory has a dynamically generated superpotential:

$$W = -\frac{\mathrm{Pf}M}{2^{N-1}\ \Lambda^{2N+1}}\ , \tag{A.4}$$

where Λ is the scale of the $N+2$ flavor theory. The superpotentials for $N_f < N+2$ can be obtained from (A.4) by integrating out extra flavors and using the scale matching relations (A.7) given below. The Pfaffian of a $2N_f$-dimensional antisymmetric matrix is defined as

$$\mathrm{Pf}M = \frac{1}{2^{N_f}\ N_f!}\ \varepsilon^{i_1i_2...i_{2N_f-1}i_{2N_f}}\ M_{i_1i_2}\ ...\ M_{i_{2N_f-1}i_{2N_f}} \equiv \frac{M^{N_f}}{2^{N_f}\ N_f!}\ ,$$

and its square equals the determinant. We use $\varepsilon^{123...(2N_f-1)(2N_f)} = +1$.

For $N_f > N+2$ the theory has a dual description in terms of an $SP(2N_f - 2N - 4)$ gauge theory [11], with $2N_F$ fundamentals and additional gauge singlets, M_{ij}, which are mapped to

the mesons of the electric theory (A.1), and a superpotential

$$W = \frac{1}{4\,\mu}\; M_{ij}\; q^i \cdot q^j \;. \tag{A.5}$$

The fields q^i are the dual quarks and are in the antifundamental representation of the $SU(2N_f)$ global flavor symmetry. The strong coupling scale Λ of the electric theory, the strong coupling scale $\bar{\Lambda}$ of the magnetic theory and the parameter μ in (A.5) obey the scale matching relation:

$$\Lambda^{3N+3-N_f}\; \bar{\Lambda}^{2N_f-3N-3} = 16\; (-)^{N_f-N-1}\; \mu^{N_f} \;. \tag{A.6}$$

Upon integrating out a flavor of $SP(2N)$ fundamentals of mass M, the scale Λ_L of the low-energy $SP(2N)$ theory with $N_f - 1$ flavors is related to the scale of the N_f-flavor theory by the $\overline{DR}$ threshold relation [26]:

$$\Lambda_L^{3N+3-(N_f-1)} = M\Lambda^{3N+3-N_f} \;. \tag{A.7}$$

Along a flat direction, such that, say, $M_{12} = Q_1 \cdot Q_2 = v^2 \neq 0$, the $SP(2N)$ theory with N_f flavors breaks to $SP(2N-2)$ with $N_f - 1$ flavors. The scale Λ_L of the low-energy $SP(2N-2)$, $N_f - 1$ flavor theory is given by the threshold relation at the mass scale of the heavy $SP(2N)$ vector bosons that transform as fundamentals of the unbroken $SP(2N-2)$ group:

$$\Lambda_L^{3(N-1)+3-(N_f-1)} = \frac{2}{v^2}\; \Lambda^{3N+3-N_f} \;. \tag{A.8}$$

Table 6: The anomalous $U(1)$ symmetries in the $[n,m]$ models and their duals.

	$U(1)_Q$	$U(1)_1$	$U(1)_2$	$U(1)_X$	$U(1)_{\mu_1}$	$U(1)_{\mu_2}$
$Q_{\alpha\dot{\alpha}}$	1	0	0	0	0	0
$L_{\alpha i}$	0	1	0	0	0	0
$R_{\dot{\alpha}a}$	0	0	1	0	0	0
Λ_1^{5-n}	2	$2n$	0	$2-2n$	0	0
Λ_2^{5-m}	2	0	$2m$	$2-2m$	0	0
X	2	0	0	0	0	0
$\mathcal{L}_{ij}$	0	2	0	0	0	0
$\mathcal{R}_{ab}$	0	0	2	0	0	0
Y_{ai}	1	1	1	0	0	0
$\frac{1}{\mu_1}V_{\dot{\alpha}i}$	1	1	0	0	-2	0
q_λ^α	-1	0	0	1	1	0
l_λ^i	0	-1	0	1	1	0
$\Lambda_2'^{\,8-2n-m}$	4	$2n$	$2m$	$4-2n-2m$	$-2n-4$	0
p_λ^λ	1	0	0	0	-1	1
$A'_{\lambda_1\lambda_2}$	-2	0	0	2	2	-2
v_λ^i	-1	-1	0	1	2	1
r_λ^a	0	0	-1	1	0	1
$\frac{1}{\mu_2}G_{\lambda a}$	-1	0	1	1	1	-2
W_{ij}	2	2	0	0	0	0
$\bar{\Lambda}_2^{4n+2m-10}$	-4	$-2n$	$-2m$	$2n+2m-4$	$2n+4$	$4n+2m-4$
$\bar{\Lambda}_1^{5-2m}$	2	0	$2m$	$2-2m$	-2	$-2m+2$
μ_1	0	0	0	0	2	0
μ_2	0	0	0	0	0	2

B The Anomalous Symmetries

Table 6 gives all anomalous $U(1)$ symmetries in the $[n,m]$ model and its first (Table 2) and second (Table 3) duals. $U(1)_X$ is an anomalous R symmetry, under which the superspace coordinate θ_α has charge unity. The nonanomalous $U(1)$ and $U(1)_R$ from Tables 1-3 are linear combinations of the symmetries in Table 6.

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