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FERMILAB-Pub-95-349-T

FERMILAB-PUB-95/349-T
NSF-ITP-95-134
hep-th/9510241

# THREE GENERATIONS IN THE FERMIONIC CONSTRUCTION

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We obtain three generation $SU(3)_c \times SU(2)_L \times U(1)_Y$ string models in all of the exactly solvable $(0,2)$ constructions sampled by fermionization. None of these examples, including those that are symmetric abelian orbifolds, rely on the $Z_2 \times Z_2$ orbifold underlying the NAHE basis. We present the first known three generation models for which the hypercharge normalization, $k_1$, takes values smaller than that obtained from an $SU(5)$ embedding, thus lowering the effective gauge coupling unification scale. All of the models contain fractional electrically charged and vectorlike exotic matter that could survive in the light spectrum.

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## 1. Introduction

Recent developments in superstring model building have focused on constructions with $(N_L, N_R)$=$(0,2)$ world-sheet supersymmetry [1,2,3,4,5]. This is in part due to the difficulty in obtaining low numbers of generation-anti-generation pairs in the simpler class of three generation $(2,2)$ compactifications [6,7,8]. Despite these technical advances, the sample of three generation models available in sufficient detail to enable phenomenological analyses have remained the examples of symmetric $(0,2)$ orbifold models, obtained in the $Z_3$ orbifold [9,10,11,12] and free fermionic constructions [13,14].

There are new developments in understanding strongly coupled string theory which could lead to a different formulation better suited to exploring low energy string theory. These developments have thus far had little direct impact on our understanding of $N$=1 string theories but this may change. We caution the reader at the outset that it is extremely unlikely that every feature of the tree superpotential and massless spectrum of any *particular* classical $N$=1 heterotic vacuum survives quantum corrections. Thus, the objective of such phenomenological analyses is not to arrive by accident at a fully realistic model but rather to sample vacua for *generic* features that would be unanticipated in traditional unified field theories.

In this paper, we resolve the problem of obtaining *odd* generation number, with generic gauge group and generic matter content, in the fermionic construction [15,16,17]. We will use the new formalism for real fermions developed in [2]. It is well-known that the free fermionic models [16,18] are equivalent to abelian orbifold models, with symmetric $Z_2 \times Z_2$ point group twists and quantized $Z_N$ Wilson lines. The real fermion construction, on the other hand, *samples* the full range of exactly solvable $(0,2)$ constructions. This includes the more generic class of asymmetric and nonabelian orbifolds [19]. It also includes an asymmetric generalization of the Gepner construction: compactifications based on tensor products of 18 right and 44 left-moving $c$<1 conformal field theory building blocks.

As a beginning, we provide fermionic realizations of seven new exact conformal field theory (cft) solutions embedding three generations of chiral superfields transforming under $SU(3)_c \times SU(2)_L \times U(1)_Y$, realized at Kac-Moody level one, along with the Higgs multiplets of the minimal supersymmetric standard model (MSSM). All of these solutions contain an anomalous gauged $U(1)$ flavor group at tree level. We can analyse the tree superpotential for all possible flat directions along which this anomalous $U(1)$ can be broken, giving families of nearby supersymmetric ground states [20]. The solutions also contain nonanomalous gauged $U(1)$ flavor groups. We find considerable freedom in the separation of the hypercharge embedding from the nonanomalous $U(1)$'s. A consistent vacuum shift to remove the anomaly and a consistent embedding of hypercharge defines a string model. We can then compute the effective superpotential to arbitrarily high order in the nonrenormalizable contact terms of these models, thus making phenomenological analysis possible.

The examples in this paper contradict some of the folklore of fermionic string model building. There is no indication of the so-called "unique" NAHE basis in the fermionization of any of our three generation examples, *including* the three examples based on free fermions, i.e., Ising and Weyl fermions, alone. It has also been speculated in the literature that no fermionic constructions exist of heterotic string models based on embeddings of the group weights and hypercharge assignments of the quarks and leptons of the MSSM with $k_1 < \frac{5}{3}$ [21]. We find several explicit counter-examples. The absence of a lower bound on $k_1$ appears in fact to be quite generic and is independent of the fermionic construction.

## 2. Review of Construction

To obtain genuinely distinct three generation models in the fermionic construction with generic gauge groups and generic matter content, we have introduced several new features in the underlying fermionic representation theory. Modular invariance of the one-loop vacuum amplitude and associativity of the vertex operator algebra restricts fermionic realizations of conformal field theory solutions to string theory as follows. The individual Majorana-Weyl world-sheet fermions must be paired into one of three possible fermionic cfts: Weyl fermions, Ising fermions, or in blocks of chiral Ising fermions. The consistent choices of spin structure for blocks of chiral Ising fermions allowed by associativity of the fermionic cfts were analyzed in [2].

### *2.1. Holomorphic Rank Reduction*

If all of the chiral Ising fermions in a block are left-moving, this corresponds to a holomorphic cft of central charge $c_m$=8, 12, 14, 15, 16, $\ldots$ 22 [2]. Such holomorphic cfts give rise to rank reduction in the string model. The earliest known example is the tachyonic ten-dimensional modular invariant discovered in [15] with 248 gauge bosons, i.e., a single $E_8$ at level two. Despite their intriguing properties rank reduced models have been difficult to work with until recently because of the lack of a straightforward prescription for identifying physical states and superpotential couplings given the one-loop partition function.

Such holomorphic cfts can be tensored together with chiral boson cfts to build $N{=}4$ supersymmetric models which do not correspond to Narain compactifications. At generic points in the moduli space the gauge group is $(U(1))^{28-c_m}$ and the dimension of the moduli space is given by $r_L r_R$, where $(r_L, r_R){=}(22-c_m, 6)$. The $N{=}4$ fermionic models with rank reduction $c_m{=}8$, 12, and 14 have been interpreted as asymmetric orbifolds [22]. The simplest example is a $Z_2$ orbifold of the toroidally compactified $E_8 \times E_8$ string, where one mods out by the outer isomorphism which interchanges the two $E_8$ lattices accompanied by a shift of half-periodicity in any cycle of the torus. All three rank reductions can be obtained by compactifying the $SO(32)$ string on an $(SO(4))^3$ torus and introducing the following Wilson lines on the gauge lattice

$$\begin{aligned}
&(1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0)\\
&(1,1,1,1,0,0,0,0,1,1,1,1,0,0,0,0)\\
&(1,1,0,0,1,1,0,0,1,1,0,0,1,1,0,0)
\end{aligned} \tag{2.1}$$

which breaks the gauge symmetry to

$$\begin{aligned}
&\left[(SO(4) \times SO(4)) \times ((SO(4) \times SO(4))\right] \times \\
&\qquad \left[(SO(4) \times SO(4)) \times (SO(4) \times SO(4))\right] \times (SO(4))^3
\end{aligned} \tag{2.2}$$

This gauge lattice has three commuting $Z_2$ isometries under interchange of repeated $(D_2)^n$ units, $n$=1,2,4, together with their conjugacy classes. Modding out by the $Z_2$ isomorphisms sequentially accompanied by shifts of half-periodicity in the $(SO(4))^3$ torus gives models with enhanced symmetry points at which part of the gauge group is realized at Kac-Moody level 2, 4, and 8, respectively.

Combining such $Z_2$ twists on the left-moving gauge lattice with twists on the right-moving world-sheet fermions gives $N{=}1$ asymmetric orbifolds of reduced rank. There is an interesting distinction between this mechanism for achieving higher level, first used in the fermionic construction by Lewellen [23], and that employed in the higher level orbifold models of [24,1]. (Note that only the asymmetric mechanism can apply in the $N{=}4$ case.) Unlike the symmetric orbifolds, the point in the moduli space with level two gauge group $G$ is *infinitely* distant from the level one $G \times G$ point and corresponds to a decompactification limit [22]. This is the large radius limit of the circle in which we embed the accompanying shift.

Holomorphic rank reduction should not be confused with field theoretic Higgsing. In the latter case, the rank of the gauge group varies *locally* within the moduli space as one varies the vev of a Higgs field transforming in the fundamental representation of the gauge group.

The case of $N{=}1$ supersymmetry allows a much richer set of possibilities than the $N{=}4$ case, giving examples both of asymmetric orbifold and heterotic Gepner model constructions. In an $N{=}1$ solution, the block of chiral Ising fermions can be split among $n_L$ left-moving and $n_R$ right-moving fermions, such that $n_L{+}n_R{=}c_m$ takes one of the allowed values listed above [2]. For the purposes of this paper, we consider fermionic realizations where all of the right-moving world-sheet fermions are Majorana-Weyl, with periodic or anti-periodic boundary conditions alone. This class of fermionic solutions already includes several new possibilities for three generation models.

Other models have a fermionic realization that includes *right-moving* chiral Ising fermions. They are the first known examples of exact cft solutions to string theory based on holomorphic tensor products of $c{<}1$ building blocks. A bosonic description of the underlying target space of such solutions is at this point unknown. We note that in abstract $(0,2)$ cft constructions obtaining a one-loop modular invariant partition function does *not* by itself provide enough information about a solution to string theory. It is essential to develop a formalism (i) that unambiguously identifies (physical) states in the partition function with string vertex operators, and (ii) constructs the vertex operator algebra yielding the complete tree superpotential including nonrenormalizable terms and couplings to singlets. The nonrenormalizable terms and the singlet couplings are crucial since they probe the moduli space of flat directions in the neighbourhood of the exactly solvable point.

The formalism used here and developed in [2] has a natural extension to heterotic tensor products of other holomorphic cfts with $c{<}1$.

### *2.2. Overlapping Embeddings*

To enlarge the scope of free fermionic representation theory we make some further modifications. We allow overlapping embeddings of the current algebra weights into fermionic charges, $Q^i_F$, where $i$ labels individual Weyl fermions, and $G$ and $G'$ are given commuting current algebras,

$$w^i_G + w^i_{G'} \;=\; Q^i_F \tag{2.3}$$

Thus, in many of our conformal field theory solutions the group weights of the hidden and the visible gauge groups actually overlap! This has no bearing on spacetime physics or equivalently on the conformal field theory, but is simply a trick that allows a free fermionic representation for many new modular invariant partition functions. Enlarging the class of allowed embeddings considerably reduces the ad-hoc restrictions on groups/weights obtained in conventional free fermionic solutions [2]. This increased flexibility in choosing the fermionic embedding of the gauge group is crucial in obtaining three generations *simultaneously* with generic gauge group and generic matter content.

As an example, consider the following embedding of the simple roots $(\beta_i, \delta;\ \alpha_j)$, of $SU(3)_c \times SU(2)_L \times SU(4)_2$, which appears in Model 4. The $SU(4)_2$ plays the role of a confining hidden sector group in this model. The roots are embedded in eleven fermionic charges as follows:

$$\begin{aligned}
\beta_1 &= (\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\
\beta_2 &= (\ 0,\ 0,\ 0,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0) \\
\delta &= (\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2},\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\
\alpha_1 &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ 0, -\tfrac{1}{2},\ \tfrac{1}{2},\ 0, -\tfrac{1}{2}) \\
\alpha_2 &= (\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ 0, -\tfrac{1}{2},\ 0,\ 0, -\tfrac{1}{2},\ \tfrac{1}{2},\ 0) \\
\alpha_3 &= (\ 0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0, -\tfrac{1}{2},\ \tfrac{1}{2})
\end{aligned} \tag{2.4}$$

There are five overlapping, but orthogonal, $U(1)$ generators, spanning the remainder of the embedding space.

### *2.3. Twist Field Current Algebra Realizations*

In addition to conventional fermion bilinear currents we also consider twist field realizations for part, or all, of the current algebra. Such currents are obtained by tensoring together 8 or 16 dimension $\frac{1}{16}$ twist operators of the block of chiral Ising cfts (tensored also with a dimension $\frac{1}{2}$ fermion operator in the first case), so as to give holomorphic operators of dimension $(1,0)$. There is also a similar construction with 4 or 8 dimension $\frac{1}{8}$ Weyl twist fields. For example,

$$J_{ijkl}(z)\ =\ j_{free}(z)\left(\sigma_i^+\sigma_j^+\sigma_k^+\sigma_l^+\ +\ \sigma_i^-\sigma_j^-\sigma_k^-\sigma_l^-\right) \tag{2.5}$$

where $i{\neq}j{\neq}k{\neq}l$ label four inequivalent pairs of fermions in the chiral Ising block, and $j_{free}$ is the product of four dimension $\frac{1}{8}$ twist fields in the free field (Weyl fermion) cft. Current algebra realizations combining twisted currents with conventional fermion bilinear currents abound in the fermionic construction. They play an essential role in fermionic realizations of higher level and non-simply laced gauge symmetry, as in the examples of [25]. But they also provide new fermionic realizations of the level one simply laced gauge groups.

One can also find examples in which all of the currents including Cartan generators have a twist field realization! In such a model, none of the gauge bosons would appear in the untwisted sector i.e., in the sector in which all of the world-sheet fermions obey NS boundary conditions. Twisted current algebra realizations are a source for *accidental* extensions of the gauge symmetry, originating in the block of chiral Ising fermions.

### 2.4. World-sheet supersymmetry

The examples in this paper were constructed using the conventional fermion trilinear generator of the $(1,0)$ world-sheet supersymmetry of the heterotic string

$$T_F(\bar{z}) \;=\; i\sum_{\mu=1}^{2}\psi_\mu\partial_{\bar{z}}X^\mu \;+\; i\sum_{k=1}^{6}\psi_{3k}\psi_{3k+1}\psi_{3k+2} \tag{2.6}$$

where the index $\mu{=}1,2$ sums over the two transverse dimensions in $D{=}4$, and we work in light-cone gauge [16]. The $N{=}1$ spacetime supersymmetry charges are embedded in the spin structure of eight right-moving fermions, which are paired into four Weyl fermions as follows, $\psi_1 + i\psi_2$, $\psi_{3k} + i\psi_{3k+3}$, $k{=}1,3,5$. The remaining 12 right-movers can be Weyl, Ising, or chiral Ising fermions. The left-moving Weyl fermions are unrestricted by world-sheet supersymmetry and are allowed to satisfy any rational boundary condition. Further generalization to rational boundary conditions on the right moving Weyl fermions is possible. Such right moving spin structures have been analyzed recently in [26], but have not as yet been incorporated into actual models.

### 2.5. Couplings

Given a conformal field theory solution we can compute arbitrary N-point functions of the vertex operators which represent massless physical states. From these string tree level S-matrix elements we can then deduce the effective field theory action in a derivative expansion [27]. The effective superpotential thus derived contains no quadratic terms, but does generally contain cubic terms as well as quartic and higher order nonrenormalizable contact terms.

To compute the effective superpotential it suffices to consider N-point cft correlators of the form [28,29]

$$\int \prod_{i=1}^{N-3} d^2 z_i \left\langle V^f_{1(-1/2)}(z_\infty)\, V^f_{2(-1/2)}(1)\, V^b_{3(-1)}(z_1) \right.$$
$$\left. V^b_{4(0)}(z_2) \ldots V^b_{N-1(0)}(z_{N-3})\, V^b_{N(0)}(0) \right\rangle \tag{2.7}$$

where $V^f_{(-1/2)}(z_i)$ is the vertex operator for the fermionic component of a chiral superfield in the ghost number $-1/2$ picture, while $V^b_{(-1)}(z_i)$, $V^b_{(0)}(z_i)$ are vertex operators for the bosonic components of chiral superfields in the ghost number $-1$ and 0 pictures, respectively. $SL(2, C)$ invariance was used, as usual, to fix three of the $z_i$ to $\infty$, 1, and 0.

For the Weyl fermion and ghost cfts, the computation of N-point correlators is straightforward. For the Ising fermion cfts, the computations are more involved but are available in the literature [29].

The N-point correlators of the cfts described by blocks of chiral Ising fermions require more work. In [2] we derived the selection rules for which chiral Ising correlators can be nonzero. These selection rules can be traced to spin one simple currents in the conformal field theory. These holomorphic dimension (1,0) operators are not true conserved currents, because they are not local with respect to at least one of the physical vertex operators which does *not* appear in the specified correlator. Nevertheless the $n$ simple charges thus defined are conserved in the correlator, otherwise the correlator vanishes.

The computer implementation of the selection rules arising from the chiral conformal field theories is still incomplete, so we caution the reader that some of the terms listed in the superpotential given in Tables 4.1-4.3 may eventually vanish. At the moment, we are limited to checking this by hand.

## 3. General Aspects of Model Building

It is helpful to keep in mind a number of phenomenological issues when building possibly semi-realistic string models. We summarize these below.

### *3.1. Gauge Embedding*

A perturbative ground state of heterotic string theory (a string model) provides an effective field theory description of physics at the string scale: $M_{str} \simeq g_{str} \times 5 \times 10^{17}$ GeV. A starting point for obtaining string models is to find an exact conformal field theory solution to heterotic string theory. In solutions with $N{=}1$ spacetime supersymmetry, the rank of the full gauge group is $\leq 22$. It may be possible to find solutions such that the full gauge group at the string scale is precisely $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$ of the MSSM, however no one has yet succeeded in constructing such a solution. In existing solutions the full gauge group is of the form:

$$G_{SM} \times G_{flavor} \times G_{hidden} \tag{3.1}$$

Here $G_{SM}$ is either the standard model gauge group or a larger nonsimple group which embeds it, $G_{flavor}$ represents new gauge interactions (typically a product of $U(1)$'s) under which the quarks and leptons of the MSSM transform in a flavor-dependent way, and $G_{hidden}$ represents new gauge interactions of particles in a hidden (or semi-hidden) sector.

Since gauged flavor symmetries and hidden sectors are useful for inducing fermion mass hierarchies and dynamical supersymmetry breaking, respectively, there is no obvious necessity in building semi-realistic string models to reduce the rank of the cft solution below 22. However, the generic class of solutions does in fact include such rank reduction.

In some previously known solutions $G_{SM}$ is just $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$. In these solutions, the conformal current algebra which realizes this gauge algebra has Kac-Moody levels $k_3 = k_2 = 1$. This has the beneficial effect of restricting the massless chiral supermultiplets in such solutions to be triplets, antitriplets, or singlets under $SU(3)_c$, and doublets or singlets under $SU(2)_L$. However, higher level embeddings of $G_{SM}$ are the more generic class.[1]

Other solutions, such as flipped $SU(5)$, take $G_{SM}$ to be a larger nonsimple or nonsemisimple group that embeds the standard model group. The known solutions of this type also have Kac-Moody level equal to one for the nonabelian factors of $G_{SM}$. None of these solutions is a good starting point for constructing a conventional GUT, and they cannot be since Kac-Moody level one excludes the possibility of massless adjoint Higgs at the string scale. Such solutions require Kac-Moody level two or greater, which in turn requires rank reduction to embed the higher-level gauge group. It should be possible to construct semi-realistic higher level solutions, but at the moment the best example is a three generation SU(5) level two model which suffers from light color sextet exotics [30].

[1] Note that level one also necessitates fractionally charged states in the string spectrum (see section 3.5). Higher level removes this restriction.

*3.2. Anomalous U(1)*

Many tree level cft solutions to string theory, including those discussed here, contain a U(1) gauge factor which is anomalous. When this occurs, the Green-Schwarz mechanism breaks the anomalous $U(1)$, at the expense of generating a Fayet-Iliopoulos $D$ term proportional to

$$D_A \;=\; \sum_i Q_i^A|\chi_i|^2 + \frac{g^2}{192\pi^2}e^{\phi}\;TrQ^A \quad , \tag{3.2}$$

where $\phi$ is the dilaton, $TrQ^A$ is the trace anomaly, and the $\chi_i$ are scalar fields with anomalous charge $Q_i^A$. This term will break supersymmetry and destabilize the vacuum [20]. The vacuum becomes stable and supersymmetry is restored when one or more of the scalar fields which carry nonzero anomalous charge acquire a vev such that the right-hand side of (3.2) vanishes. Supersymmetry is then restored provided that this vacuum shift is in a direction which is F-flat and also D-flat with respect to all non-anomalous $U(1)$'s. If we let $\chi_i$ now denote the scalar *vevs* which cause (3.2) to vanish, then the additional D and F flatness constraints are

$$D_a \;=\; \sum_i Q_i^a|\chi_i|^2 \;=\; 0, \quad <\frac{\partial W}{\partial\phi_j}>\;=\;0 \quad . \tag{3.3}$$

where $a$ labels the nonanomalous $U(1)$'s, and the $\phi_j$ are all the chiral superfields, not just those whose scalar components get vevs.

Note that the shifted vacuum is no longer a classical string vacuum, but does correspond to a consistent perturbative quantum string vacuum. Thus cft solutions which contain an anomalous $U(1)$ in some sense access a much larger class of perturbative string vacua than those that do not.

Note also that because the D term cancellation in (3.2) involves the one-loop generated anomaly, the scale of vevs in (3.2) is naturally (depending on the value of the anomaly) smaller than the string scale, by an order of magnitude or so. Since the scalars whose vevs $\chi_i$ contribute to (3.2) often carry a variety of other abelian and nonabelian quantum numbers, the vacuum shift generically breaks the original gauge group to one of smaller rank. This rank reduction is variable and can be quite large. It may be possible to perform this vacuum shift without breaking the standard model gauge group, although there is no fundamental reason why this should *always* be the case. In fact, in many of our solutions we have found considerable freedom in choosing the flat directions involved in the shift. In some, but not all vacua, it is possible to break $G_{flavor}$ completely at this stage [12,13,14].

After the vacuum shift a number of previously massless fields will acquire masses, of order $(\alpha_{str})^n M_{str}$ for some $n$, via coupling to scalar vevs. The spectrum of light fields, particularly light exotics, is often much reduced. In addition, the scalar vevs also tend to induce a number of effective Yukawa interactions for the MSSM quarks and leptons, with Yukawa couplings that are naturally suppressed by powers of $\alpha_{str}$. This combination of favorable outcomes were employed first in orbifold models [9,10,11,12], then in the free fermionic construction[13,14], to make a first pass at viable perturbative superstring phenomenology.

### *3.3. Three Generations*

One of the striking things about all known superstring contructions is the difficulty of finding vacua with precisely three chiral generations. Thus despite the plethora of perturbative superstring vacua there is a paucity of three generation constructions, and this is the main reason why the number of known semi-realistic models is still so few.

This problem has been much discussed in the free fermionic construction [31]. All of the previously known three generation solutions [13,14,32] are based on a $Z_2 \times Z_2$ orbifold of the heterotic string compactified on an $SO(12)$ six-torus with arbitrary background fields. Each successive $Z_2$ twist of this torus breaks half of the spacetime supersymmetries, and the untwisted sector contains the moduli deformations of an $(SO(4))^3$ torus. This leads to the so-called NAHE basis [13] for the basis vectors which specify the fermion spin structures.

Although the specifics of these solutions vary, the fermionic construction allows very few distinct solutions for a given gauge group *once* the NAHE embedding of three generations of chiral matter fields has been imposed. Although there has been considerable speculation that the NAHE basis is necessary to obtain three generation solutions in the free fermionic construction, this is not the case.

By changing the embedding of the standard model gauge group and the chiral matter fields we have produced a large number of three generation solutions which have no connection to the NAHE basis.

### *3.4. Gauge Coupling Unification*

A remarkable property of string theory is that it provides gauge coupling unification independently of grand unification into a simple group. At string tree level:

$$k_3 g_3^2 = k_2 g_2^2 = k_1 g_1^2 = g_{str}^2 \tag{3.4}$$

Here $k_1$ is not a Kac-Moody level, but rather a normalization factor which relates the hypercharge $q$ of a state with the hypercharge contribution to the conformal dimension $h$, of the conformal field which creates the state:

$$h = \frac{q^2}{4k_1} \tag{3.5}$$

In a given cft solution the conformal dimensions of the fields are fixed. Thus $k_1$ can be determined in any solution by, for example, declaring the quark doublet states to have their conventional hypercharge $q$=1/3, then using (3.5) to compute $k_1$. Because the total conformal dimension of a field which creates a massless string state must be $\leq 1$, we see from (3.5) that any cft solution which contains the right-handed electron multiplet of the MSSM must have $k_1 \geq 1$ [33].

String solutions which embed the standard model group into $SU(5)$ or $SO(10)$ conformal algebras have $k_1$=5/3. All previously known cft solutions have $k_1$ larger than 5/3 — usually much larger [21]. Of course $k_3$=$k_2$=1, $k_1$=5/3 makes (3.4) resemble the putative gauge coupling unification of the MSSM, *under the assumption that the visible spectrum is exactly that of the MSSM*, and at a scale $M_{str}$ which is roughly one order of magnitude higher than the $M_U \simeq 3{\times}10^{16}$ GeV suggested by low energy data. String threshold corrections to (3.4) [34,35] may explain the mismatch, although this is not the case in the simplest abelian orbifolds and free fermionic models in which these have been analyzed (see [21,36] and references within). Unfortunately, the moduli dependence of such thresholds is poorly understood in semi-realistic models and clearly deserves further analysis [36]. We can also achieve agreement between string unification and MSSM gauge coupling unification by lowering the value of $k_1$ by about 10-15% [37]. We have found the first three generation string models for which $k_1$<5/3, including an example (our Model 4) with $k_1$ = 1.458.

Another possible explanation of the mismatch in gauge coupling unification is that there is a separate grand unification scale, (either $SU(5)$ or $SO(10)$), with $SU(5)$ broken at $M_U$. Although it may be possible to construct a classical string vacuum that mimics the spectrum and couplings of a semi-realistic grand unified model (see for example [38,32]) it is unlikely that one could determine $M_U$ within string perturbation theory. Such a scenario would of course inherit the usual difficulties of grand unified models. But this is a possible option.

Lastly, it is always possible (even within some string models [21]) to arrange for suitable combinations of exotic particles at suitable mass scales, and thus change the renormalization group (RG) running of the gauge couplings to remove the mismatch. Since exotics are present in all known string models, this solution may be less unnatural than it appears. However it is very difficult to implement this scenario without arranging *large* but nearly cancelling effects in the RG equations. Thus, if this is the solution to the mismatch problem, then the fact that string tree-level unification and MSSM gauge coupling unification agree as well as they do must be regarded as an accident!

The known cft solutions all contain a fairly large number of light exotics before the anomalous $U(1)$ vacuum shift. These will wreak havoc with the gauge coupling RG equations unless almost all of them (a) acquire superheavy masses by coupling to string scale vevs or the vevs involved in the vacuum shift, or (b) assembling into approximate $SU(5)$ multiplets, which have a much smaller effect on the RG running. Our solutions have similar features, but to see whether they fare better or worse than previous models requires detailed analysis of the vacuum shift, couplings, and RG running of the effective couplings including SUSY-breaking effects.

### *3.5. Fractional Charge*

All $SU(3)_c \times SU(2)_L \times U(1)_Y$ cft solutions with $k_3{=}k_2{=}\ 1$ must contain exotics with fractional electric charge [33]. This is because, if all physical states in a string vacuum obey charge quantization, there exists a certain conformal operator which is mutually local with respect to all the physical fields. This operator must thus itself correspond to a physical field, which leads to a contradiction unless $k_1{=}5/3$ and the standard model gauge group is promoted to unbroken $SU(5)$ at level one.

This argument does not determine whether or not there are any *massless* string states with fractional charge; it may be possible to arrange for fractional charges to occur only in the *massive* modes of the string, and thus be superheavy. However in all of the known models, fractionally charged exotics do occur at the massless level. It may be possible to avoid fractionally charged exotics entirely in three generation string models with higher Kac-Moody levels, but this has never been demonstrated.

The lightest fractionally charged particle will be stable. This can create conflicts with experimental bounds from direct searches, as well as rather severe cosmological and astrophysical bounds [39]. For example, the lightest fractionally charged particle will completely dominate the energy density in the universe if its mass is greater than a few hundred Gev[40,39]. If there is an inflationary epoch and subsequent reheating, we can probably tolerate a lightest fractionally charged particle with mass greater than the reheating temperature.

In all known string models, a variety of fractionally charged exotics are seen to occur. They can be $SU(3)_c \times SU(2)_L$ singlets with hypercharges less than $\pm 2$, and they can be color triplets or Higgs with nonstandard hypercharge. These exotics have important effects on the RG running of the couplings.

*3.6. Hypercharge Embedding and Particle Identification*

In cft solutions for which $G_{SM} \times G_{flavor}$ is $SU(3)_c \times SU(2)_L \times U(1)^n$, for some $n$, particle identification is not automatic, since string theory does not label the physical states for us. For example, the three lepton doublets and the up and down type Higgs doublets all have the same $SU(3)_c \times SU(2)_L$ quantum numbers. Most known solutions also contain a number of additional weak doublet exotics.

Hypercharge disentangles these doublets somewhat, but string theory does not label hypercharge for us either, i.e. it does not tell us how to extract $U(1)_Y$ from the additional nonanomalous $U(1)$'s which go into $G_{flavor}$. In the known solutions there is usually more than one consistent embedding of $U(1)_Y$ as a linear combination of the original nonanomalous $U(1)$'s.

Because of this there are many different ways of embedding the standard model particle content within the *same* conformal field theory solution of heterotic string theory. Different choices of hypercharge and particle identification will lead to different couplings, masses, and mixings of the MSSM quarks and leptons, as well as different hypercharge assignments for the exotics. In practice we thus use phenomenological considerations as a guide for making these choices.

A given hypercharge embedding fixes $k_1$. We will constrain the hypercharge embedding by requiring that $k_1$ be reasonably close to $5/3$, and that the number of fractionally charged states is minimized. We discuss this procedure in detail for two of our models in section 4. The reason for this additional flexibility in the embedding of hypercharge is related to the fact that the solutions are *not* based on Wilson line breakings of $SU(5)$ or $SO(10)$ but instead explore the generic class of gauge embeddings.

*3.7. Rapid proton decay*

String models typically violate matter parity, allowing for the appearance of B and L violating terms in the cubic part of the effective superpotential. In particular, terms of the form

$$Q\,L\,d^c + u^c\,d^c\,d^c \tag{3.6}$$

where $Q$ denotes a quark doublet, $L$ a lepton doublet, and $u^c$, $d^c$ the conjugates of the right-handed up and down quarks, would lead to instantaneous proton decay. In addition to these cubic terms, there is also the possibility of quartic terms which can lead to unacceptably rapid proton decay.

To check a particular string model for the absence of such dangerous terms, it is insufficient to compute the effective superpotential to quartic order. This is because the dangerous B violating terms may be generated at *any* order via nonrenormalizable terms which are unsuppressed due to string scale vevs. The simplest solution to this problem gauged $U(1)_{B-L}$ as part of $G_{flavor}$ [24]. Other possibilities that have been considered in the known models are a combination of B-L and custodial $SU(2)$ along with other flavor symmetries which distinguish quarks from leptons [14,41].

*3.8. Quark and Lepton Masses*

A major challenge for any unified model is to reproduce, even qualitatively, the many observed hierarchies of masses and mixings for quarks and leptons. In known cft solutions the numerical values of the couplings in the effective superpotential are order one, and this is likely to be true rather generally in perturbative string vacua. Thus, small Yukawa couplings in the MSSM may originate from scalar vevs or fermion condensates which take values at scales other than $M_{str}$. Nonrenormalizable couplings of quarks and leptons to these vevs or condensates can then generate effective Yukawa couplings which are small.

A beautiful property of the known string models is that such a mechanism does indeed occur: the vacuum shift associated with the anomalous $U(1)$. It appears unlikely that any one mechanism will explain all of the observed hierarchies. Some previously known models [13,14] can produce a top quark Yukawa which is order one, while all the other effective Yukawas are suppressed by an order of magnitude or more. We show below that least one of our models shares this feature.

## 4. Models

To illustrate the range of options within the real fermion construction for obtaining precisely three generations of chiral matter, we have constructed a sample of seven new conformal field theory solutions embedding the standard model gauge group. Table 1 lists for each solution $c_m$, where $22-c_m$ is the rank of the full gauge group, the number, $n$, of nonanomalous $U(1)$'s in $G_{flavor}$, the number of vectorlike pairs of color triplet exotics, the number of extra weak doublets, $G_{hidden}$, and whether or not there is chiral matter transforming under $G_{hidden}$. It should be noted that these are properties of the conformal field theory solutions *before* making the vacuum shift required by the presence of an anomalous $U(1)$ and the choice of $k_1$ which together define a string model. In these solutions $G_{flavor}$ always contains an anomalous $U(1)$ and at least 5 additional $U(1)$'s. As can be seen for example from Tables 2.1 through 2.3, the quarks and leptons carry complicated, highly flavor dependent charge assignments under these extra $U(1)$'s. Some or all of these extra $U(1)$'s will be broken by the vacuum shift. It is also clear that there are particles which are both nonsinglets under $G_{hidden}$ and carry hypercharge, charges under $G_{flavor}$, or are weak doublets. However after the vacuum shift many of these particles will become superheavy, and the rank of $G_{flavor}$ (and perhaps $G_{hidden}$) is reduced. It is possible that after the vacuum shift $G_{hidden}$ is truly hidden. Note that $G_{hidden}$ need not be simple or semisimple, does not necessarily have any large nonabelian factors, and can have higher Kac-Moody levels. There is also typically chiral matter in the hidden sector. All of these features may be important for scenarios of dynamical supersymmetry breaking.

From Table 1 we see that the number of exotics in these models varies considerably from model to model. After the vacuum shift some of these exotics become superheavy, while the others will acquire TeV or intermediate scale masses after supersymmetry breaking.

Many features just described are similar to previously known solutions[9][10][12][14]. Let us now focus on the features of these solutions which are qualitatively new.

### *4.1. Three generations*

The previously known three generation solutions in the free fermionic formulation all obtain three chiral generations by using the NAHE basis. In this construction the three generations arise from three distinct sectors with different left and right-moving spin structure, corresponding to the three twisted sectors of a symmetric $Z_2 \times Z_2$ orbifold. In our solutions the three generations arise from sectors (i.e. choices of fermionic spin structure) in a variety of new ways. This can be seen from Table 3, which for each solution lists the sectors of the cft which contains the positive helicity fermionic component highest weight states of the three generations of chiral superfields. The sectors are listed as linear combinations of the basis vectors for each fermionic cft solution; the basis vectors are given in Tables A.1 to A.7 of the appendix. From these tables we can summarize the different realizations of three generations as follows:

*Free fermionic realizations:* these solutions only utilize Weyl and Ising fermions. Any rank reduction is due to Ising fermions. These examples sample symmetric orbifold models.

• NAHE: each generation comes from a distinct sector, with different left and right-moving structure.

• Models 1-3: two generations come from distinct sectors. The third comes from a sector which differs from one of these sectors only in its left-moving spin structure.

*Real fermionic realizations:* these solutions utilize Weyl and chiral Ising fermions. In Models 4,5 all of the chiral Ising fermions are left-moving. These examples are likely to have an asymmetric orbifold interpretation. In Models 6,7 four of the chiral Ising fermions are right-movers. These examples belong to the general class of asymmetric $(0, 2)$ Gepner constructions.

• Model 4: two generations come from distinct sectors, but the third generation comes from a sector which is effectively the sum of the first two sectors: adding sector 1, which contains the gravitino, simply takes the scalar component of a supermultiplet into the fermionic component, while taking sector 5 to 3*5 takes states into their CPT conjugates within the same multiplet.

• Model 5: two generations come from the same sector; the third lives in a distinct sector. In fact, the quark doublets of the two generations in the same sector differ only by a single $U(1)$ charge of $G_{flavor}$.

• Model 6: similar to Models 1-3.

• Model 7: similar to Model 5.

These results are encouraging for semi-realistic model building. As mentioned in the introduction, the real fermion construction samples the full range of exactly solvable $(0, 2)$ constructions. We have found examples with precisely three chiral generations in every such class, and with generic gauge group and matter content.

### *4.2. Hypercharge and $k_1$*

As mentioned in section 2 our construction explores the *generic* class of embeddings of the standard model gauge group allowed in string theory, as opposed to embeddings in grand unified groups like $SU(5)$ or $E_6$. In the fermionic construction, this is achieved by exploring overlapping embeddings of the standard model gauge group with embeddings of $G_{flavor}$ and $G_{hidden}$ in the fermionic charges. A consequence is to increase the number of distinct hypercharge embeddings which are possible for each conformal field theory solution we construct. In fact for some solutions there are one parameter continously varying embeddings of the hypercharge. As a result, $k_1$ can be *continuously variable* within the same conformal field theory solution.

We emphasize that this is merely a statement about the flexibility in hypercharge embedding and particle identification within these solutions, and should not be misinterpreted as the continuous variation of $k_1$ *within* a string model. In fact, a general theorem in perturbative string theory tells us that there is no continuously varying modulus that can adjust the value of $k_1$ in an $N{=}1$ heterotic string model with chiral matter [42]. Our results are consistent with this theorem. A given hypercharge embedding fixes both $k_1$ and much of the particle identification in the conformal field theory solution, thus *defining* a string model up to the vacuum shift necessary for removing the anomalous $U(1)$.

We determine an acceptable choice for $k_1$ in two steps. To find a consistent hypercharge embedding, we solve for a nonanomalous $U(1)$ for which we can identify a full 3 generations of quarks and leptons with conventional values of hypercharge in the massless spectrum. Since this definition says nothing about the Higgs, one must then check in each case whether or not there appear a pair of candidate electroweak Higgs doublets with conventional hypercharge. At this stage, for e.g. in Model 6, it is still possible that a continuous range of $k_1$ values is allowed. We now determine the hypercharge embedding by *requiring* that it is defined so as to minimize the number of fractionally charged exotics, also avoiding any hypercharge mismatch in what would otherwise be pairs of vectorlike exotics.

Let us see how this works out in particular examples. Consider Model 4, which we see from Table 2.1 has 5 nonanomalous $U(1)$ generators: $Q_1$-$Q_5$. As far as obtaining three generations of standard model quarks and leptons, there are 5 possible definitions of hypercharge:

$$
\begin{aligned}
Y_1 &= \frac{1}{3}\left(-\frac{3}{20}Q_1 + \frac{27}{320}Q_2 + \frac{3}{40}Q_3 + \frac{1}{24}Q_4 - \frac{1}{24}Q_5 + \frac{5}{192}Q_6\right) \\
Y_2 &= \frac{1}{3}\left(-\frac{3}{5}Q_1 - \frac{9}{40}Q_2 + \frac{3}{10}Q_3 + \frac{1}{6}Q_4 - \frac{1}{6}Q_5 - \frac{1}{12}Q_6\right) \\
Y_3 &= \frac{1}{3}\left(-\frac{9}{10}Q_1 - \frac{21}{160}Q_2 - \frac{3}{20}Q_3 - \frac{1}{12}Q_4 - \frac{7}{60}Q_5 + \frac{1}{96}Q_6\right) \\
Y_4 &= \frac{1}{3}\left(\frac{3}{10}Q_1 + \frac{69}{160}Q_2 - \frac{3}{5}Q_3 - \frac{1}{3}Q_4 + \frac{11}{60}Q_5 + \frac{7}{96}Q_6\right) \\
Y_5 &= \frac{1}{3}\left(\frac{9}{10}Q_1 + \frac{39}{160}Q_2 + \frac{3}{10}Q_3 + \frac{1}{6}Q_4 + \frac{1}{12}Q_5 - \frac{11}{96}Q_6\right)
\end{aligned}
$$

All 5 choices also provide at least one pair of candidate electroweak Higgs doublets, so this criterion does not distinguish between them. Two of these choices, $Y_1$ and $Y_3$, give a reduced spectrum of fractionally charged states. All of these choices except $Y_1$ have the unpleasant feature that the two exotic color triplets are not truly vectorlike, i.e. their hypercharges are not equal and opposite to those of the two extra color antitriplets. The $k_1$ values associated with these 5 choices are given respectively by

$$
k_1 = \frac{35}{24},\ \frac{46}{3},\ \frac{29}{6},\ \frac{185}{6},\ \frac{125}{6} \tag{4.1}
$$

All of these values are quite large except for the first one, associated with $Y_1$. Incidently, $k_1$=35/24 is 12.5% less than $k_1$=5/3, a value close to optimal for the scenario of improving gauge coupling unification by adjusting $k_1$.

So in the case of Model 4 we are quickly led to a unique choice of hypercharge, once we impose some phenomenological criteria.

Another interesting example is Model 6. We see from Table 2.3 that it has 10 nonanomalous $U(1)$ generators, however the first three of these clearly belong to $G_{hidden}$. Let us call the remaining seven $Q_1$-$Q_7$. There is considerable choice in the hypercharge embedding for this conformal field theory solution. Among the allowed possibilities is the following one-parameter set:

$$
\begin{aligned}
Y_3 = \frac{1}{3}\big( &- \frac{15}{16}(1+24q)Q_1 - \frac{3}{4}Q_2 + \frac{3}{20}Q_3 - \frac{2}{5}(1+10q)Q_4 \\
&+ \frac{5}{16}(1+24q)Q_5 + \frac{1}{40}(7+160q)Q_6 + qQ_7\big)
\end{aligned}
$$

where $q$ is an arbitrary real parameter.

The corresponding one-parameter set of possible $k_1$ values are given by:

$$k_1 = \frac{103}{12} + 260q + 2800q^2 \tag{4.2}$$

There are 9 weak doublet states in this conformal field theory solution from which we must identify candidates for the up and down type electroweak Higgs. The hypercharges of some (though not all) of these doublets depend on $q$. There are also three pairs of vectorlike color triplet exotics, some of whose hypercharges also depend on $q$. The hypercharges of the 9 weak doublets are given by: 3, 3, $-3$, 9, 9, $(6{+}120q)$, $-(6{+}120q)$, $(15{+}240q)$, $-(15{+}240q)$. The hypercharges of the three color triplet exotics are 4, $-(5{+}120q)$, and $(10{+}240q)$, while those of the color antitriplet exotics are $-4$, $-4$, and $(5{+}120q)$. We are led to a unique choice, $q{=}-1/40$, to avoid a hypercharge mismatch for the color triplet exotics. The corresponding value of $k_1$ is 23/6. Table 2.3 gives the spectrum corresponding to this hypercharge embedding.

*4.3. Model 5*

Rather than go into details for all of our conformal field theory solutions we will be content in the remainder to focus on one model, obtained from the solution described in Table 2.2. From Table 2.2 we see that there are 6 nonanomalous $U(1)$'s: $Q_1$-$Q_6$. There are 4 possible hypercharge definitions for this conformal field theory solution, but by the same procedure as above we are quickly led to a unique choice:

$$Y = \frac{1}{48}\left(-8Q_2 - 3Q_3 - 8Q_4 - Q_5 + Q_6\right) \tag{4.3}$$

The corresponding value of $k_1$ is 11/6, which we henceforth refer to as Model 5. This is interesting, as it is only slightly larger than the $SU(5)$ value 5/3.

It is interesting that the perturbative heterotic string vacuum corresponding to Model 5 can be obtained from two distinct fermionic realizations. The basis vectors corresponding to these two different embeddings are given in Tables A.5 and A.8. As shown in the appendix, the gauge embeddings of $SU(3)_c$ and $SU(2)_L$ in free fermionic charges are different in these two realizations. Nevertheless we have verified that the massless spectra are identical, and have checked that the superpotentials agree at least through quintic order. This demonstrates that the free fermionic realization of the gauge embedding is *not* an invariant property of the cft solution.

The second version of Model 5, Table A.8, has the property that by simply removing the final basis vector we obtain a model in which $SU(3)_c \times SU(2)_L$ is promoted to $SU(5)$. Thus we may ask the question: what happened to the conventional $k_1{=}5/3$ hypercharge $U(1)_Y \subset SU(5)$ when we broke $SU(5)$ to $SU(3)_c \times SU(2)_L$? The answer is that the fermionic charge vector which would correspond to this $U(1)_Y$ is not orthogonal to the roots of the level two $SU(4)$ hidden sector group. Thus we do not quite have an $SU(5)$ based embedding, since the true hypercharge must involve a mixture with the other abelian generators. Nevertheless the actual value of $k_1$, 11/6, is quite close to the $SU(5)$ value.

A complete listing of the nonvanishing terms in the cubic, quartic, and quintic effective superpotential of Model 5 is given in Tables 4.1-4.3. As described in section 2.5, the numerical values of the couplings can, with some effort, be computed; they are generically order one.

Let us examine how the possible patterns of quark, lepton, and Higgs masses are related to the vacuum shift associated with the anomalous $U(1)$. We immediately observe from the cubic superpotential a term $Q_3\, u_3^c\, \bar{h}_3$, which can be interpreted as giving mass to the top quark, provided that $\bar{h}$ remains light and can serve as the up type electroweak Higgs. We then examine, at the cubic level, the full Higgs mass matrix, including mixings with $L_2$ and $L_3$:

$$M = \begin{array}{c} \\ \bar{h}_1 \\ \bar{h}_2 \\ \bar{h}_3 \\ \bar{h}_4 \end{array} \begin{array}{c} \begin{array}{cccccc} h_1 & h_2 & h_3 & h_4 & L_2 & L_3 \end{array} \\ \begin{pmatrix} 0 & 0 & \chi_9 & 0 & 0 & 0 \\ \chi_2 & 0 & 0 & 0 & \chi_{11} & 0 \\ 0 & \chi_4 & 0 & 0 & \chi_6 & \chi_5 \\ 0 & \chi_{12} & 0 & 0 & 0 & \chi_{13} \end{pmatrix} \end{array} \tag{4.4}$$

By diagonalizing $MM^\dagger$ and $M^\dagger M$ we can tell which combination of fields remain light when various entries in $M$ become large after the vacuum shift. As one would expect, $\bar{h}_3$ does not remain light unless neither $\chi_4$, $\chi_5$, nor $\chi_6$ gets a superheavy vev. In this case both $L_2$ and $L_3$ remain light, but mix with $h_2$.

A reasonable requirement we could make on the vevs at this level is that two pairs of up and down type Higgs should remain light after the vacuum shift. Note that the vectorlike color triplet exotic pair remains light after the vacuum shift. An extra light Higgs pair would fill out an approximate $5,\bar{5}$ of $SU(5)$, minimizing the effect of the color exotics on the RG running of the gauge couplings [40]. One solution to this requirement is that $\chi_2$ and $\chi_{11}$ should not get superheavy vevs.

This leaves $\chi_9$, $\chi_{12}$, and $\chi_{13}$ to get vevs in the vacuum shift. We must then ask whether there is a set of vevs which includes these fields, cancels the Fayet-Iliopoulos term (3.2), and satisfies all of the F and D flatness conditions (3.3). A simple solution does exist: the vacuum shift involves appropriate vevs for the fields

$$\{\chi_9,\ \chi_{12},\ \chi_{13},\ \chi_{14},\ T_1,\ T_2,\ T_3\} \tag{4.5}$$

By examining Table 2.2 one sees that this set satisfies the D flatness conditions; we have checked that it satisfies the F flatness conditions at least through 8th order in the superpotential.

After this vacuum shift, $\bar{h}_3$, $\bar{h}_2$, $h_1$, and $h_4$ remain massless. In addition, $L_2$ remains massless, as does the combination $<\chi_{12}>L_3 - <\chi_{13}>h_2$. This shift also breaks the level two $SU(2)$ of $G_{hidden}$. If we proceed to the quartic level in this scenario, we notice the terms

$$Q_3\, d_3^c\, h_1\, \chi_{12} + L_3\, e_3^c\, h_1\, \chi_{12} \tag{4.6}$$

which give mass to the bottom quark and tau lepton. These masses are suppressed by $<\chi_{12}>/M_{str}$ relative to the top mass. If $<\chi_{13}>/<\chi_{12}>$ is not too large, we also reproduce approximate $b$-$\tau$ Yukawa unification as in $SU(5)$.

In this model there are no dangerous B violating terms through quintic order, provided that $\phi_{12}$ does not get a superheavy vev. This statement is somewhat dependent on particle identification, but in any event it does not seem difficult to avoid rapid proton decay. To have a truly viable model, we would also need large masses for the other exotics, and higher order mass terms for (at least most of) the 1st and 2nd generation quarks and leptons (as well as mixings).

## 5. Conclusions

Arriving at precisely three generations of massless chiral fermions had proven to be a notoriously unpredictable step in superstring model building. Prior to our work, the only three generation models for which both the massless spectrum and the superpotential have been computed are symmetric orbifolds with Wilson lines, which includes the free fermionic examples. The models in this paper go beyond that class. They are the first known examples of three generation models based on genuinely heterotic modular invariants, obtained by tensoring together holomorphic cfts which are not free fields. We expect this feature will be generic to other exactly solvable $(0,2)$ constructions, suggesting that there exist many new starting points for obtaining three generations.

The phenomenology of these models needs to be worked out in detail and compared with that of previously known models. Such an analysis will suggest new strategies for model building. It would be helpful to have a better understanding of string threshold effects in order to to make progress on the problem of gauge coupling unification. It is certainly possible to investigate the problem of exotics in string models embedding the MSSM. Since the number of exotics is variable, perhaps it can be reduced to zero, leaving just the MSSM matter content in the visible sector. In this event, it would be important to understand whether string threshold effects can be large giving an alternative explanation for the mismatch between the unification scales. The question of moduli dependence of such thresholds has been investigated in orbifold constructions with differing conclusions from those models investigated in the free fermionic construction [36,38]. It would also be nice to reduce the size of $G_{flavor}$, and perhaps make contact with various texture schemes [43].

A common feature of all known superstring embeddings of the MSSM is the presence of extra low-energy matter. It is intriguing that every semi-realistic example to date also has a tree-level anomalous $U(1)$. However, it is not known whether these are *essential* features of a heterotic string vacuum embedding the MSSM. In order to address this question convincingly, one must explore a *large sample* of superstring embeddings of the MSSM. The examples in this paper are only a beginning in this direction, but they sample a wide range of exactly solvable $(0,2)$ constructions.

We should mention that we have found exceptions without these features in admittedly unrealistic three generation models. We have an example of a $SO(10)$ level one model with precisely three chiral **16**'s and *no* additional vectorlike matter transforming under the $SO(10)$. Interestingly, a slight change in the fermionic construction of this solution converts it into a three generation model with $SU(5)$ realized at level two plus an exotic chiral **15**, hidden sector group $F_4$, flavor group $(U(1))^6$ but *no* anomalous $U(1)$.

Finally, we note that there are questions of phenomenological interest which would be most easily explored in pedagogical models with, for example, a *single* chiral generation of quarks and leptons. We can construct many such examples. Given the superpotential with both singlet couplings and nonrenormalizable terms included, one could investigate the absence of the $\mu$ term in the superpotential or look for specific couplings necessary for generating an intermediate scale.

**Acknowledgements**

We would like to thank the participants of the Planck'95 workshop at the ITP for many discussions, especially P. Langacker, P. Binetruy, M. Cvetic, V. Kaplunovsky, J. Louis, H.-P. Nilles, J. Polchinski, R. Schimmrigk, and M. Srednicki. We also thank K. Dienes and A. Faraggi for discussions of their work. This work was supported in part by National Science Foundation grants PHY 91-16964 and PHY 94-07194.

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## Appendix A. Details of the fermionic embeddings

In this appendix we briefly describe the specifications of the seven solutions discussed in this paper. The real fermionic construction is described in [17,2]. Tables A.1 through A.7 list the basis vectors and $k_{ij}$ matrices that define each solution. Table A.8 gives the equivalent version of Model 5. Our definition of $k_{ij}$ is as in [17]. Each basis vector specifies 20 right-moving and 44 left-moving fermion boundary conditions in Majorana-Weyl notation. A double vertical line separates the right-movers from the left-movers; a single vertical line is used to separate the left-moving Weyl fermion boundary conditions from those of the left-moving Ising or chiral Ising fermions. A "0" denotes Neveu-Schwarz boundary conditions, while a "1" denotes Ramond. The detailed map between our notation for boundary conditions and that of [17] is given by:

$$
\begin{array}{llll}
0 : 0; & 1 : -1/2; & + : +1/4; & - : -1/4; \\
e : 1/8; & x : -1/8; & t : 3/8; &
\end{array}
$$

The simple roots of $SU(3)_c \times SU(2)_L$ are embedded in the first 8 fermionic charges of the left-movers. This embedding is the same in all seven solutions:

$$
\begin{array}{l}
SU(3)_c : \\
(\tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \\
(\,0,\ 0,\ 0,\ 1,\ \tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0\,) \\
SU(2)_L : \\
(\tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2},\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,)
\end{array}
$$

We note that in the equivalent fermionic realization of Model 5 defined by Table A.8, the embedding of $SU(3)_c \times SU(2)_L$ is quite different:

$$
\begin{array}{l}
SU(3)_c : \\
(\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{3}{4},\ \tfrac{1}{2},\ 0, -\tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ 0,\ \tfrac{1}{2}, -\tfrac{1}{2},\ 0,\ 0\,) \\
(\,0,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\,) \\
SU(2)_L : \\
(\tfrac{1}{2}, -\tfrac{1}{2},\ \tfrac{1}{2}, -\tfrac{1}{2},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\,)
\end{array}
$$

The embedding of the simple roots of the nonabelian factors of $G_{hidden}$ into the fermionic charges of the left-movers is model dependent. We list these embeddings below for all seven solutions:

Model 1:

$SU(2)_2$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, 0, -1, 0 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 1, 0 )$$

Model 2:

$SO(7)$ :
$$\begin{array}{l}( 0, 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, -1, 0, 0, 0, 0, 0, 0, 0, 0 )\\( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0 )\\( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 )\end{array}$$
$SU(2)_2$ :
$$( 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 )$$
$SU(2)_2$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0 )$$
$SU(2)$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0 )$$

Model 3:

$SO(5)$ :
$$\begin{array}{l}( 0, 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, -1, 0, 0, 0, 0, 0, 0, 0, 0 )\\( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 )\end{array}$$
$SO(5)$ :
$$\begin{array}{l}( 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0 )\\( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0 )\end{array}$$
$SU(2)_2$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 )$$

Model 4:

$SU(4)_2$ :
$$\begin{array}{l}( 0, 0, 0, 0, 0, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, 0, 0, 0 )\\( \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0, -\tfrac{1}{2}, 0, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0 )\\( 0, 0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0 )\end{array}$$
$SU(2)_2$ :
$$( 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 )$$

Model 5:

$$
\begin{array}{l}
SU(4)_2: \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ 0,-\tfrac{1}{2},\ \tfrac{1}{2},\ 0,-\tfrac{1}{2},\ 0,\ 0,\ 0\,) \\
(\,\tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ 0,-\tfrac{1}{2},\ 0,\ 0,-\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0,\ 0\,) \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,-\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0\,) \\
SU(2)_2: \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0\,)
\end{array}
$$

Model 6:

$$
\begin{array}{l}
SU(2)_2: \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0,\ 0\,) \\
SU(2)_2: \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,-\tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0\,)
\end{array}
$$

Model 7:

$$
\begin{array}{l}
SO(7): \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 0,\ 0,\ 0,\ 0,-1,\ 0\,) \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,-1\,) \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1\,) \\
SU(2)_2: \\
(\,0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0\,)
\end{array}
$$

Having defined the embeddings of the nonabelian factors into the left-moving fermionic charges, the $U(1)$ embeddings are defined to span the remaining $n$ dimensional orthogonal subspace, where $n$ varies from 7 to 14 in our solutions. There is of course a great deal of freedom in the choice of a basis for these $U(1)$ embeddings. Our computer program chooses a basis by first identifying the anomalous U(1); this is then designated as $U_{n-1}$. A basis is then chosen for the remaining $n-1$ nonanomalous $U(1)$'s, $U_0$–$U_{n-2}$, such that the complete $U(1)$ basis is orthogonal. The basis is chosen such that all of the elements are integers, and the program attempts to minimize the norms, $|U_i|^2$, of all the basis vectors. The $U(1)$ charges listed in Tables 2.1 to 2.3 are obtained as follows: if $\vec{f}$ is the vector of left-moving fermionic charges of a state, then the $(i+1)$st $U(1)$ charge listed in the table would be $4\vec{f}\cdot\vec{U}_i$. The additional factor of 4 ensures that all of the charge entries will be integers.

Below we give the $U(1)$ embedding basis, $U_0$–$U_{n-1}$, for each solution. Keep in mind that the last vector listed in each case defines the anomalous $U(1)$. The integer just to the right of each basis vector is its norm; we must keep track of these in order to, for example, compute $k_1$.

Model 1:

0: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0) 1

1: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0) 1

2: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0) 1

3: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0) 1

4: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1) 1

5: ( 0 0 0 0 0 -4 -2 2 0 0 0 0 0 -3 0 0 0 0 0 0) 33

6: ( 0 0 0 0 0 8 4 -4 0 0 0 0 11-16 0 0 0 0 0 0) 473

7: ( 5 -5 0 0 0 0 0 0 6 0 0 0 0 0 0 0 0 0 0 0) 86

8: ( 3 -3 0 0 0 -2 -1 1 -5 0 0 0 8 4 0 0 0 0 0 0) 129

9: ( 31 55 86 0 0 -4 -2 2 20 0 0 0 16 8 0 0 0 0 0 0) 12126

10: (-16 8 -8 0 0 -4 45 49 20 47-47 0 16 8 0 0 0 0 0 0) 9964

11: ( 28-14 14 0 0 7 14 7-35 90 69 0-28-14 0 0 0 0 0 0) 16536

12: ( 1 19-19 39 0-29 20-29 15 6 15 0 12 6 0 0 0 0 0 0) 4992

13: ( 5 -1 1 3 0 -1 4 -1 -5 -2 -5 0 -4 -2 0 0 0 0 0 0) 128

Model 2:

0: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 1 -1 0 0 0 0 0 0) 2

1: ( 0 0 0 0 0 -1 0 1 1 0 -1 0 0 0 0 0 0 0 0 0) 4

2: ( 0 -1 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0) 2

3: ( 1 0 0 1 0 -1 0 -1 1 0 1 0 0 0 0 0 0 0 0 0) 6

4: ( -6 4 -4 2 0 0 1 -3 2 -3 -1 0 -6 -6 0 0 0 0 0 0) 168

5: ( -1 1 -1 1 0 0 1 -1 0 -1 -1 0 2 2 0 0 0 0 0 0) 16

6: ( -1 1 -1 1 0 -2 1 1 -2 3 1 0 0 0 0 0 0 0 0 0) 24

7: ( 5 -1 1 3 0 0 5 -1 -4 -1 -5 0 -2 -2 0 0 0 0 0 0) 112

Model 3:

0: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0) 1

1: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0) 1

2: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0) 1

3: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1) 1

4: ( 0 0 0 0 0 0 0 0 -1 1 1 0 0 -1 0 0 0 0 0 0) 4

5: ( 0 -1 -1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0) 2

6: ( -2 1 -1 0 0 0 0 0 0 0 0 0 -2 0 0 0 0 0 0 0) 10

7: ( 0 0 0 0 0 -1 1 2 -3 3 0 0 0 6 0 0 0 0 0 0) 60

8: ( 0 0 0 0 0 2 0 -2 -1 -1 1 0 0 1 0 0 0 0 0 0) 12

9: ( -4 2 -2 0 0 2 3 1 -4 -1 -5 0 6 -2 0 0 0 0 0 0) 120

10: ( 1 7 -7 15 0-12 7-11 4 1 5 0 6 2 0 0 0 0 0 0) 720

11: ( 5 -1 1 3 0 0 5 -1 -4 -1 -5 0 -6 -2 0 0 0 0 0 0) 144

Model 4:
0: ( 0 0 0 0 0 0 0 0 0 0 0 1 2 0) 5
1: ( 6 -6 0 0 0 0 1 1 -1 -1 -2 8 -4 0) 160
2: ( 2 -2 0 0 0 0 -3 -3 -2 -2 1 -2 1 0) 40
3: ( 0 0 -2 2 0 -2 -1 -3 4 2 5 2 -1 0) 72
4: ( 3 -3 5 -5 0 5 -2 3 2 7 4 -2 1 0) 180
5: ( 6 18 28 -4 0 -8-13 3 13-19 2 8 -4 0) 2016
6: ( 4 -2 0 2 0 -3 3 2 4 -1 -1 -4 2 0) 84
Model 5:
0: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 1) 1
1: ( 1 -1 0 0 0 0 0 0 0 0 0 0 1 0) 3
2: ( -2 2 0 0 0 0 -1 -1 -3 -3 -2 0 4 0) 48
3: ( 0 0 0 0 0 0 1 1 0 0 -1 0 0 0) 3
4: ( 1 -1 3 -3 0 3 -2 1 -3 0 -1 0 -2 0) 48
5: ( 5 -1 1 3 0 -5 -1 -2 0 -9 -3 0 -6 0) 192
6: ( 1 3 5 -1 0 -1 -1 2 4 -1 1 0 2 0) 64
Model 6:
0: ( 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0) 1
1: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0) 1
2: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1) 1
3: ( 0 0 0 0 0 0 0 0 -1 1 1 0 0 -1 0 0) 4
4: ( 0 -1 -1 0 0 0 0 0 0 0 0 0 0 0 0 0) 2
5: ( -2 1 -1 0 0 0 0 0 0 0 0 0 -2 0 0 0) 10
6: ( 0 0 0 0 0 -1 1 2 -3 3 0 0 0 6 0 0) 60
7: ( 0 0 0 0 0 2 0 -2 -1 -1 1 0 0 1 0 0) 12
8: ( -4 2 -2 0 0 2 3 1 -4 -1 -5 0 6 -2 0 0) 120
9: ( 1 7 -7 15 0-12 7-11 4 1 5 0 6 2 0 0) 720
10: ( 5 -1 1 3 0 0 5 -1 -4 -1 -5 0 -6 -2 0 0) 144
Model 7:
0: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0) 1
1: ( 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0) 1
2: ( 0 0 0 0 0 0 0 0 1 -1 0 0 0 0 0 0 0) 2
3: ( 0 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0) 2
4: ( 0 0 0 0 0 -1 -1 0 1 1 2 0 0 0 0 0) 8
5: ( 1 1 -1 3 0 -1 3 -2 -1 -1 2 0 0 0 0 0) 32
6: ( -1 1 -1 1 0 -1 -1 -2 1 1 -2 0 0 0 0 0) 16
7: ( 2 -1 1 0 0 1 -1 -2 0 0 0 0 0 0 0 0) 12
8: ( 3 -1 1 1 0 -3 1 2 1 1 -2 0 0 0 0 0) 32

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## TABLE CAPTIONS

TABLE 1:

Summary of the 7 three generation solutions. The first column lists the amount of rank reduction $c_m$, where $22-c_m$ is the rank of the full gauge group. The second column lists the number, $n$, of nonanomalous $U(1)$'s in $G_{flavor}$. The third column lists the number of vectorlike pairs of color triplet exotics, while the fourth column lists the number of extra weak doublets which are also singlets under the nonabelian part of $G_{hidden}$. The fifth column gives $G_{hidden}$, and the last column indicates whether or not there is chiral matter transforming under $G_{hidden}$.

TABLES 2.1-2.3:

The complete list of massless chiral superfields for the conformal field theory solutions which give Models 4,5, and 6, respectively. This is the light spectrum *before* the vacuum shift required by the presence of an anomalous $U(1)$. The row of integers listed for each state are its charges under the full set of nonanomalous $U(1)$'s and the anomalous $U(1)$. The charge under the anomalous $U(1)$ is the last one listed for each state. The normalization and choice of orthogonal basis for the $U(1)$'s is discussed in the appendix. The hypercharges of the states are indicated separately; the embedding of the hypercharge into the nonanomalous $U(1)$'s is given in sections 4.2 and 4.3 of the text.

TABLE 3:

For each model, we indicate the three sectors of the fermionic realization containing the positive helicity fermionic component highest weight states of the three generations of chiral superfields. The sectors are listed as linear combinations of the basis vectors for each fermionic cft solution; the basis vectors are given in Tables A.1 to A.7.

TABLES 4.1-4.3:

The complete cubic, quartic, and quintic order superpotential for Model 5. The left-handed chiral superfields are defined by Table 2.2. Note that the particle identification for down quarks and leptons is somewhat arbitrary: we have made a specific choice for purposes of illustration.

TABLE 5:

The complete cubic order superpotential for Model 4. The left-handed chiral superfields are defined by Table 2.1. Note that the particle identification is somewhat arbitrary; the cubic couplings suggest, in fact, that we should interchange the labelling of $L_2$ and $h_1$ in order to avoid interpreting the second term in $W_3$ as rapid proton decay.

TABLES A.1-A.7:

The basis vectors and $k_{ij}$ matrices which define the 7 conformal field theory solutions. The notation is explained in the appendix.

TABLE A.8:

A set of basis vectors and $k_{ij}$'s which give a fermionic realization of Model 5 equivalent to that obtained from the basis vectors in Table A.5.

TABLE 1

| | $c_m$ | $n$ | $n_{trip}$ | $n_{doub}$ | $G_{hidden}$ | hidden chiral matter? |
|---|---|---|---|---|---|---|
| Model 1: | 2 | 7 | 5 | 14 | $SU(2)_2 \times [SU(2)]^2 \times [U(1)]^5$ | no |
| Model 2: | 2 | 6 | 8 | 4 | $SO(7) \times [SU(2)_2]^2 \times [SU(2)]^4$ | yes |
| Model 3: | 2 | 6 | 4 | 8 | $[SO(5)]^2 \times SU(2)_2 \times [U(1)]^4$ | yes |
| Model 4: | 8 | 5 | 2 | 12 | $SU(4)_2 \times SU(2)_2$ | yes |
| Model 5: | 8 | 5 | 1 | 6 | $SU(4)_2 \times SU(2)_2$ | yes |
| Model 6: | 6 | 6 | 3 | 6 | $[SU(2)_2]^2 \times [U(1)]^3$ | no |
| Model 7: | 6 | 7 | 6 | 14 | $SO(7) \times SU(2)_2$ | yes |

## TABLE 2.1

Quark doublets: $(3,2)_{1/3}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$Q_1$: ( 2 8 0 -8 -28 -8 -24 )
$Q_2$: ( -2 8 -8 16 -4 -8 -24 )
$Q_3$: ( -4 8 8 -8 8 -8 -24 )

Up-type quark conjugates: $(\bar{3},1)_{-4/3}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$u_1^c$: ( 6 0 -8 -16 4 -64 -24 )
$u_2^c$: ( 4 16 0 0 24 -144 16 )
$u_3^c$: ( -2 -32 12 -4 16 -64 -24 )

Down-type quark conjugates: $(\bar{3},1)_{2/3}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$d_1^c$: ( 0 8 6 10 14 40 8 )
$d_2^c$: ( 0 8 6 10 14 40 8 )
$d_3^c$: ( 0 0 0 8 -20 32 12 )

Lepton doublets: $(1,2)_{-1}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$L_1$: ( 0 24 8 -8 32 -152 -8 )
$L_2$: ( -2 -16 -6 -2 14 -32 16 )
$L_3$: ( 4 -8 -8 0 12 -24 12 )

Lepton conjugates: $(1,1)_2$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$e_1^c$: ( 2 48 20 20 -8 -16 8 )
$e_2^c$: ( 0 48 -4 -12 -36 48 -24 )
$e_3^c$: ( -2 32 20 4 8 64 -32 )

Up-type Higgs doublets: $(1,2)_1$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$\bar{h}_1$: ( 0 -24 -8 8 -32 152 8 )
$\bar{h}_2$: ( -2 -24 8 -16 -20 152 8 )
$\bar{h}_3$: ( -6 -24 0 8 4 152 8 )

Down-type Higgs doublets: $(1,2)_{-1}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$h_1$: ( 4 -8 12 4 -28 -152 -8 )
$h_2$: ( 4 -8 -8 0 12 -24 12 )
$h_3$: ( -2 -16 -6 -2 14 -32 16 )

A vectorlike pair of color triplet exotics:
$(3,1)_{4/3}+(\bar{3},1)_{-4/3}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$t_1$: ( 2 0 4 4 8 160 -24 )
$\bar{t}_1$: ( -4 -16 -20 4 -4 -80 -16 )

A vectorlike pair of color triplet exotics:
$(3,1)_{-2/3}+(\bar{3},1)_{2/3}$
under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$t_2$: ( 2 -32 0 -24 12 96 8 )
$\bar{t}_2$: ( 0 0 0 8 -20 32 12 )

Weak doublets with fractional electric charge:
$(1,2)_0$ under $SU(3)_c{\times}SU(2)_L{\times}U(1)_Y$

$D_1$: ( 4 -8 12 -4 -8 8 -4 )
$D_2$: ( 4 -8 12 -4 -8 8 -4 )
$D_3$: ( 2 16 -14 6 6 0 0 )
$D_4$: ( 2 16 -14 6 6 0 0 )
$D_5$: ( 0 16 2 -18 18 0 0 )
$D_6$: ( 0 16 2 -18 18 0 0 )
$D_7$: ( 0 -8 4 20 16 8 -4 )
$D_8$: ( 0 -8 4 20 16 8 -4 )

Exotics with fractional electric charge:

$(6,2)_{1/2}$ under $SU(4)_2{\times}SU(2)_2{\times}U(1)_Y$

$T_1$: ( 0 0 10 -2 -10 16 -8 )
$T_2$: ( 0 0 10 -2 -10 16 -8 )

$(\bar{4},2)_{-3/4}$ under $SU(4)_2{\times}SU(2)_2{\times}U(1)_Y$

$E_1$: ( 0 4 -2 -14 2 -68 -8 )
$E_2$: ( 0 4 -2 -14 2 -68 -8 )

$(6,1)_{-1/2}$ under $SU(4)_2{\times}SU(2)_2{\times}U(1)_Y$

$S_1$: ( 0 -16 -2 2 22 32 -16 )
$S_2$: ( 0 -16 -2 2 22 32 -16 )
$S_3$: ( 0 -24 -8 0 -12 24 -12 )
$S_4$: ( 0 -24 -8 0 -12 24 -12 )

Three vectorlike pairs of $(4,1)_{-5/4}+(\bar{4},1)_{5/4}$
under $SU(4)_2{\times}SU(2)_2{\times}U(1)_Y$

$f_1$: ( 4 -12 4 4 20 -68 -8 )
$\bar{f}_1$: ( -4 12 -4 -4 -20 68 8 )
$f_2$: ( -2 -28 0 0 -12 -84 0 )
$\bar{f}_2$: ( 2 28 0 0 12 84 0 )
$f_3$: ( 0 4 -12 -12 12 -84 0 )
$\bar{f}_3$: ( 0 -4 12 12 -12 84 0 )

$(4,1)_{-1/4}$ under $SU(4)_2{\times}SU(2)_2{\times}U(1)_Y$

$F_1$: ( 2 12 -2 10 14 -44 -20 )
$F_2$: ( 2 12 -2 10 14 -44 -20 )
$F_3$: ( 2 12 -2 10 14 -44 -20 )
$F_4$: ( 2 12 -2 10 14 -44 -20 )
$F_5$: ( 2 4 -8 8 -20 -52 -16 )
$F_6$: ( 0 4 8 -16 -8 -52 -16 )

$F_7$: ( -4 4 0 8 16 -52 -16 )

$(4,1)_{3/4}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$F_8$: ( 2 4 -8 0 0 108 -12 )
$F_9$: ( 2 4 -8 0 0 108 -12 )
$F_{10}$: ( -4 -4 -6 -2 2 100 -8 )
$F_{11}$: ( -4 -4 -6 -2 2 100 -8 )

$(\bar{4},1)_{1/4}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$\bar{F}_1$: ( 4 -4 0 -8 -16 52 16 )
$\bar{F}_2$: ( 2 28 0 8 -8 -76 -4 )
$\bar{F}_3$: ( 2 28 0 8 -8 -76 -4 )
$\bar{F}_4$: ( 0 -4 -8 16 8 52 16 )
$\bar{F}_5$: ( -2 -4 8 -8 20 52 16 )
$\bar{F}_6$: ( -4 20 2 6 -6 -84 0 )
$\bar{F}_7$: ( -4 20 2 6 -6 -84 0 )

$(1,1)_1$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$H_1$: ( 4 32 -8 8 -8 32 -16 )
$H_2$: ( 2 32 8 -16 4 32 -16 )
$H_3$: ( 2 0 4 20 -32 32 -16 )
$H_4$: ( 0 0 20 -4 -20 32 -16 )
$H_5$: ( -2 32 0 8 28 32 -16 )
$H_6$: ( -2 16 -12 4 -40 16 -8 )
$H_7$: ( -4 16 4 -20 -28 16 -8 )
$H_8$: ( -4 0 12 20 4 32 -16 )
$H_9$: ( -4 -8 6 18 -30 24 -12 )
$H_{10}$: ( -4 -8 6 18 -30 24 -12 )
$H_{11}$: ( -4 -8 6 18 -30 24 -12 )
$H_{12}$: ( -4 -8 6 18 -30 24 -12 )
$H_{13}$: ( -8 16 -4 4 -4 16 -8 )

$(1,1)_{-1}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$\bar{H}_1$: ( 8 -16 4 -4 4 -16 8 )
$\bar{H}_2$: ( 4 0 -12 -20 -4 -32 16 )
$\bar{H}_3$: ( 4 -16 -4 20 28 -16 8 )
$\bar{H}_4$: ( 2 -16 12 -4 40 -16 8 )
$\bar{H}_5$: ( 2 -32 0 -8 -28 -32 16 )
$\bar{H}_6$: ( 0 8 6 10 14 -152 -8 )
$\bar{H}_7$: ( 0 8 6 10 14 -152 -8 )
$\bar{H}_8$: ( 0 0 0 8 -20 -160 -4 )
$\bar{H}_9$: ( 0 0 0 8 -20 -160 -4 )
$\bar{H}_{10}$: ( 0 0 -20 4 20 -32 16 )
$\bar{H}_{11}$: ( -2 0 -4 -20 32 -32 16 )
$\bar{H}_{12}$: ( -2 -32 -8 16 -4 -32 16 )
$\bar{H}_{13}$: ( -4 -32 8 -8 8 -32 16 )

Vectorlike pair of exotic singlets with electric charge $\pm 1$:

$s_1$: ( -6 16 0 24 -36 48 -24 )
$\bar{s}_1$: ( 4 -32 -16 -16 40 32 -16 )

$(1,3)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$T_1$: ( 6 16 4 4 32 16 -8 )
$T_2$: ( 2 0 -16 24 -12 0 0 )
$T_3$: ( -4 0 -8 24 24 0 0 )

$(1,2)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$\varphi_1$: ( 0 -24 2 -2 -22 40 -20 )
$\varphi_2$: ( 0 -24 2 -2 -22 40 -20 )
$\varphi_3$: ( 0 -24 2 -2 -22 40 -20 )
$\varphi_4$: ( 0 -24 2 -2 -22 40 -20 )
$\varphi_5$: ( -4 0 -8 -16 4 32 -16 )
$\varphi_6$: ( -4 0 -8 -16 4 32 -16 )
$\varphi_7$: ( -4 0 -8 -16 4 32 -16 )
$\varphi_8$: ( -4 0 -8 -16 4 32 -16 )

Singlets with zero hypercharge and zero anomalous charge:

$\phi_1$: ( 6 0 -8 0 -36 0 0 )
$\phi_2$: ( 4 0 8 -24 -24 0 0 )
$\phi_3$: ( 2 32 -12 -12 24 0 0 )
$\phi_4$: ( -2 32 -20 12 48 0 0 )
$\phi_5$: ( -2 0 16 -24 12 0 0 )
$\phi_6$: ( -2 -32 12 12 -24 0 0 )
$\phi_7$: ( -4 32 -4 -12 60 0 0 )
$\phi_8$: ( -6 0 8 0 36 0 0 )

Singlets with zero hypercharge and nonzero anomalous charge:

$\chi_1$: ( 10 -16 8 16 -28 16 -8 )
$\chi_2$: ( 6 -16 0 40 -4 16 -8 )
$\chi_3$: ( 4 8 14 -22 10 8 -4 )
$\chi_4$: ( 4 8 14 -22 10 8 -4 )
$\chi_5$: ( 4 8 14 -22 10 8 -4 )
$\chi_6$: ( 4 8 14 -22 10 8 -4 )
$\chi_7$: ( 4 -16 16 16 8 16 -8 )
$\chi_8$: ( 2 16 16 -8 -4 -80 40 )
$\chi_9$: ( -2 -16 4 -28 -56 -16 8 )
$\chi_{10}$: ( -4 16 -16 -16 -8 -16 8 )
$\chi_{11}$: ( -6 -16 -4 -4 -32 -16 8 )
$\chi_{12}$: ( -8 -16 12 -28 -20 -16 8 )

TABLE 2.2

Quark doublets: $(3,2)_{1/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$Q_1$: ( -2 0 8 -2 -8 16 -16 )
$Q_2$: ( 2 0 8 -2 -8 16 -16 )
$Q_3$: ( 0 -2 0 -2 20 4 -12 )

Up-type quark conjugates: $(\bar{3},1)_{-4/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$u_1^c$: ( 2 0 0 6 0 -16 -16 ) -4
$u_2^c$: ( -2 0 0 6 0 -16 -16 ) -4
$u_3^c$: ( 0 6 -8 2 -4 -28 -12 ) -4

Down-type quark conjugates: $(\bar{3},1)_{2/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$d_1^c$: ( 0 -6 -8 2 -4 -28 -12 )
$d_2^c$: ( 0 0 0 0 -12 20 4 )
$d_3^c$: ( 0 2 -8 -2 0 8 8 )

Lepton doublets: $(1,2)_{-1}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$L_1$: ( 0 4 -8 0 16 -24 -24 )
$L_2$: ( 0 2 16 -2 -12 -12 -28 )
$L_3$: ( 0 2 0 2 16 0 0 )

Lepton conjugates: $(1,1)_2$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$e_1^c$: ( 0 -2 -8 -6 4 12 -4 )
$e_2^c$: ( 4 -2 -24 -2 -4 -12 4 )
$e_3^c$: ( 0 -2 -8 -6 4 12 -4 )

Up-type Higgs doublets: $(1,2)_1$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$\bar{h}_1$: ( 0 0 -8 -4 -4 -12 4 )
$\bar{h}_2$: ( 0 -2 -16 2 0 0 0 )
$\bar{h}_3$: ( 0 -4 8 0 -16 24 24 )
$\bar{h}_4$: ( 0 -2 0 -2 8 24 -8 )

Down-type Higgs doublets: $(1,2)_{-1}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$h_1$: ( 0 0 8 4 4 12 -4 )
$h_2$: ( 0 2 0 2 16 0 0 )
$h_3$: ( 0 2 0 2 -8 -24 8 )
$h_4$: ( 0 2 16 -2 0 0 0 )

Vectorlike pair of color triplet
exotics: $(3,1)_{-2/3}+(\bar{3},1)_{2/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$t$: ( 0 0 0 0 12 -20 -4 )
$\bar{t}$: ( 0 2 -8 -2 0 8 8 )

Weak doublet exotics with fractional electric charge:
$(2,2)_0$ under $SU(2)_L \times SU(2)_2 \times U(1)_Y$

$D_1$: ( 0 0 -8 2 -4 -12 4 )
$D_2$: ( 0 0 -8 2 -4 -12 4 )
$D_3$: ( 0 0 -8 2 -4 -12 4 )
$D_4$: ( 0 0 -8 2 -4 -12 4 )

Exotics with fractional electric charge:

$(4,1)_{-1}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$
$F_1$: ( 0 0 12 0 12 0 -16 )
$F_2$: ( 0 0 12 0 12 0 -16 )

$(\bar{4},1)_1$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$
$\bar{F}_1$: ( 0 0 4 -4 -4 24 8 )
$\bar{F}_2$: ( 0 0 4 -4 -4 24 8 )
$\bar{F}_3$: ( 0 0 -12 0 0 12 -20 )
$\bar{F}_4$: ( 0 0 -12 0 0 12 -20 )
$\bar{F}_5$: ( 0 0 -12 0 0 12 -20 )
$\bar{F}_6$: ( 0 0 -12 0 0 12 -20 )

$(\bar{4},1)_{-1}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$
$\bar{F}_7$: ( 2 0 4 2 8 -12 -12 )
$\bar{F}_8$: ( 2 0 4 2 8 -12 -12 )
$\bar{F}_9$: ( -2 0 4 2 8 -12 -12 )
$\bar{F}_{10}$: ( -2 0 4 2 8 -12 -12 )

$(1,2)_1$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$
$H_1$: ( 0 -2 8 -4 -12 12 -4 )
$H_2$: ( 0 -2 8 -4 -12 12 -4 )
$H_3$: ( 0 -4 0 2 -8 24 -8 )
$H_4$: ( 0 -4 0 2 -8 24 -8 )

$(1,2)_{-1}$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$
$H_5$: ( 0 2 -8 4 12 -12 4 )
$H_6$: ( 0 0 16 2 -16 0 0 )
$H_7$: ( 0 2 -8 4 12 -12 4 )
$H_8$: ( 0 0 16 2 -16 0 0 )

Singlet exotics with electric charge +1:

$s_1$: ( 0 -4 0 -4 -8 24 -8 )
$s_2$: ( 2 0 -16 -2 -32 0 0 )
$s_3$: ( 0 -4 -16 0 8 24 -8 )
$s_4$: ( 0 -4 0 -4 -8 24 -8 )
$s_5$: ( -2 0 -16 -2 -32 0 0 )
$s_6$: ( -4 -2 -24 -2 -4 -12 4 )

Singlet exotics with electric charge $-1$:

$\bar{s}_1$: ( 0 4 16 0 16 0 0 )
$\bar{s}_2$: ( 0 4 16 0 16 0 0 )
$\bar{s}_3$: ( 0 4 16 0 -8 -24 8 )
$\bar{s}_4$: ( 0 2 24 2 4 12 -4 )
$\bar{s}_5$: ( 0 2 24 2 4 12 -4 )
$\bar{s}_6$: ( 0 -2 8 2 12 -60 20 )

Singlets with zero hypercharge and zero anomalous charge:

$\phi_1$: ( 4 0 0 0 0 0 0 )
$\phi_2$: ( 2 4 0 -2 -16 0 0 )
$\phi_3$: ( 2 4 0 -2 -16 0 0 )
$\phi_4$: ( 2 -4 16 -2 0 0 0 )
$\phi_5$: ( 2 -4 16 -2 0 0 0 )
$\phi_6$: ( 2 -4 0 2 16 0 0 )
$\phi_7$: ( 2 -4 0 2 16 0 0 )
$\phi_8$: ( 0 0 -16 4 16 0 0 )
$\phi_9$: ( 0 0 -16 4 16 0 0 )
$\phi_{10}$: ( -2 4 0 -2 -16 0 0 )
$\phi_{11}$: ( -2 4 0 -2 -16 0 0 )
$\phi_{12}$: ( -2 -4 16 -2 0 0 0 )
$\phi_{13}$: ( -2 -4 16 -2 0 0 0 )
$\phi_{14}$: ( -2 -4 0 2 16 0 0 )
$\phi_{15}$: ( -2 -4 0 2 16 0 0 )
$\phi_{16}$: ( -4 0 0 0 0 0 0 )

Hidden fields:

$(4,2)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$E_1$: ( 0 0 -4 -2 4 -24 -8 )
$E_2$: ( 0 0 -4 -2 4 -24 -8 )
$E_3$: ( 0 -2 4 0 4 0 16 )
$E_4$: ( 0 -2 4 0 -8 -12 -12 )
$E_5$: ( 0 -2 4 0 -8 -12 -12 )

$(\bar{4},2)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$\bar{E}_1$: ( 0 2 -4 0 -4 0 -16 )

$(6,1)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$S_1$: ( 2 0 -8 2 -4 -12 4 )
$S_2$: ( 0 -4 8 0 -4 -12 4 )
$S_3$: ( -2 0 -8 2 -4 -12 4 )
$S_4$: ( 2 0 8 -2 4 12 -4 )
$S_5$: ( 0 4 -8 0 4 12 -4 )
$S_6$: ( 0 2 0 2 -8 24 -8 )
$S_7$: ( 0 2 0 2 -8 24 -8 )
$S_8$: ( -2 0 8 -2 4 12 -4 )

$(1,3)_0$ under $SU(4)_2 \times SU(2)_2 \times U(1)_Y$

$T_1$: ( 2 4 0 -2 8 24 -8 )
$T_2$: ( 0 -2 8 2 12 36 -12 )
$T_3$: ( -2 4 0 -2 8 24 -8 )

Singlets with zero hypercharge and nonzero anomalous charge:

$\chi_1$: ( 2 4 0 -2 8 24 -8 )
$\chi_2$: ( 0 2 8 -6 -4 -12 4 )
$\chi_3$: ( 0 2 8 -6 -4 -12 4 )
$\chi_4$: ( 0 2 -8 -2 0 -24 -24 )
$\chi_5$: ( 0 2 -8 -2 0 -24 -24 )
$\chi_6$: ( 0 2 -24 2 28 -12 4 )
$\chi_7$: ( 0 0 16 -4 8 24 -8 )
$\chi_8$: ( 0 0 0 0 -12 -12 -28 )
$\chi_9$: ( 0 -2 8 2 12 36 -12 )
$\chi_{10}$: ( -2 4 0 -2 8 24 -8 )
$\chi_{11}$: ( 0 0 0 0 12 12 28 )
$\chi_{12}$: ( 0 0 0 0 -24 -24 8 )
$\chi_{13}$: ( 0 0 0 0 -24 -24 8 )
$\chi_{14}$: ( 0 0 -16 4 -8 -24 8 )

TABLE 2.3

Quark doublets: $(3,2)_{1/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$Q_1$: ( 0 0 0 4 0 0 -8 -4 16 24 -24 )
$Q_2$: ( 0 0 0 2 0 -4 -4 6 0 -16 -32 )
$Q_3$: ( 0 0 0 -2 0 -4 -4 -6 0 -16 -32 )

Up-type quark conjugates: $(\bar{3},1)_{-4/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$u^c_1$: ( 0 0 0 2 4 -8 -4 6 -8 16 -16 )
$u^c_2$: ( 0 0 0 0 0 0 24 0 32 -32 -16 )
$u^c_3$: ( 0 0 0 -2 4 -8 -4 -6 -8 16 -16 )

Down-type quark conjugates: $(\bar{3},1)_{2/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$d^c_1$: ( 0 0 0 2 0 12 -4 6 -8 16 -16 )
$d^c_2$: ( 0 0 0 -2 0 12 -4 -6 -8 16 -16 )
$d^c_3$: ( 0 0 0 -4 0 8 0 4 -24 -24 -24 )

Lepton doublets: $(1,2)_{-1}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$L_1$: ( 0 0 0 2 4 8 -4 6 -16 48 0 )
$L_2$: ( 0 0 0 -2 4 8 -4 -6 -16 48 0 )
$L_3$: ( 0 0 0 0 0 -8 8 0 8 0 48 )

Lepton conjugates: $(1,1)_2$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$e^c_1$: ( 0 0 0 2 -4 0 -4 6 8 -48 -48 )
$e^c_2$: ( 0 0 0 -2 -4 0 -4 -6 8 -48 -48 )
$e^c_3$: ( 0 0 0 -2 0 8 -8 2 12 -20 -28 )

Up-type Higgs doublets: $(1,2)_1$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$\bar{h}_1$: ( 0 0 0 -2 0 0 0 2 20 -20 20 )
$\bar{h}_2$: ( 0 0 0 0 0 8 -8 0 -8 0 -48 )
$\bar{h}_3$: ( 0 0 0 4 -4 -4 0 -4 32 -8 8 )

Down-type Higgs doublet: $(1,2)_{-1}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$h$: ( 0 0 0 2 0 0 0 -2 -20 20 -20 )

A vectorlike pair of weak doublet exotics:
$(1,2)_3 + (1,2)_{-3}$ under
$SU(3)_c \times SU(2)_L \times U(1)_Y$

$\bar{h}_1$: ( 0 0 0 -4 -4 4 -16 4 0 56 -8 )
$h_1$: ( 0 0 0 4 4 -4 16 -4 0 -56 8 )

Weak doublet exotics:
$(1,2)_3$ under
$SU(3)_c \times SU(2)_L \times U(1)_Y$

$\bar{h}_4$: ( 0 0 0 0 0 8 -16 0 8 -96 0 )
$\bar{h}_5$: ( 0 0 0 0 -4 4 -24 0 -24 0 0 )

A vectorlike pair of color triplet
exotics: $(3,1)_{-2/3} + (\bar{3},1)_{2/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$t_1$: ( 0 0 0 2 0 0 8 -2 4 -44 -4 )
$\bar{t}_1$: ( 0 0 0 -2 0 0 -8 2 -4 44 4 )

Two vectorlike pairs of color triplet
exotics: $(3,1)_{4/3} + (\bar{3},1)_{-4/3}$
under $SU(3)_c \times SU(2)_L \times U(1)_Y$

$t_2$: ( 0 0 0 -4 4 4 -8 4 24 -8 8 )
$\bar{t}_2$: ( 0 0 0 4 -4 -4 8 -4 -24 8 -8 )
$t_3$: ( 0 0 0 -4 -4 4 8 4 -8 -56 8 )
$\bar{t}_3$: ( 0 0 0 0 -4 -4 16 0 0 64 -16 )

Weak doublet exotics:
$(2,2)_{-1}$ under $SU(2)_L \times SU(2)^{(1)}_2 \times U(1)_Y$

$D_1$: ( 2 0 0 -2 0 0 12 2 -4 4 -4 )
$D_2$: ( 0 0 2 -2 0 0 12 2 -4 4 -4 )
$D_3$: ( 0 0 -2 -2 0 0 12 2 -4 4 -4 )
$D_4$: ( -2 0 0 -2 0 0 12 2 -4 4 -4 )

Exotics with fractional electric charge:
$(2,1)_1$ or $(1,2)_1$ under
$SU(2)^{(1)}_2 \times SU(2)^{(2)}_2 \times U(1)_Y$

$H^{(1)}_1$: ( 2 2 0 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_2$: ( 2 2 0 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_3$: ( 2 2 0 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_4$: ( 2 -2 0 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_5$: ( 2 -2 0 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_6$: ( 2 -2 0 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_7$: ( 0 2 2 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_8$: ( 0 2 2 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_9$: ( 0 2 2 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_{10}$: ( 0 2 -2 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_{11}$: ( 0 2 -2 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_{12}$: ( 0 2 -2 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_{13}$: ( 0 -2 2 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_{14}$: ( 0 -2 2 -2 2 2 -8 2 0 -32 -16 )

$H^{(2)}_{15}$: ( 0 -2 2 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_{16}$: ( 0 -2 -2 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_{17}$: ( 0 -2 -2 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_{18}$: ( 0 -2 -2 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_{19}$: ( -2 2 0 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_{20}$: ( -2 2 0 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_{21}$: ( -2 2 0 -2 0 0 -12 2 -16 16 -16 )
$H^{(1)}_{22}$: ( -2 -2 0 0 -2 6 0 0 12 12 -12 )
$H^{(1)}_{23}$: ( -2 -2 0 -2 2 2 -8 2 0 -32 -16 )
$H^{(2)}_{24}$: ( -2 -2 0 -2 0 0 -12 2 -16 16 -16 )

Exotics with unit electric charge:

$(1,2)_2$ under
$SU(2)^{(1)}_2 \times SU(2)^{(2)}_2 \times U(1)_Y$
$\delta_1$: ( 2 0 0 -2 -2 -2 -8 2 12 -20 -28 )
$\delta_2$: ( 0 0 2 -2 -2 -2 -8 2 12 -20 -28 )
$\delta_3$: ( 0 0 -2 -2 -2 -2 -8 2 12 -20 -28 )
$\delta_4$: ( -2 0 0 -2 -2 -2 -8 2 12 -20 -28 )

$(2,1)_{-2}$ or $(1,2)_{-2}$ under
$SU(2)^{(1)}_2 \times SU(2)^{(2)}_2 \times U(1)_Y$
$\bar{\delta}^{(2)}_1$: ( 2 0 0 2 2 -6 8 -2 12 44 4 )
$\bar{\delta}^{(2)}_2$: ( 2 0 0 0 2 2 16 0 -8 -24 -24 )
$\bar{\delta}^{(1)}_3$: ( 2 0 0 0 0 0 12 0 -24 24 -24 )
$\bar{\delta}^{(2)}_4$: ( 0 0 2 2 2 -6 8 -2 12 44 4 )
$\bar{\delta}^{(2)}_5$: ( 0 0 2 0 2 2 16 0 -8 -24 -24 )
$\bar{\delta}^{(1)}_6$: ( 0 0 2 0 0 0 12 0 -24 24 -24 )
$\bar{\delta}^{(2)}_7$: ( 0 0 -2 2 2 -6 8 -2 12 44 4 )
$\bar{\delta}^{(2)}_8$: ( 0 0 -2 0 2 2 16 0 -8 -24 -24 )
$\bar{\delta}^{(1)}_9$: ( 0 0 -2 0 0 0 12 0 -24 24 -24 )
$\bar{\delta}^{(2)}_{10}$: ( -2 0 0 2 2 -6 8 -2 12 44 4 )
$\bar{\delta}^{(2)}_{11}$: ( -2 0 0 0 2 2 16 0 -8 -24 -24 )
$\bar{\delta}^{(1)}_{12}$: ( -2 0 0 0 0 0 12 0 -24 24 -24 )

Singlets with electric charge +2

$\tau_1$: ( 0 0 0 2 0 -4 -36 6 24 0 0 )
$\tau_2$: ( 0 0 0 -2 0 -4 -36 -6 24 0 0 )

Singlet with electric charge $-2$

$\bar{\tau}_1$: ( 0 0 0 4 0 8 32 -4 -8 40 8 )

Singlets with electric charge +1

$s_1$: ( 0 0 0 4 -4 0 -12 4 4 -4 4 )
$s_2$: ( 0 0 0 0 0 8 -16 0 8 24 24 )
$s_3$: ( 0 0 0 0 -4 0 -12 -8 4 -4 4 )
$s_4$: ( 0 0 0 -2 0 4 -12 10 -8 -16 16 )
$s_5$: ( 0 0 0 -2 -4 -4 -8 2 -12 -44 -4 )
$s_6$: ( 0 0 0 -4 -4 -4 -8 4 8 56 40 )
$s_7$: ( 0 0 0 -6 0 4 -12 -2 -8 -16 16 )

Singlets with electric charge $-1$

$\bar{s}_1$: ( 0 4 0 6 0 -4 12 2 8 16 -16 )
$\bar{s}_2$: ( 0 4 0 2 0 -4 12 -10 8 16 -16 )
$\bar{s}_3$: ( 0 4 0 0 0 -8 16 0 -8 -24 -24 )
$\bar{s}_4$: ( 0 0 0 6 0 -4 12 2 8 16 -16 )
$\bar{s}_5$: ( 0 0 0 4 4 4 8 -4 -8 -56 -40 )
$\bar{s}_6$: ( 0 0 0 2 4 4 8 -2 12 44 4 )
$\bar{s}_7$: ( 0 0 0 2 0 -4 12 -10 8 16 -16 )
$\bar{s}_8$: ( 0 0 0 2 0 -8 8 -2 -12 20 28 )
$\bar{s}_9$: ( 0 0 0 0 4 4 16 0 16 0 -48 )
$\bar{s}_{10}$: ( 0 0 0 0 4 0 12 8 -4 4 -4 )
$\bar{s}_{11}$: ( 0 0 0 0 0 0 8 0 -16 96 -48 )
$\bar{s}_{12}$: ( 0 0 0 0 0 -8 16 0 -8 -24 -24 )
$\bar{s}_{13}$: ( 0 0 0 -4 4 0 12 -4 -4 4 -4 )
$\bar{s}_{14}$: ( 0 0 0 -4 4 -4 8 4 -24 8 40 )
$\bar{s}_{15}$: ( 0 -4 0 6 0 -4 12 2 8 16 -16 )
$\bar{s}_{16}$: ( 0 -4 0 2 0 -4 12 -10 8 16 -16 )
$\bar{s}_{17}$: ( 0 -4 0 0 0 -8 16 0 -8 -24 -24 )

Hidden fields:

$(3,3)$ under $SU(2)^{(1)}_2 \times SU(2)^{(2)}_2$
$N$: ( 0 0 0 0 0 0 0 0 0 0 0 )

$(3,1)$ and $(1,3)$, resp., under
$SU(2)^{(1)}_2 \times SU(2)^{(2)}_2$
$T_1$: ( 0 0 0 4 0 -8 -8 -4 0 -32 -16 )
$T_2$: ( 0 0 0 -4 0 8 8 4 0 32 16 )

$(2,1)$ or $(1,2)$ under
$SU(2)^{(1)}_2 \times SU(2)^{(2)}_2$
$\varphi^{(2)}_1$: ( 2 0 0 4 2 2 -8 -4 0 -32 -16 )
$\varphi^{(1)}_2$: ( 2 0 0 4 0 0 -12 -4 -16 16 -16 )
$\varphi^{(2)}_3$: ( 2 0 0 4 -2 2 4 4 -4 -28 -20 )
$\varphi^{(2)}_4$: ( 2 0 0 0 2 2 0 0 24 24 -24 )
$\varphi^{(1)}_5$: ( 2 0 0 0 0 0 -4 0 8 72 -24 )
$\varphi^{(1)}_6$: ( 2 0 0 0 0 -8 4 0 16 -48 0 )
$\varphi^{(2)}_7$: ( 2 0 0 0 -2 2 4 -8 -4 -28 -20 )
$\varphi^{(2)}_8$: ( 2 0 0 0 -2 -10 0 0 0 0 0 )
$\varphi^{(2)}_9$: ( 0 0 2 4 2 2 -8 -4 0 -32 -16 )
$\varphi^{(1)}_{10}$: ( 0 0 2 4 0 0 -12 -4 -16 16 -16 )

$\varphi_{11}^{(2)}$: ( 0 0 2 4 -2 2 4 4 -4 -28 -20 )
$\varphi_{12}^{(2)}$: ( 0 0 2 0 2 2 0 0 24 24 -24 )
$\varphi_{13}^{(1)}$: ( 0 0 2 0 0 0 -4 0 8 72 -24 )
$\varphi_{14}^{(1)}$: ( 0 0 2 0 0 -8 4 0 16 -48 0 )
$\varphi_{15}^{(2)}$: ( 0 0 2 0 -2 2 4 -8 -4 -28 -20 )
$\varphi_{16}^{(2)}$: ( 0 0 2 0 -2 -10 0 0 0 0 0 )
$\varphi_{17}^{(2)}$: ( 0 0 -2 4 2 2 -8 -4 0 -32 -16 )
$\varphi_{18}^{(1)}$: ( 0 0 -2 4 0 0 -12 -4 -16 16 -16 )
$\varphi_{19}^{(2)}$: ( 0 0 -2 4 -2 2 4 4 -4 -28 -20 )
$\varphi_{20}^{(2)}$: ( 0 0 -2 0 2 2 0 0 24 24 -24 )
$\varphi_{21}^{(1)}$: ( 0 0 -2 0 0 0 -4 0 8 72 -24 )
$\varphi_{22}^{(1)}$: ( 0 0 -2 0 0 -8 4 0 16 -48 0 )
$\varphi_{23}^{(2)}$: ( 0 0 -2 0 -2 2 4 -8 -4 -28 -20 )
$\varphi_{24}^{(2)}$: ( 0 0 -2 0 -2 -10 0 0 0 0 0 )
$\varphi_{25}^{(2)}$: ( -2 0 0 4 2 2 -8 -4 0 -32 -16 )
$\varphi_{26}^{(1)}$: ( -2 0 0 4 0 0 -12 -4 -16 16 -16 )
$\varphi_{27}^{(2)}$: ( -2 0 0 4 -2 2 4 4 -4 -28 -20 )
$\varphi_{28}^{(2)}$: ( -2 0 0 0 2 2 0 0 24 24 -24 )
$\varphi_{29}^{(1)}$: ( -2 0 0 0 0 0 -4 0 8 72 -24 )
$\varphi_{30}^{(1)}$: ( -2 0 0 0 0 -8 4 0 16 -48 0 )
$\varphi_{31}^{(2)}$: ( -2 0 0 0 -2 2 4 -8 -4 -28 -20 )
$\varphi_{32}^{(2)}$: ( -2 0 0 0 -2 -10 0 0 0 0 0 )

Singlets with zero hypercharge and zero anomalous charge:

$\phi_1$: ( 4 0 4 0 0 0 0 0 0 0 0 )
$\phi_2$: ( 4 0 -4 0 0 0 0 0 0 0 0 )
$\phi_3$: ( 0 0 0 2 0 -4 -4 6 -40 -96 0 )
$\phi_4$: ( 0 0 0 0 0 0 0 0 0 0 0 )
$\phi_5$: ( 0 0 0 0 0 0 0 0 0 0 0 )
$\phi_6$: ( 0 0 0 0 0 0 0 0 0 0 0 )
$\phi_7$: ( 0 0 0 0 0 0 0 0 0 0 0 )
$\phi_8$: ( 0 0 0 0 0 0 0 0 0 0 0 )
$\phi_9$: ( 0 0 0 -2 0 -4 -4 -6 -40 -96 0 )
$\phi_{10}$: ( -4 0 4 0 0 0 0 0 0 0 0 )
$\phi_{11}$: ( -4 0 -4 0 0 0 0 0 0 0 0 )

Singlets with zero hypercharge and nonzero anomalous charge:

$\chi_1$: ( 0 0 0 4 4 0 -12 4 4 -4 4 )
$\chi_2$: ( 0 0 0 4 0 -8 -8 -4 0 -32 -16 )
$\chi_3$: ( 0 0 0 2 -4 4 8 -2 12 44 4 )
$\chi_4$: ( 0 0 0 0 4 0 -12 -8 4 -4 4 )
$\chi_5$: ( 0 0 0 0 0 8 0 0 -24 -24 24 )
$\chi_6$: ( 0 0 0 0 0 8 0 0 -24 -24 24 )
$\chi_7$: ( 0 0 0 0 0 0 8 16 24 56 -8 )
$\chi_8$: ( 0 0 0 0 0 -8 0 0 24 24 -24 )
$\chi_9$: ( 0 0 0 0 0 -8 0 0 24 24 -24 )
$\chi_{10}$: ( 0 0 0 0 -4 0 12 8 -4 4 -4 )
$\chi_{11}$: ( 0 0 0 -2 4 -4 -8 2 -12 -44 -4 )
$\chi_{12}$: ( 0 0 0 -4 0 8 8 4 0 32 16 )
$\chi_{13}$: ( 0 0 0 -4 -4 0 12 -4 -4 4 -4 )
$\chi_{14}$: ( 0 0 0 -6 0 -4 12 14 8 16 -16 )
$\chi_{15}$: ( 0 0 0 -8 0 0 8 -8 24 56 -8 )
$\chi_{16}$: ( 0 0 0 -10 0 -4 12 2 8 16 -16 )

TABLE 3

| | | |
|---|---|---|
| Model 1: | A,B,A' | $V_4$, $V_5$, $V_4{+}2V_3{+}2V_5{+}2V_6$ |
| Model 2: | A,B,A' | $V_4$, $V_5$, $V_4{+}2V_3{+}2V_5{+}2V_6$ |
| Model 3: | A,B,A' | $V_4$, $V_5$, $V_4{+}2V_3{+}2V_5{+}2V_6$ |
| Model 4: | A,B,A+B | $V_4$, $V_5$, $V_1{+}V_4{+}3V_5$ |
| Model 5: | A,A,B | $V_4$, $3V_5$, $3V_5$ |
| Model 6: | A,B,A' | $V_4$, $V_5$, $V_4{+}2V_3{+}2V_5{+}2V_6$ |
| Model 7: | A,A,B | $V_4{+}2V_3{+}2V_5{+}2V_6$, $V_5$, $V_5$ |

$$
\begin{aligned}
W_3 = {} & Q_3\ u^c_3\ \bar{h}_3 + Q_3\ d^c_2\ h_3 + L_2\ \bar{h}_2\ \chi_{11} + L_2\ \bar{h}_3\ \chi_6 \\
& + L_3\ \bar{h}_3\ \chi_5 + L_3\ \bar{h}_4\ \chi_{13} + L_3\ h_3\ s_4 + e^c_1\ h_1\ h_3 \\
& + \bar{h}_1\ \bar{h}_2\ \bar{s}_4 + \bar{h}_1\ h_3\ \chi_9 + \bar{h}_2\ \bar{h}_4\ \bar{s}_3 + \bar{h}_2\ h_1\ \chi_2 \\
& + \bar{h}_3\ h_2\ \chi_4 + \bar{h}_4\ h_2\ \chi_{12} + h_2\ h_3\ s_1 + h_3\ h_4\ s_3 \\
& + F_1\ \bar{F}_1\ \chi_{14} + F_2\ \bar{F}_2\ \chi_{14} + \bar{F}_1\ \bar{F}_7\ S_3 + \bar{F}_1\ \bar{F}_9\ S_1 \\
& + \bar{F}_1\ H_5\ E_5 + \bar{F}_2\ \bar{F}_8\ S_3 + \bar{F}_2\ \bar{F}_{10}\ S_1 + \bar{F}_2\ H_7\ E_5 \\
& + s_1\ \bar{s}_2\ \chi_{14} + s_2\ \bar{s}_1\ \phi_{15} + s_2\ \bar{s}_2\ \phi_{14} + s_4\ \bar{s}_1\ \chi_{14} \\
& + s_5\ \bar{s}_1\ \phi_7 + s_5\ \bar{s}_2\ \phi_6 + \phi_1\ \phi_{10}\ \phi_{15} + \phi_1\ \phi_{11}\ \phi_{14} \\
& + \phi_1\ S_3\ S_8 + \phi_2\ \phi_7\ \phi_{16} + \phi_2\ \phi_9\ \phi_{12} + \phi_3\ \phi_6\ \phi_{16} \\
& + \phi_3\ \phi_8\ \phi_{12} + \phi_4\ \phi_8\ \phi_{11} + \phi_4\ \phi_9\ \phi_{10} + \phi_5\ S_3\ S_5 \\
& + \phi_6\ \chi_{10}\ \chi_{13} + \phi_7\ \chi_{10}\ \chi_{12} + \phi_8\ \chi_7\ \chi_{13} + \phi_9\ \chi_7\ \chi_{12} \\
& + \phi_{13}\ S_1\ S_5 + \phi_{14}\ \chi_1\ \chi_{13} + \phi_{15}\ \chi_1\ \chi_{12} + \phi_{16}\ S_1\ S_4 \\
& + E_1\ E_3\ S_7 + E_2\ E_3\ S_6 + E_3\ E_5\ S_5 + E_5\ \bar{E}_1\ \chi_{11} \\
& + S_1\ S_2\ \chi_{10} + S_2\ S_3\ \chi_1
\end{aligned}
$$

TABLE 4.1

$$
\begin{aligned}
W_4 = {} & Q_2\ d^c_3\ h_3\ \phi_{15} + Q_2\ h_3\ \bar{t}\ \phi_{14} + Q_3\ d^c_3\ h_1\ \chi_{12} + u^c_2\ d^c_3\ \bar{t}\ \phi_4 \\
& + d^c_1\ d^c_2\ \bar{t}\ \bar{s}_2 + L_3\ e^c_3\ h_1\ \chi_{12} + L_3\ \bar{h}_2\ \phi_5\ \phi_{11} + L_3\ h_2\ s_2\ \phi_{12} \\
& + e^c_1\ \bar{s}_4\ \phi_9\ \chi_{13} + e^c_3\ \bar{s}_5\ \phi_8\ \chi_{12} + \bar{h}_2\ h_2\ \phi_5\ \phi_{10} + \bar{h}_2\ h_3\ F_2\ \bar{F}_1 \\
& + \bar{h}_2\ h_4\ \phi_6\ \phi_{11} + \bar{h}_2\ h_4\ \phi_7\ \phi_{10} + \bar{h}_2\ h_4\ S_1\ S_8 + \bar{h}_4\ h_3\ \phi_2\ \phi_{15} \\
& + \bar{h}_4\ h_3\ \phi_3\ \phi_{14} + \bar{h}_4\ h_3\ S_3\ S_4 + h_2\ D_2\ \bar{F}_1\ E_4 + h_2\ D_4\ \bar{F}_2\ E_4 \\
& + h_4\ D_1\ \bar{F}_4\ E_3 + h_4\ D_2\ \bar{F}_3\ E_3 + t\ \bar{t}\ S_2\ S_7 + t\ \bar{t}\ \chi_9\ \chi_{13} \\
& + D_1\ D_3\ T_2\ \chi_3 + D_2\ D_4\ S_4\ S_8 + D_3\ D_4\ \phi_4\ T_3 + D_3\ D_4\ \phi_{12}\ T_1 \\
& + F_1\ \bar{F}_7\ s_5\ \chi_{11} + F_1\ \bar{F}_9\ s_2\ \chi_{11} + \bar{F}_1\ \bar{F}_4\ \bar{s}_6\ S_7 + \bar{F}_2\ \bar{F}_6\ \bar{s}_6\ S_7 \\
& + H_1\ H_7\ S_3\ S_4 + H_2\ H_5\ S_3\ S_4 + s_1\ \bar{s}_1\ S_1\ S_3 + s_4\ \bar{s}_4\ \chi_6\ \chi_{13} \\
& + s_4\ \bar{s}_6\ S_5\ S_7 + s_5\ \bar{s}_6\ T_1\ T_2 + \phi_2\ \phi_{15}\ E_3\ \bar{E}_1 + \phi_3\ \phi_9\ S_2\ S_8 \\
& + \phi_3\ \phi_{14}\ E_3\ \bar{E}_1 + \phi_9\ \phi_{11}\ S_2\ S_4 + \phi_9\ S_2\ S_7\ \chi_2 + \phi_9\ S_4\ S_8\ \chi_{13} \\
& + \phi_9\ \chi_2\ \chi_9\ \chi_{13} + E_2\ E_4\ S_7\ \chi_{11} + E_3\ E_4\ S_3\ T_1 + E_3\ \bar{E}_1\ S_3\ S_4 \\
& + S_2\ S_7\ \chi_5\ \chi_{11} + \chi_5\ \chi_9\ \chi_{11}\ \chi_{13}
\end{aligned}
$$

TABLE 4.2

$$
\begin{aligned}
W_5 = {} & Q_1\ \bar{t}\ L_3\ \phi_4\ \varphi_4 + Q_1\ d_3^c\ h_2\ \phi_4\ \varphi_4 + Q_2\ \bar{t}\ L_3\ \phi_{12}\ \varphi_4 + Q_2\ \bar{t}\ D_1\ \bar{F}_9\ E_3 \\
& + Q_2\ d_3^c\ h_2\ \phi_{12}\ \varphi_4 + Q_2\ d_3^c\ D_2\ \bar{F}_9\ E_3 + Q_3\ u_3^c\ L_1\ \bar{h}\ \bar{h} + Q_3\ d_2^c\ L_1\ \bar{h}\ h_3 \\
& + u_2^c\ \bar{t}\ \bar{t}\ S_2\ S_4 + L_1\ h\ e_2^c\ \bar{h}\ h_3 + L_1\ h_4\ s_3\ \bar{h}\ h_3 + L_1\ \bar{h}_2\ \bar{F}_1\ H_4\ E_3 \\
& + L_1\ \bar{h}_2\ \bar{F}_3\ H_2\ E_3 + L_1\ \bar{h}\ \phi_2\ \phi_9\ \phi_{12} + L_1\ \bar{h}\ \phi_3\ \phi_8\ \phi_{12} + L_1\ \bar{h}\ E_5\ \bar{E}_1\ \varphi_1 \\
& + L_1\ \bar{h}\ S_1\ S_2\ \chi_{10} + h\ e_2^c\ h_3\ \phi_1\ \phi_{16} + h\ s_3\ \bar{h}_2\ \bar{s}_3\ \chi_2 + h\ s_3\ h_2\ \varphi_2\ \chi_2 \\
& + h\ s_3\ L_3\ \varphi_3\ \chi_2 + h\ h_3\ e_3^c\ \phi_2\ \phi_{14} + h\ L_3\ s_1\ \phi_{10}\ \phi_{12} + h\ D_1\ \bar{F}_5\ E_3\ \chi_3 \\
& + h_4\ \bar{h}_1\ D_1\ D_1\ T_2 + h_4\ \bar{h}\ \bar{F}_5\ H_3\ E_2 + h_4\ \bar{h}\ \bar{F}_6\ H_3\ E_1 + h_4\ \bar{h}\ \bar{F}_7\ H_1\ E_2 \\
& + h_4\ \bar{h}\ \bar{F}_8\ H_1\ E_1 + h_4\ \bar{h}\ \phi_8\ E_2\ \bar{E}_1 + h_4\ \bar{h}\ \phi_9\ E_1\ \bar{E}_1 + h_4\ D_2\ \bar{F}_3\ \phi_9\ E_4 \\
& + h_4\ D_2\ \bar{F}_6\ E_4\ \varphi_1 + e_1^c\ \bar{F}_9\ H_1\ \phi_3\ E_3 + e_1^c\ \bar{s}_3\ \phi_7\ S_3\ S_5 + e_1^c\ \bar{s}_3\ \phi_{15}\ S_1\ S_5 \\
& + e_2^c\ \bar{h}_4\ h_3\ \bar{s}_5\ \varphi_4 + e_2^c\ D_1\ D_4\ \bar{s}_3\ T_2 + e_2^c\ D_2\ D_3\ \bar{s}_3\ T_2 + e_2^c\ \bar{s}_5\ E_3\ \bar{E}_1\ \varphi_4 \\
& + s_3\ \bar{h}_1\ h_2\ \bar{s}_4\ \varphi_2 + s_3\ L_2\ h_2\ \varphi_1\ \varphi_2 + s_3\ \bar{F}_1\ \bar{F}_2\ \bar{s}_3\ S_3 + s_3\ \bar{F}_1\ \bar{F}_9\ \bar{s}_3\ S_1 \\
& + s_3\ \bar{F}_1\ H_1\ \bar{s}_3\ E_5 + s_3\ \bar{F}_3\ \bar{F}_4\ \bar{s}_3\ S_3 + s_3\ \bar{F}_3\ \bar{F}_{10}\ \bar{s}_3\ S_1 + s_3\ \bar{F}_3\ H_3\ \bar{s}_3\ E_5 \\
& + s_3\ \bar{s}_3\ \phi_1\ S_3\ S_8 + s_3\ \bar{s}_3\ \phi_5\ S_3\ S_5 + s_3\ \bar{s}_3\ \phi_{13}\ S_1\ S_5 + s_3\ \bar{s}_3\ \phi_{16}\ S_1\ S_4 \\
& + s_3\ \bar{s}_3\ E_3\ E_5\ S_5 + s_3\ \bar{s}_5\ \phi_8\ \varphi_3\ \chi_2 + s_3\ \bar{s}_5\ \phi_9\ \varphi_2\ \chi_2 + s_3\ \bar{s}_5\ \varphi_1\ \varphi_3\ \chi_4 \\
& + \bar{h}_1\ \bar{h}_2\ \bar{s}_5\ \phi_6\ \phi_{10} + \bar{h}_1\ \bar{h}\ \bar{F}_2\ \bar{F}_{10}\ S_5 + \bar{h}_1\ \bar{h}\ \bar{F}_4\ \bar{F}_9\ S_5 + \bar{h}_1\ L_3\ \bar{F}_5\ H_4\ E_3 \\
& + \bar{h}_1\ L_3\ \phi_{12}\ S_1\ S_6 + \bar{h}_1\ D_1\ F_1\ E_3\ S_6 + \bar{h}_1\ D_1\ \bar{F}_5\ \bar{s}_5\ E_3 + \bar{h}_2\ \bar{h}_4\ \bar{s}_5\ \varphi_4\ \chi_2 \\
& + \bar{h}_2\ D_4\ \bar{F}_1\ \bar{s}_2\ E_4 + \bar{h}_2\ D_4\ \bar{F}_1\ \bar{s}_4\ E_2 + \bar{h}\ D_1\ \bar{F}_4\ E_2\ \chi_{10} + \bar{h}\ D_1\ \bar{F}_{10}\ E_2\ \chi_1 \\
& + \bar{h}\ D_2\ \bar{F}_4\ E_1\ \chi_{10} + \bar{h}\ D_2\ \bar{F}_{10}\ E_1\ \chi_1 + \bar{h}\ D_3\ \bar{F}_2\ E_2\ \chi_{10} + \bar{h}\ D_3\ \bar{F}_9\ E_2\ \chi_1 \\
& + \bar{h}\ D_4\ \bar{F}_2\ E_1\ \chi_{10} + \bar{h}\ D_4\ \bar{F}_9\ E_1\ \chi_1 + \bar{h}_4\ h_2\ \phi_2\ \phi_{12}\ \varphi_4 + \bar{h}_4\ h_2\ \phi_4\ \phi_{10}\ \varphi_4 \\
& + \bar{h}_4\ L_3\ \phi_3\ \phi_{12}\ \varphi_4 + \bar{h}_4\ L_3\ \phi_4\ \phi_{11}\ \varphi_4 + \bar{h}_4\ D_1\ \bar{F}_9\ \phi_3\ E_3 + \bar{h}_4\ D_2\ \bar{F}_9\ \phi_2\ E_3 \\
& + h_2\ h_2\ s_2\ S_2\ S_8 + h_2\ h_3\ \bar{F}_1\ \bar{F}_8\ S_2 + h_2\ h_3\ \bar{F}_1\ H_8\ E_2 + h_2\ h_3\ s_2\ \phi_{14}\ \chi_7 \\
& + h_2\ h_3\ s_5\ \phi_6\ \chi_7 + h_2\ D_2\ \bar{F}_3\ E_3\ \chi_8 + h_2\ D_4\ \bar{F}_1\ \bar{E}_1\ S_2 + h_2\ D_4\ H_6\ S_2\ S_5 \\
& + h_3\ L_3\ s_2\ \phi_{15}\ \chi_7 + h_3\ L_3\ s_5\ \phi_7\ \chi_7 + h_3\ D_3\ H_6\ \phi_{15}\ T_1 + h_3\ D_4\ \bar{F}_1\ E_2\ \chi_9 \\
& + h_3\ D_4\ H_6\ \phi_{14}\ T_1 + D_1\ D_4\ \phi_1\ \phi_{13}\ T_3 + D_1\ D_4\ \phi_5\ \phi_{16}\ T_1 + D_2\ D_3\ \phi_1\ \phi_{13}\ T_3 \\
& + D_2\ D_3\ \phi_5\ \phi_{16}\ T_1 + D_4\ D_4\ \phi_4\ S_5\ S_8 + D_4\ D_4\ \phi_{12}\ S_4\ S_5 + D_4\ D_4\ S_2\ S_4\ T_3 \\
& + D_4\ D_4\ S_2\ S_8\ T_1 + F_1\ \bar{F}_1\ \varphi_4\ \varphi_4\ \chi_7 + F_1\ H_1\ s_5\ E_3\ S_4 + F_2\ \bar{F}_3\ \varphi_4\ \varphi_4\ \chi_7 \\
& + \bar{F}_1\ \bar{F}_2\ S_3\ \varphi_4\ \chi_7 + \bar{F}_1\ \bar{F}_4\ \phi_{15}\ S_5\ \varphi_3 + \bar{F}_1\ \bar{F}_5\ \bar{s}_3\ \phi_7\ S_3 + \bar{F}_1\ \bar{F}_5\ \bar{s}_3\ \phi_{15}\ S_1 \\
& + \bar{F}_1\ \bar{F}_6\ \bar{s}_3\ \phi_6\ S_3 + \bar{F}_1\ \bar{F}_6\ \bar{s}_3\ \phi_{14}\ S_1 + \bar{F}_1\ \bar{F}_9\ S_1\ \varphi_4\ \chi_7 + \bar{F}_1\ \bar{F}_{10}\ \phi_7\ S_5\ \varphi_3 \\
& + \bar{F}_1\ H_1\ \phi_3\ \phi_{15}\ E_4 + \bar{F}_1\ H_1\ E_5\ \varphi_4\ \chi_7 + \bar{F}_1\ H_3\ \phi_4\ \bar{E}_1\ S_3 + \bar{F}_1\ H_3\ E_2\ S_2\ S_7 \\
& + \bar{F}_1\ H_3\ E_2\ \varphi_3\ \chi_9 + \bar{F}_3\ \bar{F}_4\ S_3\ \varphi_4\ \chi_7 + \bar{F}_3\ \bar{F}_7\ \bar{s}_3\ \phi_7\ S_3 + \bar{F}_3\ \bar{F}_7\ \bar{s}_3\ \phi_{15}\ S_1 \\
& + \bar{F}_3\ \bar{F}_8\ \bar{s}_3\ \phi_6\ S_3 + \bar{F}_3\ \bar{F}_8\ \bar{s}_3\ \phi_{14}\ S_1 + \bar{F}_3\ \bar{F}_{10}\ S_1\ \varphi_4\ \chi_7 + \bar{F}_3\ H_3\ \phi_3\ \phi_{15}\ E_4 \\
& + \bar{F}_3\ H_3\ E_5\ \varphi_4\ \chi_7 + \bar{F}_5\ \bar{F}_9\ \phi_3\ E_3\ E_3 + \bar{F}_5\ H_1\ \phi_3\ \phi_{13}\ E_3 + \bar{F}_5\ H_4\ \phi_8\ E_3\ \chi_3 \\
& + \bar{F}_6\ \bar{F}_9\ \phi_2\ E_3\ E_3 + \bar{F}_6\ H_1\ \phi_2\ \phi_{13}\ E_3 + \bar{F}_9\ H_1\ s_4\ \phi_2\ E_3 + \bar{F}_9\ H_5\ \phi_2\ \phi_9\ E_3 \\
& + \bar{F}_9\ H_5\ \phi_3\ \phi_8\ E_3 + H_3\ H_6\ \phi_4\ S_3\ S_5 + H_3\ H_6\ \phi_{14}\ \varphi_3\ T_1 + H_3\ H_6\ \phi_{15}\ \varphi_2\ T_1 \\
& + H_3\ H_6\ S_2\ S_3\ T_1 + s_2\ s_6\ \bar{s}_1\ \bar{s}_5\ \phi_6 + s_2\ \bar{s}_1\ \phi_4\ \phi_9\ \phi_{16} + e_3^c\ \bar{s}_3\ \phi_8\ S_2\ S_7 \\
& + e_3^c\ \bar{s}_3\ \phi_9\ S_2\ S_6 + s_4\ \bar{s}_3\ \phi_6\ S_3\ S_5 + s_4\ \bar{s}_3\ \phi_{14}\ S_1\ S_5 + s_5\ \bar{s}_1\ \phi_1\ \phi_9\ \phi_{12} \\
& + s_6\ \bar{s}_1\ \phi_5\ S_1\ S_6 + \phi_1\ \phi_{16}\ E_5\ \bar{E}_1\ \varphi_1 + \phi_1\ \phi_{16}\ S_1\ S_2\ \chi_{10} + \phi_1\ S_3\ S_8\ \varphi_4\ \chi_7 \\
& + \phi_2\ \phi_9\ \phi_{12}\ \varphi_4\ \chi_7 + \phi_2\ E_1\ E_3\ S_3\ T_2 + \phi_3\ \phi_8\ \phi_{12}\ \varphi_4\ \chi_7 + \phi_3\ \phi_{15}\ E_3\ E_4\ S_5 \\
& + \phi_3\ \phi_{15}\ E_4\ \bar{E}_1\ \varphi_1 + \phi_4\ \phi_8\ \phi_{11}\ \varphi_4\ \chi_7 + \phi_4\ \phi_9\ \phi_{10}\ \varphi_4\ \chi_7 + \phi_4\ S_1\ S_8\ \varphi_4\ \chi_{10} \\
& + \phi_4\ S_3\ S_8\ \varphi_4\ \chi_1 + \phi_5\ \phi_{11}\ S_2\ S_7\ \chi_6 + \phi_5\ \phi_{11}\ \varphi_3\ \chi_6\ \chi_9 + \phi_5\ S_3\ S_4\ \varphi_4\ T_3 \\
& + \phi_5\ S_3\ S_5\ \varphi_4\ \chi_7 + \phi_5\ S_3\ S_8\ \varphi_4\ T_1 + \phi_5\ S_7\ S_8\ \varphi_3\ \chi_6 + \phi_6\ \phi_{16}\ S_1\ S_6\ \chi_3 \\
& + \phi_8\ \phi_{12}\ S_1\ S_6\ \chi_3 + \phi_{12}\ S_1\ S_4\ \varphi_4\ \chi_{10} + \phi_{12}\ S_3\ S_4\ \varphi_4\ \chi_1 + \phi_{13}\ S_1\ S_4\ \varphi_4\ T_3 \\
& + \phi_{13}\ S_1\ S_5\ \varphi_4\ \chi_7 + \phi_{13}\ S_1\ S_8\ \varphi_4\ T_1 + \phi_{16}\ S_1\ S_4\ \varphi_4\ \chi_7 + E_1\ E_3\ S_7\ \varphi_4\ \chi_7 \\
& + E_2\ E_3\ S_6\ \varphi_4\ \chi_7 + E_3\ E_5\ S_4\ \varphi_4\ T_3 + E_3\ E_5\ S_5\ \varphi_4\ \chi_7 + E_3\ E_5\ S_8\ \varphi_4\ T_1 \\
& + E_3\ \bar{E}_1\ \varphi_4\ \chi_3\ \chi_9 + E_5\ \bar{E}_1\ \varphi_1\ \varphi_4\ \chi_7 + S_1\ S_2\ \varphi_4\ \chi_7\ \chi_{10} + S_2\ S_3\ \varphi_4\ \chi_1\ \chi_7
\end{aligned}
$$

TABLE 4.3

$$
\begin{aligned}
W_3 = {} & Q_1\ u^c_2\ \bar{h}_3 + Q_1\ d^c_1\ L_2 + Q_1\ d^c_2\ h_3 + Q_2\ u^c_2\ \bar{h}_2 \\
& + Q_3\ u^c_2\ \bar{h}_1 + Q_3\ d^c_3\ h_2 + Q_3\ L_3\ \bar{t}_2 + u^c_2\ t_2\ s_1 \\
& + L_2\ D_3\ H_4 + L_2\ D_5\ H_3 + L_3\ D_1\ H_{13} + L_3\ D_5\ H_{10} \\
& + L_3\ D_6\ H_9 + L_3\ D_7\ H_7 + e^c_1\ \bar{t}_1\ t_2 + \bar{h}_1\ h_1\ \phi_7 \\
& + \bar{h}_1\ D_5\ \bar{H}_7 + \bar{h}_1\ D_6\ \bar{H}_6 + \bar{h}_2\ h_1\ \phi_4 + \bar{h}_2\ D_3\ \bar{H}_7 \\
& + \bar{h}_2\ D_4\ \bar{H}_6 + h_2\ D_2\ H_{13} + h_2\ D_5\ H_{12} + h_2\ D_6\ H_{11} \\
& + h_2\ D_8\ H_7 + h_3\ D_4\ H_4 + h_3\ D_6\ H_3 + t_1\ \bar{t}_1\ \chi_8 \\
& + D_1\ D_8\ \chi_{10} + D_2\ D_7\ \chi_{10} + D_3\ D_6\ \phi_6 + D_4\ D_5\ \phi_6 \\
& + T_1\ E_2\ \bar{F}_4 + T_2\ E_1\ \bar{F}_4 + S_1\ \bar{F}_1\ \bar{F}_7 + S_2\ \bar{F}_1\ \bar{F}_6 \\
& + S_3\ \bar{F}_3\ \bar{F}_5 + S_4\ \bar{F}_2\ \bar{F}_5 + f_1\ \bar{F}_1\ H_{13} + f_1\ \bar{F}_4\ H_7 \\
& + f_1\ \bar{F}_5\ H_6 + \bar{f}_1\ F_5\ \bar{H}_4 + \bar{f}_1\ F_6\ \bar{H}_3 + \bar{f}_1\ F_7\ \bar{H}_1 \\
& + f_2\ \bar{F}_1\ H_5 + f_2\ \bar{F}_4\ H_2 + f_2\ \bar{F}_5\ H_1 + \bar{f}_2\ F_5\ \bar{H}_{13} \\
& + \bar{f}_2\ F_6\ \bar{H}_{12} + \bar{f}_2\ F_7\ \bar{H}_5 + f_3\ \bar{F}_1\ H_8 + f_3\ \bar{F}_4\ H_4 \\
& + f_3\ \bar{F}_5\ H_3 + \bar{f}_3\ F_5\ \bar{H}_{11} + \bar{f}_3\ F_6\ \bar{H}_{10} + \bar{f}_3\ F_7\ \bar{H}_2 \\
& + F_8\ \bar{F}_5\ \bar{H}_9 + F_9\ \bar{F}_5\ \bar{H}_8 + F_{10}\ \bar{F}_1\ \bar{H}_7 + F_{11}\ \bar{F}_1\ \bar{H}_6 \\
& + H_9\ \bar{H}_{10}\ \chi_6 + H_{10}\ \bar{H}_{10}\ \chi_5 + H_{11}\ \bar{H}_{10}\ \chi_4 + H_{12}\ \bar{H}_{10}\ \chi_3 \\
& + s_1\ \bar{s}_1\ \chi_8 + T_1\ T_2\ \chi_{12} + T_1\ T_3\ \chi_9 + \phi_4\ \chi_1\ \chi_{12} \\
& + \phi_7\ \chi_2\ \chi_9
\end{aligned}
$$

TABLE 5

$V_0$: (11111111111111111111||1111111111111111111111111111111111111111|1111)
$V_1$: (11100100100100100100||0000000000000000000000000000000000000000|0000)
$V_2$: (00000000000000000000||0000000011111110000000000000000000000000|0000)
$V_3$: (11100100010010001001||++++++++11110011001100000000000000000000|0011)
$V_4$: (11001001001001100100||1111111100001111110000001100000000000000|0000)
$V_5$: (11010010010010100100||++−−++−−0000−−++00++++001111000000000000|0000)
$V_6$: (11001001100100001001||1111000000++++00++++00000000111111110000|0011)
$V_7$: (11010010100100001010||0000000000001111110000110000111100001100|1010)
$V_8$: (00000000000000000011||0000000000000000000000110000110011000011|1001)

$$
\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 & 0 \\
0 & 0 & -1/2 & -1/4 & -1/2 & +1/4 & 0 & 0 & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & -1/4 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & 0 & 0 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & -1/2 & -1/2 & +1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2
\end{pmatrix}
$$

TABLE A.1

$V_0$: (11111111111111111111||1111111111111111111111111111111111111111|1111)
$V_1$: (11100100100100100100||0000000000000000000000000000000000000000|0000)
$V_2$: (00000000000000000000||0000000011111110000000000000000000000000|0000)
$V_3$: (11100100010010001001||++++++++11110011001100000000000000000000|0011)
$V_4$: (11001001001001100100||1111111000011111100000011000000000000000|0000)
$V_5$: (11010010010010100100||++−−++−−0000−−++00++++001111000000000000|0000)
$V_6$: (11001001100100001001||1111000000++++00++++00000000111111110000|0011)
$V_7$: (11010010100100001010||0000000000001111110000110000111100001100|1010)

$$\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & -1/2 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 \\
0 & 0 & -1/2 & -1/4 & -1/2 & +1/4 & 0 & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & -1/4 & 0 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & 0 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2 & -1/2 & -1/4 & 0
\end{pmatrix}$$

TABLE A.2

$V_0$: (11111111111111111111||1111111111111111111111111111111111111111|1111)
$V_1$: (11100100100100100100||0000000000000000000000000000000000000000|0000)
$V_2$: (00000000000000000000||0000000011111111000000000000000000000000|0000)
$V_3$: (11100100010010001001||++++++++11110011001100000000110011001111|0101)
$V_4$: (11001001001001100100||1111111100001111110000001100000000000000|0000)
$V_5$: (11010010010010100100||++−−++−−0000−−++00++++001111000000000000|0000)
$V_6$: (11001001100100001001||1111000000++++00++++00000000111111110000|0101)
$V_7$: (11010010100100001010||0000000000001111110000110000111100001100|0110)

$$
\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & -1/2 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 \\
0 & 0 & -1/2 & +1/4 & -1/2 & +1/4 & -1/2 & -1/2 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & -1/4 & 0 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & 0 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/2 & -1/2 & -1/4 & 0
\end{pmatrix}
$$

TABLE A.3

$V_0$: (11111111111111111111||1111111111111111111111111111|1111111111111111)
$V_1$: (11100100100100100100||0000000000000000000000000000|0000000000000000)
$V_2$: (00000000000000000000||0000000011111110000000000000|0000000000000000)
$V_3$: (11100100010010001001||++++++++11110011001100000011|1111111100000000)
$V_4$: (11100100010010010010||1111111100111100110000001100|0000000000000000)
$V_5$: (11010010010010100100||++−−++−−00++−−0000++++111100|0000000000000000)
$V_6$: (11001001100100001001||111100000000++++++++00000011|1111000011110000)
$V_7$: (00011011011011000000||0000000000111100110000110000|1100110011001100)
$V_8$: (11001001001001100100||1111111100111100110000110000|1010101010101010)

$$\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & -1/2 & 0 & -1/2 & 0 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & +1/4 & -1/2 & -1/2 & 0 \\
0 & 0 & -1/2 & 0 & 0 & -1/4 & 0 & -1/2 & -1/2 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & -1/2 & 0 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & +1/4 & 0 & -1/2 & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & +1/4 & 0 & -1/2
\end{pmatrix}$$

TABLE A.4

$V_0$: (11111111111111111111||1111111111111111111111111111|1111111111111111)
$V_1$: (11100100100100100100||0000000000000000000000000000|0000000000000000)
$V_2$: (00000000000000000000||0000000011111110000000000000|0000000000000000)
$V_3$: (11100100010010001001||++++++++11110011001100000011|1111111100000000)
$V_4$: (11100100010010010010||1111111001111001100000001100|0000000000000000)
$V_5$: (11001001100100001001||111100000000++++++++00000011|0000000000000000)
$V_6$: (11010010010010100100||++−−++−−00++−−0000++++111100|1111000011110000)
$V_7$: (00011011011011000000||000000000111100110000110000|1100110011001100)
$V_8$: (11001001001001100100||111111100111100110000110000|1010101010101010)

$$\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & -1/2 & -1/2 & 0 & 0 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & -1/4 & 0 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 & 0 \\
0 & 0 & -1/2 & 0 & 0 & 0 & -1/4 & -1/2 & -1/2 \\
0 & 0 & 0 & +1/4 & 0 & -1/2 & +1/4 & -1/2 & -1/2 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & +1/4 & -1/2 & 0 \\
0 & -1/2 & -1/2 & 0 & 0 & +1/4 & -1/4 & 0 & -1/2
\end{pmatrix}$$

TABLE A.5

$V_0$: (11111111111111111111||11111111111111111111111111111111|111111111111)
$V_1$: (11100100100100100100||00000000000000000000000000000000|000000000000)
$V_2$: (00000000000000000000||00000000111111100000000000000000|000000000000)
$V_3$: (11100100010010001001||++++++++111100110011000000001111|111111000000)
$V_4$: (11001001001001100100||11111110000111111000000011000000|000000000000)
$V_5$: (11010010010010100100||++−−++−−0000−−++00++++0011110000|000000000000)
$V_6$: (11001001100100001001||1111000000++++00++++000000001111|110000111100)
$V_7$: (11010010100100001010||00000000000011111100001100001100|101100110010)
$V_8$: (00000000000000000011||00000000000000000000001100000011|101010101001)

$$\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & -1/2 & +1/4 & -1/2 & -1/2 & -1/2 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & -1/4 & 0 & 0 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & 0 & 0 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & -1/2 & -1/2 & +1/4 & 0 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2
\end{pmatrix}$$

TABLE A.6

$V_0$: (11111111111111111111||11111111111111111111111111111111|111111111111)
$V_1$: (11100100100100100100||00000000000000000000000000000000|000000000000)
$V_2$: (00000000000000000000||00000000111111110000000000000000|000000000000)
$V_3$: (11100100010010001001||++++++++111100110011000000001111|111111000000)
$V_4$: (11001001001001100100||11111111000011111100000011000000|000000000000)
$V_5$: (11010010010010100100||++−−++−−0000−−++00++++0011110000|000000000000)
$V_6$: (11001001100100001001||1111000000++++00++++000000001111|110000111100)
$V_7$: (00000000000000000011||00000000000011111100000011001111|101010101001)
$V_8$: (11010010100100001010||00000000000000000000001111111111|101100110010)

$$\begin{pmatrix}
0 & 0 & 0 & -1/4 & 0 & +1/4 & +1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & 0 & -1/2 \\
0 & 0 & -1/2 & +1/4 & 0 & 0 & -1/4 & -1/2 & 0 \\
0 & 0 & -1/2 & +1/4 & -1/2 & +1/4 & -1/2 & 0 & -1/2 \\
0 & -1/2 & -1/2 & 0 & 0 & -1/4 & -1/4 & -1/2 & 0 \\
0 & -1/2 & 0 & 0 & 0 & -1/2 & -1/2 & -1/2 & -1/2 \\
0 & 0 & 0 & +1/4 & 0 & +1/4 & 0 & -1/2 & 0 \\
0 & 0 & 0 & 0 & -1/2 & -1/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/2 & +1/4 & 0 & 0
\end{pmatrix}$$

TABLE A.7

$V_0$: (11111111111111111111||1111111111111111111111111111|1111111111111111)
$V_1$: (11100100100100100100||0000000000000000000000000000|0000000000000000)
$V_2$: (00000000000000000000||0000000011111110000000000000|0000000000000000)
$V_3$: (11100100010010010010||1111111001111001100000001100|0000000000000000)
$V_4$: (11001001100100001001||111100000000++++++++00000011|0000000000000000)
$V_5$: (11010010010010100100||tteeeexx00++--0000++++++1100|1111111100000000)
$V_6$: (11001001001001100100||1111111001111001100000110000|1111000011110000)
$V_7$: (00000000000000011011||0000000000000000000000001111|1100110011001100)
$V_8$: (00011011011011000000||0000000001111001100000110000|1010101010101010)

$$\begin{pmatrix}
0 & 0 & 0 & 0 & +1/4 & -3/8 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 & -1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & -1/2 & 0 & -1/4 & 0 & 0 & -1/2 & -1/2 \\
0 & 0 & -1/2 & 0 & -1/2 & -1/2 & -1/2 & -1/2 & -1/2 \\
0 & 0 & 0 & -1/2 & -1/2 & 0 & -1/2 & 0 & 0 \\
0 & -1/2 & 0 & 0 & -1/2 & -1/8 & -1/2 & -1/2 & 0 \\
0 & -1/2 & -1/2 & 0 & +1/4 & +1/8 & -1/2 & -1/2 & 0 \\
0 & 0 & -1/2 & -1/2 & 0 & +1/4 & 0 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & -1/2 & -1/8 & 0 & 0 & -1/2
\end{pmatrix}$$

TABLE A.8