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Ekman pumping in astrophysical bodies

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We examine the dynamics of a rotating viscous fluid following an abrupt change in the angular velocity of the solid bounding surface. We include the effects of a density stratification and compressibility which are important in astrophysical objects such as neutron stars. We confirm and extend the conclusions of previous studies that stratification restricts the Ekman pumping process to a relatively thin layer near the boundary, leaving much of the interior fluid unaffected. We find that finite compressibility further inhibits Ekman pumping by decreasing the extent of the pumped layer and by increasing the time for spin-up. Elsewhere we show that the results of this paper are important for interpreting the spin period discontinuities ("glitches") observed in rotating neutron stars.

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1 Introduction

The approach to solid-body rotation of a fluid inside a rotating boundary is a familiar phenomenon with many applications. For instance, not only can we directly observe this phenomenon in the laboratory, but it may also play an important role in solar models, neutron stars and other environments. Greenspan & Howard (1963) give a fundamental analysis of the linearized version of this problem by considering a rotating axisymmetric container filled with a viscous incompressible fluid. They examine behaviour of the fluid after the angular velocity of the container is suddenly changed by a small amount. Their solution consists of three distinct, time-separated phases: boundary layer formation, Ekman pumping and viscous relaxation. The bulk of the fluid spin-up (or down) occurs through Ekman pumping. Subsequent studies of the effect a stratification in density has on the Ekman pumping process (Walin 1969; Sakurai 1969) chose parameters suited to laboratory conditions, e.g. an incompressible fluid with a temperature dependent density, and found the Ekman pumping process inhibited.

In this investigation, we are primarily concerned with the possible astrophysical applications of the theory of stratified rotating fluids. For instance, when a rotating neutron star undergoes a sudden slight increase in its rotational frequency, as observed in pulsar "glitches", we can describe the fluid dynamics interior to the crust by the theory of rotating fluids. The parameter regime appropriate to this problem is sufficiently different from those considered previously to require a reanalysis of the dynamical equations and their associated assumptions. For instance, one needs to consider the full continuity equation, rather than the incompressible limit.

2 Ekman pumping

2.1 Fluid dynamics

To investigate the response of the fluid in a rotating container, we examine the usual, simple model of a cylinder of height $2L_*$ and radius r_{c*} rotating with angular velocity $\mathbf{\Omega}_*$ (here and elsewhere an asterisk subscript indicates a dimensional variable or operator; quantities without this subscript are dimensionless.) When the angular velocity of the container is abruptly changed by a small amount, the differential rotation between the fluid and the top and bottom of the cylinder generates the "Ekman pumping" process. Unlike the previous studies, we do not assume fluid incompressibility by allowing for an equation of state which relates the mass-energy density ρ_* to the pressure p_* and to the composition. For a fluid with a viscosity ν_*, the Navier–Stokes equations of motion in a frame rotating with angular velocity $\mathbf{\Omega}_*$ are

$$\rho_* \left(\frac{\partial \mathbf{v}_*}{\partial t_*} + \mathbf{v}_* \cdot \mathbf{\nabla}_* \mathbf{v}_* + 2(\mathbf{\Omega}_* \times \mathbf{v}_*) \right) = -\mathbf{\nabla}_* p_* + \rho_* \mathbf{g}_* + \frac{1}{2} \rho_* \mathbf{\nabla}_* \Omega_*^2 r_*^2 + \rho_* \nu_* \nabla_*^2 \mathbf{v}_*, \quad (1)$$

where $\mathbf{g}_*$ is,

$$\mathbf{g}_* = \mathbf{\nabla}_* \Phi_* = -g_* \mathbf{e}_z, \quad (2)$$

and Φ_* the gravitational potential, r_* is the cylindrical radius, and we take $\mathbf{\Omega}_* = \Omega_* \mathbf{e}_z$ with $\mathbf{e}_z$ the unit vector in the z-direction. As long as r_{c*} is not too large the centrifugal acceleration is small compared to the gravitational acceleration and can be neglected. More precisely, we assume that finite Froude number effects can be ignored, i.e. $F \equiv 4\Omega_*^2 r_{c*}/g_* \ll 1$, where F is the Froude number.

In the state of rotational equilibrium the velocity is zero in the rotating frame while the pressure p_{s*} and density ρ_{s*} are functions only of z_*, the direction of gravity, since we are neglecting terms of order F. The Navier–Stokes equation for the equilibrium system is

$$\frac{\partial}{\partial z_*} p_{s*} = -\rho_{s*} g_*, \tag{3}$$

We now look at a perturbed system in which the angular velocity is suddenly changed by a relatively small amount $\Delta\Omega_*$. The resulting pressure and density are

$$\begin{aligned} p_* &= p_{s*}(z_*) + \delta p_*(r_*, z_*, t_*) & (4)\\ \rho_* &= \rho_{s*}(z_*) + \delta\rho_*(r_*, z_*, t_*) & (5)\end{aligned}$$

To first order in δp_*, $\delta\rho_*$ and $v_*(r_*, z_*, t_*)$ we have

$$\frac{\partial \mathbf{v}_*}{\partial t_*} + 2\Omega_* \mathbf{e}_z \times \mathbf{v}_* = -\frac{1}{\rho_{s*}} \nabla_* \delta p_* - \frac{1}{\rho_{s*}} \delta\rho_* \, \mathbf{g}_* + \nu_* \nabla_*^2 \mathbf{v}_*. \tag{6}$$

We non-dimensionalize the equations by writing variables and operators as a dimensional constant times a non-dimensional variable or operator as follows,

$$\begin{aligned} \mathbf{v}_* &\equiv (L_* \Delta\Omega_*)\, \mathbf{v} \\ t_* &\equiv (E^{1/2} 2\Omega_*)^{-1}\, t \\ \mathbf{r}_* &\equiv L_*\, \mathbf{r} \\ \mathbf{z}_* &\equiv L_*\, z\, \mathbf{e}_z \\ \delta p_* &\equiv (2\Omega_* \rho_{0*} L_*^2 \Delta\Omega_*) \delta p \\ \delta\rho_* &\equiv (2\Omega_* \rho_{0*} \mathrm{L}_* \Delta\Omega_* / g_*) \delta\rho \\ \rho_{s*} &\equiv \rho_{0*}\, \rho_s \\ \nabla_* &\equiv (1/L_*) \nabla \end{aligned}$$

where ρ_{0*} is a fiducial value for the equilibrium density. We also introduce the dimensionless viscosity, or Ekman number,

$$E = \frac{\nu_*}{2\Omega_* L_*^2}. \tag{7}$$

The Navier–Stokes equation for the perturbations is now

$$E^{1/2}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{e}_z \times \mathbf{v} = -\frac{1}{\rho_s}\boldsymbol{\nabla}\delta p - \frac{1}{\rho_s}\delta\rho\, \mathbf{e}_z + E\nabla^2\mathbf{v}, \tag{8}$$

or in terms of the individual cylindrical components,

$$E^{1/2}\frac{\partial u}{\partial t} - v = -\frac{\partial}{\partial r}\frac{\delta p}{\rho_s} + E\left(\nabla^2 - \frac{1}{r^2}\right)u \tag{9}$$

$$E^{1/2}\frac{\partial v}{\partial t} + u = E\left(\nabla^2 - \frac{1}{r^2}\right)v \tag{10}$$

$$E^{1/2}\frac{\partial w}{\partial t} = -\frac{1}{\rho_s}\frac{\partial}{\partial z}\delta p - \frac{\delta\rho}{\rho_s} + E\nabla^2 w, \tag{11}$$

where (u, v, w) are the velocities in the (r, θ, z) directions. We need two more equations in order to complete the formulation of the problem, an equation of state and the continuity equation.

We describe the fluid in terms of the pressure and the concentrations of its constituent elements. Within the context of neutron stars these elements are mainly electrons, protons and neutrons. The equation of state then relates the density to these quantities, $\rho_* = \rho_*(p_*, Y_i)$ where Y_i is the concentration of the i-th particle species[1].

The nature of the restoring force and the corresponding Brunt–Väisälä frequency is most readily calculated in the Lagrangian, as opposed to the Eulerian, formulation of the perturbations. We use δq_* for an Eulerian perturbation of a quantity q_*, the difference between the actual and non-perturbed values of that quantity at a given point in space and time. A Lagrangian perturbation Δq_* describes the change from the non-perturbed value an element of fluid experiences as it travels from one point

[1] In the core of an equilibrium neutron star the Y_i are the concentrations that minimize the free energy through nuclear and weak interaction reactions. In the perturbations considered here, the fluctuation time scales are short compared to those for the weak interactions to adjust the ratio of neutrons to proton. The values of the Y_i can thus be considered as fixed properties of the matter. If the equilibrium values of Y_i give a stable stratification, buoyant forces will cause perturbations to oscillate with the Brunt–Väisälä frequency.

to another. The two pertubations are related by a displacement vector field, $\boldsymbol{\xi}_*$,

$$\Delta q_* = \delta q_* + \boldsymbol{\xi}_* \cdot \boldsymbol{\nabla}_* q_{0*}, \tag{12}$$

where $q_{0*}(\mathbf{r})$ is the non-perturbed quantity. The displacement vector field $\boldsymbol{\xi}_*$ is related to the velocity by,

$$\mathbf{v}_* = \frac{\partial}{\partial t_*}\boldsymbol{\xi}_*. \tag{13}$$

In non-dimensional notation $\boldsymbol{\xi}_* = (L_*\Delta\Omega_*/2\Omega_*)\,\boldsymbol{\xi}$ and

$$\mathbf{v} = E^{1/2}\frac{\partial}{\partial t}\boldsymbol{\xi}. \tag{14}$$

To relate the density and pressure perturbations, consider a fluid displacement in which some quantity Y is held constant i.e. $\Delta Y = 0$. The Lagrangian perturbations $\Delta\rho_*$ and Δp_* are then related by

$$\Delta\rho_* = \left(\frac{\partial\rho_*}{\partial p_*}\right)_Y \Delta p_* \equiv \frac{1}{c_{Y*}^2}\Delta p_*. \tag{15}$$

If the fluid is displaced adiabatically so that the entropy and composition are fixed, then c_{Y*} is the usual sound speed. We characterizes the equilibrium relationship between the density and the pressure, by

$$\frac{\partial\rho_{s*}/\partial z_*}{\partial p_{s*}/\partial z_*} = \left(\frac{\partial\rho_*}{\partial p_*}\right)_{\mathrm{eq}} \equiv \frac{1}{c_{\mathrm{eq}*}^2}. \tag{16}$$

Notice that the equilibrium condition,

$$\frac{\partial}{\partial z_*}p_{s*} = -\rho_{s*}g_*, \tag{17}$$

gives

$$\frac{1}{c_{\mathrm{eq}*}^2} = -\frac{1}{\rho_{s*}g_*}\frac{\partial}{\partial z_*}\rho_{s*} = \frac{-1}{g_*}\frac{\partial}{\partial z_*}\ln\rho_{s*} = \frac{-1}{g_*L_*}\frac{\partial}{\partial z}\ln\rho_s. \tag{18}$$

With (12)-(16) we can relate $\delta\rho_*$ and δp_*:

$$\begin{aligned}\delta\rho_* &= \Delta\rho_* - \boldsymbol{\xi}_*\cdot\boldsymbol{\nabla}\rho_{s*} &(19)\\ &= \frac{1}{c_{Y*}^2}\delta p_* + \left(\frac{1}{c_{\mathrm{eq}*}^2} - \frac{1}{c_{Y*}^2}\right)\rho_{s*}g_*\xi_{z*}, &(20)\end{aligned}$$

with ξ_{z*} the z-component of $\boldsymbol{\xi}_*$. Once again, non-dimensionalizing we obtain,

$$\delta\rho = \left(\frac{g_* L_*}{c_{Y*}^2}\right)\delta p + \left(\frac{N_*^2}{4\Omega_*^2}\right)\rho_s \xi_z \equiv \kappa_Y \delta p + N^2 \rho_s \xi_z, \tag{21}$$

where the Brunt–Väisälä frequency N_* is

$$N_*^2 \equiv g_*^2 \left(\frac{1}{c_{\mathrm{eq}*}^2} - \frac{1}{c_{Y*}^2}\right), \tag{22}$$

and the two dimensionless parameters $\kappa_Y \equiv g_* L_*/c_{Y*}^2$ and $N = N_*/2\Omega_*$ are the "constant-Y compressibility" and the normalized Brunt–Väisälä frequency, respectively. In previous studies κ_Y was assumed to be negligible, but in self gravitating astronomical bodies κ_Y can be of order unity or much larger. Returning to our example of the neutron star, for instance, we can estimate the size of κ_Y. Using the values $g_* \approx 10^{14}\mathrm{cm/sec}^2$, $L_* \approx 10^6\mathrm{cm}$ and $c_{Y*} \approx 10^9\mathrm{cm/sec}$ (Epstein 1988), we obtain $\kappa_Y \approx 10^2$. N characterizes the influence of density srtatification on Ekman pumping.

The final equation is the continuity equation for the perturbations,

$$\frac{\Delta\rho_*}{\rho_{s*}} = -\boldsymbol{\nabla}_* \cdot \boldsymbol{\xi}_* \tag{23}$$

$$= -\nabla_{r*} \cdot \xi_{r*} - \frac{\partial \xi_{z*}}{\partial z_*}. \tag{24}$$

With (12), the continuity equation becomes,

$$\delta\rho_* + \xi_{z*}\frac{\partial \rho_{s*}}{\partial z_*} + \rho_{s*}\nabla_{r*} \cdot \xi_{r*} + \rho_{s*}\frac{\partial \xi_{z*}}{\partial z_*} = 0. \tag{25}$$

Using (16) and (20), and taking the time derivative of (25), we get

$$\frac{1}{\rho_{s*} c_{Y*}^2}\frac{\partial}{\partial t_*}\delta p_* - \frac{g_*}{c_{Y*}^2} w_* + \frac{\partial w_*}{\partial z_*} + \nabla_{r*} \cdot u_* = 0. \tag{26}$$

In non-dimensionalized units this is

$$E^{1/2}\Omega^2 \frac{\partial}{\partial t}\frac{\delta p}{\rho_s} - \kappa_Y w + \frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(ru) = 0, \tag{27}$$

where we have introduced the dimensionless angular velocity,

$$\Omega \equiv \frac{2L_*\Omega_*}{c_{Y*}}. \tag{28}$$

We can now rearrange the complete set of perturbation equations in a more convenient form. We use (21) to eliminate $\delta\rho$ in (11) and take the time derivative to obtain,

$$E\frac{\partial^2 w}{\partial t^2} = -N^2 w - E^{1/2}\frac{1}{\rho_s}\frac{\partial}{\partial z}\frac{\partial}{\partial t}\delta p - E^{1/2}\kappa_Y\frac{\partial}{\partial t}\frac{\delta p}{\rho_s} + E^{3/2}\nabla^2\frac{\partial}{\partial t}w. \tag{29}$$

We use

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}\delta p = \frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_s}\right) + \left(\frac{\delta p}{\rho_s}\right)\frac{\partial}{\partial z}\ln\rho_s \tag{30}$$

with (18), to change (29) to

$$E\frac{\partial^2 w}{\partial t^2} = -N^2 w - E^{1/2}\frac{\partial}{\partial z}\frac{\partial}{\partial t}\frac{\delta p}{\rho_s} - E^{1/2}(\kappa_Y - \kappa_{\rm eq})\frac{\partial}{\partial t}\frac{\delta p}{\rho_s} + E^{3/2}\nabla^2\frac{\partial}{\partial t}w. \tag{31}$$

where the "equilibrium compressibility" is

$$\kappa_{\rm eq} \equiv \frac{g_* L_*}{c^2_{\rm eq*}}. \tag{32}$$

Since the constant-Y and equilibrium compressibilities are comparable, we write

$$\Delta\kappa \equiv \kappa_{\rm eq} - \kappa_Y = \frac{N_*^2 L_*}{g_*} \ll 1. \tag{33}$$

The variable δp only occurs in the combination $\delta p/\rho_s$, so we define $\delta P \equiv \delta p/\rho_s$. The final equations are now

$$E^{1/2}\frac{\partial}{\partial t}u - v = -\frac{\partial}{\partial r}\delta P + E\left(\nabla^2 - \frac{1}{r^2}\right)u \tag{34}$$

$$E^{1/2}\frac{\partial}{\partial t}v + u = E\left(\nabla^2 - \frac{1}{r^2}\right)v \tag{35}$$

$$E\frac{\partial^2 w}{\partial t^2} = -N^2 w - E^{1/2}\frac{\partial}{\partial z}\frac{\partial}{\partial t}\delta P + E^{1/2}\Delta\kappa\frac{\partial}{\partial t}\delta P + E^{3/2}\nabla^2\frac{\partial}{\partial t}w. \tag{36}$$

$$E^{1/2}\Omega^2\frac{\partial}{\partial t}\delta P - \kappa_Y w + \frac{\partial}{\partial z}w + \frac{1}{r}\frac{\partial}{\partial r}(ru) = 0. \qquad (37)$$

Note the harmonic restoring force provided by the Brünt–Väisälä term in (36). The above four equations, (34)–(37), describe the evolution of the four unknowns, v and δP. We have dimensionless parameters, E, N^2, $\Delta\kappa$, κ_Y and Ω. In order to reduce the parameter space we consider only the slow rotation limit, $\Omega \ll 1$. Furthermore, since $\Delta\kappa \ll 1$, we will not consider this term in what follows. Both Ω and $\Delta\kappa$ are easily included in the general solution, but we have found that they have little effect on the numerical results.

The presence of the compressibility κ_Y distinguishes this set of equations from earlier studies (Walin 1969; Sakurai 1969; Clark *et al.* 1971). Previous studies chose to emphasize the effects of temperature on the density of the fluid. Specifically, the density was considered a function of the temperature and the stratification was a result of a temperature gradient which was imposed by the boundary conditions. The dynamical significance of the stratification and Brunt–Väisälä frequency arose through the effects of temperature diffusion and the heat equation. This approach is not appropriate to the astrophysical cases with which we are primarily concerned. In neutron stars, for example, thermal effects have a negligible result on the fluid dynamics, whereas compressibility is quite significant. We, therefore, focus on the dependence on the equation of state.

2.2 Boundary values and initial conditions

To obtain a unique solution to (34)–(37), we need to specify both the boundary and initial conditions to our problem. There are, in essence, two approaches to take at this point. The most complete method is to state that initially the fluid rotates uniformly with the cylinder, and solve for the behaviour of the fluid after the angular velocity of the cylinder changes with the Laplace transformation technique

(Greenspan & Howard 1963). A simpler, and more physically elucidating approach, although less rigorous, used by other researchers in the field (Walin 1969; Sakurai 1969; Barcilon & Pedlosky 1967) entails recognizing that different physical processes take place on widely different time scales in different regions of the fluid. We will follow this latter approach.

If the Ekman number E, or dimensionless viscosity, is sufficiently small, the behaviour of the fluid following an abrupt change in rotation rate of the container can be viewed as three distinct physical processes which occur on timescales Ω_*^{-1}. $E^{-1/2}\Omega_*^{-1}$ and $E^{-1}\Omega_*^{-1}$. The most rapid process is the formation of a viscous boundary layer. Following the impulsive change of rotation of the cylinder, a viscous Rayleigh shear layer forms on the upper and lower surfaces in a time scale on the order of a rotation time ($t_{b*} \approx \Omega_*^{-1}$). Within this region the gradient in the azimuthal velocity results in an imbalance between the centrifugal and pressure gradient forces causing fluid to flow radially. This radial flow in the boundary layer establishes a secondary flow where fluid in the interior is pulled into the boundary layer to replace the flow in the Ekman layer, creating an opposing radial flow in the interior fluid that satisfies continuity requirements. This Ekman pumping spins the interior of the fluid up in a time scale of order $E^{-1/2}\Omega_*^{-1}$. With our choice of dimensionless variables this corresponds to a dimensionless time, $t_E \approx 1$. Finally, residual oscillations decay in the viscous diffusion time $t_{v*} \approx E^{-1}\Omega_*^{-1}$.

Since the principal goal of this investigation is to understand the effects of the stratification and compressibility on the Ekman pumping in the interior of the fluid, we expand (34)-(37) in powers of $E^{1/2}$ and isolate the equations relating to Ekman pumping. The initial velocity distribution for the Ekman pumping equation is equivalent to the final velocity distribution of the boundary layer which forms during the first phase. Following Walin(1969), we formulate the boundary condition in terms of the continuity of the velocity perpendicular to the Ekman boundary just outside the boundary layer,

$$w(z=\pm 1) = \mp \frac{E^{1/2}}{\sqrt{2}}(\nabla \times \mathbf{v})_z \tag{38}$$

$$= \mp \frac{E^{1/2}}{\sqrt{2}} \frac{1}{r} \frac{\partial}{\partial r}(rv). \tag{39}$$

The boundary condition on the sidewalls at r_c is that the radial velocity goes to zero, i.e. $u(r_c) = 0$.

It is critical to the dynamics of Ekman pumping that the vertical velocity at the boundary layer is $O(E^{1/2})$. This standard result (see, e.g. Pedlosky 1979) can be understood by scaling arguments. The imbalance between the centrifugal forces and pressure gradient forces in the boundary layer drives the Ekman pumping process. The thickness λ of the boundary layer is $O(E^{1/2})$ since the viscous terms in the dimensionless Navier–Stokes equation is $E\nabla^2 \approx E/\lambda^2 = O(1)$. The mass flux *within* the boundary layer is $\dot{M}_\lambda \propto \lambda = O(E^{1/2})$. The net mass flux $\dot{M}_z \propto w$ *perpendicular* to the boundary layer is of the same order as $\dot{M}_\lambda$ giving $w = O(E^{1/2})$.

3 Solutions

To solve (34)–(37) perturbatively we expand each fluid variable q as a series $q = q_0 + E^{1/2}q_1 + Eq_2 + \cdots$. Collecting terms of a given power of $E^{1/2}$, we obtain a set of equations governing each order in the expansion.

We find that the $O(1)$ equations are,

$$v_0 = \frac{\partial}{\partial r}\delta P_0 \tag{40}$$

$$u_0 = 0 \tag{41}$$

$$w_0 = 0 \tag{42}$$

$$-\kappa_Y w_0 + \frac{\partial}{\partial z} w_0 + \frac{1}{r}\frac{\partial}{\partial r}(r u_0) = 0, \tag{43}$$

and the $O(E^{1/2})$ equations are,

$$v_1 = \frac{\partial}{\partial r}\delta P_1 \tag{44}$$

$$\frac{\partial}{\partial t} v_0 = -u_1 \tag{45}$$

$$N^2 w_1 = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\delta P_0 \tag{46}$$

$$-\kappa_Y w_1 + \frac{\partial}{\partial z} w_1 + \frac{1}{r}\frac{\partial}{\partial r}(r u_1) = 0. \tag{47}$$

We define $\phi \equiv -\partial \delta P_0/\partial t$, so that (40) and (45) become

$$u_1 = \frac{\partial}{\partial r}\phi \tag{48}$$

and (46) and (47) are now, respectively,

$$N^2 w_1 = \frac{\partial}{\partial z}\phi \tag{49}$$

$$\frac{-\kappa_Y}{N^2}\left(\frac{\partial}{\partial z}\phi\right) + \frac{\partial}{\partial z}\frac{1}{N^2}\left(\frac{\partial}{\partial z}\phi\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}\phi) = 0. \tag{50}$$

Assuming N^2 varies slowly over z, we neglect its derivatives and simplify (50) to

$$\frac{\partial^2}{\partial z^2}\phi - \kappa_Y \frac{\partial}{\partial z}\phi + N^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) = 0. \tag{51}$$

After taking the time derivative, the boundary condition, (39), is

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\phi = \pm \frac{N^2}{\sqrt{2}}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right), \quad \text{at } z = \pm 1. \tag{52}$$

By setting $\phi = Z(z)\,R(r)\,T(t)$, (51) becomes,

$$\frac{1}{Z}\frac{d^2}{dz^2}Z - \frac{\kappa_Y}{Z}\frac{d}{dz}Z + \frac{N^2}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}R\right) = 0. \tag{53}$$

The solutions to the spatial functions are

$$Z = Ae^{\beta_+ z} + Be^{\beta_- z} \tag{54}$$

$$R = J_0(kr), \tag{55}$$

where

$$\beta_{\pm} = \frac{1}{2}\left(\kappa_Y \pm (\kappa_Y{}^2 + 4k^2N^2)^{\frac{1}{2}}\right) \tag{56}$$

The symmetry of the boundary condition, $w_1(z=0)=0$, relates A and B:

$$A = -\frac{\beta_-}{\beta_+}B. \tag{57}$$

The constant B is arbitrary, and we choose it so that $Z(1)=1$. This leads to

$$B = \left(e^{\beta_-} - \frac{\beta_-}{\beta_+}e^{\beta_+}\right)^{-1}. \tag{58}$$

The possible values of k are determined by the boundary condition at the sidewall at $r = r_c$, i.e. $u_1(r_c) = 0$. From (48) and (55) we see that this condition corresponds to $J_1(k_m r_c) = 0$ for $m = 0, 1, 2, \ldots$. The first zeroes of J_1 are $k_m r_c =$ 0, 3.8317..., 7.0156.... The solution $k_m = 0$ has $\mathbf{v} = 0$ everywhere and is of no interest.

We utilize the boundary condition to determine the time dependence of ϕ. Putting our solution for R and Z into (39), we obtain the differential equation,

$$\frac{d}{dt}T = \frac{-k^2N^2}{\sqrt{2}}\frac{Z(1)}{\frac{dZ}{dz}(1)}T, \tag{59}$$

whose solution is,

$$T(t) = e^{-\omega t}, \tag{60}$$

where,

$$\omega = \frac{k^2N^2}{\sqrt{2}}\left(A\beta_+e^{\beta_+} + B\beta_-e^{\beta_-}\right)^{-1}, \tag{61}$$

with A and B defined as above.

There are two interesting limiting cases. The first is that of no stratification $N \to 0$; the second is that of an incompressible fluid $\kappa_Y \to 0$. Let us consider the first of these which gives,

$$\beta_+ \approx \kappa_Y\left(1 + \frac{k_m^2N^2}{\kappa_Y{}^2}\right) \tag{62}$$

$$\beta_- \approx -\frac{k_m^2 N^2}{\kappa_Y} \tag{63}$$

$$B \approx 1 + \frac{k_m^2 N^2}{\kappa_Y}(1 + e^{\kappa_Y}/\kappa_Y) \tag{64}$$

$$A \approx \frac{k_m^2 N^2}{\kappa_Y{}^2} \tag{65}$$

and,

$$\omega \approx \frac{\kappa_Y}{\sqrt{2}(e^{\kappa_Y} - 1)}. \tag{66}$$

Equation (66) shows that for large compressibilities the Ekman spin-up time scale ω^{-1} grows exponentially with κ_Y. The second limiting case, $\kappa_Y \to 0$, gives $\beta_\pm \approx \pm k_m N$ and

$$\omega \approx \frac{k_m N \cosh k_m N}{\sqrt{2} \sinh k_m N}. \tag{67}$$

This matches the $\kappa_Y = 0$, $N \neq 0$ solution which was obtained by Walin (1969).

We are now in a position to write the complete solution for the quantities ϕ and v_0;

$$\phi = \sum_{m=1}^{\infty} C_m Z_m(z) J_0(k_m r) e^{-\omega_m t}. \tag{68}$$

The velocity, v_0, is found from the relationship,

$$\frac{\partial}{\partial r}\phi = -\frac{\partial}{\partial t} v_0, \tag{69}$$

which gives,

$$v_0(r, z, t) = -\sum_{m=1}^{\infty} \frac{k_m}{\omega_m} C_m Z_m(z) J_1(k_m r) e^{-\omega_m t} + v_\infty(r, z). \tag{70}$$

The last term represents the final velocity due to Ekman pumping. If we take the frame of reference as that rotating with the cylinder, the final velocity at the boundary of the interior fluid is given by

$$v_\infty(r, z = \pm 1) = 0 \tag{71}$$

$$v_\infty(r_c, z) = 0. \tag{72}$$

We determine C_m from the initial state of the fluid, which we choose as depending only on the radial coordinate,

$$v_0(r, z, t = 0) = -r = -\sum_{m=1}^{\infty} \frac{k_m C_m}{\omega_m} J_1(k_m r). \tag{73}$$

The coefficients, given by the standard equation for a Fourier–Bessel series, are

$$\frac{k_m C_m}{\omega_m} = \frac{2}{k_m J_2(k_m)}, \tag{74}$$

and the final velocity is

$$v_\infty(r, z) = -2 \sum_{m=1}^{\infty} \frac{1}{k_m J_2(k_m)} J_1(k_m r) \left[1 - Z_m(z)\right], \tag{75}$$

where we have chosen $r_c = 1$ ($r_{c*} = L_*$).

4 Discussion

The time dependence of the Ekman pumping process is exponential with characteristic time $1/\omega$. We plot the value of ω as a function of $k_m N$ in figure 1 for different values of the parameter κ_Y. Larger N, corresponding to greater density stratification gives larger ω and reduced characteristic time. That is, a strongly stratified fluid spins up much quicker than a non-stratified fluid. On the other hand, an increased value of the compressibility κ_Y slows the pumping process for a given $k_m N$. The spin-up time ω^{-1} decreases with increased stratification because stratification isolates much of the fluid from the pumping process.

Figure 1. The spin-up characteristic time, ω, as a function of $k_m N$ for varying values of κ_Y.

Figures 2 and 3 show that the rotation state at the end of the Ekman pumping stage is not that of a solid body. The ordinate $Z(z)$ is proportional to the final azimuthal velocity, with $Z = 1$ being the largest possible spin-up. Larger values of $k_m N$ leave more of the internal fluid unaffected by the Ekman pumping process. In contrast, in a homogeneous fluid, $N = 0$, Ekman pumping brings the entire fluid to an angular velocity equal to that of the boundary. The compressibility κ_Y further decreases the amount of pumped fluid, as we can see by comparing figure 2, for $\kappa_Y = 0$, with figure 3, for $\kappa_Y = 10$.

Figure 2. The final azimuthal velocity as a function of depth for an arbitrary value of the radius. A value of $Z = 1$ is complete spin-up, while $Z = 0$ is no spin-up, with $\kappa_Y = 0$.

Figure 3. As in figure 2, but with $\kappa_Y = 10$.

Compressibility thus decreases the efficacy of Ekman pumping both by lengthening the spin-up time and by decreasing the amount of affected fluid. To convey a clearer picture of how strong the effect of κ_Y is, we plot in figure 4 the final angular velocity of the fluid at its central ($z = 0$) layer as a function of $k_m N$ for different values of κ_Y.

Though there is little change between $\kappa_Y = 0$ and $\kappa_Y = 1$, the internal final angular velocity is strongly suppressed as κ_Y increases to 10.

Figure 4. The final velocity of the central layer of the fluid ($z = 0$), as a function of $k_m N$, for different values of κ_Y.

In figure 5 we plot the average spin-up of the fluid $\langle Z \rangle$ as a function of the normalized Brunt–Väisälä frequency N for the two lowest order modes, k_1 and k_2, for $r_c = 1$ (i.e. $r_{c*} = L_*$). We see that even modest values of N prevent most of the fluid from spinning up during the Ekman pumping phase. The state of the fluid after a time scale of $t_* \approx E^{-1/2}\Omega_*^{-1}$ is, thus, one of non-uniform rotation. The process of viscous diffusion, which operates in a time $t_{\nu*} \approx E^{-1}\Omega_*^{-1}$ eventually brings the fluid into solid-body rotation.

Figure 5. The average final spin-up of the fluid as a function of the stratification. The top two curves (solid and dashed lines) were calculated for an incompressible fluid, $\kappa_Y = 0$, while for the bottom two curves (dotted and dash–dotted lines) $\kappa_Y = 10$. With a highly compressible fluid ($\kappa_Y = 10$) even very small values of N result in very little spin-up from Ekman pumping.

The case of spherical geometry was studied by Clark *et al.* (1971), where they found that the solution for a sphere is qualitatively similar to that of the cylinder. That is, the final state of non-uniform rotation also exists in the sphere, but the geometry of the layer that gets Ekman pumped is modified.

We find a particularly interesting application of these phenomena is the response of the interior of a rotating neutron star to a glitch, a sudden small change in the rotational velocity. Within the star there exists a significant stratification due to the strong gravitational field and the equilibrium concentrations of protons, neutrons and electrons. Reisenegger & Goldreich (1992) estimated a value of $N_* \approx 500\mathrm{s}^{-1}$ for a neutron star. For a canonical value of $\Omega_* \approx 100\mathrm{s}^{-1}$, we obtain $N \approx 2.5$. This is large enough to have a significant effect on the length of time the core of the star needs to come into rotational equilibrium. We explore these issues in a forthcoming paper.

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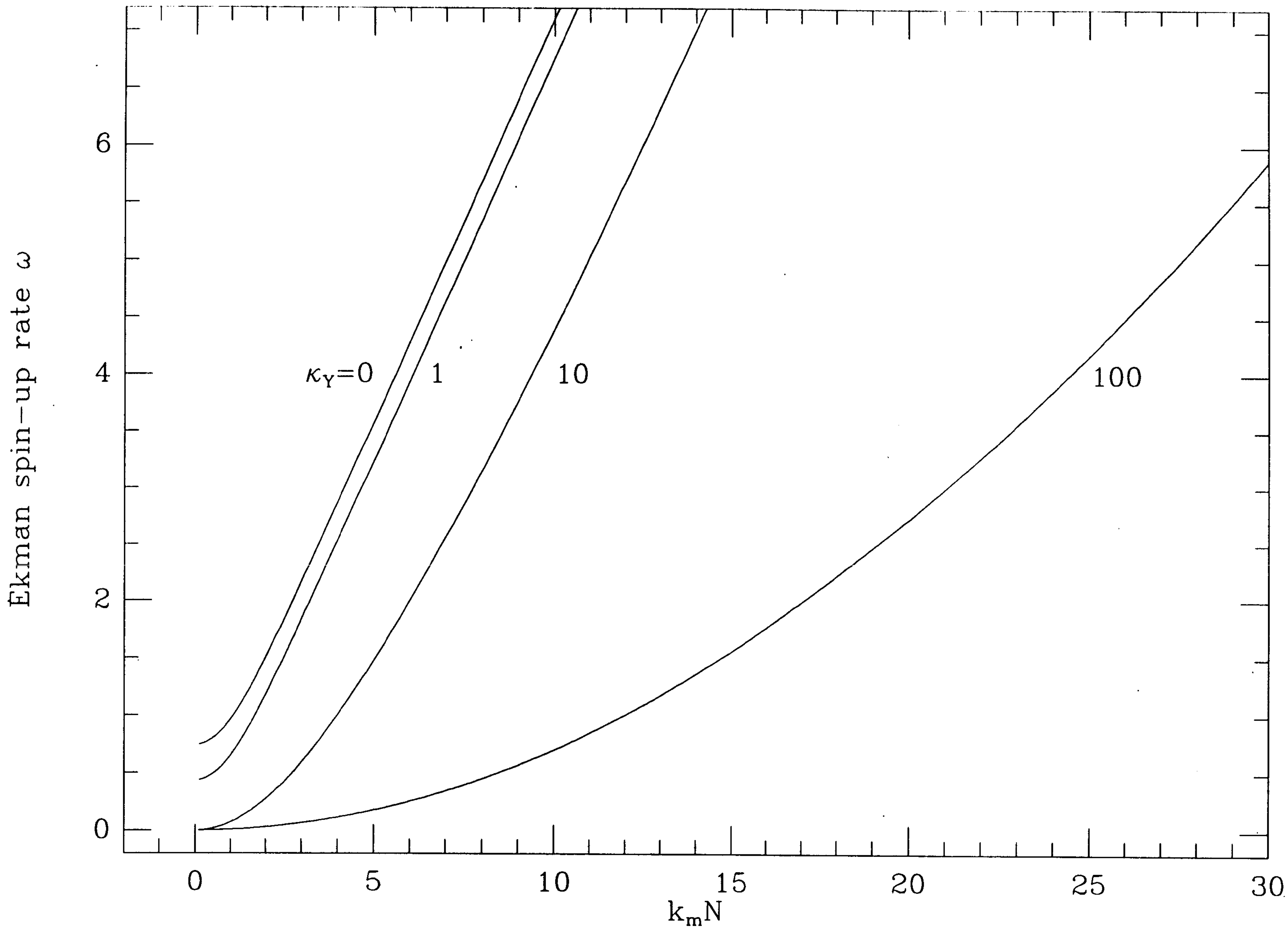


Figure 1

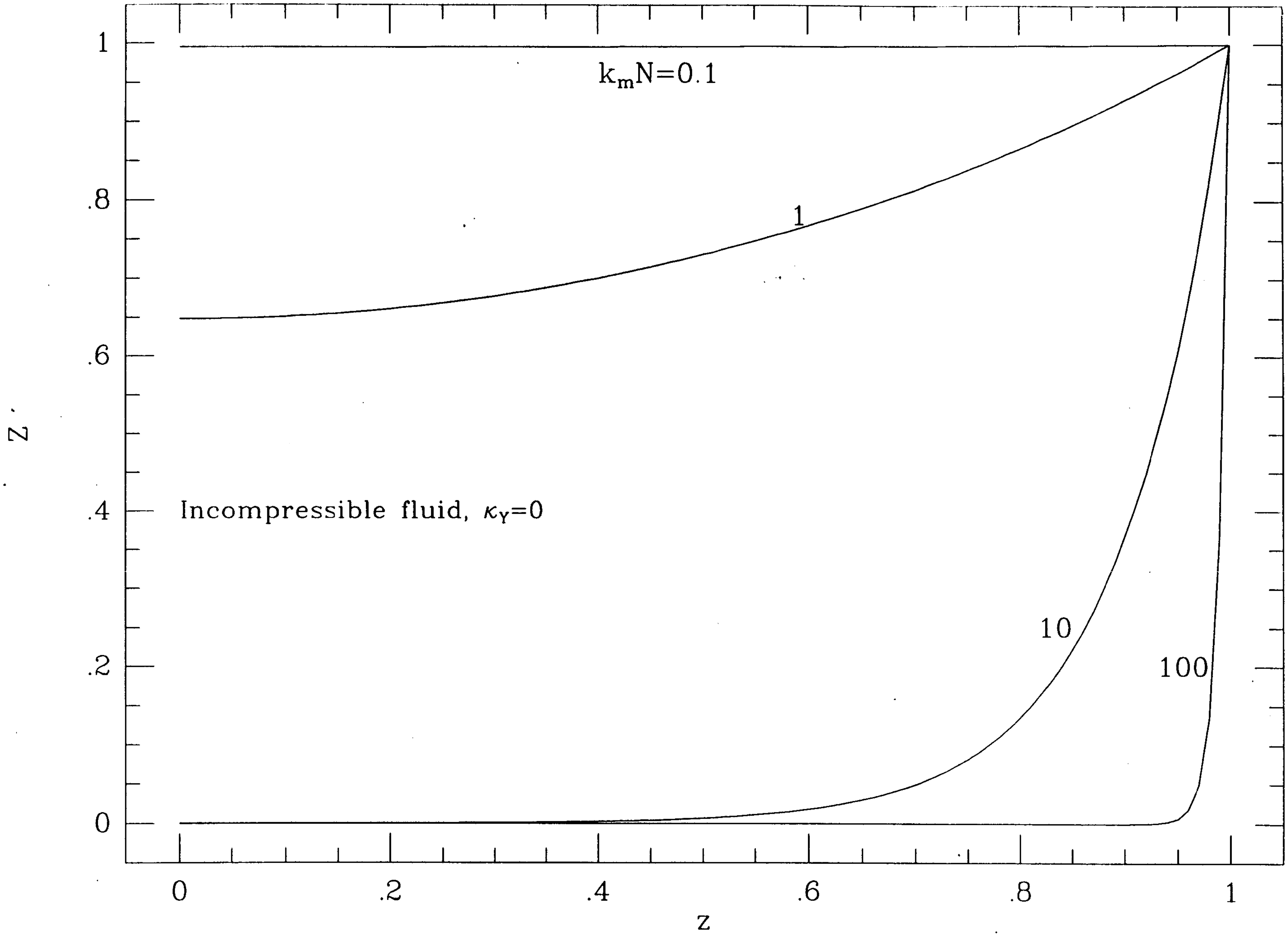
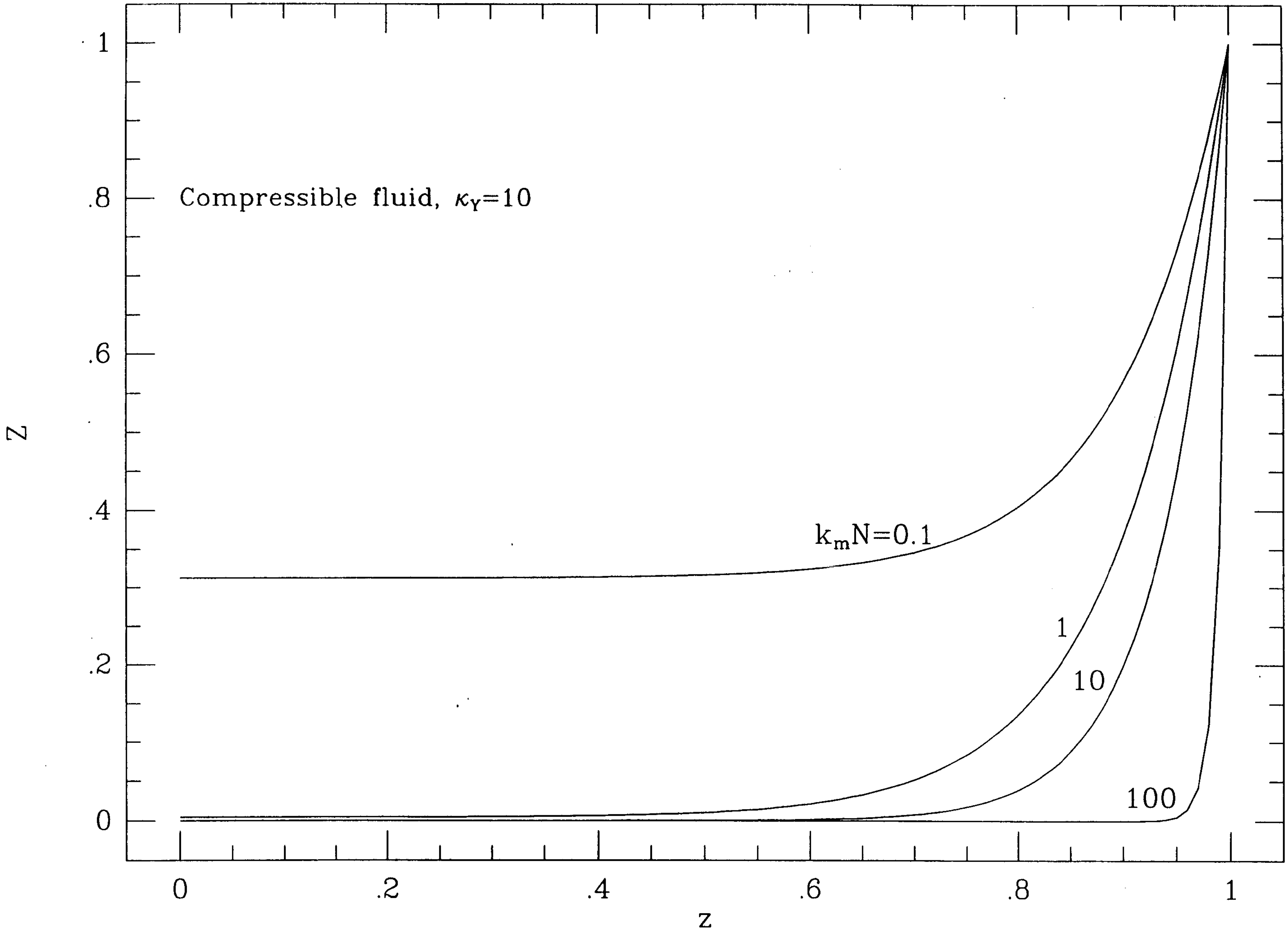


Figure 2



Compressible fluid, κ_Y=10
$k_m N$=0.1
1
10
100
Z
1
.8
.6
.4
.2
0
z
0
.2
.4
.6
.8
1

Figure 3

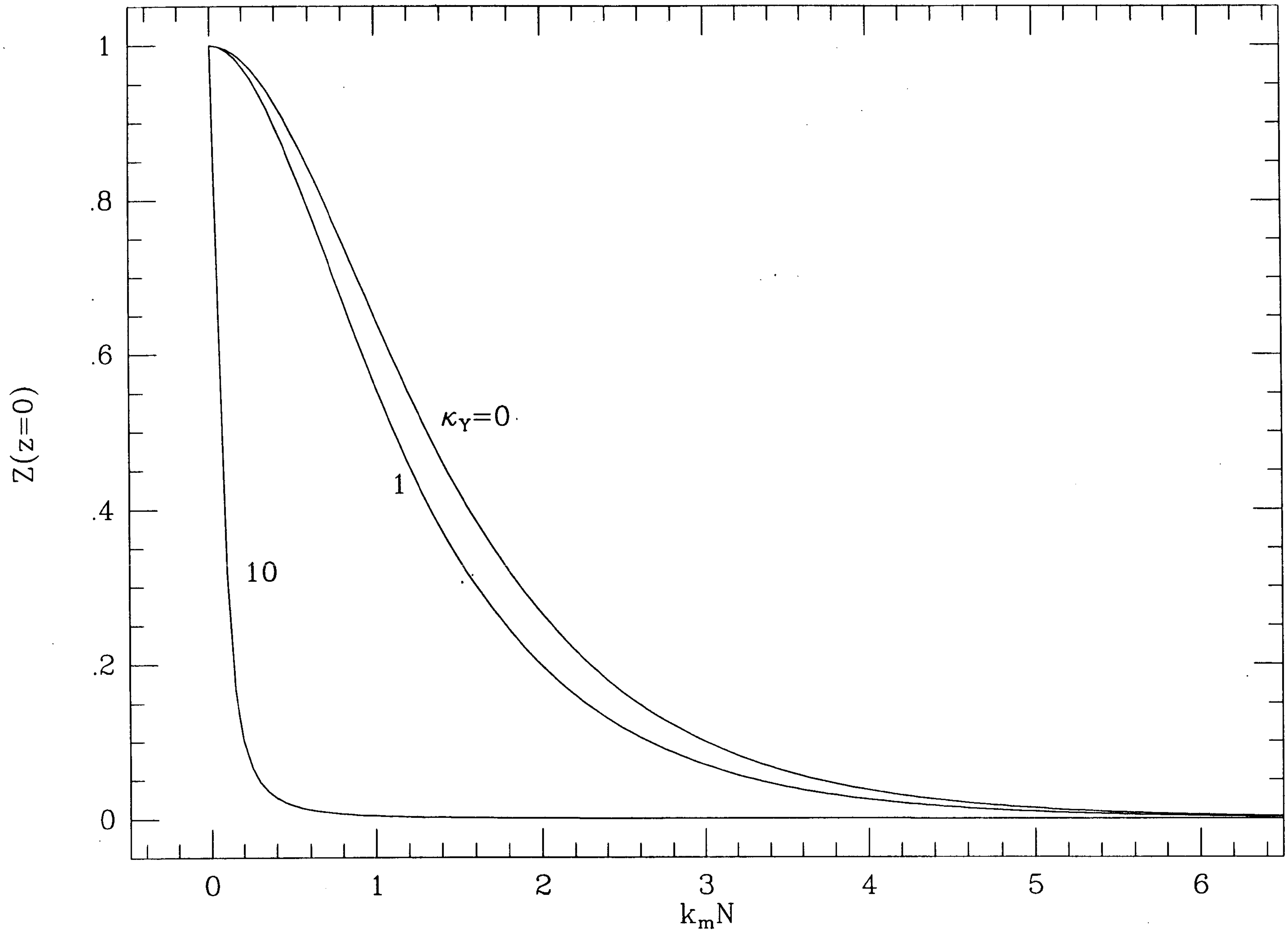


Figure 4

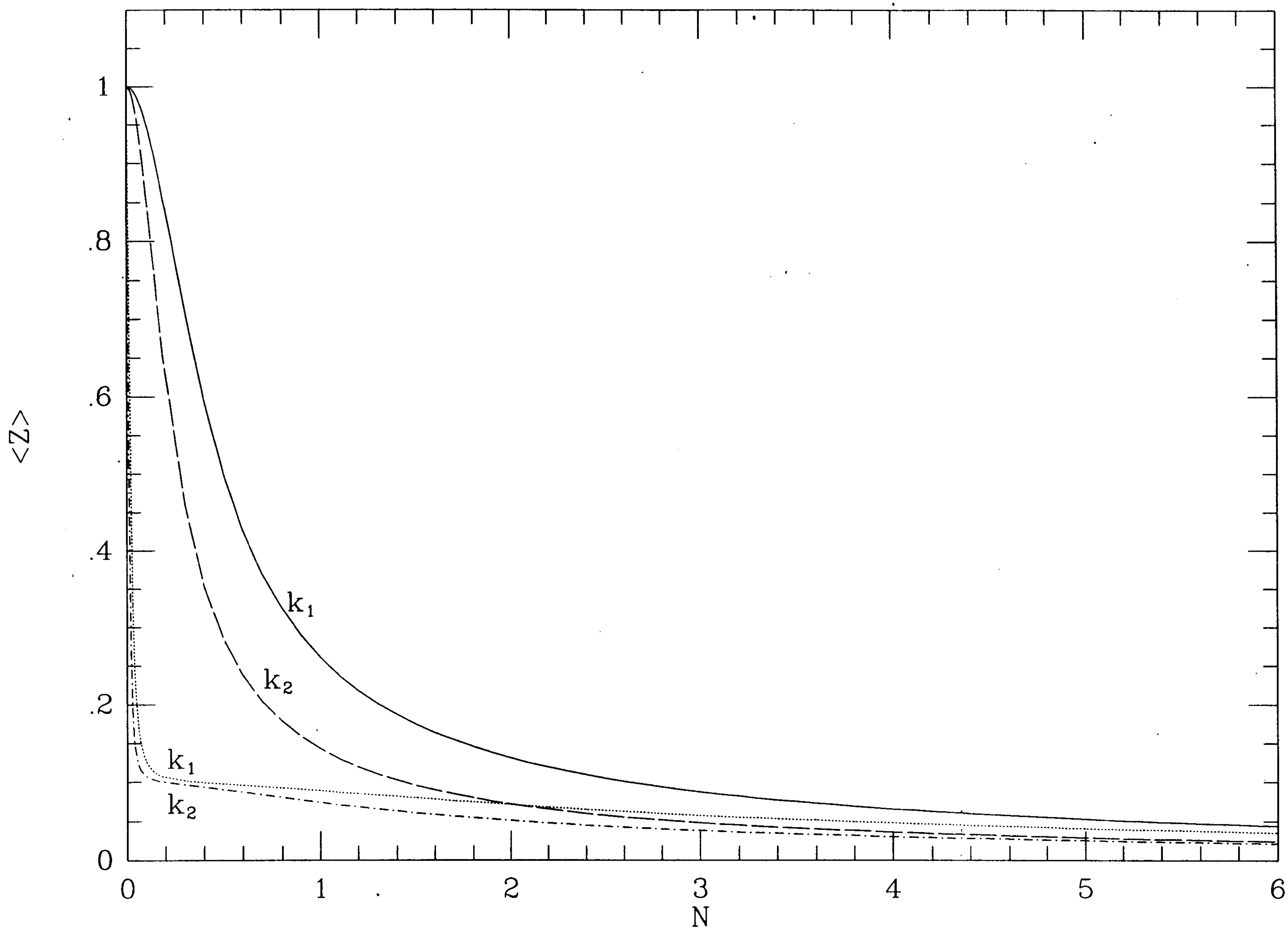


Figure 5