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Relating spectral indices to tensor and scalar amplitudes in inflation

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Within an expansion in slow-roll inflation parameters, we derive second-order expressions relating the ratio of tensor to scalar density perturbations and the spectral index of the scalar spectrum. We find that "corrections" to previously derived formulae can dominate if the tensor to scalar ratio is small. For instance, if $VV''/(V')^2 \neq 1$ or if $m_{Pl}^2/(4\pi)\ |V'''/V'| \gtrsim 1$, where $V(\phi)$ is the inflaton potential and m_{Pl} is the Plank mass, then the previously used simple relations between the indices and the tensor to scalar ratio fails. This failure occurs in particular for natural inflation, Coleman-Weinberg inflation, and "chaotic" inflation.

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I. INTRODUCTION

In slow-roll inflation the energy density of the Universe is dominated by the potential energy density of some scalar field ϕ, known as the *inflaton* field. During slow-roll inflation, scalar density perturbations and gravitational mode perturbations are produced as the inflaton field evolves. The amplitude of the scalar density perturbation as it crosses the Hubble radius after inflation is defined as

$$\left(\frac{\delta\rho}{\rho}\right)_\lambda^{\rm HOR} \equiv \frac{m}{\sqrt{2}} A_S(\lambda), \tag{1}$$

where the constant m equals 2/5 (or 4) if the perturbation re-enters during the matter (or radiation) dominated era. In addition to the scalar density perturbations, slow-roll inflation produces metric fluctuations, h, and the amplitude of the dimensionless strain on scale λ when it crosses the Hubble radius after inflation is defined by

$$\left| k^{3/2} h \right|_\lambda^{\rm HOR} \equiv A_G(\lambda). \tag{2}$$

Both the scalar density perturbations and the tensor modes contribute to temperature fluctuations in the cosmic background radiation (CBR). On large angular scales ($\theta \gg 1^\circ$, corresponding to the horizon at the last scattering surface) CBR fluctuations are proportional to the sum of the squares of the two modes:[1]

$$[\Delta T(\theta)/T]^2 \propto S(\theta) + T(\theta), \tag{3}$$

where with the normalization above,

$$S(\theta) = \frac{m^2}{2} A_S^2(\phi); \qquad T(\theta) = A_G^2(\phi). \tag{4}$$

Here we will be interested in scales that crossed the Hubble radius during the matter-dominated epoch when $m = 2/5$.

It is convenient to parameterize the scalar and tensor spectrum by their spectral indices:

$$\begin{aligned} 1 - n_S &\equiv d\ln\left[m^2 A_S^2(\lambda)/2\right]/d\ln\lambda = d\ln[S(\lambda)]/d\ln\lambda \\ n_T &\equiv d\ln[A_G^2(\lambda)]/d\ln\lambda = d\ln[T(\lambda)]/d\ln\lambda, \end{aligned} \tag{5}$$

[1]There are three interrelated scales used to characterize sizes: λ, θ, and ϕ. The length scale λ is related to the angular scale θ by $\lambda = \theta(34.4''\Omega_0 h)^{-1}$Mpc; i.e., θ is the angle subtended on the sky today by comoving scale λ at the surface of last scattering of the CBR. This comoving scale λ crossed the Hubble radius during inflation when the value of the scalar field was ϕ.

where λ is the physical wavelength that a given scale would have today if it evolved linearly, and is given by $\lambda = [a(t_0)/a(t)]\ H^{-1}(t)$, where t_0 is the time today, and $a(t)$ and $H(t) = \dot{a}(t)/a(t)$ are the scale factor and Hubble expansion rate, respectively, when the scale left the Hubble radius at time t.

With the prospect of measurements of anisotropy in the cosmic microwave background at different angular scales, it may soon be possible to determine the relative contributions of scalar and tensor components to CBR fluctuations [1], and thus provide information about the scalar potential driving inflation [2]. Several attempts have already been made to develop a method to isolate the scalar and tensor components of the signal [3]. This work assumed a relationship between the ratio of the tensor and scalar contribution to the temperature fluctuations and the spectral indices of the form

$$T/S \equiv R \simeq -7n_T \simeq 7(1 - n_S + \zeta) \tag{6}$$

$$\simeq 7(1 - n_S), \tag{7}$$

where $\zeta \equiv (2V'/(3H^2))'$, V is the inflaton potential, prime denotes $d/d\phi$, and H is the expansion rate at the time when the scale λ crossed out of the Hubble radius during inflation. It is claimed in Ref. [3] that ζ is small in generic models of inflation, and that confirmation of $n_T \approx 1 - n_S$ would be support for inflation and provide detailed information about the first instants of the Universe. In this paper, we discuss the relationship between R, n_T and $1 - n_S$ within an expansion of the scalar and tensor amplitudes in terms of "slow-roll" parameters.[2] We confirm Eq. (6) as the lowest-order result in this expansion if $|dR/d\ln\lambda| \gg |d^2R/d\ln\lambda^2|$, and we derive the next-order terms in the expansion as well. We also discuss when the approximation in Eq. (7) is accurate, i.e., what is the relative magnitude of the ζ contribution compared to $1 - n_S$.

We will now discuss the equations relating T and S to the spectral indices. In the Hamilton–Jacobi treatment of the field equations for inflation, the scalar field ϕ is used as a time variable, and the field equations are [4]

$$[H'(\phi)]^2 - \frac{3}{2}\kappa^2 H^2(\phi) = -\frac{1}{2}\kappa^4 V(\phi); \qquad \kappa^2\dot{\phi} = -2H'(\phi), \tag{8}$$

where dot denotes a time derivative, and $\kappa^2 = 8\pi/m_{Pl}^2$ with m_{Pl} the Planck mass. The scalar and tensor perturbations are calculated in an expansion in slow-roll parameters.

[2]We derive these relations for the density fluctuation amplitudes rather than the actual measured temperature fluctuations. This will account for the difference between 6.25 and 7 in the expressions for $1 - n_S$ and n_T.

These parameters involve various derivatives of $H(\phi)$. In de Sitter space $H(\phi)$ is a constant ($\dot{\phi} = 0$). However, in slow-roll inflation $H(\phi)$ varies with time, albeit slowly. The familiar first-order result for A_G and A_S is

$$A_S(\phi) = -\frac{\sqrt{2}\kappa^2}{8\pi^{3/2}} \frac{H^2(\phi)}{H'(\phi)}; \qquad A_G(\phi) = \frac{\kappa}{4\pi^{3/2}} H(\phi). \tag{9}$$

However there are corrections to this result that depend upon the slow-roll expansion parameters, ϵ, η, and ξ, which are defined as

$$\epsilon(\phi) = \frac{2}{\kappa^2}\left[\frac{H'(\phi)}{H(\phi)}\right]^2, \tag{10}$$

$$\eta(\phi) = \frac{2}{\kappa^2}\frac{H''(\phi)}{H(\phi)} = \epsilon + \frac{H(\phi)}{2H'(\phi)}\epsilon', \tag{11}$$

$$\xi(\phi) = \frac{2}{\kappa^2}\frac{H'''(\phi)}{H'(\phi)} = \eta + \frac{H(\phi)}{H'(\phi)}\eta'. \tag{12}$$

These three parameters depend upon $H'^2(\phi)$, $H''(\phi)$, or $H'''(\phi)/H'(\phi)$, second order in derivatives. In the slow-roll approximation ϵ and η are less than one. The parameter ξ can be much larger than one, however.[3] The expressions for A_G and A_S given previously in Eq. (9) are correct to first order in $\{\epsilon,\ \eta\}$, which we will refer to as the "first-order" results. To second order, Stewart and Lyth showed [5]:

$$A_S(\phi) = -\frac{\sqrt{2}\kappa^2}{8\pi^{3/2}} \frac{H^2(\phi)}{H'(\phi)} \left[1 - (2C+1)\epsilon + C\eta\right] \tag{13}$$

$$A_G(\phi) = \frac{\kappa}{4\pi^{3/2}} H(\phi) \left[1 - (C+1)\epsilon\right], \tag{14}$$

where $C \equiv -2 + \ln 2 + \gamma \simeq -0.73$ and $\gamma = 0.577$ is Euler's constant. It is important to realize that Eq. (13) is a double expansion in ϵ and η. In deriving this expression, terms of order ϵ^2, $\epsilon\eta$ and η^2, etc., have been neglected in the square bracket. This expression is therefore consistent to second-order as long as $|\eta| \sim \epsilon$. However, because η depends on the derivative of ϵ, this is usually not the case. It is possible that the second-order terms in one expansion variable might be larger than the first-order terms in the other variable. For example, if $\eta^2 \gg \epsilon$, then it is consistent to second-order to *neglect* the ϵ term in brackets in Eq. (13). We will see later that this occurs for many inflationary models of interest.

[3] Even if η is smaller than one, the derivative of η can be large, resulting in $|\xi| > 1$. This turns out to be the case for a wide range of parameters in Coleman-Weinberg inflation.

Let us now find the expressions for R, $1-n_S$, and n_T to second order in the slow-roll parameters. We can find R directly from Eqs. (13), and (14). Using the relation [2,6]

$$\frac{d\ln\lambda}{d\phi} = \frac{\kappa^2}{2}\frac{H}{H'}\left[1-\epsilon\right], \tag{15}$$

we can also find the spectral indices. The complete second-order expressions then are

$$R = \frac{25}{2}\frac{A_G^2}{A_S^2} \simeq \frac{25}{4}2\epsilon\left[1-2C(\eta-\epsilon)\right] \tag{16}$$

$$n_T = 2\epsilon\left[1+(2C+3)\epsilon-2(C+1)\eta\right] \tag{17}$$

$$1-n_S = 2\epsilon\left[2-\frac{\eta}{\epsilon}+4(C+1)\epsilon-(5C+3)\eta+C\xi(1+2(C+1)\epsilon-C\eta)\right]. \tag{18}$$

As mentioned above, ξ can be of order or greater than one, and is therefore not an expansion variable as are ϵ and η.[4] We have therefore included terms of order $\epsilon^2\xi$ and $\epsilon\eta\xi$, since they can contribute to $1-n_S$ to second-order (i.e. they can have magnitudes similar to those terms of order ϵ^2 or $\epsilon\eta$). This leads to the additional terms $4\epsilon^2\xi C(C+1)$ and $-2\epsilon\xi\eta C^2$ in the expression for $1-n_S$, which were not included in Ref. [5].

Now that we have the complete expression to second order, it is easy to isolate the first-order terms:[5]

$$\begin{aligned} R &= \frac{25}{4}2\epsilon \\ n_T &= 2\epsilon \\ 1-n_S &= 2\epsilon\left[1+\left(1-\frac{\eta}{\epsilon}+C\xi\right)\right]. \end{aligned} \tag{19}$$

The term corresponding to ζ in Eq. (19) is $2\epsilon(1-\eta/\epsilon)$. If $\eta=\epsilon$ and $|\xi|\ll 1$, then $1-n_S = 2\epsilon = R/6.25$, and Eq. (7) is correct to first order. It is straightforward to show that $\zeta \equiv (2V'/(3H^2))' = (6.25)^{-1}R(\eta/\epsilon-1)-2\epsilon\xi/3$. Clearly in terms of the slow-roll parameters, ζ can be of the same order as $1-n_S$, R, or n_T when $\eta\neq\epsilon$. Eq. (19) can now be rewritten as

$$R = \frac{6.25\,(1-n_S+\zeta)}{1+\xi(C-1/3)}. \tag{20}$$

We can now see that Eq. (6) holds only when $|\xi|\ll 1$.

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[4] Although $|\xi|$ can be much larger than one, it must be less than $1/\epsilon$: $|\xi|\ll 1/\epsilon$. This is because $|\eta\ddot{\phi}/(H\dot{\phi})| = |\xi\epsilon+\eta^2|\ll 1$ in order for Eq. (13) and Eq. (14) to be the correct second-order expressions. See Eq.(58) of Ref. [5].

[5] We include the η/ϵ and ξ terms in the first-order expression for $1-n_S$ because they *can* be greater than or of order one.

The ζ term will contribute to the value of $1 - n_S$ if $\eta \neq \epsilon$ or if $|\xi| \gtrsim |\eta/\epsilon| \gtrsim 1$. As an example, if $|\eta|/\epsilon \gg 1$ and $|\xi| \ll 1$ (as is the case for natural inflation), then $1 - n_S \simeq -2\eta \simeq -R^{-1}dR/d\ln\lambda$, as we will see in Section III. In this case, $1 - n_S$ depends on the *derivative* of R divided by R, rather than on R.

Finally, note that the term proportional to ξ in Eq. (19) comes from the derivative of a second-order term in the expansion for A_S. Thus, it cannot be derived from the first-order result. However, if A_S is expanded to higher orders, no higher-order derivative terms will appear. This is because there are no η', η'', etc., terms to any order in the exact expression for A_S [5].

II. EXAMPLES

Let us now illustrate this formalism by a couple of examples. Most of these examples have already been worked out by the authors given in the reference section. They are given here not only to illustrate the formalism developed in the last section, but also to make the case that Eq. (7) does not hold for a collection of popular inflation models.

Only in a few cases, such as power-law inflation, can one find an analytic solution for $H(\phi)$ and its derivatives. It is more useful to have an expression for the slow-roll parameters in terms of the inflaton potential and its derivatives. This is a difficult task however, since the slow-roll parameters cannot be unambiguously expressed in terms of the inflaton potential and its derivatives. This is because $V^{(5)}/V'''$ and higher-order derivatives contribute to the slow-roll parameter expressions, even though there is no restriction on their magnitudes. This point is elaborated in the Appendix.

Starting with the field equations [Eq. (8)], we can express the slow-roll parameters in a form involving $V(\phi)$ and its derivatives appearing in various combinations involving

$$\left(\frac{d^aV(\phi)}{d\phi^a}\right)^b \left(\frac{d^cV(\phi)}{d\phi^c}\right)^{-d}; \qquad ab - cd = 2. \tag{21}$$

To second order in $\{\alpha,\ \beta,\ \gamma,\ \delta\}$ defined as

$$\kappa^2\alpha \equiv (V'/V)^2; \quad \kappa^2\beta \equiv V''/V; \quad \kappa^2\gamma \equiv V'''/V'; \quad \kappa^2\delta \equiv V''''/V'', \tag{22}$$

the expressions for $\{\epsilon,\ \eta,\ \xi\}$ when $|\xi| \ll 1$ are[6]

$$\begin{aligned}
\epsilon &= \frac{1}{2}\alpha - \alpha\left[\frac{\alpha}{3} - \frac{\beta}{3}\right] \\
\eta &= -\frac{1}{2}\alpha + \beta + \alpha\left[\frac{2\alpha}{3} - \frac{4\beta}{3} + \frac{\gamma}{3}\right] + \beta\left[\frac{\beta}{3}\right] \\
\xi &= \frac{3}{2}\alpha - 3\beta + 2\gamma - \alpha\left[\frac{20\alpha}{3} - \frac{50\beta}{3} + \frac{13\gamma}{3}\right] - \beta\left[9\beta - \frac{8\gamma}{3} - \frac{2\delta}{3}\right].
\end{aligned} \tag{23}$$

These expressions are derived in the Appendix. We can see that $|\xi| \ll 1$ is satisfied when $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$. The more general solutions for ϵ, η and ξ when γ and δ are larger than or are of order one are given also in the Appendix.

Note that the first-order expressions for $\{\epsilon,\ \eta,\ \xi\}$ in the expansion in terms of $\{\alpha,\ \beta,\ \gamma\ \delta\}$ can be found by ignoring all the terms in square brackets.

We can then substitute the above expression into Eq. (16)-(18) to give R and the spectral indices directly in terms of $\{\alpha,\ \beta,\ \gamma,\ \delta\}$. To second order the expressions are

$$\begin{aligned}
R &= \frac{25}{4}\alpha + \frac{25}{4}\alpha\left[\frac{2}{3}(3C-1)(\alpha-\beta)\right] \\
n_T &= \alpha + \alpha\left[\frac{11}{6}\alpha - \frac{4}{3}\beta + 2C(\alpha-\beta)\right] \\
1-n_S &= 3\alpha - 2\beta + \alpha\left[\left(\frac{5}{6}+6C\right)\alpha + (1-8C)\beta + \left(2C-\frac{2}{3}\right)\gamma\right] - \frac{2}{3}\beta\left[\beta\right].
\end{aligned} \tag{24}$$

Again, the first order expressions can be obtained by setting the terms in the square brackets to zero.

It is shown in the Appendix that if $\gamma > 1$ and δ is arbitrarily large, to first order R and n_T are the same as those given in Eq. (24). The scalar spectral index, however, differs:

$$1-n_S = 3\alpha - 2\beta + \frac{2}{3}(3C-1)\alpha\gamma + \frac{2}{3}(C-1)\alpha\beta\delta. \tag{25}$$

Here derivatives of order $V^{(5)}/V'''$ and higher have been neglected. This new expression will be important for certain regimes in Coleman-Weinberg, scale-invariant, and other models of inflation for which $|\gamma| \gg 1$ or $|\beta\delta| \gg 1$.

The procedure we will follow is straightforward. For a given potential, we calculate $\{\alpha,\ \beta,\ \gamma,\ \delta\}$ using Eq. (22) and check to see if $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$. Then if we wish we can calculate the slow-roll parameters using Eq. (23), and then find R and the spectral

[6]The parameters α, β, γ and δ are not the slow-roll parameters. This is important because the slow-roll parameters ϵ, η and ξ cannot be expressed unambiguously in terms of them.

indices using Eq. (16)-(18), or we can substitute $\{\alpha,\ \beta,\ \gamma,\ \delta\}$ directly into Eq. (24). We now turn to the examples.

A. Power-law inflation

As a first example, we consider the second-order corrections for power-law inflation, with potential $V(\phi) = M^4 \exp(-2\phi/\phi_0)$. For power-law inflation, it is simple to show that $\alpha = \beta = \gamma = \delta = 4/(\kappa^2\phi_0^2)$. For power-law inflation the second-order terms in $\{\epsilon,\ \eta,\ \xi\}$ all vanish, and to second order $\epsilon = \eta = \xi = 2/(\kappa^2\phi_0^2)$. Therefore $|\xi| \ll 1$ and to second order

$$\begin{aligned} R &= \frac{25}{2}\epsilon = \frac{25}{2}\frac{2}{\kappa^2\phi_0^2} \\ n_T &= 2\epsilon[1+\epsilon] = 2\frac{2}{\kappa^2\phi_0^2}\left[1+\frac{2}{\kappa^2\phi_0^2}\right] \\ 1-n_S &= 2\epsilon[1+\epsilon] = 2\frac{2}{\kappa^2\phi_0^2}\left[1+\frac{2}{\kappa^2\phi_0^2}\right]. \end{aligned} \tag{26}$$

So the relationship between R, n_T, and $1 - n_S$ to second order is

$$6.25 n_T = 6.25(1-n_S) = R\left[1+\frac{2}{25}R\right]. \tag{27}$$

For the exponential potential the second-order corrections are of order $8R\%$ of the first-order term. The magnitude of the second-order corrections increase with R, and can become important. We can also express the second-order corrections in terms of the parameters of the potential by writing $2R/25 = 2/(\kappa^2\phi_0^2) = m_{Pl}{}^2/(4\pi\phi_0^2)$.

Only for the exponential potential of power-law inflation is $\epsilon = \eta$. This results in $\zeta \simeq 0$.

These results for power-law inflation are summarized in Table 1.

B. Chaotic and hybrid inflation

Now let's consider a potential commonly used in chaotic inflation [7]:

$$V = v\phi^p, \tag{28}$$

where v and p are constants. The mass dimension of v is $4 - p$, and p is an integer. We denote as ϕ_N the value of ϕ corresponding to the value of the scalar field when the length scale of interest crossed outside the Hubble radius during inflation. The expansion parameters $\{\alpha,\ \beta,\ \gamma,\ \delta\}$ are easily found, giving to second order

$$\begin{aligned}
\epsilon &= \frac{1}{2\kappa^2\phi_N^2}\left[p^2 - \frac{2p^3}{3\kappa^2\phi_N^2}\right] \\
\eta &= \frac{1}{2\kappa^2\phi_N^2}\left[p(p-2) - \frac{2p^2(p-3)}{3\kappa^2\phi_N^2}\right] \\
\xi &= \frac{1}{2\kappa^2\phi_N^2}\left[(p-2)(p-4) - \frac{2p(p^2-9p+28)}{3\kappa^2\phi_N^2}\right].
\end{aligned} \tag{29}$$

Since $\eta \simeq \xi$, the condition $|\xi| \ll 1$ is satisfied. It will be convenient to define the dimensionless ratio $r_N = 1/(2\kappa^2\phi_N^2) = {m_{Pl}}^2/(16\pi\phi_N^2)$. The ratio r_N can be related to the number of e-folds to the end of inflation. For $p = 4$, $r_N \simeq 1/(16N)$ and for $p = 2$, $r_N = 1/(8N)$. Since for the scales of interest to us $N \sim 50$, $r_N \sim 1.25 \times 10^{-3}$ for $p = 4$ and $r_N \sim 2.5 \times 10^{-3}$ for $p = 2$. The expressions to second order are

$$\begin{aligned}
R &= \frac{25}{4} 2r_N p^2 \left[1 - \frac{4}{3} r_N p(1-3C)\right] \\
n_T &= 2r_N p^2 \left[1 + r_N p^2 + \frac{4}{3} r_N p(2+3C)\right] \\
1 - n_S &= 2r_N p^2 \left[1 + \frac{2}{p} + \frac{1}{3} r_N (3p^2 + p(14+12C) - 12(1-2C))\right].
\end{aligned} \tag{30}$$

Note that our expansion is valid in this example, because $|\xi| \lesssim r_N \ll 1$ for $p \leq 4$. For $p = 2$, the terms in the square brackets for R, n_T and $1 - n_S$ are $[1 - 8.5r_N]$, $[1 + 3.5r_N]$, and $[2 - 2.3r_N]$ respectively, while for $p = 4$ the terms are $[1 - 17r_N]$, $[1 + 15r_N]$ and $[1.5 + 13r_N]$. Therefore the largest second-order corrections is $17r_N \sim 2.1\%$ for $p = 4$.

Now turn to the first-order results. To first order, the relationship between R, n_T, and $1 - n_S$ is

$$R = 6.25 n_T = 6.25(1 - n_S)/(1 + 2/p). \tag{31}$$

If we take $p = 2, 4$ and ∞, then $R \simeq 3.1(1-n_S)$, $4.2(1-n_S)$ and $6.25(1-n_S)$, respectively. Thus, the contribution of the ζ term depends upon p, and is negligible only for $p \gg 1$.

Note that the tensor to scalar ratio for this model need not be *very* small, since $R = (25/2)r_N p^2$, which for $p = 2$ and 4 is $R = 0.125$ and 0.25. In any case, in slightly more complicated inflation models it is possible to have inflation during an epoch when the potential is approximately power law, but to modify the relation between r_N and N.

As an example of such a modified model, we examine a hybrid inflationary model [8] of two scalar fields with potential

$$V(\sigma, \phi) = \frac{1}{4\lambda}\left(M^2 - \lambda\sigma^2\right)^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{2}\phi^2\sigma^2. \tag{32}$$

When $\phi > \phi_C \equiv M/g$, the global minimum is at $\sigma = 0$. Assuming that $\sigma \simeq 0$, the effective potential during inflation is

$$V(\phi) \simeq \frac{M^4}{4\lambda} + \frac{m^2}{2}\phi^2. \tag{33}$$

Except for the addition of a constant term, the potential looks like the $p = 2$ version of chaotic inflation.

We will assume that $M^2 > 2\lambda m^2/g^2$, which gives a parameter regime for which double inflation occurs. During the first epoch of inflation $V \simeq m^2\phi^2/2$, and during the second epoch $V \simeq M^4/(4\lambda)$. The value of the field at the transition between these two epochs is given by $\phi_T = M^2/(\sqrt{2\lambda}m)$. We will also assume that $M^3 \ll \sqrt{\lambda} gm\, m_{Pl}^2$ (or $2\phi_T^2\phi_C^2 \ll m_{Pl}^4$) so that inflation ends immediately after $\phi \simeq \phi_C$ [8].

In order to satisfy the slow-roll conditions during inflation, $\phi_T/m_{Pl} \gg 1/\sqrt{12\pi}$. Assuming that $\sigma \simeq 0$, the slow-roll parameters are

$$\begin{aligned}
\kappa^2\epsilon &= \frac{2\phi_N^2}{(\phi_N^2+\phi_T^2)^2}\left[1 - \frac{4}{3}\frac{\phi_N^2-\phi_T^2}{\kappa^2(\phi_N^2+\phi_T^2)^2}\right] \\
\kappa^2\eta &= \frac{2\phi_T^2}{(\phi_N^2+\phi_T^2)^2}\left[1 + \frac{2}{3}\frac{1}{\kappa^2\phi_T^2} - \frac{16}{3}\frac{\phi_N^2}{\kappa^2(\phi_N^2+\phi_T^2)^2}\right] \\
\kappa^2\xi &= -\frac{6\phi_T^2}{(\phi_N^2+\phi_T^2)^2}\left[1 + \frac{6}{\kappa^2\phi_T^2} - \frac{40}{9}\frac{\phi_N^2}{\phi_T^2}\frac{(\phi_N^2+5\phi_T^2)}{\kappa^2(\phi_N^2+\phi_T^2)^2}\right].
\end{aligned} \tag{34}$$

Note that for $\phi_N \gg \phi_T$ the results for $p = 2$ chaotic inflations obtains. We will be interested in the opposite limit, $\phi_N \ll \phi_T$. In this case

$$\begin{aligned}
\epsilon &= \frac{\phi_N^2}{\phi_T^2}\frac{2}{\kappa^2\phi_T^2}\left[1 + \frac{4}{3\kappa^2\phi_T^2}\right] \\
\eta &= \frac{2}{\kappa^2\phi_T^2}\left[1 + \frac{2}{3}\frac{1}{\kappa^2\phi_T^2}\right] \\
\xi &= -\frac{6}{\kappa^2\phi_T^2}\left[1 + \frac{6}{\kappa^2\phi_T^2}\right].
\end{aligned} \tag{35}$$

In this case $|\epsilon| \ll \{|\eta|,\ |\xi|\}$ but $|\xi| \ll 1$ since $\eta \simeq -\xi/3$. The expressions to second order are

$$\begin{aligned}
R &= \frac{25}{\kappa^2\phi_T^4}\frac{\phi_N^2}{}\left[1 + \frac{4(1-3C)}{3\kappa^2\phi_T^2}\right] \\
n_T &= \frac{4\,\phi_N^2}{\kappa^2\phi_T^4}\left[1 - \frac{16(2+3C)}{3\kappa^2\phi_T^2}\right] \\
1 - n_S &= -\frac{4}{\kappa^2\phi_T^4}\left[1 + \frac{2}{3\kappa^2\phi_T^2}\right].
\end{aligned} \tag{36}$$

To illustrate the point that it is possible to have $R \sim 0$ with $1 - n_S$ relatively large, we can work to first order in $\{\epsilon,\ \eta,\ \xi\}$ and first order in ϕ_N/ϕ_T. In these limits

$$R \sim n_T \sim 0; \qquad 1 - n_S \sim -2\eta \sim -8r_T, \tag{37}$$

where we have defined $r_T = 1/(2\kappa^2\phi_T^2)$ in the same manner as we have defined r_N, so that the slow-roll condition is satisfied for $r_T \ll 3/4$). Note that $n_S > 1$ in this case, because $|\eta/\epsilon| \gg 1$, and because the second derivative of the inflaton potential is positive: $V'' > 0$.

The number of e-folds from the end of inflation is

$$N = -\frac{8\pi}{m_{Pl}^2}\int\frac{Vd\phi}{V'} = \frac{1}{8r_N}\left[1 + \frac{r_N}{r_T}\ln(r_C/r_N) - \frac{r_N}{r_C}\right], \tag{38}$$

where r_C is defined to be $r_C \equiv 1/(2k^2\phi_C^2) > r_T$. Since we are interested in $N \sim 50$, then $r_C \gg r_N$.

We first choose an example whereby the ζ-term completely dominates, but for which R is very small. We take $M = 10^{14}$GeV, $m = 7 \times 10^8$GeV and $\lambda = g = 1$. For this model, the number of e-folds in the second inflationary epoch is 105. Therefore, the observable universe would have density perturbations only from the constant potential epoch for reasonable reheat temperatures. For this model, $r_T \simeq .029$. For the scale leaving 50 e-folds before the end of inflation, $R = 1.3 \times 10^{-5}$, $n_T = 2.0 \times 10^{-6}$ and $n_S = 1.2$ to first order. The value for R ignoring ζ is $6.25(1 - n_S) = -1.4$, which is not only negative but is 11×10^6% larger than the correct result. Clearly $R \neq 6.25(1 - n_S)$ because the ζ term overwhelmingly dominates: $\eta/\epsilon = r_N/r_T = 1.1 \times 10^5$. We can also find a value for m such that $R \sim 1$ but the ζ-term still contributes non-negligibly. We take $M = 10^{14}$GeV, $m = 9.8 \times 10^8$GeV and $\lambda = g = 1$. For this model, the number of e-folds in the second inflationary epoch is 52 and $r_T = .055$. For a scale leaving 50 e-folds before the end of inflation, $R = 1.6$, $n_T = 0.26$ and $n_S = 1.4$ to first order. Thus the value for R ignoring ζ is $6.25(1 - n_S) = 2.5$, which is 72% larger than the result. We see that even when R is near one, the correction terms can contribute nearly 100%.

C. Natural inflation

In natural inflation the potential is $V(\phi) = \Lambda^4\left[1 + \cos(\phi/f)\right]$, with $f \sim m_{Pl} \gg \Lambda$. Inflation occurs when $|\phi| \ll f$. The spectral indices and R are easily found to second order in $\{\alpha,\ \beta,\ \gamma\}$. Since the expressions are unwieldy, we expand the trigonometric functions in ϕ/f. To lowest order in ϕ/f but second order in $\{\alpha,\ \beta,\ \gamma\}$, the expressions for R and the spectral indices are

$$\begin{aligned} R &= \frac{25}{4}\left(1+\frac{3C-1}{3\kappa^2 f^2}\right)\frac{1}{4\kappa^2 f^2}\left(\frac{\phi_N^2}{f^2}\right) \\ n_T &= \left(1+\frac{3C+2}{3\kappa^2 f^2}\right)\frac{1}{4\kappa^2 f^2}\left(\frac{\phi_N^2}{f^2}\right) \\ 1-n_S &= \left(1-\frac{1}{6\kappa^2 f^2}\right)\frac{1}{\kappa^2 f^2} \end{aligned} \tag{39}$$

where again ϕ_N is the value of ϕ for the length scale of interest. Note that $2\beta \simeq \gamma \simeq \delta$, so that $|\xi| \ll 1$. It is clear that since $|\phi_N|/f \ll 1$, $R \sim n_T \ll 1$, but $1-n_S$ can be substantial.

Now let's consider the ζ contribution in the first-order result. To first order

$$\epsilon = \frac{1}{8\kappa^2 f^2}\left(\frac{\phi_N^2}{f^2}\right)\left(1-\frac{1}{3\kappa^2 f^2}\right); \qquad \eta = -\frac{1}{2\kappa^2 f^2}\left(1-\frac{1}{6\kappa^2 f^2}\right), \tag{40}$$

so $\eta|/\epsilon \gg 1$ is large, and in fact the ζ contribution, proportional to $1-\eta/\epsilon$, dominates. Thus, to first order, $1-n_S$ is *independent* of R, although $6.25 n_T \sim R$ is still valid. We remind the reader that for this model $R \ll 1$.

D. Scale-invariant inflation

We now consider the potential found in a scale-invariant theory [10]:

$$V(\phi) = \Lambda^4\left[1+\frac{\phi-\tilde{\phi}}{\tilde{\phi}}\exp(\phi/\tilde{\phi})\right], \tag{41}$$

where Λ and $\tilde{\phi} > 0$ are positive constants with mass dimension 4 and 1, respectively. This potential has a global minimum at $\phi = 0$, and slow-roll inflation occurs for $\phi = \phi_N$ when $|\phi_N/\tilde{\phi}| \gg 1$ in the regions $\phi_N > 0$ or $\phi_N < 0$. In the region of positive ϕ, the results resemble power-law inflation:

$$\frac{4}{25}R = n_T = 1-n_S = \frac{1}{\kappa^2\tilde{\phi}^2}; \qquad \phi/\tilde{\phi} \gg 1 \tag{42}$$

Note that the results do not depend upon ϕ_N (to leading order). Thus R, n_T, and n_S are truly constant.

The results in the other region of inflation ($\phi/\tilde{\phi} \ll -1$) are more interesting:

$$R = \frac{25}{4}\frac{1}{\kappa^2\tilde{\phi}^2}\left(\frac{\phi_N}{\tilde{\phi}}\right)^2 \exp(-2|\phi|/\tilde{\phi})$$

$$
\begin{aligned}
n_T &= \frac{1}{\kappa^2\tilde{\phi}^2}\left(\frac{\phi_N}{\tilde{\phi}}\right)^2 \exp(-2|\phi|/\tilde{\phi}) \\
1-n_S &= \frac{2}{\kappa^2\tilde{\phi}^2}\left(\frac{|\phi_N|}{\tilde{\phi}}\right) \exp(-|\phi|/\tilde{\phi}).
\end{aligned}
\tag{43}
$$

Because $\gamma \simeq \delta = 1/(\kappa^2\tilde{\phi}^2)$, then $\kappa^2\tilde{\phi}^2 \gg 1$ in order that Eq. (43) is valid. The second-order corrections are $1/(\kappa^2\tilde{\phi}^2)$ for $1-n_S$ and $-1/(\kappa^2\tilde{\phi}^2)|\phi_N|/\tilde{\phi}\exp(-|\phi_N|/\tilde{\phi})$ for R and n_T. In this case $R \sim 6.25 n_T$, but $1-n_S \propto \sqrt{R}$. In addition, note that R is small. One can imagine $1-n_S$ large enough to be detectable, but $R \sim 0$.

Because $|\alpha\gamma| \ll \beta$, when $\gamma > 1$ the scalar index is $1-n_S \simeq -2\beta$ to first order, as given above in Eq. (43).

E. Coleman-Weinberg inflation

Finally, we examine the Coleman-Weinberg potential in the context of new inflation. The potential is

$$
V(\phi) = B\sigma^4/2 + B\phi^4[\ln(\phi^2/\sigma^2) - 1/2]. \tag{44}
$$

The global minimum of this potential is at $\phi = \sigma$. Inflation occurs for $0 < \phi/\sigma \ll 1$ when the potential is nearly flat: $V \simeq B\sigma^4/2$. We can calculate the slow-roll parameters in this model with the assumption that $V \sim B\sigma^4/2$ in the denominator of α and β. The spectral indices and R are to lowest order in ϕ_N/σ

$$
R = \frac{25}{4} n_T = \frac{25}{4}\frac{64}{\kappa^2\sigma^2}\left(\frac{\phi_N}{\sigma}\right)^6 \left[\ln\left(\phi_N^2/\sigma^2\right)\right]^2 \tag{45}
$$

$$
1-n_S = \frac{48}{\kappa^2\sigma^2}\left(\frac{\phi_N}{\sigma}\right)^2 \left|\ln\left(\phi_N^2/\sigma^2\right)\right|. \tag{46}
$$

Note that $\gamma = 3\delta = 6/(\kappa^2\phi_N^2)$ in this limit, so that $\kappa^2\phi_N^2 \gg 6$ in order for these expressions to be valid. To first order then, $1-n_S \propto R^{1/3}$ in this model.

Again, the fact that $n_T \neq 1-n_S$ can be traced to a large value of ζ, i.e., $|\eta/\epsilon| \neq 1$. Let's look at the slow-roll parameters with the assumption that $\phi_N/\sigma \ll 1$:

$$
\begin{aligned}
\kappa^2\epsilon &= \frac{32}{\sigma^2}\left(\frac{\phi_N}{\sigma}\right)^6 \left[\ln\left(\phi_N^2/\sigma^2\right)\right]^2; \quad \kappa^2\eta = \frac{24}{\sigma^2}\left(\frac{\phi_N}{\sigma}\right)^2 \ln\left(\phi_N^2/\sigma^2\right) \\
\kappa^2\xi &= \frac{12}{\sigma^2}\left(\frac{\sigma}{\phi_N}\right)^2.
\end{aligned}
\tag{47}
$$

Table 1: Magnitude of corrections to spectral relations when $|\xi| \ll 1$ ($x \equiv \phi_N/\tilde{\phi}$).

Inflation Model	Second-order correction	Relative contribution of ζ term: $\|1-\eta/\epsilon\|$
Power-law	$8R\%$	0
Chaotic ($p \leq 4$)	$\leq 2.1\%$	$2/p$: 100% ($p=2$); 50% ($p=4$)
Hybrid ($r_N \ll r_T$)	$\leq 2.1\%$	100%
Hybrid ($r_N \gg r_T$)	r_T	$(\phi_T/\phi_N)^2 \gg 1$
Natural	${m_{Pl}}^2/(16\pi f^2)$	$4f^2/\phi_N^2 \gg 1$
Scale-Invariant ($x \gg 1$)	$1/(\kappa^2\tilde{\phi}^2)$	0
Scale-Invariant ($x \ll -1$)	$\lesssim 1/(\kappa^2\tilde{\phi}^2)$	$2\|x\|^{-1}e^{\|x\|} \gg 1$
Coleman-Weinberg	$\lesssim (3/(2\pi))(m_{Pl}/\phi_N)^2$	$(3/4)(\sigma/\phi_N)^4/\|\ln(\phi_N^2/\sigma^2)\| \gg 1$

Note that $|\xi| \gg |\eta| \gg |\epsilon|$, so that the ζ contribution is dominant: $\eta/\epsilon \simeq (3/4)(\sigma/\phi_N)^4/\ln(\phi_N/\sigma)^2 \ll -1$. Therefore the second-order contribution will be of order $3/(2\pi)\,(m_{Pl}/\phi_N)^2$ for $1-n_S$, and $(3/\pi)\,(m_{Pl}^2\phi_N^2/\sigma^4)\ln(\phi_N^2/\sigma^2)$ for R and n_T.

Because $|\alpha\gamma| \ll \beta$, when $\gamma > 1$ the scalar index is $1-n_S \simeq -2\beta$ to first order, as given above in Eq. (46).

A summary of the results of this section is given in Table 1.

III. First-Order Expression relating n_S, R and derivatives of R

Using Eqs. (10)-(12) and Eq. (15), we can express η and ξ in terms of ϵ only.

$$\eta = \epsilon + \frac{1-\epsilon}{2\epsilon}\frac{d\epsilon}{d\ln\lambda} \tag{48}$$

$$\xi = \eta + \frac{1-\epsilon}{\epsilon}\frac{d\eta}{d\ln\lambda} \tag{49}$$

$$= \eta + \frac{1-\epsilon}{\epsilon}\left(\frac{d\epsilon}{d\ln\lambda} - \frac{1}{2\epsilon^2}\left(\frac{d\epsilon}{d\ln\lambda}\right)^2 + \frac{1-\epsilon}{2\epsilon}\frac{d^2\epsilon}{d\ln\lambda^2}\right). \quad (50)$$

Using Eq. (19), we then find that the most general first-order expression relating n_S, R and derivatives of R with respect to $\ln\lambda$ is

$$6.25(1-n_S) \simeq R\left[1 - \frac{6.25}{R^2}\left(\frac{dR}{d\ln\lambda} - C\frac{d^2R}{d\ln\lambda^2}\right)\right]. \quad (51)$$

This shows explicitly in terms of measurable quantities when the old formula fails. If $R^{-2}|dR/d\ln\lambda - Cd^2R/d\ln\lambda^2| \gg 1$, then the "correction term" dominates and $6.25(1 - n_S) \neq R$. Thus even if $R(\lambda)$ changes slowly changes with scale, if $R \ll 1$, "corrections" to the previous formula can dominate. Note that the $\eta/\epsilon - 1$ and ξ terms are the $dR/d\ln\lambda$ and $d^2R/d\ln\lambda^2$ terms, respectively.

IV. CONCLUSIONS

In this paper, we derive the contribution of scalar and tensor perturbations from inflation to second order in slow-roll parameters. We find that the previously derived formula fails when $\eta \neq \epsilon$ or $|\xi| \gtrsim 1$. In particular, it fails for natural inflation and Coleman-Weinberg inflation, where $|\eta| \gg \epsilon$, and for "chaotic" ϕ^2 inflation, where $|\eta| \ll \epsilon$. For natural inflation, a type of scale-invariant inflation, and Coleman-Weinberg inflation, to first order $1 - n_S \simeq const$, $1 - n_S \propto \sqrt{R}$ and $1 - n_S \propto R^{1/3}$, respectively. Thus the relationship between R and $1 - n_S$ is in general not linear. We have shown that this occurs when $V|V''|/(V')^2 \neq 1$ or $m_{Pl}^2/(4\pi)\ |V'''/V'| \gtrsim 1$.

While completing this paper, we received a paper by Liddle and Turner [11] on the same subject.

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Appendix

In this appendix, we derive the first-order results for ϵ, η and ξ when $|\xi| \gtrsim 1$, and the second-order results when $|\xi| \ll 1$. The latter example is used in Section II of this paper.

We define the function $f \equiv H'/H$. Then using Eq. (8), which can be rewritten as

$$H^2 = \frac{\kappa^2 V}{3}\left(1 + \frac{\epsilon}{3}\right), \tag{A.1}$$

and $\epsilon' = 2f(\eta - \epsilon)$, "$f$" becomes

$$f = \frac{1}{2}\frac{V'}{V}\left[1 + \frac{\eta - \epsilon}{3}\right]. \tag{A.2}$$

The slow-roll parameters can then be determined in terms of f and its derivatives: $\epsilon = 2f^2/\kappa^2$, $\eta = 2(f^2 + f')/\kappa^2$ and $\xi = 2(f^2 + 3f' + f''/f)/\kappa^2$. In addition, the derivatives of η and σ are $\eta' = f(\xi - \eta)$ and $\xi' = f(\eta/\epsilon)\ (\sigma - \xi)$, where

$$\sigma \equiv \frac{2}{\kappa^2}\frac{H''''}{H''}. \tag{A.3}$$

The mixed second-order expressions for ϵ, η and ξ as a function of σ and $\{\alpha,\ \beta,\ \gamma\}$ (as defined in Eq. (22)) are

$$\epsilon = \frac{1}{2}\alpha\left(1 + \frac{2}{3}\{\eta - \epsilon\}\right) \tag{A.4}$$

$$\eta = \beta\left(1 + \left\{\frac{\eta - \epsilon}{3}\right\}\right) - \frac{1}{2}\alpha\left[1 - \frac{1}{3}\xi + \left\{\eta - \frac{2}{3}\epsilon + \frac{\xi}{9}(\epsilon - \eta)\right\}\right] \tag{A.5}$$

$$\begin{aligned}\xi = {} & 2\gamma - 3\beta\left[1 - \frac{\xi}{3} + \left\{\frac{2\eta - \epsilon}{3}\right\}\right] + \frac{3\alpha}{2}\left[1 + \xi\left(-\frac{2}{3} - \frac{1}{9}\frac{\eta}{\epsilon} + \frac{\xi}{27}\right) + \frac{1}{9}\sigma\frac{\eta}{\epsilon}\right. \\ & \left. + \left\{\frac{2}{3}(2\eta - \epsilon) + \frac{\xi}{27}\left(4\epsilon - 5\eta - \eta\frac{\eta}{\epsilon}\right) + \frac{\sigma\eta}{27}\left(-1 + \frac{\eta}{\epsilon}\right)\right\}\right].\end{aligned} \tag{A.6}$$

In the above equations, the first-order expressions can be obtained by setting the terms in curly brackets $\{\}$ equal to zero. Note that we cannot solve for ξ (and therefore η and ϵ) until σ is determined.

We calculate σ from f and its derivatives:

$$\frac{\kappa^2\sigma}{2} = f^2 + 3f' + \frac{f''}{f} + \frac{f}{f^2 + f'}\left[2ff' + 3f'' + \frac{f'''}{f} + \frac{ff''}{f^2}\right]. \tag{A.7}$$

Because the expression for σ to second-order is too long and lends no new insight, we calculate σ to first-order only. We note that the second-order corrections are of order ϵ, β and $\alpha\xi$. As is similar to the mixed expressions for ϵ, η and ξ in Eq. (A.4)-(A.6), several terms on the right-hand side of Eq. (A.7) will contain the factors σ and σ'. We substitute in the result $\sigma' = f\ (\xi/\eta)(\tau - \sigma)$, where $\tau = (2/\kappa^2)H^{(5)}/H'''$. Then, combining all of the σ terms and keeping only the largest ones, we find that

$$\begin{aligned} \sigma &= \frac{2}{\eta}\left(1+\frac{\alpha^2\xi}{36\eta}\right)^{-1}\left[\alpha\gamma\left[-4+\xi\right]+2\beta\gamma+\beta\delta+\frac{\alpha\xi\ \tau}{12}+\right. \\ & \alpha^2\left[-\frac{47}{8}+\frac{\xi}{216}\left(873+126\frac{\eta}{\epsilon}+27\left(\frac{\eta}{\epsilon}\right)^2-9\frac{\xi}{\epsilon}-24\ \xi\frac{\eta}{\epsilon}+3\xi^2\right)\right]. \\ & \left.+\alpha\beta\left[\frac{29}{2}+\frac{\xi}{216}\left(-1404-144\frac{\eta}{\epsilon}+84\xi\right)\right]+\beta^2\left[-\frac{15}{2}+\frac{3}{2}\xi\right]\right], \end{aligned} \qquad (A.8)$$

The quantity τ is the higher order derivative term mentioned at the beginning of Section II, and is of order $\kappa^{-2}V^{(5)}/V'''$ as long as higher order derivatives are unimportant.

As an example, if $|\xi| \ll 1$,

$$\sigma = \frac{2}{\eta}\left[-4\alpha\gamma+2\beta\gamma+\beta\delta-\frac{47}{8}\alpha^2+\frac{29}{2}\alpha\beta-\frac{15}{2}\beta^2\right]. \qquad (A.9)$$

Note that $\eta\sigma$ is second order in small quantities, so that the "$\alpha\eta\sigma/\epsilon$" term in Eq. (A.6) is second order. We now calculate ϵ, η and ξ in this limit from Eq. (A.4) - (A.6). For the second-order terms, we substitute in Eq. (A.9) and the first-order expressions for ϵ, η and ξ, which are $\epsilon = \alpha/2$, $\eta = \beta - \alpha/2$ and $\xi = 2\gamma - 3\beta + 3\alpha/2$. The final results (when $|\xi| \ll 1$) are given in Eq. (23).

To determine the slow-roll parameter ξ when $1 \lesssim |\xi| \ll 1/\epsilon$, we substitute Eq. (A.8) into Eq. (A.6) along with the first-order expressions

$$\begin{aligned} \epsilon &= \frac{\alpha}{2} \\ \eta &= \beta - \frac{\alpha}{2}\left(1-\frac{\xi}{3}\right). \end{aligned} \qquad (A.10)$$

Eq. (A.6) then becomes

$$\begin{aligned} \xi &= \frac{3}{2}\alpha - 3\beta + 2\gamma - \frac{5}{6}\alpha\,\xi + \frac{2}{3}\beta\,\xi \\ &+ \frac{2\,(-3\,\alpha+6\,\beta+\alpha\,\xi)}{-9\,\alpha+18\,\beta+3\,\alpha\,\xi+\alpha^2\,\xi/2}\left(\beta\,\delta-4\,\alpha\,\gamma+2\,\beta\,\gamma+\alpha\,\gamma\,\xi-\frac{\alpha\,\xi^2}{12}+\frac{\alpha\xi\ \tau}{12}\right. \\ &+ \alpha^2\left[-\frac{47}{8}+\frac{43\,\xi}{12}+\frac{2\xi^2}{9}-\frac{\xi^3}{108}\right]+\alpha\,\beta\left[\frac{29}{2}-\frac{31\,\xi}{6}+\frac{\xi^2}{9}\right] \\ &+ \left.\beta^2\left[-\frac{15}{2}+\frac{2\,\xi}{3}\right]\right). \end{aligned} \qquad (A.11)$$

We now rearrange Eq. (A.11) to get an algebraic equation for ξ in terms of α, β, γ and δ. In doing so, we keep only the largest terms, keeping in mind that $\alpha \ll 1$, $|\beta| \ll 1$ and $|\alpha\,\xi| \ll 1$ in order that the original slow-roll expansion is valid. We do not assume anything about γ or δ, however. Therefore, we keep the largest of all terms that include γ, δ and τ. (For example, we keep terms of order β, γ and ξ and neglect terms of order $\alpha\beta$, $\alpha\gamma$, $(\alpha\xi)\gamma$, $(\alpha\xi)\xi$ and $(\alpha\xi)^2\xi$). The solution to Eq. (A.11) then is

$$\xi = \frac{1}{1-\alpha\tau/18}\left(\frac{3}{2} - 3\beta + 2\gamma + \frac{2}{3}\beta\delta\right), \tag{A.12}$$

where we have implicitly assumed that τ does not depend on ξ. If we assume that $|\alpha\tau|/18 \ll 1$, $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$, then to first order, $\xi \simeq 3\,\alpha/2 - 3\beta + 2\,\gamma$, as found previously. Note that if $|\gamma| \gtrsim 1$, $|\beta\delta| \ll |\gamma|$ and $|\alpha\tau|/18 \ll 1$, then $\xi = 2\gamma$ to first order. In this case, $|\gamma| \ll 1/(2\alpha)$ in order that the original expansion be valid.

Because we have solved for ξ to first order only in general, we can only calculate R and the spectral indices to first order. Using Eq. (A.10), the slow-roll parameters are $\epsilon = \alpha/2$ and $\eta = \beta - \alpha/2 + \alpha(\gamma + \beta\delta/3)/3$ to first order. Using Eq. (19), the expressions for R and the spectral indices are

$$\begin{aligned} R &= 6.25\alpha = 6.25 n_T \\ 1 - n_S &= 3\alpha - 2\beta + \frac{2}{3}(3C-1)\alpha\gamma + \frac{2}{3}(C-1)\alpha\beta\delta \end{aligned} \tag{A.13}$$

to first order. This is similar to the first order result obtained when $|\xi| \ll 1$, and differs only in the appearance of the γ and δ terms. Note that when $|\beta|/\alpha \gg |\gamma| > 1$ and $|\beta|/\alpha \gg |\beta\delta|$, the first-order results are the same as when $|\beta|/\alpha \gg 1$, $|\gamma| \ll 1$ and $|\beta\delta| \ll 1$: $1 - n_S = -2\beta$.