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Multigluon Helicity Amplitudes Involving a Quark Loop

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Abstract

We apply the solution to the recursion relation for the double-off-shell quark current to the problem of computing one loop amplitudes with an arbitrary number of gluons. We are able to compute amplitudes for photon-gluon scattering, electron-positron annihilation to gluons, and gluon-gluon scattering via a quark loop in the case of like-helicity gluons. In addition, we present the result for the one-loop gluon-gluon scattering amplitude when one of the gluons has opposite helicity from the others.

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I. INTRODUCTION

In this paper we will extend the discussion of "simple" one-loop amplitudes involving an arbitrary number of gauge bosons begun in Ref. [1] to the case of QCD. In particular, we will evaluate the one-loop corrections to the following processes:

$$\gamma g \to gg\dots g \tag{1}$$

$$e^+e^- \to gg\dots g \tag{2}$$

and

$$gg \to gg\dots g, \tag{3}$$

for the case of like-helicity gluons. In addition, we will consider (3) in the case where one of the gluons has opposite helicity from the rest. In this paper we will consider only those contributions to (3) that arise from diagrams containing a closed quark loop.

In the indicated helicity configurations, all of the above processes vanish at tree level. Therefore, their one-loop corrections should be especially simple. In particular, these corrections must be ultraviolet and infrared finite: there is no counterterm to absorb any ultraviolet divergence, and no tree graphs with an additional soft gluons to handle an infrared divergence. Furthermore, since the amplitude for $q\bar{q}$ annihilation to n like-helicity gluons vanishes at tree level [2], the Cutkosky rules my be used to demonstrate that these particular one-loop diagrams do not contain any cuts in the complex plane.

There are three major ideas which make this calculation feasible. First, we treat the gauge bosons in the theory on an equal footing with the fermions by utilizing the multispinor representation for the gauge field [3–8]. With a proper choice for the spinor basis, and a clever selection of the gauge boson polarization spinors, great simplifications are obtained in the expressions involved in the computation of Feynman diagrams. We utilize Weyl-van der Waerden spinors for this purpose. A summary of our conventions and notations may be found in Ref. [9]. Second, the many terms in a calculation are conveniently organized into gauge-invariant sub-groups by the color factorization property of QCD amplitudes [10]. Finally, we will use recursive methods [9,11,12] to obtain expressions for the sums of large groups of Feynman diagrams. In particular, we define currents which are the sum of all tree graphs containing exactly n gauge bosons in some given helicity configuration. Explicit closed-form solutions for these currents exist for certain special helicity configurations. Currents with two off-shell particles will play an important role in our discussion of loop amplitudes.

This paper is organized as follows. In Sec. II, we will discuss the double-off-shell quark current. This current consists of a single quark line with both ends off shell plus n on-shell gluons. All of the amplitudes discussed in this paper are derived from this current by joining the two ends of the quark line and performing the appropriate integration. Since we have obtained explicit solutions for this current in the case of n like-helicity gluons, we are able to compute the corresponding loop amplitudes in a straightforward manner. We present our computation for some of these amplitudes in Sec. III. The favorable form of the solution for the double-off-shell quark current allows us to evaluate the integrals exactly for arbitrary n. We obtain compact expressions for the amplitudes for processes (1)–(3) in the case of like-helicity gluons, and a somewhat more complicated expression for the case (3) containing a single opposite-helicity gluon. In the cases where these results overlap those previously obtained by Bern, Kosower, and Dixon [13,14] using string-based methods, we find agreement. In particular, our computation proves the conjecture made in Ref. [14], as well as producing additional new amplitudes. We conclude with a few closing remarks in section IV.

II. THE DOUBLE-OFF-SHELL QUARK CURRENTS

In this section we will present recursion relations for currents consisting of a quark line plus n gluons attached all possible ways. In contrast to the earlier quark currents presented by Berends and Giele [11], both ends of the quark line will be off shell. The generalizations from the Berends and Giele relations are straightforward. We are able to solve these recursion relations for the case of an arbitrary number of like-helicity gluons.

We adopt the convention that, in any given diagram, all of the momenta are directed inward. The gluons will carry momenta $k_1, k_2, \ldots, k_n$ and color indices $a_1, a_2, \ldots, a_n$. We will denote the quark momentum by $\mathcal{P}$ and its color by i. The antiquark will have momentum $\mathcal{Q}$ and color j. Let us denote the complete double-off-shell quark current by $\widehat{\Psi}_{ji}(\mathcal{Q}; 1, \ldots, n; \mathcal{P})$. Momentum conservation implies that not all of the momenta independent; indeed, we have

$$\begin{aligned}\mathcal{P} &= -[\mathcal{Q} + k_1 + \cdots + k_n] \\ &\equiv -[\mathcal{Q} + \kappa(1,n)].\end{aligned} \tag{4}$$

We will often suppress "$\mathcal{P}$" or "$\mathcal{Q}$" from the argument list of $\widehat{\Psi}$ when convenient.

The factorization of the current into color and kinematical pieces found by Berends and Giele [11] for the quark current with only one off-shell particle still holds:

$$\widehat{\Psi}_{ji}(\mathcal{Q}; 1, \ldots, n; \mathcal{P}) = g^n \sum_{\mathcal{P}(1\ldots n)} (\Omega[1,n])_{ji} \Psi(\mathcal{Q}; 1, \ldots, n; \mathcal{P}). \tag{5}$$

In Eq. (5) we have used the notation

$$\Omega[1,n] \equiv T^{a_1} T^{a_2} \cdots T^{a_n}, \tag{6}$$

where the T^a's are color matrices in the fundamental representation of the gauge group. Because of our definitions, the complete current $\widehat{\Psi}$ is a symmetric function of its gluon arguments, whereas the order of the arguments appearing in Ψ is important. Hence, the current Ψ is often referred to as the color-ordered quark current.

The recursion relation satisfied by the color-ordered double-off-shell quark current may be obtained in two different ways for each possible chirality of the quark line. The first way is to begin with a current containing an off-shell antiquark and on-shell quark. The quark may then be taken off mass shell to produce the desired double-off-shell current. Derived in this manner, we obtain the following recursion relation for the left-handed quark current:

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{Q}; 1, \ldots, n; \mathcal{P}) = -\sqrt{2} \sum_{j=0}^{n-1} \Psi_{\alpha\beta}(\mathcal{Q}; 1, \ldots, j) \bar{\mathcal{J}}^{\dot{\beta}\beta}(j+1, \ldots, n) \frac{[\mathcal{Q} + \kappa(1,n)]_{\beta\dot{\alpha}}}{[\mathcal{Q} + \kappa(1,n)]^2}, \tag{7}$$

where $\mathcal{J}(1,\ldots,n)$ refers to the the color-ordered n-gluon current (see Appendix). The other option is to begin with a current containing an off-shell quark and an on-shell antiquark. This gives

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{Q};1,\ldots,n;\mathcal{P}) = \sqrt{2}\sum_{j=1}^{n}\frac{[\kappa(1,n)+\mathcal{P}]_{\alpha\dot{\beta}}}{[\kappa(1,n)+\mathcal{P}]^2}\bar{\mathcal{J}}^{\dot{\beta}\beta}(1,\ldots,j)\Psi_{\beta\dot{\alpha}}(j{+}1,\ldots,n;\mathcal{P}) \tag{8}$$

The solutions to (7) and (8) are equivalent, although it may be easier to use one form instead of the other in certain situations.

The recursion relations for the right-handed quark current are precisely what one would expect given (7) and (8). We may write either

$$\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{Q};1,\ldots,n;\mathcal{P}) = -\sqrt{2}\sum_{j=0}^{n-1}\bar{\Psi}^{\dot{\alpha}\beta}(\mathcal{Q};1,\ldots,j)\mathcal{J}_{\beta\dot{\beta}}(j{+}1,\ldots,n)\frac{[\bar{\mathcal{Q}}+\bar{\kappa}(1,n)]^{\dot{\beta}\alpha}}{[\mathcal{Q}+\kappa(1,n)]^2} \tag{9}$$

or

$$\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{Q};1,\ldots,n;\mathcal{P}) = \sqrt{2}\sum_{j=1}^{n}\frac{[\bar{\kappa}(1,n)+\bar{\mathcal{P}}]^{\dot{\alpha}\beta}}{[\kappa(1,n)+\mathcal{P}]^2}\mathcal{J}_{\beta\dot{\beta}}(1,\ldots,j)\bar{\Psi}^{\dot{\beta}\alpha}(j{+}1,\ldots,n;\mathcal{P}). \tag{10}$$

The left- and right-handed currents are connected by the crossing relation

$$\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{Q};1,2,\ldots,n;\mathcal{P}) = (-1)^{n+1}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\Psi_{\beta\dot{\beta}}(\mathcal{P};n,n{-}1,\ldots,1;\mathcal{Q}). \tag{11}$$

It is not difficult to solve the recursion relations (7) and (9) in the case of like-helicity gluons. The gauge choice suited to this case is the one given in Ref. [11], namely

$$\epsilon_{\alpha\dot{\alpha}} = \frac{u_\alpha(h)\bar{u}_{\dot{\alpha}}(k_j)}{\langle j\ h\rangle} \tag{12}$$

for the jth gluon. The parameter h appearing in (12) is an arbitrary null vector which satisfies

$$h\cdot k_j \neq 0 \tag{13}$$

for all of the gluon momenta. For the current, we find that

$$u^\alpha(h)\Psi_{\alpha\dot{\alpha}}(\mathcal{Q};1^+,\ldots,n^+) = u^\alpha(h)[\mathcal{Q}+\kappa(1,n)]_{\alpha\dot{\alpha}}Y(\mathcal{Q};1,\ldots,n) \tag{14}$$

where the scalar function Y is given by

$$Y(Q) = \frac{-i}{Q^2} \tag{15a}$$

for zero gluons and

$$Y(Q;1,\ldots,n) = \frac{-i(-\sqrt{2})^n}{\langle h|1,\ldots,n|h\rangle} \sum_{j=1}^{n} u^\beta(h) \Pi_\beta{}^\gamma(Q,1,\ldots,j) u_\gamma(h) \tag{15b}$$

for one or more gluons. The details of the derivation are similar to those given in Ref. [11]. The function Π appearing in (15b) is given by

$$\Pi_\beta{}^\gamma(Q,1,\ldots,j) \equiv \frac{k_{j\beta\dot{\gamma}}[\bar{Q}+\bar{\kappa}(1,j)]^{\dot{\gamma}\gamma}}{[Q+\kappa(1,j-1)]^2[Q+\kappa(1,j)]^2}. \tag{16}$$

Note that is is possible to use momentum conservation to eliminate Q from Y in favor of $\mathcal{P}$, and then to use (8) to obtain the complete expression for $\Psi_{\alpha\dot{\alpha}}$ [*i.e.* without contraction with $u^\alpha(h)$]. However, since that expression will not be required in the following, we will not present it here.

Using the same gauge choice, the right-handed current is found to be

$$\bar{\Psi}^{\dot{\alpha}\alpha}(Q;1^+,\ldots,n^+)u_\alpha(h) = \bar{Q}^{\dot{\alpha}\alpha}u_\alpha(h)Y(Q;1,\ldots,n), \tag{17}$$

utilizing the same function Y. It is a simple matter to verify that the solutions (14) and (17) satisfy the crossing relation (11).

The forms given above are adequate if one wishes to study tree-level processes. Loop amplitudes, however, involve the integral of Y (multiplied by some other factors) over Q. Unless the other factors contain inverse powers of Q, we see that such integrals will diverge. It is easy to argue that the amplitudes for the processes we are considering in this paper are ultraviolet finite (in particular, because they vanish at tree-level). Hence, any divergences are spurious. To circumvent this difficulty, we introduce a "regulated" current, obtained by using the recursion relation to replace $Y(Q;1,\ldots,n)$ with an expression containing one more propagator. This new expression is then continued to d dimensions and simplified to the extent possible. This is the same method as the one employed in Ref. [1] to solve the corresponding problem in QED loop amplitudes, and the steps required to derive the regulated current are similar. Hence, we will only present the result:

$$
Y(\mathcal{Q};1,\ldots,n) = \frac{-i(-\sqrt{2})^n}{\langle h|1,\ldots,n|h\rangle} \left\{ \sum_{j=1}^{n} \frac{u^\beta(h) k_{j\beta\dot\gamma} [\bar{Q} + \bar\kappa(1,j)]^{\dot\gamma\gamma} u_\gamma(h)}{\{[Q+\kappa(1,j-1)]^2 - \mu^2\}\{[Q+\kappa(1,j)]^2 - \mu^2\}} \right.
$$
$$
\left. + \sum_{j=1}^{n-1} \frac{\mu^2 u^\beta(h) \kappa_{\beta\dot\gamma}(j+1,n) \bar{k}_j^{\dot\gamma\gamma} u_\gamma(h)}{\{[Q+\kappa(1,j-1)]^2 - \mu^2\}\{[Q+\kappa(1,j)]^2 - \mu^2\}\{[Q+\kappa(1,n)]^2 - \mu^2\}} \right\}. \tag{18}
$$

The new quantities Q and μ^2 appearing in (18) are a result of continuing the vector $\mathcal{Q}$ to d dimensions. They are defined as follows [15]: Q is a 4-dimensional vector consisting of the usual space-time components of $\mathcal{Q}$, while μ^2 is the $(d-4)$-dimensional invariant formed from the "extra" components of $\mathcal{Q}$,

$$
-\mu^2 \equiv \mathcal{Q}^2 - Q^2. \tag{19}
$$

All of the other momenta and polarization vectors remain in 4 dimensions, since they are external quantities [16]. The first term of (18) is what one would obtain by simply regulating (15b). The presence of an additional term reveals that it is crucial to regulate *before* reducing the number of propagators, to ensure that no illegal operations are performed on divergent quantities. For the processes considered in this paper, the integrals obtained using (18) are convergent. If this had not been the case, then it would be necessary to apply the recursion relation two (or more) times before regulating the expression.

III. CONSTRUCTION OF LOOP AMPLITUDES

In this section we will consider processes which may be studied by joining the two ends of the double off-shell quark current together at a vertex.

A. The process $\gamma g \to gg\cdots g$

The simplest amplitude which we may imagine forming from the double-off-shell quark current is (1). Figure 1 illustrates the contributions to this process. The photon is attached at the point where the two ends of the quark current come together. We may write the integrand corresponding to this process as

$$\widehat{I}(Q;1,\ldots,n-1,n_\gamma) = \sum_{\mathcal{P}(1\ldots n-1)} (-ie_q)\gamma^\xi \widehat{\Psi}_{ji}(Q;1,\ldots,n-1)\epsilon_\xi(n). \quad (20)$$

We write "n_γ" in the argument list of $\widehat{I}$ to indicate that the photon has momentum k_n. The polarization vector for the photon is $\epsilon(n)$. We take all external momenta to flow into the diagram, $\kappa(1,n) = 0$. Physical processes are obtained by crossing. The quark circulating in the loop of this diagram has charge e_q. Note that Fig. 1 represents all of the one-loop contributions to this process since photons do not couple directly to gluons.

Inserting the color factorization (5) for the quark current produces

$$\widehat{I}(Q;1,\ldots,n-1,n_\gamma) = -ie_q\, g^{n-1} \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n-1]\}\ \mathrm{Tr}\,\{\Psi(Q;1,\ldots,n-1)\ \not\epsilon(n)\}. \quad (21)$$

Equation (21) is valid for an arbitrary combination of helicities. The quark currents discussed in Sec. II all have like-helicity gluons, so that is the case we will discuss here. To obtain the amplitude with a positive helicity photon and $n-1$ positive helicity gluons, we make the gauge choice given in (12), not only for the gluons, but for the photon as well. We will leave h as a free parameter. To obtain the amplitude with a negative helicity photon and $n-1$ positive helicity gluons, we use (12) for the gluons and set $h = k_n$. For the photon, we write

$$\epsilon_{\alpha\dot\alpha}(n^-) = \frac{u_\alpha(k_n)\bar u_{\dot\alpha}(h)}{\langle n\ h\rangle^*}. \quad (22)$$

Because the computations are similar, we will concentrate on the positive helicity photon case, and simply quote the result for a negative helicity photon.

Before proceeding, however, let us pause to define a color-ordered integrand and a color-ordered amplitude. If we write

$$\widehat{I}(Q;1^+,\ldots,(n-1)^+,n_\gamma^+) \equiv -ie_q\, g^{n-1} \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n-1]\}\, I(Q;1^+,\ldots,(n-1)^+,n_\gamma^+)) \quad (23)$$

then, obviously

$$I(Q;1^+,\ldots,(n-1)^+,n_\gamma^+) = \mathrm{Tr}\,\{\Psi(Q;1^+,\ldots,(n-1)^+)\ \not\epsilon(n^+)\}. \quad (24)$$

The actual amplitude is obtained from (23) by integrating over Q and supplying the factor -1 for closing the loop. Likewise, we may obtain a color-ordered amplitude from (24) in analogous manner.

Let us convert the trace over Dirac matrices that appears in (21) to spinor notation, and substitute for the photon polarization. Since we are considering the massless limit, in which the two possible chiralities for the circulating quark are decoupled, we obtain two terms:

$$I(Q;1^+,\ldots,(n{-}1)^+,n_\gamma^+) = \frac{\sqrt{2}}{\langle n\ h\rangle}\Big\{\bar{u}_{\dot\alpha}(k_n)\bar{\Psi}^{\dot\alpha\alpha}(Q;1^+,\ldots,(n{-}1)^+)u_\alpha(h) + u^\alpha(h)\Psi_{\alpha\dot\alpha}(Q;1^+,\ldots,(n{-}1)^+)\bar{u}^{\dot\alpha}(k_n)\Big\}. \tag{25}$$

Notice that the undotted index of the quark current always appears contracted with the gauge spinor, as promised in Sec. II. Hence, we may use (14) and (17) for the spinor structure of the currents:

$$I(Q;1^+,\ldots,(n{-}1)^+,n_\gamma^+) = 2\sqrt{2}\,\frac{\bar{u}_{\dot\alpha}(k_n)\bar{Q}^{\dot\alpha\alpha}u_\alpha(h)}{\langle n\ h\rangle}Y(Q;1,\ldots,n{-}1). \tag{26}$$

We have utilized the Weyl equation plus over-all momentum conservation for the diagram to combine the two contributions from (25) into a single term.

Since (26) does not contain any extra inverse powers of Q, we must use the regulated form of Y given in (18), and perform a d-dimensional integration. The integrals which arise during this procedure have been discussed in Ref. [1] in connection with n-photon scattering amplitude. Hence, we will proceed immediately to the result for the color-ordered amplitude, which reads

$$\mathcal{A}(1^+,\ldots,(n{-}1)^+,n_\gamma^+) = \frac{(-\sqrt{2})^n}{48\pi^2}\sum_{\ell=1}^{n-2}\frac{u^\beta(h)\kappa_{\beta\dot\beta}(\ell{+}1,n{-}1)\bar{k}_\ell^{\dot\beta\delta}u_\delta(h)}{\langle h|1,\ldots,n{-}1|h\rangle\langle n\ h\rangle}\,\bar{u}_{\dot\alpha}(k_n)[\bar\kappa(1,\ell{-}1)-\bar\kappa(\ell{+}1,n)]^{\dot\alpha\alpha}u_\alpha(h). \tag{27}$$

In order to prove the gauge invariance of the final result, we will consider

$$\widetilde{\mathcal{A}}(1^+,\ldots,(n{-}1)^+,n_\gamma^+) \equiv \sum_{C(1\ldots n-1)}\mathcal{A}(1^+,\ldots,(n{-}1)^+,n_\gamma^+). \tag{28}$$

Because the color factor appearing in the full amplitude is invariant under cyclic permutations of $\{1,\ldots,n{-}1\}$, we may replace the color-ordered amplitude $\mathcal{A}$ by $\widetilde{\mathcal{A}}/(n{-}1)$ without altering the value of the final result. In general, to prove gauge invariance, we must consider all of the terms which have the same color structure. Thus, it will be necessary to make the indicated replacement to demonstrate that the expression we obtain is independent of the gauge spinor. The cyclic symmetry present in this case is more restrictive than the permutation symmetry which occurred in the study of the n-photon scattering amplitude. As a consequence, although a portion of the derivation presented here is the same as Ref. [1], there are certain key differences.

We begin by multiplying the summand by $\langle n{-}1\ 1\rangle / \langle n{-}1\ 1\rangle$ and applying the Schouten identity to the combination $u^\beta(h)\langle n{-}1\ 1\rangle$. We thus obtain

$$
\begin{aligned}
&\widetilde{\mathcal{A}}(1^+,\ldots,(n{-}1)^+,n_\gamma^+) = \\
&\quad \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{\ell=1}^{n-2} \frac{u^\beta(k_1)\kappa_{\beta\dot\beta}(\ell{+}1,n{-}1)\bar{k}_\ell^{\dot\beta\delta}u_\delta(h)}{\langle h|1,\ldots,n{-}1|1\rangle\langle n\ h\rangle}\, \bar{u}_{\dot\alpha}(k_n)[\bar\kappa(1,\ell{-}1)-\bar\kappa(\ell{+}1,n)]^{\dot\alpha\alpha}u_\alpha(h) \\
&\quad + \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{\ell=1}^{n-2} \frac{u^\beta(k_{n-1})\kappa_{\beta\dot\beta}(\ell{+}1,n{-}2)\bar{k}_\ell^{\dot\beta\delta}u_\delta(h)}{\langle n{-}1|1,\ldots,n{-}1|h\rangle\langle n\ h\rangle}\, \bar{u}_{\dot\alpha}(k_n)[\bar\kappa(1,\ell{-}1)-\bar\kappa(\ell{+}1,n)]^{\dot\alpha\alpha}u_\alpha(h).
\end{aligned}
\tag{29}
$$

We take advantage of the sum over cyclic permutations by relabeling the momenta in the second term of (29) as follows:

$$
1 \to 2 \to 3 \to \cdots \to n{-}1 \to 1. \tag{30}
$$

The effect of this relabeling is to produce

$$
\begin{aligned}
\widetilde{\mathcal{A}}_2 = \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{\ell=1}^{n-2} &\frac{u^\beta(k_1)\kappa_{\beta\dot\beta}(\ell{+}2,n{-}1)\bar{k}_{\ell+1}^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n{-}1,1|h\rangle\langle n\ h\rangle} \\
&\times \bar{u}_{\dot\alpha}(k_n)[\bar\kappa(2,\ell) - \bar\kappa(\ell{+}2,n{-}1) - \bar{k}_1]^{\dot\alpha\alpha}u_\alpha(h).
\end{aligned}
\tag{31}
$$

A little bit of algebra converts this expression to

$$
\begin{aligned}
\widetilde{\mathcal{A}}_2 = -\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{\ell=2}^{n-2} &\frac{u^\beta(k_1)\kappa_{\beta\dot\beta}(\ell{+}1,n{-}1)\bar{k}_\ell^{\dot\beta\delta}u_\delta(h)}{\langle h|1,\ldots,n{-}1|1\rangle\langle n\ h\rangle} \\
&\times \bar{u}_{\dot\alpha}(k_n)[\bar\kappa(1,\ell{-}1) - \bar\kappa(\ell{+}1,n{-}1) - 2\bar{k}_1]^{\dot\alpha\alpha}u_\alpha(h).
\end{aligned}
\tag{32}
$$

When (32) is recombined with the first term of (29), all but the $\ell = 1$ term of that sum cancels, and we are left with

$$\widetilde{\mathcal{A}}(1^+,\ldots,(n-1)^+,n_\gamma^+) = -\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{C(1\ldots n-1)}\frac{\bar{u}_{\dot\alpha}(k_n)\bar{k}_1^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(2,n-1)\bar{k}_1^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\ h\rangle}$$
$$-\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{C(1\ldots n-1)}\sum_{\ell=1}^{n-2}\frac{2\,\bar{u}_{\dot\alpha}(k_n)\bar{k}_1^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(\ell+1,n-1)\bar{k}_\ell^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\ h\rangle}. \quad (33)$$

In the QED case, the first term of (33) vanishes when the sum over permutations of $\{2,\ldots,n-1\}$ is performed. In this case, however, we have but a cyclic symmetry to work with, and this term must be retained.

Working within the allowed symmetry, we note that the denominators appearing in (33) are invariant under cyclic permutations of $\{1,\ldots,n-1\}$. We will exploit this symmetry to recast the sum appearing in the second term, by relabeling the successive terms in the sum on ℓ so that k_ℓ always becomes k_1, $k_{\ell+1}$ always becomes k_2, etc. In the $\ell = 2$ term, k_1 becomes k_{n-1}. For $\ell = 3$, k_1 becomes k_{n-2}, and so on through $\ell = n-2$, in which k_1 becomes k_3. Consequently, we may rewrite (33) as

$$\widetilde{\mathcal{A}}(1^+,\ldots,(n-1)^+,n_\gamma^+) = -\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{C(1\ldots n-1)}\frac{\bar{u}_{\dot\alpha}(k_n)\bar{k}_1^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(2,n-1)\bar{k}_1^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\ h\rangle}$$
$$-\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{C(1\ldots n-1)}\sum_{j=3}^{n-1}\frac{2\,\bar{u}_{\dot\alpha}(k_n)\bar{k}_j^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(2,j-1)\bar{k}_1^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\ h\rangle}. \quad (34)$$

The next step is to use the anticommutation properties of the σ-matrices to write

$$\begin{aligned}2\,\bar{u}_{\dot\alpha}(k_n)\bar{k}_j^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(2,j-1)\bar{k}_1^{\dot\beta\delta}u_\delta(h) &= \bar{u}_{\dot\alpha}(k_n)\bar{k}_j^{\dot\alpha\beta}\kappa_{\beta\dot\beta}(2,j-1)\bar{k}_1^{\dot\beta\delta}u_\delta(h)\\ &\quad - \bar{u}_{\dot\alpha}(k_n)\bar{\kappa}^{\dot\alpha\beta}(2,j-1)k_{j\beta\dot\beta}\bar{k}_1^{\dot\beta\delta}u_\delta(h)\\ &\quad + 2\,k_j\cdot\kappa(2,j-1)\ \bar{u}_{\dot\beta}(k_n)\bar{k}_1^{\dot\beta\delta}u_\delta(h).\end{aligned} \quad (35)$$

It is a simple exercise to demonstrate that the contribution generated from the last term of (35) exactly cancels the first term of (34). We will designate the contributions from the first and second terms (35) $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ respectively.

We will consider $\widetilde{\mathcal{A}}_2$ first, since we will perform the fewest operations on it. From (35) and (34) we see that

$$\widetilde{\mathcal{A}}_2 = \frac{(-\sqrt{2})^n}{48\pi^2}\sum_{C(1\ldots n-1)}\sum_{j=3}^{n-1}\frac{\bar{u}_{\dot\alpha}(k_n)\bar{\kappa}^{\dot\alpha\beta}(2,j-1)k_{j\beta\dot\beta}\bar{k}_1^{\dot\beta\delta}u_\delta(h)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\ h\rangle}. \quad (36)$$

If we supply the factor $\langle 1\ n\rangle/\langle 1\ n\rangle$ and perform a bit of spinor algebra, we find

$$\widetilde{\mathcal{A}}_2 = \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=3}^{n-1} \frac{\bar{k}_j^{\dot\beta\beta}\kappa_{\beta\dot\alpha}(2,j-1)\bar{k}_n^{\dot\alpha\delta}k_{1\delta\dot\beta}}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}. \tag{37}$$

We will save (37) for later cancellation.

Turning to $\widetilde{\mathcal{A}}_1$ and supplying a factor $\langle 1\ n\rangle/\langle 1\ n\rangle$ as we did for the other term, we find that it may be written

$$\widetilde{\mathcal{A}}_1 = -\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=3}^{n-1} \frac{\bar{u}_{\dot\beta}(k_1)\bar{\kappa}^{\dot\beta\beta}(1,j-1)k_{j\beta\dot\alpha}\bar{k}_n^{\dot\alpha\alpha}u_\alpha(k_1)}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}. \tag{38}$$

We now break $\widetilde{\mathcal{A}}_1$ into two pieces, utilizing momentum conservation and the Weyl equation to write

$$\begin{aligned} k_{j\beta\dot\alpha}\bar{k}_n^{\dot\alpha\alpha}u_\alpha(k_1) &= -k_{j\beta\dot\alpha}\bar{\kappa}^{\dot\alpha\alpha}(2,n-1)u_\alpha(k_1) \\ &= -k_{j\beta\dot\alpha}\bar{\kappa}^{\dot\alpha\alpha}(2,j-1)u_\alpha(k_1) - k_{j\beta\dot\alpha}\bar{\kappa}^{\dot\alpha\alpha}(j+1,n-1)u_\alpha(k_1). \end{aligned} \tag{39}$$

The second contribution in the last line of (39) vanishes if $j = n-1$. Thus, we may write

$$\begin{aligned} \widetilde{\mathcal{A}}_1 =& \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=3}^{n-1}\sum_{\ell=2}^{j-1} \frac{\bar{u}_{\dot\beta}(k_1)\bar{\kappa}^{\dot\beta\beta}(1,j-1)k_{j\beta\dot\alpha}\bar{k}_\ell^{\dot\alpha\alpha}u_\alpha(k_1)}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle} \\ &+\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=2}^{n-2}\sum_{\ell=j+1}^{n-1} \frac{\bar{u}_{\dot\beta}(k_1)\bar{\kappa}^{\dot\beta\beta}(1,j-1)k_{j\beta\dot\alpha}\bar{k}_\ell^{\dot\alpha\alpha}u_\alpha(k_1)}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}, \end{aligned} \tag{40}$$

where the extension of the second sum to include $j = 2$ may be done without producing a compensating term since the contribution in question vanishes. We have also chosen to write the implicit κ-sums from (39) explicitly as sums over ℓ.

The next step is to interchange the order of the double sum appearing in the first term of (40). If we subsequently interchange the names of the two dummy summation variables, we see that the summation ranges of the two terms are identical. Performing this manipulation and rearranging the spinor products a bit we find

$$\begin{aligned} \widetilde{\mathcal{A}}_1 =& \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=2}^{n-2}\sum_{\ell=x+1}^{n-1} \frac{\bar{k}_j^{\dot\alpha\alpha}k_{1\alpha\dot\beta}\bar{\kappa}^{\dot\beta\beta}(1,\ell-1)k_{\ell\beta\dot\alpha}}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle} \\ &+\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\dots n-1)} \sum_{j=2}^{n-2}\sum_{\ell=j+1}^{n-1} \frac{\bar{k}_j^{\dot\alpha\alpha}\kappa_{\alpha\dot\beta}(2,j-1)\bar{k}_1^{\dot\beta\beta}k_{\ell\beta\dot\alpha}}{\langle 1|2,\dots,n-1|1\rangle}\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}. \end{aligned} \tag{41}$$

To make the terms of (41) look even more nearly alike, we write

$$k_{1\alpha\dot{\beta}} = \kappa_{\alpha\dot{\beta}}(1, j-1) - \kappa_{\alpha\dot{\beta}}(2, j-1) \tag{42}$$

in the first term and

$$\bar{k}_1^{\dot{\beta}\beta} = \bar{\kappa}^{\dot{\beta}\beta}(1, \ell-1) - \bar{\kappa}^{\dot{\beta}\beta}(2, \ell-1) \tag{43}$$

in the second term. Two of the four terms thus generated cancel, leaving only

$$\begin{aligned} \widetilde{\mathcal{A}}_1 = & \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-2} \sum_{\ell=x+1}^{n-1} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1, j-1) \bar{\kappa}^{\dot{\beta}\beta}(1, \ell-1) k_{\ell\beta\dot{\alpha}}}{\langle 1|2, \ldots, n-1|1\rangle} \frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle} \\ & - \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(2, j-1) \bar{\kappa}^{\dot{\beta}\beta}(2, \ell-1) k_{\ell\beta\dot{\alpha}}}{\langle 1|2, \ldots, n-1|1\rangle} \frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}. \end{aligned} \tag{44}$$

We will designate the first term of (44) $\widetilde{\mathcal{A}}_{1\mathrm{A}}$ and save it in its present form. The other contribution will be referred to as $\widetilde{\mathcal{A}}_{1\mathrm{B}}$. We begin operations on it by using the cyclic sum to shift all of the dummy indices down by one unit, with $k_1 \to k_{n-1}$. The result reads

$$\widetilde{\mathcal{A}}_{1\mathrm{B}} = -\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{\bar{k}_{j-1}^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1, j-2) \bar{\kappa}^{\dot{\beta}\beta}(1, \ell-2) (k_{\ell-1})_{\beta\dot{\alpha}}}{\langle 1|2, \ldots, n-1|1\rangle} \frac{\langle n-1\ h\rangle}{\langle n-1|n|h\rangle}. \tag{45}$$

Shifting the summations by 1 unit produces

$$\widetilde{\mathcal{A}}_{1\mathrm{B}} = -\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-3} \sum_{\ell=j+1}^{n-2} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1, j-1) \bar{\kappa}^{\dot{\beta}\beta}(1, \ell-1) k_{\ell\beta\dot{\alpha}}}{\langle 1|2, \ldots, n-1|1\rangle} \frac{\langle n-1\ h\rangle}{\langle n-1|n|h\rangle}. \tag{46}$$

Since

$$\frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle} - \frac{\langle n-1\ h\rangle}{\langle n-1|n|h\rangle} = -\frac{\langle n-1\ 1\rangle}{\langle n-1|n|1\rangle}, \tag{47}$$

it is desirable to extend the summation range appearing in (46) to match the summation range of the first term in (44). This time, the compensating contribution does not vanish, but instead reads

$$\widetilde{\mathcal{A}}_{1\mathrm{X}} \equiv \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-2} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1, j-1) \bar{\kappa}^{\dot{\beta}\beta}(1, n-2) (k_{n-1})_{\beta\dot{\alpha}}}{\langle 1|2, \ldots, n-1|1\rangle} \frac{\langle n-1\ h\rangle}{\langle n-1|n|h\rangle}. \tag{48}$$

If we apply the cyclic sum to (48), shifting the dummy momentum labels up by 1 unit, and then shift the sum over j, we find that

$$\widetilde{\mathcal{A}}_{1X} = \frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=3}^{n-1} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(2,j-1) \bar{\kappa}^{\dot{\beta}\beta}(2,n-1) k_{1\beta\dot{\alpha}}}{\langle 1|2,\ldots,n-1|1\rangle} \frac{\langle 1\ h\rangle}{\langle 1|n|h\rangle}. \tag{49}$$

A straightforward application of momentum conservation plus the Weyl equation reveals that $\widetilde{\mathcal{A}}_{1X}$ is exactly the term required to cancel $\widetilde{\mathcal{A}}_2$ [see Eq. (37)]. Thus, the entire result reads

$$\widetilde{\mathcal{A}}(1^+,\ldots,(n-1)^+,n_\gamma^+) = -\frac{(-\sqrt{2})^n}{48\pi^2} \sum_{C(1\ldots n-1)} \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1,j) \bar{\kappa}^{\dot{\beta}\beta}(1,\ell) k_{\ell\beta\dot{\alpha}}}{\langle 1|2,\ldots,n|1\rangle}, \tag{50}$$

implying the amplitude

$$\widehat{\mathcal{A}}(1^+,\ldots,(n-1)^+,n_\gamma^+) = \frac{i}{48\pi^2}(-e_q\sqrt{2})(-g\sqrt{2})^{n-1} \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n-1]\} \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{\bar{k}_j^{\dot{\alpha}\alpha} \kappa_{\alpha\dot{\beta}}(1,j) \bar{\kappa}^{\dot{\beta}\beta}(1,\ell) k_{\ell\beta\dot{\alpha}}}{\langle 1|2,\ldots,n|1\rangle}. \tag{51}$$

It is easy to repeat the above calculation for a negative helicity photon and $n-1$ positive helicity gluons. We obtain

$$\widehat{\mathcal{A}}(1^+,\ldots,(n-1)^+,n_\gamma^-) = \frac{i}{48\pi^2}(-e_q\sqrt{2})(-g\sqrt{2})^{n-1} \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n-1]\} \times \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{k_{j\alpha\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}\beta}(1,j) \kappa_{\beta\dot{\beta}}(1,\ell) \bar{k}_\ell^{\dot{\beta}\alpha}}{\langle 1|2,\ldots,n-1|1\rangle} \frac{\langle 1\ n-1\rangle^*}{\langle 1|n|n-1\rangle^*}. \tag{52}$$

This calculation proceeds in almost exactly the same manner as the positive helicity photon case, except that the step corresponding to Eq. (34) should be skipped.

The amplitudes presented in Eqs. (51) and (52) should reduce to the amplitudes for n-photon scattering in the $U(1)$ limit. This limit is simple to obtain: simply replace all of the color matrices by the unit matrix. We have verified that the above expressions do indeed reduce to the values for the n-photon amplitude reported in Ref. [1].

B. The process $e^+e^- \to gg\cdots g$

The lowest order diagrams for electron-positron annihilation to gluons are obtained from Fig. 1 by attaching an electron-positron pair to the photon. There are no additional diagrams because leptons do not couple directly to gluons. This process is closely related to the one examined in the last section: in the limit where the e^+e^- pair becomes collinear it is simply some (divergent) factor times the amplitude obtained above. Thus, it is not surprising that the entire discussion of the last section may be applied to this case with very little modification.

For a negative helicity positron of momentum p, a positive helicity electron of momentum q, and n positive helicity gluons we find

$$\widehat{\mathcal{A}}(p^-;1^+,\ldots,(n)^+;q^+) = \\ -\frac{iee_q}{24\pi^2}(-g\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} tr\left\{\Omega[1,n]\right\} \sum_{j=2}^{n-1} \sum_{\ell=j+1}^{n} \frac{k_{j\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}\beta}(1,j)\kappa_{\beta\dot{\beta}}(1,\ell)\bar{k}_\ell^{\dot{\beta}\alpha}}{\langle 1|2,\ldots,n|1\rangle\langle p\ q\rangle} \frac{\langle 1\ n\rangle^*}{\langle 1|q|n\rangle^*} \tag{53}$$

(all momenta directed inward). To obtain the amplitude for a positive helicity positron of momentum p, a negative helicity electron of momentum q, and n positive helicity gluons, interchange p and q in the above expression.

C. The process $gg \to gg\cdots g$

We now turn to the process of gluon-gluon scattering. At present, we have only been able to evaluate those diagrams which contain a quark loop. In principle, the diagrams involving a gluon loop should be obtainable from a gluon current with two off-shell gluons. However, the complicated form of the recursion relation for this object [9] makes this path difficult to follow. Nevertheless, it is possible to use supersymmetry [17] to obtain some of the subamplitudes where a gluon loop replaces the quark loop.

Fig. 2 illustrates the process in terms of the double off-shell quark current and the gluon current (single off-shell particle). To avoid over-counting we will "anchor" the nth gluon and write the integrand as

$$\hat{I}(Q;1,\ldots,n) = \sum_{\mathcal{P}(1\ldots n-1)} \sum_{t=1}^{n-1} \frac{1}{t!(n-t-1)!}(-ig)(T^x)_{ij}\gamma^\xi \hat{\Psi}_{ji}(Q;1,\ldots,t)\widehat{J}^x_\xi(t+1,\ldots,n). \quad (54)$$

In specifying that the sum on t begin at 1 instead of 0, we have dropped the (vanishing) tadpole diagram. Insertion of the color factorizations for the quark and gluon currents [Eqs. (5) and (A1)] produces

$$\hat{I}(Q;1,\ldots,n) = -2ig^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{t=1}^{n-1}\sum_{s=t}^{n-1} tr\,\{\Omega[1,t]T^x\}\; tr\{T^x\Omega[t+1,s]T^{a_n}\Omega[s+1,n-1]\}$$
$$\times \mathrm{Tr}\,\{\Psi(Q;1,\ldots,t)\, \not{J}(t+1,\ldots,s,n,s+1,\ldots,n-1)\}. \quad (55)$$

Let us examine the color factor appearing in (55):

$$C \equiv tr\,\{\Omega[1,t]T^x\}\; tr\{T^x\Omega[t+1,s]T^{a_n}\Omega[s+1,n-1]\} \quad (56)$$

We simplify it by applying the completeness relation for $SU(N)$ to do the implied sum on x. The result,

$$C = \frac{1}{2} tr\,\{\Omega[s+1,n-1]\Omega[1,s]T^{a_n}\} - \frac{1}{2N} tr\,\{\Omega[1,t]\}\; tr\,\{\Omega[s+1,n-1]\Omega[t+1,s]T^{a_n}\} \quad (57)$$

suggests that the manner in which we chose to label the gluons when writing down (54) (*i.e.* the gluons labeled 1 through t as a part of the quark current, and the remaining gluons labeled $t+1$ through s, n, and then $s+1$ through $n-1$ as part of the gluon current) is not the best way to proceed. Consider the contribution to the integrand from the second term of (57). The structure of this term suggests that we write

$$\hat{I}_2(Q;1,\ldots,n) = \frac{ig^n}{N} \sum_{\mathcal{P}(1\ldots n-1)} \sum_{t=1}^{n-1}\sum_{v=t}^{n-1} tr\,\{\Omega[1,t]\}\; tr\{\Omega[t+1,n]\}$$
$$\times \mathrm{Tr}\,\{\Psi(Q;1,\ldots,t)\, \not{J}(v+1,\ldots,n-1,n,t+1,\ldots,v)\}, \quad (58)$$

making use of the sum over permutations to simplify the color factor. In this form, it is obvious that we may apply the cyclic sum identity [11]

$$\sum_{C(t+1\ldots n)} J_\xi(t+1,\ldots,n) = 0 \tag{59}$$

to show that all except the $t = n-1$ contribution vanishes. However, the color factor for this piece is

$$tr\,\{\Omega[1,n-1]\}\; tr\{T^{a_n}\} = 0. \tag{60}$$

Hence, the color-suppressed contribution to the amplitude vanishes. Alternatively, we could have considered an extended $U(N)$ gauge theory. The observation that the gauge boson coupling the quark loop to the gluon current can not have $U(1)$ quantum numbers in the extended theory guarantees that the $U(N)$ and $SU(N)$ results coincide. Then, the absence of a $1/N$ term in the $U(N)$ completeness relation implies the absence of such a term in the amplitude.

Turning to the remaining contribution from (57), we see that the order of the color matrices suggests that we write

$$\begin{aligned}\hat{I}(Q;1,\ldots,n) = -ig^n \sum_{\mathcal{P}(1\ldots n-1)} & tr\,\{\Omega[1,n]\} \\ & \times \sum_{y=1}^{n-1}\sum_{z=0}^{y-1} \mathrm{Tr}\,\{\Psi(Q;z+1,\ldots,y)\; \not{J}(y+1,\ldots,n,1,\ldots,z)\}.\end{aligned} \tag{61}$$

Equation (61) may be taken as a general starting point for *all* gluon-gluon scattering diagrams containing a single quark loop. It is valid for any helicity combination. All that is required to utilize it is a knowledge of the color-ordered currents appearing on the right hand side.

At this stage we define the color-ordered integrand and color-ordered amplitude. If we write

$$\hat{I}(Q;1,\ldots,n) \equiv -ig^n \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n]\}\, I(Q;1,\ldots,n) \tag{62}$$

then, obviously

$$I(Q;1,\ldots,n) = \sum_{y=1}^{n-1}\sum_{z=0}^{y-1} \mathrm{Tr}\,\{\Psi(Q;z+1,\ldots,y)\; \not{J}(y+1,\ldots,n,1,\ldots,z)\}. \tag{63}$$

As before, the complete amplitude is obtained from (62) by integrating over $\mathcal{Q}$ and supplying the factor -1 for closing the loop. The color-ordered amplitude is obtained from (63) in the same manner.

We now specialize to the first case for which we have obtained explicit results: n like-helicity gluons. In spinor notation, (63) becomes

$$I(\mathcal{Q};1,\ldots,n) = \sqrt{2}\sum_{y=1}^{n-1}\sum_{z=0}^{y-1}\Big\{\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{Q};z{+}1,\ldots,y)\mathcal{J}_{\alpha\dot{\alpha}}(y{+}1,\ldots,n,1,\ldots,z)$$
$$+\Psi_{\alpha\dot{\alpha}}(\mathcal{Q};z{+}1,\ldots,y)\bar{\mathcal{J}}^{\dot{\alpha}\alpha}(y{+}1,\ldots,n,1,\ldots,z)\Big\}. \qquad (64)$$

We choose the gauge indicated by (12). Inserting the solution (A2) for the gluon current we find

$$I(\mathcal{Q};1,\ldots,n) = \sum_{y=1}^{n-1}\sum_{z=0}^{y-1}\frac{-(-\sqrt{2})^{n-y+z}}{\langle h|y{+}1,\ldots,n,1,2,\ldots,z|h\rangle}$$
$$\times\Big\{u^{\beta}(h)[\kappa(y{+}1,n)+\kappa(1,z)]_{\beta\dot{\alpha}}\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{Q};z{+}1,\ldots,y)u_{\alpha}(h)$$
$$-u^{\alpha}(h)\Psi_{\alpha\dot{\alpha}}(\mathcal{Q};z{+}1,\ldots,y)[\bar{\kappa}(y{+}1,n)+\bar{\kappa}(1,z)]^{\dot{\alpha}\beta}u_{\beta}(h)\Big\}. \qquad (65)$$

Once again, as promised in Sec. II, all occurrences of the double-off-shell quark current have its undotted index contracted into the gauge spinor. Using Eqs. (14) and (17) to insert the spinor structure of the quark currents, plus over-all momentum conservation for the diagram, we see that the two contributions to (65) are actually equal. That is,

$$I(\mathcal{Q};1,\ldots,n) = \sum_{y=1}^{n-1}\sum_{z=0}^{y-1}(-\sqrt{2})^{n-y+z}\frac{2u^{\beta}(h)\kappa_{\beta\dot{\alpha}}(z{+}1,y)\bar{\mathcal{Q}}^{\dot{\alpha}\alpha}u_{\alpha}(h)}{\langle h|y{+}1,\ldots,n{-}1|n\rangle\langle n|1,2,\ldots,z|h\rangle}Y(\mathcal{Q};z{+}1,\ldots,y). \qquad (66)$$

Since there are no extra powers of $\mathcal{Q}$ in the denominator of (66), we should use the regulated form for Y given in Eq. (18) and perform a d-dimensional integration. The integrals that occur are the same as the ones discussed in Ref. [1], and so we proceed immediately to the integrated expression for the color-ordered amplitude

$$\mathcal{A}(1^{+},\ldots,n^{+}) = -\frac{(-\sqrt{2})^{n}}{48\pi^{2}}\sum_{x=1}^{n-2}\sum_{y=x+1}^{n-1}\sum_{z=0}^{x-1}u^{\alpha}(h)\kappa_{\alpha\dot{\alpha}}(x{+}1,y)\bar{k}_{x}^{\dot{\alpha}\beta}u_{\beta}(h)$$
$$\times\frac{u^{\gamma}(h)\kappa_{\gamma\dot{\gamma}}(z{+}1,y)[\bar{\kappa}(z{+}1,x{-}1)-\bar{\kappa}(x{+}1,y)]^{\dot{\gamma}\delta}u_{\delta}(h)}{\langle n|1,\ldots,z|h\rangle\langle h|z{+}1,\ldots,y|h\rangle\langle h|y{+}1,\ldots,n{-}1|n\rangle}. \qquad (67)$$

The expression in Eq. (67) must be gauge invariant by itself: the color factor for this process contains no symmetries which would cause different terms within the permutation sum to mix.

We begin our demonstration that (67) is indeed independent of the gauge spinor by isolating a portion of the numerator:

$$\mathcal{N}_2 \equiv -u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(z{+}1,y)[\bar\kappa(z{+}1,x{-}1) - \bar\kappa(x{+}1,y)]^{\dot\gamma\delta}u_\delta(h). \tag{68}$$

By writing $\kappa(z{+}1,y) = \kappa(z{+}1,x{-}1) + k_x + \kappa(x{+}1,y)$ we obtain

$$\begin{aligned}\mathcal{N}_2 =& -u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(x{+}1,y)\bar\kappa^{\dot\gamma\delta}(z{+}1,x{-}1)u_\delta(h) + u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(z{+}1,x{-}1)\bar\kappa^{\dot\gamma\delta}(x{+}1,y)u_\delta(h)\\ &-u^{\gamma}(h)k_{x\gamma\dot\gamma}\bar\kappa^{\dot\gamma\delta}(z{+}1,x{-}1)u_\delta(h) + u^{\gamma}(h)k_{x\gamma\dot\gamma}\bar\kappa^{\dot\gamma\delta}(x{+}1,y)u_\delta(h)\\ =& -u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(x,y)\bar\kappa^{\dot\gamma\delta}(z{+}1,x{-}1)u_\delta(h) - u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(x{+}1,y)\bar\kappa^{\dot\gamma\delta}(z{+}1,x)u_\delta(h).\end{aligned} \tag{69}$$

Since the two terms appearing in the last line of (69) differ by only a shift in x of 1 unit, for the moment we need only discuss operations on one of the two terms.

Considering then, the contribution from the first term, we have

$$\begin{aligned}\mathcal{A}_1 \equiv& -\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{x=1}^{n-2}\sum_{y=x+1}^{n-1}\sum_{z=0}^{x-1} u^{\alpha}(h)\kappa_{\alpha\dot\alpha}(x{+}1,y)\bar k_x^{\dot\alpha\beta}u_\beta(h)\\ &\times\frac{u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(x,y)\bar\kappa^{\dot\gamma\delta}(z{+}1,x{-}1)u_\delta(h)}{\langle n|1,\ldots,z|h\rangle\langle h|z{+}1,\ldots,y|h\rangle\langle h|y{+}1,\ldots,n{-}1|n\rangle}.\end{aligned} \tag{70}$$

The sum on z is very easy to perform because of the identity

$$\sum_{i=a}^{b-1}\frac{\langle i\; i{+}1\rangle}{\langle i|h|i{+}1\rangle} = \frac{\langle a\; b\rangle}{\langle a|h|b\rangle}, \tag{71}$$

a direct consequence of the Schouten identity. Note that we identify k_0 with k_n in applying (71) to (70), as implied by its denominator structure. Interchanging the sum on z with the implicit κ-sum involving z and using (71) yields

$$\mathcal{A}_1 = -\frac{(-\sqrt{2})^n}{48\pi^2}\sum_{x=1}^{n-2}\sum_{y=x+1}^{n-1} u^{\alpha}(h)\kappa_{\alpha\dot\alpha}(x{+}1,y)\bar k_x^{\dot\alpha\beta}u_\beta(h)\frac{u^{\gamma}(h)\kappa_{\gamma\dot\gamma}(x,y)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)}{\langle n|1,\ldots,y|h\rangle\langle h|y{+}1,\ldots,n{-}1|n\rangle}. \tag{72}$$

The sum on y is a little more involved since there are two implicit κ-sums involving y. The relevant structure to examine reads

$$\sigma_y \equiv \sum_{y=x+1}^{n-1} \frac{\langle y\ y{+}1\rangle}{\langle y|h|y{+}1\rangle}[u^\alpha(h)\kappa_{\alpha\dot\alpha}(x,y)\bar k_x^{\dot\alpha\beta}u_\beta(h)][u^\gamma(h)\kappa_{\gamma\dot\gamma}(x,y)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)]$$

$$= \sum_{y=x+1}^{n-1}\sum_{a=x}^{n-1}\sum_{b=x}^{y} \frac{\langle y\ y{+}1\rangle}{\langle y|h|y{+}1\rangle}[u^\alpha(h)k_{a\alpha\dot\alpha}\bar k_x^{\dot\alpha\beta}u_\beta(h)][u^\gamma(h)k_{b\gamma\dot\gamma}\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)]. \tag{73}$$

Interchanging the summations and using (71) to do the sum on y produces

$$\sigma_y = \frac{1}{\langle h\ n\rangle}\sum_{a=x}^{n-1}\sum_{b=x}^{a-1}[u^\alpha(k_n)k_{a\alpha\dot\alpha}\bar k_x^{\dot\alpha\beta}u_\beta(h)][u^\gamma(h)k_{b\gamma\dot\gamma}\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)]$$

$$+\frac{1}{\langle h\ n\rangle}\sum_{a=x}^{n-1}\sum_{b=a}^{n-1}[u^\alpha(h)k_{a\alpha\dot\alpha}\bar k_x^{\dot\alpha\beta}u_\beta(h)][u^\gamma(k_n)k_{b\gamma\dot\gamma}\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)]. \tag{74}$$

We extend the sum on b appearing in the second term of (74) to the range $b \in [x, n{-}1]$. This allows both sums to be performed in that piece. The compensating term has precisely the same range as the first term of (74), and may be combined with it by applying the Schouten identity. The result of doing all of this is

$$\sigma_y = -\sum_{a=x}^{n-1} u^\beta(h)k_{x\beta\dot\alpha}\bar k_a^{\dot\alpha\gamma}\kappa_{\gamma\dot\gamma}(x,a{-}1)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)$$

$$+\frac{1}{\langle h\ n\rangle}[u^\alpha(h)\kappa_{\alpha\dot\alpha}(x,n{-}1)\bar k_x^{\dot\alpha\beta}u_\beta(h)][u^\gamma(k_n)\kappa_{\gamma\dot\gamma}(x,n{-}1)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)]. \tag{75}$$

The second term of (75) vanishes, since

$$\begin{aligned} u^\gamma(k_n)\kappa_{\gamma\dot\gamma}(x,n{-}1)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n) &= -u^\gamma(k_n)\kappa_{\gamma\dot\gamma}(1,x{-}1)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n) \\ &= -\langle n\ n\rangle\kappa^2(1,x{-}1) \\ &= 0. \end{aligned} \tag{76}$$

Inserting the non-vanishing contribution from (75) into (72) and including the piece generated from the second term of (69) gives

$$\mathcal{A}(1^+,\ldots,n^+) = \frac{(-\sqrt2)^n}{48\pi^2}\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}\frac{u^\beta(h)k_{x\beta\dot\alpha}\bar k_a^{\dot\alpha\gamma}\kappa_{\gamma\dot\gamma}(x,a{-}1)\bar\kappa^{\dot\gamma\delta}(1,x{-}1)u_\delta(k_n)}{\langle n|1,\ldots,n|h\rangle}$$

$$+\frac{(-\sqrt2)^n}{48\pi^2}\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}\frac{u^\beta(h)k_{x\beta\dot\alpha}\bar k_a^{\dot\alpha\gamma}\kappa_{\gamma\dot\gamma}(x{+}1,a{-}1)\bar\kappa^{\dot\gamma\delta}(1,x)u_\delta(k_n)}{\langle n|1,\ldots,n|h\rangle} \tag{77}$$

for the color-ordered amplitude.

To execute the final stage of the reduction, it is sufficient to consider only the numerators of (77). Hence, we define

$$\mathcal{N} \equiv \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(x, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x-1) u_{\delta}(k_n)$$
$$+ \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(x+1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x) u_{\delta}(k_n). \quad (78)$$

Our first action is to write $\kappa(x, a-1) = \kappa(1, a-1) - \kappa(1, x-1)$ to obtain

$$\mathcal{N} = \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x-1) u_{\delta}(k_n)$$
$$- \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} \kappa^2(1, x-1)\, u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} u_{\gamma}(k_n)$$
$$+ \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(x+1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x) u_{\delta}(k_n). \quad (79)$$

Using the Schouten identity on the first term of (79) produces

$$\mathcal{N} = -\sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} \langle n\ h \rangle\, \bar{u}_{\dot{\alpha}}(k_x) \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x-1) u_{\delta}(k_x)$$
$$+ \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(k_n) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x-1) u_{\delta}(h)$$
$$- \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} \kappa^2(1, x-1)\, u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} u_{\gamma}(k_n)$$
$$+ \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(x+1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x) u_{\delta}(k_n). \quad (80)$$

Since the first term of (80) produces a gauge-invariant contribution to (77), we are led to suspect that the last three terms sum to zero. This is indeed the case, as we shall now demonstrate.

Let $\mathcal{Z}$ equal the last three terms of (80). We begin by extending the factor of $\kappa(x+1, a-1)$ appearing in the last term of $\mathcal{Z}$ to $\kappa(1, a-1)$ and compensating. The result is

$$\mathcal{Z} = \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(k_n) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x-1) u_{\delta}(h)$$
$$- \sum_{x=1}^{n-2} \kappa^2(1, x-1)\, u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{\kappa}^{\dot{\alpha}\gamma}(x+1, n) u_{\gamma}(k_n)$$
$$+ \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} \kappa_{\gamma\dot{\gamma}}(1, a-1) \bar{\kappa}^{\dot{\gamma}\delta}(1, x) u_{\delta}(k_n)$$
$$- \sum_{x=1}^{n-2} \sum_{a=x+1}^{n-1} \kappa^2(1, x)\, u^{\beta}(h) k_{x\beta\dot{\alpha}} \bar{k}_a^{\dot{\alpha}\gamma} u_{\gamma}(k_n). \quad (81)$$

We now interchange the order of the sum on x and the implicit κ-sum involving x that appears in the third term of (81), obtaining

$$\begin{aligned}\mathcal{Z} = &-\sum_{x=1}^{n-2}\left[\kappa^2(1,x-1)+\kappa^2(1,x)\right]u^\beta(h)k_{x\beta\dot\alpha}\bar\kappa^{\dot\alpha\gamma}(x+1,n)u_\gamma(k_n)\\ &-\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}u^\delta(h)\kappa_{\delta\dot\gamma}(1,x-1)\bar\kappa^{\dot\gamma\gamma}(1,a-1)k_{a\gamma\dot\alpha}\bar k^{x\dot\alpha\beta}u_\beta(k_n)\\ &+\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}u^\beta(h)\kappa_{\beta\dot\alpha}(x,a-1)\bar k_a^{\dot\alpha\gamma}\kappa_{\gamma\dot\gamma}(1,a-1)\bar k_x^{\dot\gamma\delta}u_\delta(k_n).\end{aligned} \tag{82}$$

In the second term of (82) we extend $\kappa(1,x-1)$ to $\kappa(1,a-1)$ and compensate:

$$\begin{aligned}\mathcal{Z} = &-\sum_{x=1}^{n-2}\left[\kappa^2(1,x-1)+\kappa^2(1,x)\right]u^\beta(h)k_{x\beta\dot\alpha}\bar\kappa^{\dot\alpha\gamma}(x+1,n)u_\gamma(k_n)\\ &-\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}\kappa^2(1,a-1)\,u^\beta(h)k_{a\beta\dot\alpha}\bar k_x^{\dot\alpha\gamma}u_\gamma(k_n)\\ &+\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}u^\delta(h)\kappa_{\delta\dot\gamma}(x,a-1)\bar\kappa^{\dot\gamma\gamma}(1,a-1)k_{a\gamma\dot\alpha}\bar k_x^{\dot\alpha\beta}u_\beta(k_n)\\ &+\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}u^\beta(h)\kappa_{\beta\dot\alpha}(x,a-1)\bar k_a^{\dot\alpha\gamma}\kappa_{\gamma\dot\gamma}(1,a-1)\bar k_x^{\dot\gamma\delta}u_\delta(k_n).\end{aligned} \tag{83}$$

When the sum over x is performed in the second term of (83), it is seen that it partially cancels the first term. The last two terms may be combined by noting that

$$\begin{aligned}\bar\kappa^{\dot\gamma\gamma}(1,a-1)k_{a\gamma\dot\alpha}+\bar k_a^{\dot\gamma\gamma}\kappa_{\gamma\dot\alpha}(1,a-1) &= 2k_a\cdot\kappa(1,a-1)\,\delta_{\dot\alpha}^{\dot\gamma}\\ &= \left[\kappa^2(1,a)-\kappa^2(1,a-1)\right]\delta_{\dot\alpha}^{\dot\gamma}.\end{aligned} \tag{84}$$

Thus

$$\begin{aligned}\mathcal{Z} = &-\sum_{x=1}^{n-2}\kappa^2(1,x)\,u^\beta(h)k_{x\beta\dot\alpha}\bar\kappa^{\dot\alpha\gamma}(x+1,n)u_\gamma(k_n)\\ &+\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}\left[\kappa^2(1,a)-\kappa^2(1,a-1)\right]u^\delta(h)\kappa_{\delta\dot\gamma}(x,a-1)\bar k_x^{\dot\gamma\gamma}u_\gamma(k_n).\end{aligned} \tag{85}$$

The two terms in the double sum appearing in (85) may be combined by shifting the sum over a by 1 unit in one of the two pieces, yielding

$$\begin{aligned}\mathcal{Z} = &-\sum_{x=1}^{n-2}\kappa^2(1,x)\,u^\beta(h)k_{x\beta\dot\alpha}\bar\kappa^{\dot\alpha\gamma}(x+1,n)u_\gamma(k_n)\\ &-\sum_{x=1}^{n-2}\sum_{a=x+1}^{n-1}\kappa^2(1,a)\,u^\delta(h)k_{a\delta\dot\gamma}\bar k_x^{\dot\gamma\gamma}u_\gamma(k_n).\end{aligned} \tag{86}$$

When we perform the sum over x in the second term of (86), we see that it exactly cancels the first term, giving $\mathcal{Z} = 0$, as promised.

Thus, the entire amplitude is generated from the first term of (80), with the result

$$\widehat{\mathcal{A}}(1^+,\ldots,n^+) = \frac{i}{48\pi^2}(-g\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} tr\,\{\Omega[1,n]\} \sum_{x=2}^{n-2}\sum_{y=x+1}^{n-1} \frac{\bar{k}_x^{\dot{\alpha}\alpha}\kappa_{\alpha\dot{\beta}}(1,x)\bar{\kappa}^{\dot{\beta}\beta}(1,y)k_{y\beta\dot{\alpha}}}{\langle 1|2,\ldots,n|1\rangle}. \tag{87}$$

As a check of (87), we note that when we set the color factor equal to unity and (numerically) perform the permutation sum, we recover the n-photon result reported in Ref. [1].

Recently, Bern, Dixon, and Kosower [14] published a conjecture for a particular color-ordered subamplitude for the scattering of n like-helicity gluons via a *gluon* loop. It is possible to connect the result derived here for diagrams containing a quark loop to their result by use of supersymmetry identities [17]. The bottom line is that the result (87) should agree with the conjecture of Ref. [14] up to a trivial multiplicative factor. We have verified that the two expressions do indeed agree. Thus, our calculation may be taken as the proof of this conjecture.

Since we know the expression for the gluon current appearing in (61) when the gluon labeled "n" carries negative helicity (see Appendix), we may repeat the calculation to obtain $\widehat{\mathcal{A}}(1^+,\ldots,(n-1)^+,n^-)$. After a somewhat lengthy calculation we obtain

$$\widehat{\mathcal{A}}(1^+,\ldots,(n-1)^+,n^-) = -\frac{i}{48\pi^2}(-g\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{tr\,\{\Omega[1,n]\}}{\langle 1|2,\ldots,n|1\rangle}\,\Xi(1^+,\ldots,(n-1)^+,n^-) \tag{88}$$

where

$$\begin{aligned}
&\Xi(1^+,\ldots,(n-1)^+,n^-) = \\
&\quad\sum_{x=2}^{n-2} u^\gamma(k_n)\kappa_{\gamma\dot{\gamma}}(x,n)\bar{k}_x^{\dot{\gamma}\delta}u_\delta(k_n)[k_n+\kappa(1,x)]^2 \\
&\qquad\times\left\{\frac{\langle n-1\ 1\rangle^*}{\langle n-1|n|1\rangle^*} + \sum_{s=2}^{x} u^\lambda(k_n)\Pi_\lambda{}^\tau(n,1,\ldots,s)u_\tau(k_n)\right\} \\
&\quad+\sum_{x=3}^{n-2} u^\gamma(k_n)\kappa_{\gamma\dot{\gamma}}(x,n)\bar{k}_x^{\dot{\gamma}\delta}u_\delta(k_n) \\
&\qquad\times\left\{\frac{\bar{u}_{\dot{\alpha}}(k_1)}{\langle n\ 1\rangle^*}[\bar{k}_n+\bar{\kappa}(1,x-1)]^{\dot{\alpha}\beta}u_\beta(k_n) + [k_n+\kappa(1,x-1)]^2\sum_{s=2}^{x-1} u^\lambda(k_n)\Pi_\lambda{}^\tau(n,1,\ldots,s)u_\tau(k_n)\right\}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{x=2}^{n-2} u^{\gamma}(k_n)\kappa_{\gamma\dot\gamma}(x,n)\bar{k}_x^{\dot\gamma\delta}u_\delta(k_n)\ \kappa^2(x,n) \\
&\quad\times\left\{\frac{\langle 1\ n{-}1\rangle^*}{\langle 1|n|n{-}1\rangle^*}+\sum_{r=x}^{n-2}u^{\lambda}(k_n)\Pi_{\lambda}{}^{\tau}(n,n{-}1,\ldots,r)u_\tau(k_n)\right\} \\
&-\sum_{x=2}^{n-3} u^{\gamma}(k_n)\kappa_{\gamma\dot\gamma}(x,n)\bar{k}_x^{\dot\gamma\delta}u_\delta(k_n) \\
&\quad\times\left\{\frac{\bar{u}_{\dot\alpha}(k_{n-1})}{\langle n\ n{-}1\rangle^*}\bar\kappa^{\dot\alpha\beta}(x{+}1,n)u_\beta(k_n)+\kappa^2(x{+}1,n)\sum_{r=x+1}^{n-2}u^{\lambda}(k_n)\Pi_{\lambda}{}^{\tau}(n,n{-}1,\ldots,r)u_\tau(k_n)\right\} \\
&-\sum_{x=2}^{n-3}\sum_{r=x+2}^{n-1}\sum_{s=1}^{x-1}u^{\gamma}(k_n)\kappa_{\gamma\dot\gamma}(x,r{-}1)\bar{k}_x^{\dot\gamma\delta}u_\delta(k_n)\frac{1}{\langle r\ s\rangle}u^{\lambda}(k_n)\Pi\!\Pi_{\lambda}{}^{\tau}(r,\ldots,n,1,\ldots,s)u_\tau(k_n) \\
&\quad\times u^{\beta}(k_r)\Big[\kappa_{\beta\dot\alpha}(x,r{-}1)\bar\kappa^{\dot\alpha\alpha}(s{+}1,x{-}1)+\kappa_{\beta\dot\alpha}(x{+}1,r{-}1)\bar\kappa^{\dot\alpha\alpha}(s{+}1,x)\Big]u_\alpha(k_s) \\
&-\sum_{x=2}^{n-3}\sum_{a=x+1}^{n-2}\sum_{r=a+1}^{n-1}\sum_{s=1}^{a-1}\frac{1}{\langle r\ s\rangle}u^{\lambda}(k_n)\Pi\!\Pi_{\lambda}{}^{\tau}(r,\ldots,n,1,\ldots,s)u_\tau(k_n) \\
&\quad\times u^{\delta}(k_n)k_{x\delta\dot\gamma}\bar{k}_a^{\dot\gamma\gamma}\Big[\kappa_{\gamma\dot\beta}(x,a)\bar\kappa^{\dot\beta\alpha}(s{+}1,x{-}1)+\kappa_{\gamma\dot\beta}(x{+}1,a)\bar\kappa^{\dot\beta\alpha}(s{+}1,x)\Big]u_\alpha(k_s)
\end{aligned}
\tag{89}
$$

and the function $\Pi\!\Pi$ (read "double pi") is a particular combination of Π's defined in the Appendix. Not all of the terms of (89) contribute when $n = 4$ or 5.

Note that $\Pi(n,1,\ldots,n{-}2)$ appears in the first term of (89). This function contains $[k_n+\kappa(1,n{-}2)]^{-2} = k_{n-1}^{-2}$, which is singular for an on-shell gluon. However, there is a factor of k_{n-1}^2 available in the prefactor to cancel this singularity. The difficulty with $\Pi(n,n{-}1,\ldots,2)$ in the third term is resolved in the same manner.

Eq. (89) agrees with the Bern and Kosower results for $n = 4$ and $n = 5$ [13]. This is the first calculation including the cases $n \geq 6$. Additional checks are possible, however, by comparing to the previously obtained results containing photons. Since the $\gamma g \to gg\ldots g$ scattering amplitude may be generated from an all-gluon amplitude simply by replacing a color matrix by the identity, we should have the relation

$$
\sum_{C(1\ldots n-1)} \mathcal{A}(1,\ldots,n{-}1,n_\gamma) = \sum_{C(1\ldots n-1)} \mathcal{A}(1,\ldots,n{-}1,n) \tag{90}
$$

connecting the two different color-ordered amplitudes, independent of the helicities of the gauge bosons. In the like-helicity case, (90) is obviously satisfied since the color-ordered amplitudes themselves are identical [*cf.* Eqs. (51) and (87)]. In the case of a negative helicity gluon becoming a negative helicity photon however [Eq. (89) versus Eq. (51)], the sum indicated in (90) must actually be performed to see the equality. We have verified that the required agreement is indeed present. One further check is provided by replacing all of the gluons by photons. Again, when the appropriate additional sums are performed, we find agreement with the results reported in Ref. [1].

IV. CONCLUSIONS

In this paper we have applied the double-off-shell quark current containing n like-helicity gluons to the problem of one loop QCD corrections. We have found that it is a relatively simple matter to obtain the corrections to those amplitudes that vanish at tree-level. This is not surprising, since such amplitudes should have a relatively simple form, with no cuts in the complex plane. We have obtained compact expressions for the helicity amplitudes for photon-gluon scattering, electron-positron annihilation to gluons, and gluon-gluon scattering via a quark loop in the case of like-helicity gluons. In addition, the gluon-gluon amplitude with a single gluon of opposite helicity in addition to an arbitrary number of like-helicity gluons has been computed, albeit in a somewhat more complicated form.

Of course, any realistic cross-section computation involves a complete set of helicity amplitudes, since the gluons are never observed directly in the final state. In principle, it is a straightforward matter to obtain these amplitudes from the recursion relations, although a significant amount of computational labor is required. A particularly important question to investigate is the mechanism for explicitly canceling the infrared divergences at loop level against the appropriate tree diagrams. This is an absolute necessity if one is to develop numerical methods for evaluating complete cross-sections.

A second issue which should be examined is the validity of using the massless limit for the quarks. Although this limit seems reasonable for the light flavors, the large mass of the t-quark may translate into significant (if not dominant) mass effects. Thus, an efficient means of handling massive fermions, preferably within a recursive framework, should be sought.

In spite of these loose ends, much progress has been made in the evaluation of one-loop QCD processes. Amplitudes that are unapproachable from a direct attack starting with only the Feynman rules for the theory have been obtained, and in a far more compact form than would be expected given the myriad of Feynman diagrams involved. There are additional lessons to be learned from continued investigation utilizing the powerful combination of the spinor representation, color factorization, and recursion relations.

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APPENDIX: CURRENTS WITH ONE OFF SHELL PARTICLE

For convenience, we record here some of the solutions to the Berends and Giele recursion relations for currents with one off-shell particle [11].

For the n-gluon current, we have the color factorization

$$\widehat{\mathcal{J}}^x(1,\ldots,n) = 2g^{n-1} \sum_{\mathcal{P}(1\ldots n)} tr(\Omega[1,n]T^x)\mathcal{J}(1,\ldots,n), \tag{A1}$$

valid for an arbitrary helicity configuration. Note that there is no zero-gluon current.

We now turn to specific color-ordered currents relevant to the amplitudes computed here. For the case of n like-helicity gluons, Berends and Giele find [11]

$$\mathcal{J}_{\alpha\dot{\alpha}}(1^+,\ldots,n^+) = (-\sqrt{2})^{n-1}\frac{u_\alpha(h)u^\beta(h)\kappa_{\beta\dot{\alpha}}(1,n)}{\langle h|1,\ldots,n|h\rangle}, \tag{A2}$$

where we use the same gauge momentum h for all of the gluons.

We also require color-ordered gluon currents containing a single gluon of opposite helicity. In this case the form of the expression depends on the location of the opposite helicity gluon in the argument list of $\mathcal{J}$. For a lone negative helicity gluon we write:

$$\mathcal{J}_{\alpha\dot{\alpha}}(n^-) = \frac{u_\alpha(k_n)\bar{u}_{\dot{\alpha}}(h)}{\langle n\ h\rangle^*}, \tag{A3}$$

that is, its gauge momentum is h. All of the positive helicity gluons use the momentum k_n of the negative helicity gluon as their gauge momentum. If the negative helicity gluon is first in the list, we have

$$\mathcal{J}_{\alpha\dot{\alpha}}(n^-, 1^+, \ldots, z^+) = (-\sqrt{2})^z \frac{u_\alpha(k_n)u^\beta(k_n)[k_n + \kappa(1,z)]_{\beta\dot{\alpha}}}{\langle n|1,\ldots,z|n\rangle} \sum_{s=1}^{z} u^\gamma(k_n)\Pi_\gamma{}^\delta(n,1,\ldots,s)u_\delta(k_n), \tag{A4}$$

while if the negative helicity gluon is last we find

$$\begin{aligned}\mathcal{J}_{\alpha\dot{\alpha}}((y{+}1)^+, \ldots, (n{-}1)^+, n^-) = -(-\sqrt{2})^{n-y-1} &\frac{u_\alpha(k_n)u^\beta(k_n)\kappa_{\beta\dot{\alpha}}(y{+}1,n)}{\langle n|y{+}1,\ldots,n{-}1|n\rangle} \\ &\times \sum_{r=y+1}^{n-1} u^\gamma(k_n)\Pi_\gamma{}^\delta(n,n{-}1,\ldots,r)u_\delta(k_n).\end{aligned} \tag{A5}$$

The function Π appearing here is defined exactly as in (16) with one exception. When precisely two massless momenta appear as arguments of Π, this function is singular. For this situation, we define

$$u^\alpha(k_n)\Pi_\alpha{}^\beta(n,j)u_\beta(k_n) \equiv \frac{\langle h\ j\rangle^*}{\langle h|n|j\rangle^*}, \qquad k_j^2 = k_n^2 = 0. \tag{A6}$$

This special definition does not apply if $k_n^2 \neq 0$, as Π is a well-defined quantity in that case.

Finally, we present the expression for the case of the negative helicity gluon appearing somewhere in the middle of the argument list:

$$\begin{aligned}&\mathcal{J}_{\alpha\dot{\alpha}}((y{+}1)^+, \ldots, (n{-}1)^+, n^-, 1^+, \ldots, z^+) = \\ &\qquad -(-\sqrt{2})^{n-y+z-1} \frac{u_\alpha(k_n)u^\beta(k_n)[\kappa(y{+}1,n) + \kappa(1,z)]_{\beta\dot{\alpha}}}{\langle n|y{+}1,\ldots,n{-}1|n\rangle\langle n|1,\ldots,z|n\rangle} \\ &\qquad\qquad \times \sum_{r=y+1}^{n-1}\sum_{s=1}^{z} \frac{\langle r|n|s\rangle}{\langle r\ s\rangle} u^\gamma(k_n)\Pi\Pi_\gamma{}^\delta(r,\ldots,n{-}1,n,1,\ldots,s)u_\delta(k_n)\end{aligned} \tag{A7}$$

where we have defined

$$
\begin{aligned}
I\!\!\Pi(r,\ldots,n{-}1,n,1,\ldots,s) \equiv\; & \Pi(r,\ldots,n{-}1,n,1,\ldots,s) \\
& -\Pi(r{+}1,\ldots,n{-}1,n,1,\ldots,s) \\
& -\Pi(s,s{-}1,\ldots,1,n,n{-}1,\ldots,r) \\
& +\Pi(s{-}1,s{-}2,\ldots,1,n,n{-}1,\ldots,r).
\end{aligned} \tag{A8}
$$

The version of Eq. (A4) with $h = k_1$ was presented by Berends and Giele in Ref. [11]. The remaining color-ordered currents containing a single opposite-helicity gluon closely resemble the results obtained for the modified gluon current in Ref. [9]. This is not surprising, since one way to obtain this current is to begin with a double-off-shell gluon current with all like helicities. One may then put one of the two off-shell gluons on shell, assigning it negative helicity. Indeed, the inductive proof that (A7) satisfies the Berends and Giele recursion relation is virtually identical to the proof given in Ref. [9], the differences lying entirely within the terms covered by (A6). This proof is easily adapted to handle these terms.

It should be noted that even though $I\!\!\Pi(n{-}1,1,n)$ is apparently dependent upon the gauge momentum h, it actually contains

$$
\begin{aligned}
u^\gamma(k_n)\Pi_\gamma{}^\delta(n,1)u_\delta(k_n) - u^\gamma(k_n)\Pi_\gamma{}^\delta(n,n{-}1)u_\delta(k_n) &= \frac{\langle h\ 1\rangle^*}{\langle h|n|1\rangle^*} - \frac{\langle h\ n{-}1\rangle^*}{\langle h|n|n{-}1\rangle^*} \\
&= \frac{\langle n{-}1\ 1\rangle^*}{\langle n{-}1|n|1\rangle^*},
\end{aligned} \tag{A9}
$$

which is independent of h. Consequently, the expressions given in (89) and (A7) are also independent of h.

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