Fermi National Accelerator Laboratory

Fermilab–Pub–93/327-T

One Loop Multiphoton Helicity Amplitudes

Gregory Mahlon*

Fermi National Accelerator Laboratory

P.O. Box 500

Batavia, IL 60510

(November 1993)

Abstract

We use the solutions to the recursion relations for double-off-shell fermion currents to compute helicity amplitudes for n-photon scattering and electron-positron annihilation to photons in the massless limit of QED. The form of these solutions is simple enough to allow *all* of the integrations to be performed explicitly. For n-photon scattering, we find that unless $n = 4$, the amplitudes for the helicity configurations $(+++\cdots+)$ and $(-++\cdots+)$ vanish to one-loop order.

11.80.Cr, 12.20.Ds

I. INTRODUCTION

Much progress has been made during the last decade or so in the evaluation of tree-level processes containing multiple gauge bosons, both in QCD and the high energy limit of the Weinberg-Salam-Glashow model. Three major techniques have aided this progress. First, the multispinor representation of a gauge field [1–6] allows us to treat the gauge bosons in a theory on equal footing with the fermions by replacing the single Lorentz index on the polarization vectors with a pair of spinor indices. With the proper choice of basis, the gauge boson polarizations factorize on these indices, allowing long strings of Dirac matrices to be written as several short ones. Second, the color factorization of QCD amplitudes [7] helps to organize the many terms in a calculation into gauge-invariant sub-groups, each proportional to a different color structure. A final ingredient is the introduction of currents and the recursion relations that they satisfy [8–11]. The currents are defined to be the sum of all tree graphs containing exactly n gauge bosons (and possibly a single scalar or spinor line) with one off-shell particle. These currents satisfy relatively simple recursion relations, allowing expressions involving many gauge bosons to be built from those involving fewer. Tree-level amplitudes may be obtained from the currents either by putting the off-shell particle on shell, or by combining two or more currents using the vertices of the theory in question. Of note is the existence of explicit closed-form solutions for these currents for certain special helicity configurations.

Recently, recursion relations for currents with two off-shell particles have been obtained [10,12,13]. With the presence of a second off-shell particle, the possibility of forming a one-loop amplitude from tree-level currents exists. The purpose of this paper is to pursue that idea in the case of massless QED. Application of this method to QCD will be discussed elsewhere [14].

The processes we will consider are

$$\gamma\gamma \to \gamma\gamma\ldots\gamma \qquad (1)$$

and

$$e^+e^- \to \gamma\gamma\ldots\gamma. \tag{2}$$

The Feynman diagrams describing both (1) and (2) may be built from a double-off-shell fermion current: that is, a fermion line which radiates n photons all possible ways and has both ends off shell. We have been able to solve the recursion relation for this current in the case of like-helicity photons. As a result, we can compute the one-loop amplitudes for (1) involving the helicity configurations $(++ + \cdots ++)$ and $(-++\cdots ++)$. For (2) we are able to obtain amplitudes for either helicity of the fermion line, with n like-helicity photons.

The processes listed above are especially well-suited to be starting points in an investigation of how to extend the use of recursive calculations from tree level to one-loop processes. In the indicated helicity configurations, both processes vanish at tree level. As a consequence, both processes must be ultraviolet and infrared finite at the one-loop level. Furthermore, it follows from the vanishing of (2) at tree level for like-helicity photons, plus the Cutkosky rules, that these particular one-loop diagrams do not possess any cuts. Thus, the results for these diagrams should be relatively simple. One might hope that the steps required to reach this simple endpoint could be made similarly simple, in contrast to the intermediate steps involved in a conventional calculation which contain logarithmic terms that ultimately cancel. Indeed, we shall see that this is the case.

Our discussion is organized as follows. In Sec. II, we review the double-off-shell fermion currents of Ref. [12]. These currents form the basis of the loop amplitudes presented in this paper. Of special note is the solution for the current containing n like-helicity photons. We examine the one-loop n-photon scattering amplitude in Sec. III. Because of the favorable form of the double-off-shell fermion current appearing in this amplitude, we are able to evaluate the integrals exactly for arbitrary n. We find the surprising result that *all* of the photon-photon scattering amplitudes vanish for $n \geq 5$ in the two helicity configurations we are able to evaluate. The one-loop contribution to (2) is the topic of Sec. IV. Once again, we find that the relatively simple form of the double-off-shell fermion current allows us to perform all of the integrations exactly, producing a fairly compact result for arbitrary n. We conclude with a few closing remarks in Sec. V.

II. THE DOUBLE-OFF-SHELL FERMION CURRENTS

In this section we will review the recursion relations and solutions for the double-off-shell fermion currents presented in Ref. [12]. A summary of our conventions and notation for the Weyl–van der Waerden spinors used in this discussion may be found in Ref. [11].

We define the n-particle double off-shell fermion current to consist of the sum of all tree graphs containing exactly n photons attached to a single fermion line all possible ways. Both ends of the fermion line are off shell. All momenta are directed inward. We will denote the momenta of the photons by $k_1, k_2, \ldots, k_n$. The off-shell positron has momentum $\mathcal{P}$, while the momentum of the off-shell electron is $\mathcal{Q}$. Momentum conservation relates the momenta via

$$\mathcal{P} + k_1 + k_2 + \cdots + k_n + \mathcal{Q} \equiv \mathcal{P} + \kappa(1,n) + \mathcal{Q} = 0 \tag{3}$$

Since the helicity of the fermion is conserved in the massless limit, we have two different (but related) double off-shell fermion currents. Let us denote the left-handed current by $\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n; \mathcal{Q})$ and the right-handed current by $\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{P}; 1, \ldots, n; \mathcal{Q})$. Because of (3), the argument lists are overspecified. When convenient, we will suppress either $\mathcal{P}$ or $\mathcal{Q}$. Because of the way we have defined these currents, the order of the photon arguments is irrelevant.

In Ref. [12] we find that the left-handed double-off-shell fermion current satisfies the following recursion relation:

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; 1, \ldots, n) = -e\sqrt{2} \sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!} \Psi_{\alpha\dot{\beta}}(\mathcal{P}; 1, \ldots, n-1) \bar{\epsilon}^{\dot{\beta}\beta}(n) \frac{[\mathcal{P}+\kappa(1,n)]_{\beta\dot{\alpha}}}{[\mathcal{P}+\kappa(1,n)]^2}. \tag{4}$$

The notation $\mathcal{P}(1 \cdots n)$ indicates a sum over all permutations of the n photons. The zero-photon current is just a propagator for the fermion:

$$\Psi_{\alpha\dot{\alpha}}(\mathcal{P}; \mathcal{Q}) = \frac{-i\mathcal{P}_{\alpha\dot{\alpha}}}{\mathcal{P}^2} = \frac{i\mathcal{Q}_{\alpha\dot{\alpha}}}{\mathcal{Q}^2} \tag{5}$$

(This is one expression in which it is important to include both $\mathcal{P}$ and $\mathcal{Q}$ in the argument list to avoid ambiguity.)

The right-handed current satisfies the recursion relation

$$\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{P};1,\ldots,n) = -e\sqrt{2}\sum_{\mathcal{P}(1\ldots n)} \frac{1}{(n-1)!}\bar{\Psi}^{\dot{\alpha}\beta}(\mathcal{P};1,\ldots,n-1)\epsilon_{\beta\dot{\beta}}(n)\frac{[\bar{\mathcal{P}}+\bar{\kappa}(1,n)]^{\dot{\beta}\alpha}}{[\mathcal{P}+\kappa(1,n)]^2}. \tag{6}$$

It is closely related to the left-handed current via the crossing relation

$$\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{P};1,\ldots,n;\mathcal{Q}) = (-1)^{n+1}\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\Psi_{\beta\dot{\beta}}(\mathcal{Q};1,\ldots,n;\mathcal{P}). \tag{7}$$

In Ref. [12], we were able to obtain a solution for the left-handed current in the case where all of the photons have the same helicity. In this situation, the photon polarizations may be written as [8]

$$\epsilon_{\alpha\dot{\alpha}}(j^+) = \frac{u_\alpha(g)\bar{u}_{\dot{\alpha}}(k_j)}{\langle k\ g\rangle} \tag{8}$$

for the jth photon. In Eq. (8) g is an arbitrary null momentum, partially reflecting the gauge freedom associated with QED. The choice of g does not affect any physics result. In general, a different value of g may be chosen for each photon; however, the choice of Eq. (8) leads to the useful property

$$\bar{\epsilon}^{\dot{\alpha}\alpha}(j^+)\epsilon_{\alpha\dot{\alpha}}(\ell^+) = 0 \tag{9}$$

for any pair of positive helicity polarization spinors.

The solution to the recursion relation given in Eq. (4) using the gauge choice (8) was found in Ref. [12] to be

$$u^\alpha(g)\Psi_{\alpha\dot{\alpha}}(\mathcal{P};1^+,\ldots,n^+) = \\ -i(-e\sqrt{2})^n\sum_{\mathcal{P}(1\ldots n)}\frac{u^\alpha(g)[\mathcal{P}+\kappa(1,n)]_{\alpha\dot{\alpha}}}{\langle g|1,\ldots,n|g\rangle}\sum_{j=1}^{n}u^\beta(g)\Pi_\beta{}^\gamma(\mathcal{P},1,\ldots,j)u_\gamma(g), \tag{10}$$

where

$$\Pi_\beta{}^\gamma(\mathcal{P},1,\ldots,j) \equiv \frac{k_{j\beta\dot{\beta}}[\bar{\mathcal{P}}+\bar{\kappa}(1,j-1)]^{\dot{\beta}\gamma}}{[\mathcal{P}+\kappa(1,j-1)]^2[\mathcal{P}+\kappa(1,j)]^2}. \tag{11}$$

Note that the zero-photon current (5) does not fit into the form (10). Also, although the combination $u^\alpha(g)\Psi_{\alpha\dot\alpha}$ takes a convenient form, we have not been able to find a similarly compact expression for $\Psi_{\alpha\dot\alpha}$ itself. Usually, this is not a serious problem.

From the left-handed current given in (10) and the crossing relation (7), it is not hard to see that

$$\bar\Psi^{\dot\alpha\alpha}(\mathcal{P};1^+,\ldots,n^+)u_\alpha(g) = \\ -i(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n)} \frac{\bar{\mathcal{P}}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n|g\rangle} \sum_{j=1}^{n} u^\beta(g)\Pi_\beta{}^\gamma(\mathcal{P},1,\ldots,j)u_\gamma(g), \tag{12}$$

III. PHOTON-PHOTON SCATTERING

We begin the application of the double-off-shell current to loop processes with a discussion of photon-photon scattering. Photon-photon interactions have long been of theoretical interest, the first complete computation for four photons being performed by Karplus and Neuman [15]. Helicity-projected amplitudes, including finite mass effects for the electron, were first obtained by Costantini, De Tollis, and Pistoni [16]. Gastmans and Wu [4] use this process to illustrate the use of spinor helicity methods at loop level, presenting the limit where the electron mass may be ignored. In this section, we will not only reproduce the massless limit results, but we will present amplitudes involving more than four photons, albeit for a restricted set of helicity configurations.

A. Preliminaries

Figure 1 illustrates how to utilize the double-off-shell fermion current in the amplitude for the self-interaction of n photons. If we sum over all permutations of the n photons, we overcount by a factor of n (the n cyclic permutations of the photons do not produce distinct Feynman diagrams, as may be seen by shifting the loop momentum). For our purposes, it is convenient circumvent this problem by excluding the nth photon from the permutation sum, rather than including it and dividing by n. Applying the QED Feynman rules to Fig. 1 we obtain

$$\mathcal{A}(1,\ldots,n) = -\int \frac{d^4\mathcal{P}}{(2\pi)^4}\,\Lambda(1,\ldots,n) \tag{13}$$

where

$$\Lambda(1,\ldots,n) = -ie\ \mathrm{Tr}\left[\,\not{\epsilon}(n)\Psi(\mathcal{P};1,\ldots,n{-}1)\right]. \tag{14}$$

In order to write (14) in spinor form, we break the trace into left- and right-handed contributions by inserting $1 = \frac{1}{2}(1+\gamma_5) + \frac{1}{2}(1-\gamma_5)$. Thus, we obtain

$$\Lambda(1,\ldots,n) = -ie\sqrt{2}\,[\bar{\epsilon}^{\dot{\alpha}\alpha}(n)\Psi_{\alpha\dot{\alpha}}(\mathcal{P};1,\ldots,n{-}1) + \epsilon_{\alpha\dot{\alpha}}(n)\bar{\Psi}^{\dot{\alpha}\alpha}(\mathcal{P};1,\ldots,n{-}1)]. \tag{15}$$

Equation (15) is valid for any combination of photon helicities. We need only know the expressions for the currents appearing on the right hand side. We now focus on the two amplitudes that our knowledge of $\Psi(\mathcal{P};1^+,\ldots,n^+)$ allows us to obtain. Our first option is to use Eq. (8) for all of the photons and so compute $\mathcal{A}(1^+,\ldots,n^+)$. Note that this gauge choice leaves g as an arbitrary parameter which must cancel in the final result. Our other option is to write

$$\epsilon_{\alpha\dot{\alpha}}(n^-) = \frac{u_\alpha(k_n)\bar{u}_{\dot{\alpha}}(h)}{\langle n\ h\rangle^*} \tag{16}$$

for the nth photon, and set $g = k_n$ in the other polarizations [8]. This gives us access to $\mathcal{A}(1^+,\ldots,(n{-}1)^+,n^-)$. The indicated gauge choice has the virtue of satisfying not only (9), but also

$$\bar{\epsilon}^{\dot{\alpha}\alpha}(j^+)\epsilon_{\alpha\dot{\alpha}}(n^-) = 0. \tag{17}$$

The arbitrary null vector h should not appear in the final result. We will concentrate our discussion on $\mathcal{A}(1^+,\ldots,n^+)$. The computation of $\mathcal{A}(1^+,\ldots,(n{-}1)^+,n^-)$ proceeds in essentially the same manner.

B. The momentum integral

It would seem that all we have to do at this stage is to insert the expressions for the polarizations and the currents and perform the momentum integration. However, this is not quite correct. A naïve application of the solutions to the recursion relations to (15) yields

$$\Lambda(1^+,\ldots,n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{2\bar{u}_{\dot\alpha}(k_n)\bar{\mathcal{P}}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \sum_{j=1}^{n-1} \frac{u^\beta(g)k_{j\beta\dot\beta}[\bar{\mathcal{P}}+\bar\kappa(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{[\mathcal{P}+\kappa(1,j-1)]^2[\mathcal{P}+\kappa(1,j)]^2}, \tag{18}$$

the factor of 2 resulting from equal contributions by the left- and right-handed fermion lines. If this expression were inserted into (13), power counting would indicate a superficial quadratic divergence. Actually, the divergence is only logarithmic, as the coefficient of the highest power of $\mathcal{P}$ is proportional to $\langle g\ g\rangle$, which vanishes. Since it is well-known that the amplitude for light-by-light scattering is finite, any divergences apparently present were introduced in the intermediate steps. In order to cancel these divergences and obtain the correct finite result, we must first examine their origin. The process of reducing the double-off-shell current into the form given in (10) involves repeated use of identities such as

$$k_j = [\mathcal{P} + \kappa(1,j)] - [\mathcal{P} + \kappa(1,j-1)] \tag{19}$$

to convert the expression inside the permutation sum from one with a single term and n propagators into one with n terms and two propagators. The two terms on the right hand side of (19) each contain one more power of $\mathcal{P}$ than the one on the right hand side. If the integral corresponding to the left hand side of (19) is finite, then the combination of the two integrals on the right is also finite. However, it is possible that both terms on the right diverge when considered separately, a finite quantity being obtained only from their combination. In that case, to obtain the correct result from the right hand side, we must treat the two pieces in an identical manner. In particular, shifts of the integration momentum in one term relative to the other term are forbidden. Because of the way the "fragments" from (19) recombine to form the final result (10) [see the discussion following

Eq. (29)], a straightforward integration of (18) involves such forbidden shifts. Thus, simply regulating the integral obtained from (18) does not give the correct result.

Instead, we must turn to the recursion relation to let us "back-up" one stage in the reduction. That is, we use (4) and (6) to write the integrand (15) in terms of the $(n-2)$–photon current instead of the $(n-1)$–photon current. The result of this procedure is

$$\Lambda(1^+,\ldots,n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{2\bar{u}_{\dot\alpha}(k_n)\bar{\mathcal{P}}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-2|g\rangle\langle n\ g\rangle} \frac{\bar{u}_{\dot\delta}(k_{n-1})[\bar{\mathcal{P}}+\bar{\kappa}(1,n-1)]^{\dot\delta\delta}u_\delta(g)}{\langle n-1\ g\rangle[\mathcal{P}+\kappa(1,n-1)]^2} \times \sum_{j=1}^{n-2} \frac{u^\beta(g)k_{j\beta\dot\beta}[\bar{\mathcal{P}}+\bar{\kappa}(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{[\mathcal{P}+\kappa(1,j-1)]^2[\mathcal{P}+\kappa(1,j)]^2}. \tag{20}$$

Power counting of Eq. (20) implies a possible linear divergence; however, the coefficients of both the linearly and logarithmically divergent pieces are proportional to $\langle g\ g\rangle$ and thus vanish. The expression is actually convergent.

Since (20) converges, we may impose a regulator on it, and then carry out the reduction to two denominators on the regulated expression. We should like to employ dimensional regularization, and so must consider how to continue (20) to d dimensions. Fortunately, it is sufficient to extend only the internal momentum $\mathcal{P}$, leaving the external momenta and polarization vectors in 4 dimensions [17]. It is not possible to translate an object like $\bar{\mathcal{P}}^{\dot\alpha\alpha}$ into d dimensions, as it corresponds to $\frac{1}{2}(1+\gamma_5)\not{\mathcal{P}}\frac{1}{2}(1-\gamma_5)$. However, note that every occurrence of $\bar{\mathcal{P}}^{\dot\alpha\alpha}$ in (20) may be rewritten as a Lorentz dot product with a polarization, forming $\mathcal{P}\cdot\epsilon(\ell^+)$ with various values of ℓ. We may extend this form to d dimensions.

To facilitate a quick return to multispinor notation, we decompose the n-dimensional vector $\mathcal{P}$ into a 4-dimensional piece P and a $(n-4)$-dimensional piece m [18],

$$\mathcal{P} = P + m. \tag{21}$$

Only the usual 4 space-time components of P are non-zero, while only the "extra" components of m are non-zero. Hence, m dotted into any 4 dimensional vector vanishes. We set $m^2 \equiv -\mu^2$, and adopt μ as the radial integration variable in the $(n-4)$-dimensional subspace. The integration measure appearing in (13) is replaced by

$$\int \frac{d^d\mathcal{P}}{(2\pi)^d} = \int \frac{d^4 P}{(2\pi)^4} \frac{-\epsilon(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^\infty d\mu^2\ (\mu^2)^{-1-\epsilon}. \tag{22}$$

Since every occurrence of $\mathcal{P}$ in the numerator of (20) is as $\mathcal{P}\cdot\epsilon(\ell^+)$, the continuation to d dimensions simply implies

$$\mathcal{P}\cdot\epsilon(\ell^+) \to P\cdot\epsilon(\ell^+). \tag{23}$$

Furthermore, the denominators simply pick up an extra term, $-\mu^2$. Thus, we obtain

$$\begin{aligned}\Lambda(1^+,\dots,n^+) = (-e\sqrt{2})^n \sum_{\mathcal{P}(1\dots n-1)} \frac{2\bar{u}_{\dot\alpha}(k_n)\bar{P}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\dots,n-2|g\rangle\langle n\ g\rangle} \frac{\bar{u}_{\dot\delta}(k_{n-1})[\bar{P}+\bar{\kappa}(1,n-1)]^{\dot\delta\delta}u_\delta(g)}{\langle n-1\ g\rangle\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ \times \sum_{j=1}^{n-2} \frac{u^\beta(g)k_{j\beta\dot\beta}[\bar{P}+\bar{\kappa}(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}}.\end{aligned} \tag{24}$$

We now attempt to reduce (24) to a form containing just two propagators, like (18).

We begin by multiplying by $\langle n-2\ n-1\rangle/\langle n-2\ n-1\rangle$ and writing

$$\langle n-2\ n-1\rangle\langle g\ j\rangle = \langle n-2\ g\rangle\langle n-1\ j\rangle - \langle n-1\ g\rangle\langle n-2\ j\rangle \tag{25}$$

(the Schouten identity) to obtain

$$\begin{aligned}&\Lambda(1^+,\dots,n^+) = \\ &\quad 2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\dots n-1)} \frac{\bar{u}_{\dot\alpha}(k_n)\bar{P}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\dots,n-1|g\rangle\langle n\ g\rangle} \\ &\qquad \times \sum_{j=1}^{n-2} \frac{u^\delta(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}\bar{k}_{n-1}^{\dot\delta\beta}k_{j\beta\dot\beta}[\bar{P}+\bar{\kappa}(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,n-1)]^2-\mu^2\}\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}} \\ &\quad -2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\dots n-1)} \frac{\bar{u}_{\dot\alpha}(k_n)\bar{P}^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\dots,n-2|g\rangle\langle n\ g\rangle} \frac{\bar{u}_{\dot\delta}(k_{n-1})[\bar{P}+\bar{\kappa}(1,n-1)]^{\dot\delta\delta}u_\delta(g)}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ &\qquad \times \sum_{j=1}^{n-3} \frac{\langle n-2\ j\rangle}{\langle n-2\ n-1\rangle} \frac{\bar{u}_{\dot\beta}(k_j)[\bar{P}+\bar{\kappa}(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}}.\end{aligned} \tag{26}$$

We now focus on the following portion of the first term:

$$\mathcal{N}_1 \equiv u^\delta(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}\bar{k}_{n-1}^{\dot\delta\beta}k_{j\beta\dot\beta}[\bar{P}+\bar{\kappa}(1,j)]^{\dot\beta\gamma}u_\gamma(g). \tag{27}$$

By clever application of identities similar to (19), we may rewrite $\mathcal{N}_1$ as

$$\begin{aligned}\mathcal{N}_1 = \quad & [P+\kappa(1,n-1)]^2\, u^{\delta}(g) k_{j\delta\dot\beta}[\bar P+\bar\kappa(1,j)]^{\dot\beta\gamma} u_\gamma(g) \\ & - [P+\kappa(1,j)]^2\, u^{\delta}(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}[\bar P+\bar\kappa(1,j-1)]^{\dot\delta\gamma} u_\gamma(g) \\ & + [P+\kappa(1,j-1)]^2\, u^{\delta}(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}[\bar P+\bar\kappa(1,j)]^{\dot\delta\beta} u_\beta(g) \\ & - u^{\delta}(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}\bar\kappa^{\dot\delta\beta}(j+1,n-2) k_{j\beta\dot\beta}[\bar P+\bar\kappa(1,j)]^{\dot\beta\gamma} u_\gamma(g). \end{aligned} \tag{28}$$

If the regulator were not present, the quadratic factors appearing in the first three terms of (28) would each produce terms containing only two propagators. Since this is precisely what we want to occur, we add and subtract the appropriate terms. Thus, when we combine (28) with (26) we obtain

$$\begin{aligned}\Lambda(1^+,\ldots,n^+) = & \\ & 2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\bar u_{\dot\alpha}(k_n)\bar P^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \\ & \qquad \times\sum_{j=1}^{n-2}\frac{u^{\delta}(g)k_{j\delta\dot\beta}[\bar P+\bar\kappa(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}} \\ & -2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\bar u_{\dot\alpha}(k_n)\bar P^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle}\frac{u^{\delta}(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ & \qquad \times\sum_{j=1}^{n-2}\left\{\frac{[\bar P+\bar\kappa(1,j-1)]^{\dot\delta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}}-\frac{[\bar P+\bar\kappa(1,j)]^{\dot\delta\beta}u_\beta(g)}{\{[P+\kappa(1,j)]^2-\mu^2\}}\right\} \\ & -2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\bar u_{\dot\alpha}(k_n)\bar P^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle}\frac{u^{\delta}(g)[P+\kappa(1,n-1)]_{\delta\dot\delta}}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ & \qquad \times\sum_{j=1}^{n-3}\frac{\bar\kappa^{\dot\delta\beta}(j+1,n-2)k_{j\beta\dot\beta}[\bar P+\bar\kappa(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}} \\ & -2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\bar u_{\dot\alpha}(k_n)\bar P^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-2|g\rangle\langle n\ g\rangle}\frac{\bar u_{\dot\delta}(k_{n-1})[\bar P+\bar\kappa(1,n-1)]^{\dot\delta\delta}u_\delta(g)}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ & \qquad \times\sum_{j=1}^{n-3}\frac{\langle n-2\ j\rangle}{\langle n-2\ n-1\rangle}\frac{\bar u_{\dot\beta}(k_j)[\bar P+\bar\kappa(1,j)]^{\dot\beta\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}} \\ & -2\mu^2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\bar u_{\dot\alpha}(k_n)\bar P^{\dot\alpha\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle}\frac{1}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \\ & \qquad \times\sum_{j=1}^{n-2}\frac{u^{\delta}(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j+1,n-1)u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}}. \end{aligned} \tag{29}$$

The sum on j appearing in the second term of (29) may be performed trivially, with only the endpoint terms surviving. One of the resultant terms vanishes when the permutation sum is performed; the other term may be used to extend the sum appearing in the first term to include $j = n-1$. This converts the first term into precisely the result that we would have obtained by regulating (18) directly. The third and fourth terms of (29) may be shown to cancel by writing out the implicit sum $\kappa(j+1, n-2) = k_{j+1} + \cdots + k_{n-2}$ and using the permutation sum to relabel the successive terms thus generated [11]. The fifth term does not combine with anything else: it represents the effect of imposing the regulator prior to reducing the integrand.

Thus, the integrals we must consider are

$$\mathcal{A}_1 \equiv -2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-1} \int \frac{d^d\mathcal{P}}{(2\pi)^d} \frac{\bar{u}_{\dot{\alpha}}(k_n)\bar{P}^{\dot{\alpha}\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \times \frac{u^\delta(g)k_{j\delta\dot{\beta}}[\bar{P}+\bar{\kappa}(1,j)]^{\dot{\beta}\gamma}u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}}, \tag{30}$$

generated from the first two terms of (29), and

$$\mathcal{A}_2 \equiv 2(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-2} \int \frac{d^d\mathcal{P}}{(2\pi)^d} \frac{\bar{u}_{\dot{\alpha}}(k_n)\bar{P}^{\dot{\alpha}\alpha}u_\alpha(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \frac{\mu^2}{\{[P+\kappa(1,n-1)]^2-\mu^2\}} \times \frac{u^\delta(g)k_{j\delta\dot{\gamma}}\bar{\kappa}^{\dot{\gamma}\gamma}(j+1,n-1)u_\gamma(g)}{\{[P+\kappa(1,j-1)]^2-\mu^2\}\{[P+\kappa(1,j)]^2-\mu^2\}}, \tag{31}$$

from the last term of (29).

We will begin with $\mathcal{A}_1$ since it is the simplest. A single Feynman parameter is sufficient to combine the denominators, producing

$$\Delta_1 = (1-x)\{[P+\kappa(1,j-1)]^2 - \mu^2\} + x\{[P+\kappa(1,j)]^2 - \mu^2\}. \tag{32}$$

Expanding and rearranging this expression, we find that

$$\Delta_1 = [P + \kappa(1,j-1) + xk_j]^2 - \mu^2. \tag{33}$$

We now apply the momentum shift implied by (33) to the numerator of (30). Since the combination

$$\begin{aligned} u^\delta(g)k_{j\delta\dot{\beta}}[\bar{P}+\bar{\kappa}(1,j)]^{\dot{\beta}\gamma}u_\gamma(g) \;&\to\; (1-x)u^\delta(g)k_{j\delta\dot{\beta}}k_j^{\dot{\beta}\gamma}u_\gamma(g) \\ &= (1-x)\,k_j^2\,\langle g\ g\rangle = 0, \end{aligned} \tag{34}$$

appears, we conclude that the first integral vanishes.

We now turn to the second integral, which contains the entire result. Introducing three Feynman parameters produces the denominator

$$\Delta_2 = x\{[P+\kappa(1,j-1)]^2-\mu^2\} + y\{[P+\kappa(1,j)]^2-\mu^2\} + z\{[P+\kappa(1,n-1)]^2-\mu^2\}, \tag{35}$$

which may be rewritten as

$$\Delta_2 = [P + (1-z)\kappa(1,j-1) + yk_j - zk_n]^2 + K^2 - \mu^2 \tag{36}$$

where

$$K^2 \equiv z(1-z)[\kappa(1,j-1) + k_n]^2 + 2yzk_j \cdot [\kappa(1,j-1) + k_n]. \tag{37}$$

Shifting the momentum in the manner implied by (36) and doing the integral over the four space-time dimensions yields

$$\begin{aligned}
\mathcal{A}(1^+,\ldots,n^+) &= \\
&\frac{i}{8\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-2} \frac{u^\delta(g) k_{j\delta\dot\gamma} \bar\kappa^{\dot\gamma\gamma}(j+1,n-1) u_\gamma(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \\
&\times \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz\ \delta(1-x-y-z)\, \bar u_{\dot\alpha}(k_n)[(1-z)\bar\kappa(1,j-1) + y\bar k_j]^{\dot\alpha\alpha} u_\alpha(g) \\
&\times \frac{\epsilon(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \int_0^\infty d\mu^2\ \frac{(\mu^2)^{-\epsilon}}{K^2-\mu^2}.
\end{aligned} \tag{38}$$

Because an explicit factor of ϵ appears in the numerator of (38), any pieces of the integral over μ^2 which are finite as $\epsilon \to 0$ are irrelevant. Thus, we write

$$\int_0^\infty d\mu^2 \frac{(\mu^2)^{-\epsilon}}{K^2-\mu^2} = -\frac{1}{\epsilon} + \mathcal{O}(1). \tag{39}$$

Inserting (39) into (38) and performing the now trivial Feynman parameter integrals yields

$$\begin{aligned}
\mathcal{A}(1^+,\ldots,n^+) = \frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-2} \frac{u^\delta(g) k_{j\delta\dot\gamma} \bar\kappa^{\dot\gamma\gamma}(j+1,n-1) u_\gamma(g)}{\langle g|1,\ldots,n-1|g\rangle\langle n\ g\rangle} \\
\times\ \bar u_{\dot\alpha}(k_n)[\bar\kappa(j+1,n-1) - \bar\kappa(1,j-1)]^{\dot\alpha\alpha} u_\alpha(g).
\end{aligned} \tag{40}$$

All that remains at this stage is to eliminate the occurrences of the gauge momentum g.

C. Gauge invariance of the result

We begin by multiplying the expression in (40) by $\langle n{-}1\ 1\rangle/\langle n{-}1\ 1\rangle$ and applying the Schouten identity to obtain

$$\begin{aligned}\mathcal{A}(1^+,\ldots,n^+) &= \frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j{+}1,n{-}2)u_\gamma(k_{n-1})}{\langle n{-}1|1,\ldots,n{-}1|g\rangle\langle n\ g\rangle} \\ &\qquad \times\ \bar u_{\dot\alpha}(k_n)[\bar\kappa(j{+}1,n{-}1)-\bar\kappa(1,j{-}1)]^{\dot\alpha\alpha}u_\alpha(g) \\ &+\frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-2} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j{+}1,n{-}1)u_\gamma(k_1)}{\langle g|1,\ldots,n{-}1|1\rangle\langle n\ g\rangle} \\ &\qquad \times\ \bar u_{\dot\alpha}(k_n)[\bar\kappa(j{+}1,n{-}1)-\bar\kappa(1,j{-}1)]^{\dot\alpha\alpha}u_\alpha(g) \end{aligned} \tag{41}$$

We take advantage of the permutation sum to cyclicly relabel the momenta in the second term as follows:

$$1 \to n{-}1 \to n{-}2 \to \cdots \to 2 \to 1 \tag{42}$$

This converts the second term into

$$\begin{aligned}\mathcal{A}_2 &= \frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{u^\delta(g)(k_{n-1})_{\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(1,n{-}2)u_\gamma(k_{n-1})}{\langle g|n{-}1,1,2,\ldots,n{-}2|n{-}1\rangle\langle n\ g\rangle}\bar u_{\dot\alpha}(k_n)\bar\kappa^{\dot\alpha\alpha}(1,n{-}2)u_\alpha(g) \\ &+\frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=2}^{n-2} \frac{u^\delta(g)(k_{j-1})_{\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j,n{-}2)u_\gamma(k_{n-1})}{\langle g|n{-}1,1,2,\ldots,n{-}2|n{-}1\rangle\langle n\ g\rangle} \\ &\qquad \times\ \bar u_{\dot\alpha}(k_n)[\bar\kappa(j,n{-}2)-\bar k_{n-1}-\bar\kappa(1,j{-}2)]^{\dot\alpha\alpha}u_\alpha(g). \end{aligned} \tag{43}$$

A little bit of algebra allows us to rewrite this as

$$\begin{aligned}\mathcal{A}_2 &= -\frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j{+}1,n{-}2)u_\gamma(k_{n-1})}{\langle n{-}1|1,\ldots,n{-}1|g\rangle\langle n\ g\rangle} \\ &\qquad \times\ \bar u_{\dot\alpha}(k_n)[\bar\kappa(j{+}1,n{-}1)-\bar\kappa(1,j{-}1)]^{\dot\alpha\alpha}u_\alpha(g) \\ &+\frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j{+}1,n{-}2)u_\gamma(k_{n-1})}{\langle n{-}1|1,\ldots,n{-}1|g\rangle\langle n\ g\rangle}\bar u_{\dot\alpha}(k_n)[2\bar k_{n-1}]^{\dot\alpha\alpha}u_\alpha(g) \\ &-\frac{i}{48\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{2k_{n-1}\cdot k_n\langle g\ n{-}1\rangle\langle n{-}1\ n\rangle^*}{\langle n{-}1|1,2,\ldots,n{-}2|n{-}1\rangle\langle n\ g\rangle}. \end{aligned} \tag{44}$$

The first term of (44) precisely cancels $\mathcal{A}_1$ [*i.e.* the first term of (41)]. We apply the useful identity [19]

$$\sum_{\mathcal{P}(1\ldots m)} \frac{1}{\langle p|1,\ldots,m|q\rangle} = \frac{\langle p\; q\rangle^{m-1}}{\prod\limits_{i=1}^{m}\langle p|i|q\rangle}. \tag{45}$$

to see that the last term vanishes for $n > 3$. (We already know that the amplitude vanishes for $n \leq 3$, and so lose no information by assuming that $n > 3$.) Consequently, the entire amplitude may be written as

$$\mathcal{A}(1^+,\ldots,n^+) = -\frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j+1,n-2)(k_{n-1})_{\gamma\dot\beta}\bar u^{\dot\beta}(k_n)}{\langle 1|2,\ldots,n-1|1\rangle\langle n\; g\rangle} \tag{46}$$

for $n \geq 4$.

In order to eliminate the last occurrence of the gauge momentum g, it is necessary to re-order the terms a bit. To this end, we introduce the identity

$$\bar\kappa^{\dot\gamma\gamma}(j+1,n-2)u_\gamma(k_{n-1}) = \sum_{\ell=j+1}^{n-2} \bar\kappa^{\dot\gamma\gamma}(j+1,\ell)u_\gamma(k_j)\langle j\; n-1\rangle \frac{\langle \ell\; \ell+1\rangle}{\langle \ell|j|\ell+1\rangle}. \tag{47}$$

Eq. (47) is most easily proven by performing the sum on ℓ appearing on the right hand side, noting that

$$\frac{\langle 1\; 2\rangle}{\langle 1|P|2\rangle} + \frac{\langle 2\; 3\rangle}{\langle 2|P|3\rangle} = \frac{\langle 1\; 3\rangle}{\langle 1|P|3\rangle}, \tag{48}$$

a result that follows from the Schouten identity. Application of (47) to (46) produces

$$\mathcal{A}(1^+,\ldots,n^+) = \\ -\frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \sum_{\ell=j+1}^{n-2} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar\kappa^{\dot\gamma\gamma}(j+1,\ell)u_\gamma(k_j)u^\beta(k_j)(k_{n-1})_{\beta\dot\beta}\bar u^{\dot\beta}(k_n)}{\langle 1|2,\ldots\ell|j\rangle\langle j|\ell+1,\ldots,n-1|1\rangle\langle n\; g\rangle}. \tag{49}$$

Let us examine the denominator of (49), which may be written

$$\Delta = \langle j|\ell+1,\ell+2,\ldots,n-1,1,2,\ldots,\ell|j\rangle\langle n\; g\rangle. \tag{50}$$

Since $j < \ell$, we may express (50) in the form

$$\Delta = \langle j|\ell+1,\ell+2,\ldots,n-1,1,2,\ldots,j-1|j\rangle\langle j|j+1,\ldots,\ell|j\rangle\langle n\; g\rangle. \tag{51}$$

Because the numerator of (49) is symmetric under permutations of $\{j+1,\ldots,\ell\}$, we may use (45) to deduce that the permutation sum causes every term in the sum over ℓ to vanish except for $\ell = j+1$. Thus,

$$\mathcal{A}(1^+,\ldots,n^+) =$$
$$-\frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar{k}_{j+1}^{\dot\gamma\gamma}u_\gamma(k_j)\ \bar{u}_{\dot\beta}(k_n)\bar{k}_{n-1}^{\dot\beta\beta}u_\beta(k_j)}{\langle j|j{+}2,j{+}3,\ldots,n{-}1,1,2,\ldots j{-}1|j\rangle\langle j|j{+}1|j\rangle\langle n\ g\rangle}$$
$$= -\frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \sum_{j=1}^{n-3} \frac{u^\delta(g)k_{j\delta\dot\gamma}\bar{k}_{j+1}^{\dot\gamma\gamma}u_\gamma(k_{j+2})\ \bar{u}_{\dot\beta}(k_n)\bar{k}_{n-1}^{\dot\beta\beta}u_\beta(k_j)}{\langle j\ j{+}2\rangle\langle 1|2,3,\ldots,n{-}1|1\rangle\langle n\ g\rangle}, \tag{52}$$

where we have done some minor rearranging to obtain the last line.

The denominator appearing in (52) is very nearly symmetric under cyclic permutations of $\{1,2,\ldots,n{-}1\}$. Let us take advantage of this near-cyclic symmetry to relabel the successive terms appearing in the sum over j so that k_j always becomes k_1, k_{j+1} becomes k_2, and k_{j+2} becomes k_3. The only portion of (52) which has a form which varies after such relabeling is $\bar{k}_{n-1}^{\dot\beta\beta}$. In the $j=1$ term, $\bar{k}_{n-1}^{\dot\beta\beta}$ remains unchanged. For $j=2$, $\bar{k}_{n-1}^{\dot\beta\beta}$ becomes $\bar{k}_{n-2}^{\dot\beta\beta}$, and so on through $j=n-3$, in which $\bar{k}_{n-1}^{\dot\beta\beta}$ becomes $\bar{k}_3^{\dot\beta\beta}$. Hence, we see that (52) is equivalent to

$$\mathcal{A}(1^+,\ldots,n^+) = -\frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{u^\delta(g)k_{1\delta\dot\gamma}\bar{k}_2^{\dot\gamma\gamma}u_\gamma(k_3)\ \bar{u}_{\dot\beta}(k_n)\bar{\kappa}^{\dot\beta\beta}(3,n{-}1)u_\beta(k_1)}{\langle 1\ 3\rangle\langle 1|2,3,\ldots,n{-}1|1\rangle\langle n\ g\rangle}. \tag{53}$$

Momentum conservation for the scattering amplitude implies that

$$\kappa(3,n{-}1) = -k_1 - k_2 - k_n. \tag{54}$$

Utilizing this relation plus a little more spinor algebra yields

$$\mathcal{A}(1^+,\ldots,n^+) = \frac{i}{24\pi^2}(-e\sqrt{2})^n \sum_{\mathcal{P}(1\ldots n-1)} \frac{\langle g\ 1\rangle\langle 2\ 1\rangle^*\langle n\ 2\rangle^*}{\langle 1|3,\ldots,n{-}1|1\rangle\langle n\ g\rangle}. \tag{55}$$

Because the numerator of (55) is symmetric under permutations of $\{3,\ldots,n{-}1\}$, we may conclude from (45) that the permutation sum will cause the amplitude to vanish unless $n=4$:

$$\mathcal{A}(1^+,\ldots,n^+) = 0, \qquad n \geq 5. \tag{56}$$

Thus, we obtain the extraordinary result that the *only* non-vanishing n-photon amplitude in the case of like helicities is the set of $n=4$ box diagrams. We will comment further on this result in section III D.

Although we have finished the derivation for $n > 4$, we still have not proven that the amplitude is independent of g when $n = 4$. We shall now do so. For $n = 4$, Eq. (55) reads

$$\mathcal{A}(1^+, 2^+, 3^+, 4^+) = \frac{ie^4}{6\pi^2} \sum_{\mathcal{P}(123)} \frac{\langle g\ 1\rangle\langle 2\ 1\rangle^*\langle 4\ 2\rangle^*}{\langle 1\ 3\rangle\langle 3\ 1\rangle\langle 4\ g\rangle}. \tag{57}$$

Since the denominator is symmetric under the interchange $1 \leftrightarrow 3$, we may easily reduce the sum over all permutations of $\{1,2,3\}$ to a sum over only cyclic permutations of $\{1,2,3\}$:

$$\mathcal{A}(1^+, 2^+, 3^+, 4^+) = \frac{ie^4}{6\pi^2} \sum_{\mathcal{C}(123)} \frac{\langle 4\ 2\rangle^*}{\langle 1\ 3\rangle\langle 3\ 1\rangle\langle 4\ g\rangle} [\langle g\ 1\rangle\langle 2\ 1\rangle^* + \langle g\ 3\rangle\langle 2\ 3\rangle^*]. \tag{58}$$

Momentum conservation and the Weyl equation allows us to rewrite the combination in the square brackets as

$$\begin{aligned} u^\alpha(g)(k_1 + k_3)_{\alpha\dot\alpha}\bar{u}^{\dot\alpha}(k_2) &= -u^\alpha(g)k_{4\alpha\dot\alpha}\bar{u}^{\dot\alpha}(k_2) \\ &= \langle 4\ g\rangle\langle 2\ 4\rangle^*. \end{aligned} \tag{59}$$

Inserting (59) into (58) produces the g-independent expression

$$\mathcal{A}(1^+, 2^+, 3^+, 4^+) = \frac{ie^4}{6\pi^2} \sum_{\mathcal{C}(123)} \frac{\langle 4\ 2\rangle^*\langle 2\ 4\rangle^*}{\langle 1\ 3\rangle\langle 3\ 1\rangle} \tag{60}$$

Further judicious use of momentum conservation and the Weyl equation reveals that the three terms of (60) are actually equal to each other. Thus, with a bit more algebra, we may write

$$\mathcal{A}(1^+, 2^+, 3^+, 4^+) = \frac{ie^4}{2\pi^2} \frac{\langle 1\ 2\rangle^*\langle 3\ 4\rangle^*}{\langle 1\ 2\rangle\langle 3\ 4\rangle}. \tag{61}$$

In this form it is clear that the square of the amplitude is constant (up to an unobservable phase), and agrees with the result for a massless fermion loop given by Gastmans and Wu [4].

A similar calculation for $\mathcal{A}(1^+, \ldots, (n-1)^+, n^-)$ yields

$$\mathcal{A}(1^+, 2^+, 3^+, 4^-) = \frac{ie^4}{2\pi^2} \frac{\langle 1\ 2\rangle^*\langle 2\ 3\rangle^*\langle 3\ 1\rangle}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle^*}, \tag{62a}$$

$$\mathcal{A}(1^+, \ldots, (n-1)^+, n^-) = 0, \qquad n \geq 5. \tag{62b}$$

D. Discussion

We now turn to a brief discussion of the surprising results in Eqs. (56) and (62b). Our first observation is that we do not expect *all* of the photon-photon helicity amplitudes to vanish for $n \geq 5$, unless n is odd (Furry's theorem). When there are two or more negative helicity photons present in the diagram, it is possible to cut the diagram in such a way so as to make the factors corresponding to each piece non-zero. Then, the Cutkosky rules imply the existence of a non-vanishing imaginary part, unless some special symmetry intervenes. For example, in the case of the photon-photon amplitudes for odd n, charge conjugation symmetry forces such a cancellation. For even n, no such symmetry is obvious.

Our second observation is that this result is confined to the one-loop order of perturbation theory. Indeed, it is easy to see that the two-loop correction to $\mathcal{A}(1^+, \ldots, 6^+)$ does not vanish. The two types of diagrams contributing to this amplitude are illustrated in Fig. 2. Of the possible diagrams containing two fermion loops, only those which contain three photons on each loop survive (Fig. 2a). Furry's theorem causes any diagram with an odd number of photons attached to a loop to vanish. Hence, either the photons are divided equally, as in Fig. 2a, or there is only one external photon on one of the loops and five on the other. But this latter arrangement consists of the massless vacuum polarization renormalization factor multiplied by the one-loop six-photon result: this, of course, vanishes. The diagrams like Fig. 2a may be viewed as a pair of $n = 4$ photon-photon scattering diagrams with one off-shell photon. These diagrams do not vanish when *all* of the photons are on shell; taking one of the photons off shell cannot change this situation. Thus, this contribution to the amplitude is non-zero. Furthermore, it should have a pole in the limit where the photon connecting the two loops becomes soft. The only other available diagrams are variants of Fig. 2b. It is apparent that these diagrams do not have the same pole structure. Consequently, they can not cancel the contribution from Fig. 2a. Hence, we expect a non-vanishing result for the two-loop six-photon amplitude with like helicity photons.

IV. ONE-LOOP CORRECTIONS TO $e^+e^- \to \gamma\gamma\cdots\gamma$

We now turn to electron-positron annihilation into photons. The one loop corrections to $e^+e^- \to \gamma\gamma$ have been known for some time [20], and helicity amplitudes for the related process of Compton scattering are present in the literature [21] for full QED including finite mass effects. Helicity amplitudes for the massless limit, however, seem to be absent. We will present results for an arbitrary number of like-helicity photons in this limit.

A. Preliminaries

There are two basic types of diagrams contributing to the one loop corrections to electron-positron annihilation to n photons, as illustrated in Figs. 3 and 4. We will refer to Fig. 3 as the light-by-light contribution, since it contains the light-by-light scattering process as a sub-diagram, with one of the photons off shell. We will refer to Fig. 4 as the "jellyfish" contribution since the off-shell photon in Fig. 4 spans varying numbers of on-shell photons. The evaluation of these two diagrams is relatively easy.

B. The light-by-light contribution

Figure 3 illustrates the basic structure of those Feynman diagrams which contain the light-by-light scattering process as a sub-diagram. The expressions relevant to this contribution may be generated from (14) by making the replacement

$$\epsilon_\nu(n) \to -ie\bar{\psi}(p;1,\ldots,i)\gamma_\nu\psi(i+1,\ldots,j;q) \tag{63}$$

and introducing the appropriate sums to include all possible divisions of the n photons into three groups as well as all possible permutations of the n photons. That is, equation (14) becomes

$$\Lambda(p;1,\ldots,n;q) =$$
$$-e^2 \sum_{\mathcal{P}(1\ldots n)} \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{1}{i!(j-i)!(n-j)!} \bar{\psi}(p;1,\ldots,i)\gamma_\nu \psi(i+1,\ldots,j;q) \mathrm{Tr}\,[\gamma^\nu \Psi(\mathcal{P};j+1,\ldots,n)]. \tag{64}$$

In equation (64), $\bar{\psi}$ and ψ stand for fermion currents with only one end of the fermion line off shell. Solutions for these objects are discussed in Refs. [8] and [11]. The incoming positron has momentum p, while the incoming electron has momentum q. The incoming photons have momenta $k_1,\ldots,k_n$ as usual. This expression is valid for all combinations of photon and fermion helicity.

Since the known double-off-shell fermion current contains only like helicity photons, there are two helicity amplitudes for this process which are readily accessible, corresponding to the two possible helicities of the fermion line. For concreteness, we will discuss the amplitude containing a left-handed positron. The computation of the other amplitude follows the same pattern. Alternatively, one may apply charge-conjugation symmetry to obtain the result.

Specializing then, to the left-handed positron case, and converting (64) to spinor form produces

$$\Lambda(p^-;1,\ldots,n;q^+) =$$
$$-(-e\sqrt{2})^2 \sum_{\mathcal{P}(1\ldots n)} \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{1}{i!(j-i)!(n-j)!} \bar{\psi}^\alpha(p^-;1,\ldots,i)(\sigma_\nu)_{\alpha\dot{\alpha}} \psi^{\dot{\alpha}}(i+1,\ldots,j;q^+)$$
$$\times \mathrm{Tr}\,[(\sigma^\nu)_{\beta\dot{\beta}} \bar{\Psi}^{\dot{\beta}\beta}(\mathcal{P};j+1,\ldots,n) + (\bar{\sigma}^\nu)^{\dot{\beta}\beta} \Psi_{\beta\dot{\beta}}(\mathcal{P};j+1,\ldots,n)]. \tag{65}$$

Recall that the solutions for Ψ and $\bar{\Psi}$ presented in Eqs. (10) and (12) require that a factor of the gauge spinor be contracted into the undotted index. The sum on ν appearing in (65) will cause the current $\bar{\psi}$ to appear in this position. From Refs. [8] and [11] we know that

$$\bar{\psi}^\alpha(p^-;1^+,\ldots,i^+) = (-e\sqrt{2})^i \sum_{\mathcal{P}(1\ldots i)} \frac{u^\alpha(p)\langle p\ g\rangle}{\langle p|1,\ldots,i|g\rangle}. \tag{66}$$

Thus, the natural choice for the gauge momentum is $g = p$. The expression in (66) vanishes for $i \neq 0$ when $g = p$, and reads

$$\bar{\psi}^\alpha(p^-) = u^\alpha(p) \tag{67}$$

for $i = 0$.

The remaining steps in evaluating the contribution from (65) involve straightforward spinor algebra and an integration like that described in Sec. III. Therefore, we will immediately present the result:

$$\mathcal{A}_1(p^-;1^+,\ldots,n^+;q^+) = \frac{-i}{24\pi^2}(-e\sqrt{2})^{n+2} \sum_{\mathcal{P}(1\ldots n)} \frac{u^\alpha(p)k_{1\alpha\dot\alpha}\bar{k}_2^{\dot\alpha\beta}k_{3\beta\dot\beta}\bar{\kappa}^{\dot\beta\gamma}(1,3)u_\gamma(p)}{\langle 1|2,3|1\rangle\,\langle p|4,\ldots,n|q\rangle\,\kappa^2(1,3)}. \tag{68}$$

The structure of Eq. (68) reflects the fact that the results obtained for photon-photon scattering [in particular, Eq. (56)] continue to hold even when one of the photons has a non-zero mass-squared [Note that the structure on the right-hand side of (63) is transverse in the gauge utilized in this calculation]. Thus, the only diagrams which actually contribute to $\mathcal{A}_1$ have exactly three external photons attached to the fermion loop. For e^+e^- annihilation to two photons, $\mathcal{A}_1 \equiv 0$.

The corresponding amplitude with a right-handed positron is given by

$$\mathcal{A}_1(p^+;1^+,\ldots,n^+;q^-) = \frac{i}{24\pi^2}(-e\sqrt{2})^{n+2} \sum_{\mathcal{P}(1\ldots n)} \frac{u^\alpha(q)k_{1\alpha\dot\alpha}\bar{k}_2^{\dot\alpha\beta}k_{3\beta\dot\beta}\bar{\kappa}^{\dot\beta\gamma}(1,3)u_\gamma(q)}{\langle 1|2,3|1\rangle\,\langle p|4,\ldots,n|q\rangle\,\kappa^2(1,3)}. \tag{69}$$

C. The "jellyfish" contribution

Figure 4 illustrates the basic structure of this group of Feynman diagrams. The direct application of ordinary QED Feynman rules to produces

$$\begin{aligned}\mathcal{A}_2(p^-;1,\ldots,n;q^+) = \\ ie^2 \sum_{\mathcal{P}(1\ldots n)} \sum_{i=0}^{n}\sum_{j=i}^{n} \frac{1}{i!(j-i)!(n-j)!}\bar\psi(p;1,\ldots,i)\tfrac{1}{2}(1-\gamma^5)\gamma^\nu \\ \times\Psi(\mathcal{P};i{+}1,\ldots,j)\gamma_\nu\tfrac{1}{2}(1+\gamma^5)\psi(j{+}1,\ldots,n;q)\frac{1}{\mathcal{K}^2}\end{aligned} \tag{70}$$

where we have inserted the appropriate projection operator to describe a left-handed positron. The spanning photon in Fig. 4 has momentum $\mathcal{K}$. The momentum of the off-shell positron in Ψ is given by

$$\mathcal{P} \equiv \mathcal{K} + p + \kappa(1,i). \tag{71}$$

We again choose $g = p$ to ensure the desired contraction of the gauge spinor into the undotted index of Ψ or $\bar{\Psi}$. This also reduces the three currents appearing in (70) to two. Making this gauge selection, translating to spinor notation, and performing the sum on ν yields

$$\mathcal{A}_2(p^-;1^+,\ldots,n^+;q^+) = \\ -2ie^2 \sum_{\mathcal{P}(1\ldots n)} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \psi_{\dot\alpha}((j+1)^+,\ldots,n^+;q^+) \bar{\Psi}^{\dot\alpha\alpha}(\mathcal{P};1^+,\ldots,j^+) u_\alpha(p) \frac{1}{\mathcal{K}^2}. \qquad (72)$$

The $j = 0$ term in Eq. (72) is special: it contains the fermion self-energy as a subgraph. The contribution from this term is proportional to the tree-level result. Since the tree-level diagram vanishes for this helicity combination, we may drop $j = 0$ from the sum in (72). We now insert the expressions for the currents in the remaining terms of (72) to obtain

$$\mathcal{A}_2(p^-;1^+,\ldots,n^+;q^+) = \\ -(-e\sqrt{2})^{n+2} \sum_{\mathcal{P}(1\ldots n)} \sum_{j=1}^{n} \sum_{\ell=1}^{j} \int \frac{d^4\mathcal{K}}{(2\pi)^4} \frac{u^\beta(p)[\kappa(j+1,n)+q]_{\beta\dot\alpha} \bar{\mathcal{K}}^{\dot\alpha\alpha} u_\alpha(p)}{\langle p|1,\ldots,j|p\rangle\langle p|j+1,\ldots,n|q\rangle} \\ \times \frac{u^\gamma(p) k_{\ell\gamma\dot\gamma} [\bar{\mathcal{K}}+\bar{p}+\bar{\kappa}(1,\ell)]^{\dot\gamma\delta} u_\delta(p)}{\mathcal{K}^2[\mathcal{K}+p+\kappa(1,\ell-1)]^2[\mathcal{K}+p+\kappa(1,\ell)]^2}. \qquad (73)$$

While it is possible to do the integral at this stage, it is advantageous to simplify the integrand as much as possible first. To this end, we perform the sum on j, producing

$$\mathcal{A}_2(p^-;1^+,\ldots,n^+;q^+) = \\ (-e\sqrt{2})^{n+2} \sum_{\mathcal{P}(1\ldots n)} \sum_{\ell=1}^{n} \int \frac{d^4\mathcal{K}}{(2\pi)^4} \frac{u^\alpha(p)\mathcal{K}_{\alpha\dot\alpha}[\bar{p}+\bar{\kappa}(1,\ell-1)]^{\dot\alpha\beta} k_{\ell\beta\dot\gamma}[\bar{\mathcal{K}}+\bar{p}+\bar{\kappa}(1,\ell)]^{\dot\gamma\gamma} u_\gamma(p)}{\langle p|1,\ldots,n|q\rangle \mathcal{K}^2[\mathcal{K}+p+\kappa(1,\ell-1)]^2[\mathcal{K}+p+\kappa(1,\ell)]^2}. \qquad (74)$$

Next, we would like to write

$$p + \kappa(1,\ell) = [\mathcal{K} + p + \kappa(1,\ell)] - \mathcal{K} \qquad (75)$$

in order to begin the process of canceling some of the denominators. However, this would break up a convergent integral into divergent bits. We must regulate this expression first, as discussed in the Sec. III B. As was true in that case, every occurrence of the loop momentum may be written as a Lorentz dot product with one of the polarization vectors. Thus, Eq. (74) may be continued to d dimensions without difficulty. The subsequent algebra to actually cancel the denominators is very similar to that performed in Eqs. (27)–(31). Hence, we shall immediately quote the result:

$$\begin{aligned}
\mathcal{A}_2(p^-;1^+,\ldots,n^+;q^+) &= \\
&-(-e\sqrt{2})^{n+2}\sum_{\mathcal{P}(1\ldots n)}\sum_{\ell=1}^{n+1}\frac{1}{\langle p|1,\ldots,n|q\rangle} \\
&\qquad\times\int\frac{d^d\mathcal{K}}{(2\pi)^d}\frac{u^\alpha(p)k_{\ell\alpha\dot\alpha}[\bar{K}+\bar{p}+\bar\kappa(1,\ell)]^{\dot\alpha\gamma}u_\gamma(p)}{\{[K+p+\kappa(1,\ell-1)]^2-\mu^2\}\{[K+p+\kappa(1,\ell)]^2-\mu^2\}} \\
&+(-e\sqrt{2})^{n+2}\sum_{\mathcal{P}(1\ldots n)}\sum_{\ell=2}^{n}\frac{1}{\langle p|1,\ldots,n|q\rangle}\int\frac{d^d\mathcal{K}}{(2\pi)^d}\frac{1}{[K^2-\mu^2]} \\
&\qquad\times\frac{\mu^2\,u^\alpha(p)k_{\ell\alpha\dot\alpha}[\bar{p}+\bar\kappa(1,\ell)]^{\dot\alpha\gamma}u_\gamma(p)}{\{[K+p+\kappa(1,\ell-1)]^2-\mu^2\}\{[K+p+\kappa(1,\ell)]^2-\mu^2\}}
\end{aligned} \tag{76}$$

where $k_{n+1}\equiv q$. The first integral appearing in (76) contains the same denominator structure as the integral in (30). Since the numerator of this term contains the same factor that caused (30) to vanish [see Eq. (34)], this integral vanishes as well. The second integral may be evaluated with little difficulty, yielding

$$\mathcal{A}_2(p^-;1^+,\ldots,n^+;q^+)=\frac{-i}{32\pi^2}(-e\sqrt{2})^{n+2}\sum_{\mathcal{P}(1\ldots n)}\sum_{\ell=1}^{n}\frac{u^\alpha(p)k_{\ell\alpha\dot\alpha}[\bar{p}+\bar\kappa(1,\ell)]^{\dot\alpha\gamma}u_\gamma(p)}{\langle p|1,\ldots,n|q\rangle}. \tag{77}$$

A similar calculation utilizing a right-handed positron produces

$$\mathcal{A}_2(p^+;1^+,\ldots,n^+;q^-)=\frac{i}{32\pi^2}(-e\sqrt{2})^{n+2}\sum_{\mathcal{P}(1\ldots n)}\sum_{\ell=1}^{n}\frac{u^\alpha(q)k_{\ell\alpha\dot\alpha}[\bar{p}+\bar\kappa(1,\ell)]^{\dot\alpha\gamma}u_\gamma(q)}{\langle p|1,\ldots,n|q\rangle}. \tag{78}$$

The total one-loop amplitudes are, of course, given by the sums of (68) and (77) or (69) and (78) depending upon the helicity of the fermion line.

V. CONCLUSIONS

In this paper we have seen a simple way to compute the one-loop corrections to QED helicity amplitudes that vanish at tree level. We know from analyticity considerations that the expressions should be relatively simple: in particular, there should be no cuts. By using the solutions to the recursion relations for currents that contain two off shell particles, it is possible to evaluate these amplitudes easily, without the production of extraneous logarithmic contributions which cancel in the end. The amount of labor involved is essentially independent of the number of photons.

We have obtained helicity amplitudes for n-photon scattering as well as electron-positron annihilation to n photons for the case of like helicity photons. We have also obtained n-photon scattering amplitudes for the case where one of the photon helicities is the opposite of the rest. We find that the only non-vanishing n-photon scattering amplitudes for these helicity combinations are for $n = 4$. This is a new and surprising result.

In principle, the extension of these methods to more complicated helicity combinations is straightforward, although somewhat more computational labor is required. It is likely that the quest for closed-form expressions for arbitrary n will have to end, however. Instead, one should focus on using the recursion relations as a powerful guide in simplifying the integrand as much as possible before attacking the actual integration. Indeed, the reductions illustrated here suggest that except for the most complicated helicity configuration (*i.e.* equal numbers of positive and negative helicity photons), there is much to be learned from this point of view.

ACKNOWLEDGMENTS

I would like to thank T.–M. Yan, C.–S. Lam, R.T. Ward, B. Dimm, S. Parke, W.T. Giele and D. Summers for useful discussions during the course of this work. Additional thanks to T.–M. Yan and W.T. Giele for reading and commenting on the manuscript prior to release. I would also like to thank B. Dimm for the use of FeynDiagram to generate the figures for this paper.

This work was started at Cornell University and supported in part by the National Science Foundation.

REFERENCES

* Electronic address: gdm@fnth010.fnal.gov

[1] J. Schwinger, *Particles, Sources, and Fields* (Addison-Wesley, Redwood City, CA, 1970, Vol. I; Ann. Phys. (N.Y.) **119**, 192 (1979).

[2] P. De Causmaecker, R. Gastmans, W. Troost, and T.T. Wu, Phys. Lett. **105B**, 215 (1981); P. De Causmaecker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. **B206**, 53 (1982); F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, W. Troost and T.T. Wu, Nucl. Phys. **B206**, 61 (1982); F.A. Berends, P. De Causmaecker, R. Gastmans, R. Kleiss, W. Troost and T.T. Wu, Nucl. Phys. **B239**, 382 (1984); **B239**, 395 (1984); **B264**, 243 (1986); **B264**, 265 (1986).

[3] Z. Xu, D.H. Zhang, and Z. Chang, Tsinghua University Report No. TUTP-84/3, 84/4, and 84/5a, 1984 (unpublished); Nucl. Phys. **B291**, 392 (1984).

[4] Many of the results for processes containing six or fewer particles are collected in R. Gastmans and T.T. Wu, *The Ubiquitous Photon: Helicity Method for QED and QCD* (Oxford University Press, New York, 1990).

[5] The excellent review by Mangano and Parke provides a guide to the various approaches to and extensive literature on the subject: M. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991).

[6] Additional references to the vast literature on this subject may be found within Refs. [4] and [5].

[7] F.A. Berends and W.T. Giele, Nucl. Phys. **B294**, 700 (1987); M. Mangano, S. Parke, and Z. Xu, Nucl. Phys. **B298**, 653 (1988); M. Mangano, Nucl. Phys. **B309**, 461 (1988).

[8] F.A. Berends and W.T. Giele, Nucl. Phys. **B206**, 759 (1988).

[9] C. Dunn and T.–M. Yan, Nucl. Phys. **B352**, 402 (1991).

[10] C. Dunn, Ph.D. thesis, Cornell University, 1990.

[11] G. Mahlon and T.–M. Yan, Phys. Rev. **D47**, 1776 (1993).

[12] G. Mahlon, Phys. Rev. **D47**, 1812 (1993).

[13] G. Mahlon, T.–M. Yan, and C. Dunn, Phys. Rev. **D48**, 1337 (1993).

[14] G. Mahlon, in preparation.

[15] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950); **83**, 776 (1951).

[16] V. Costantini, B. De Tollis, and G. Pistoni, Nuovo Cimento, **2A**, 733 (1971).

[17] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972).

[18] W.T. Giele, private communication.

[19] M. Mangano, Nucl. Phys. **B309**, 461 (1988).

[20] L.M. Brown and R.P. Feynman, Phys. Rev. **80**, 231 (1952); I. Harris and L.M. Brown, Phys. Rev. **105**, 1656 (1957).

[21] K.A. Milton, W.–Y. Tsai, and L.L. DeRaad, Jr., Phys. Rev. **D6**, 1411 (1972); **D6**, 1428 (1972); A. Góngora–T and R.G. Stuart, Z. Phys. **C42**, 617 (1989); H. Veltman, Phys. Rev. **D40**, 2810 (1989).

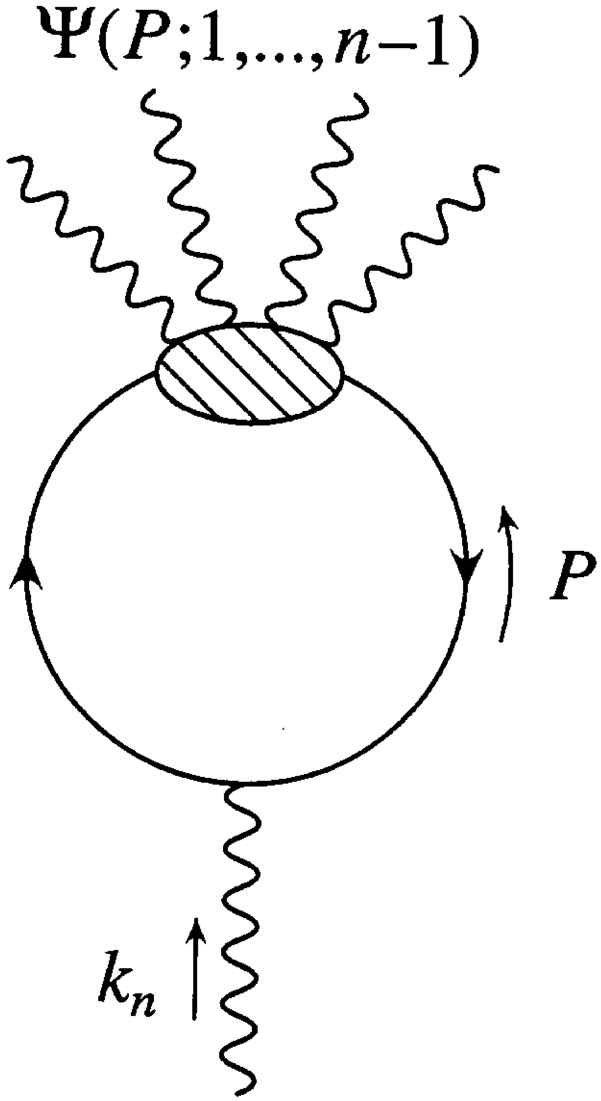
FIGURES

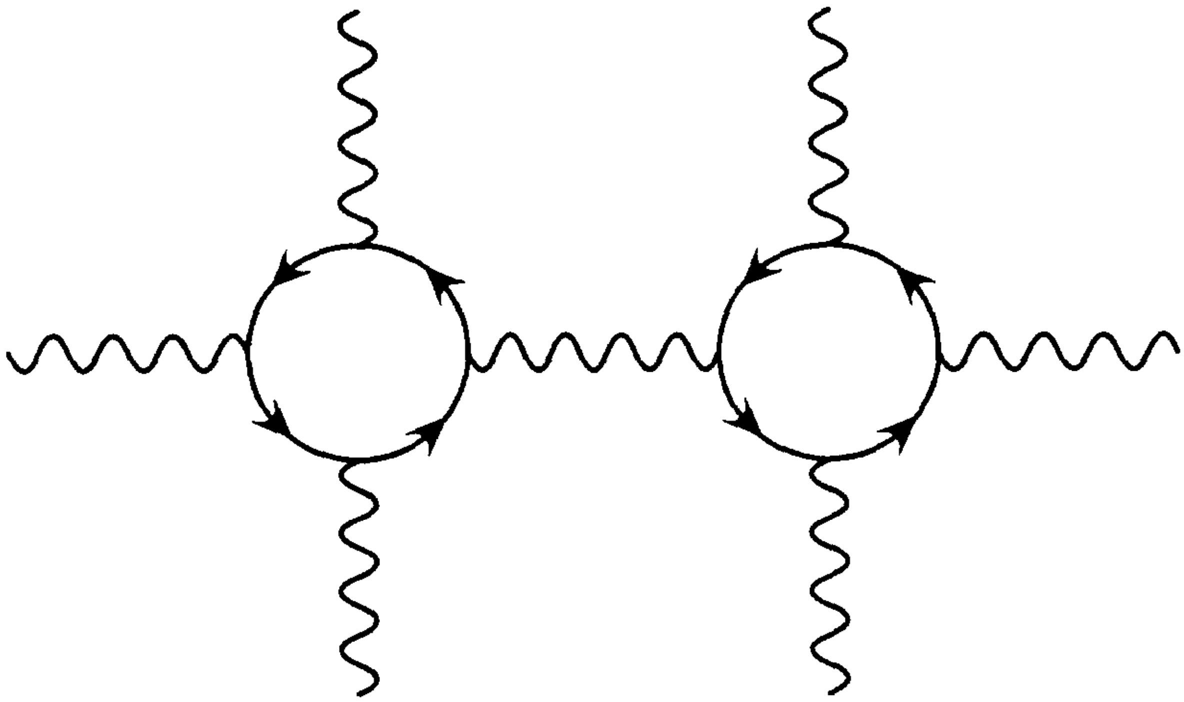
FIG. 1. The basic diagram for n-photon scattering. The blob represents the sum of all possible tree graphs with $n-1$ photons attached to a fermion line.

FIG. 2. Two of the two-loop diagrams for $\gamma\gamma \to \gamma\gamma\gamma\gamma$.

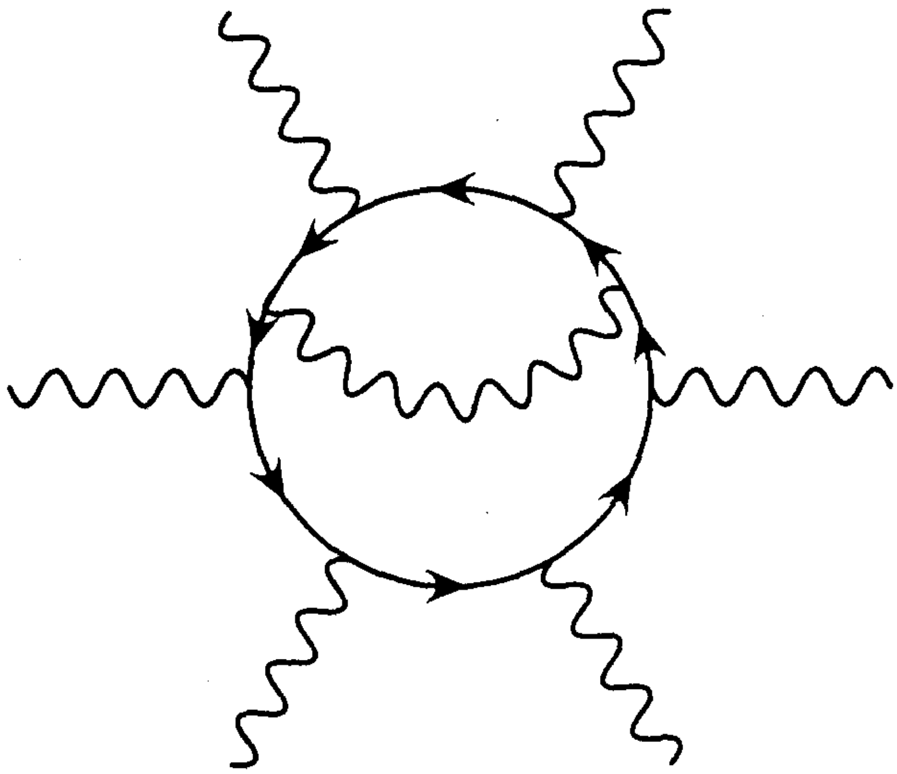
FIG. 3. The light-by-light diagram for the process $e^+e^- \to \gamma\gamma\ldots\gamma$.

FIG. 4. The "jellyfish" diagram for the process $e^+e^- \to \gamma\gamma\ldots\gamma$.

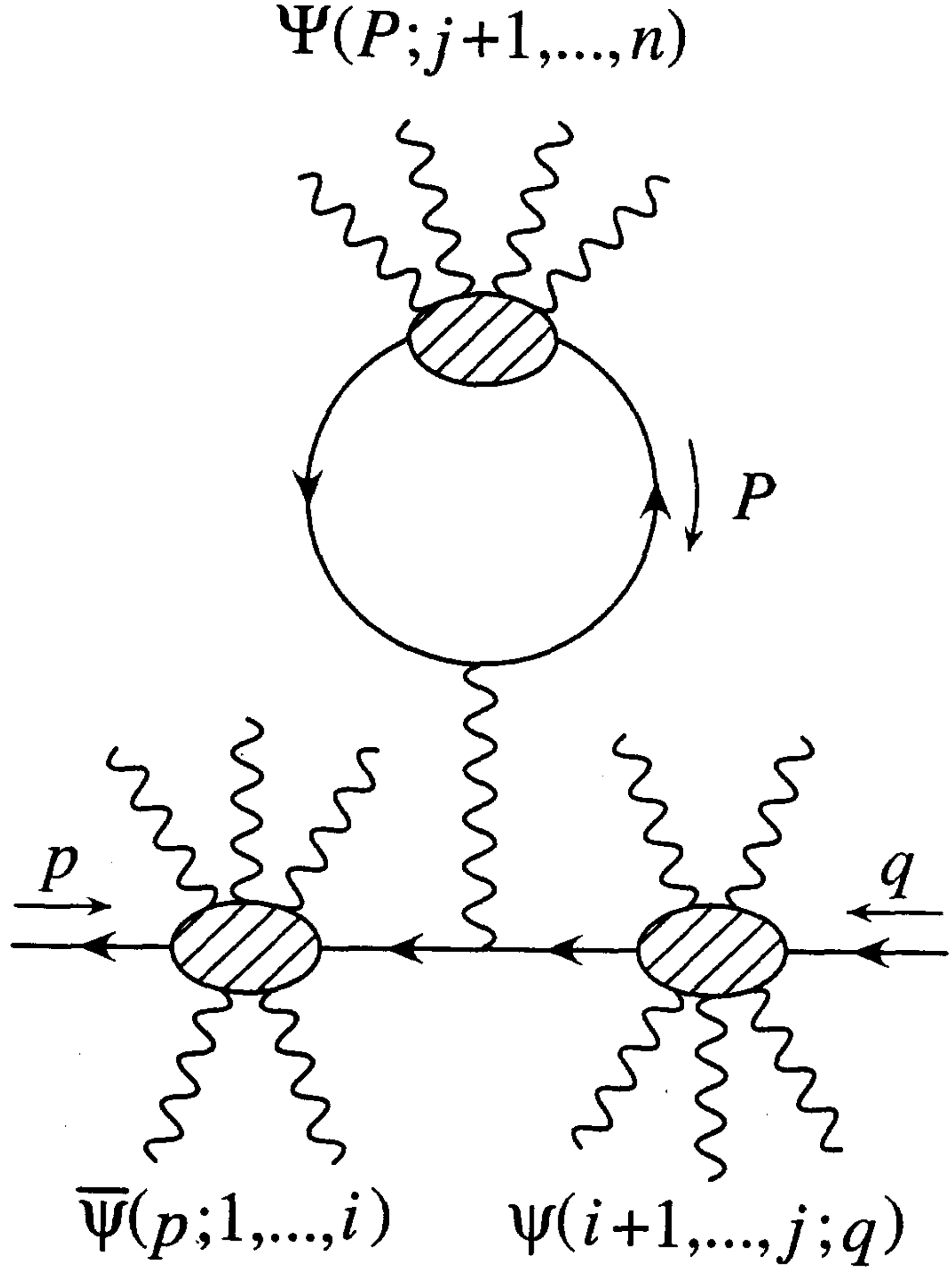

Ψ(P;1,...,n−1)
P
k_n



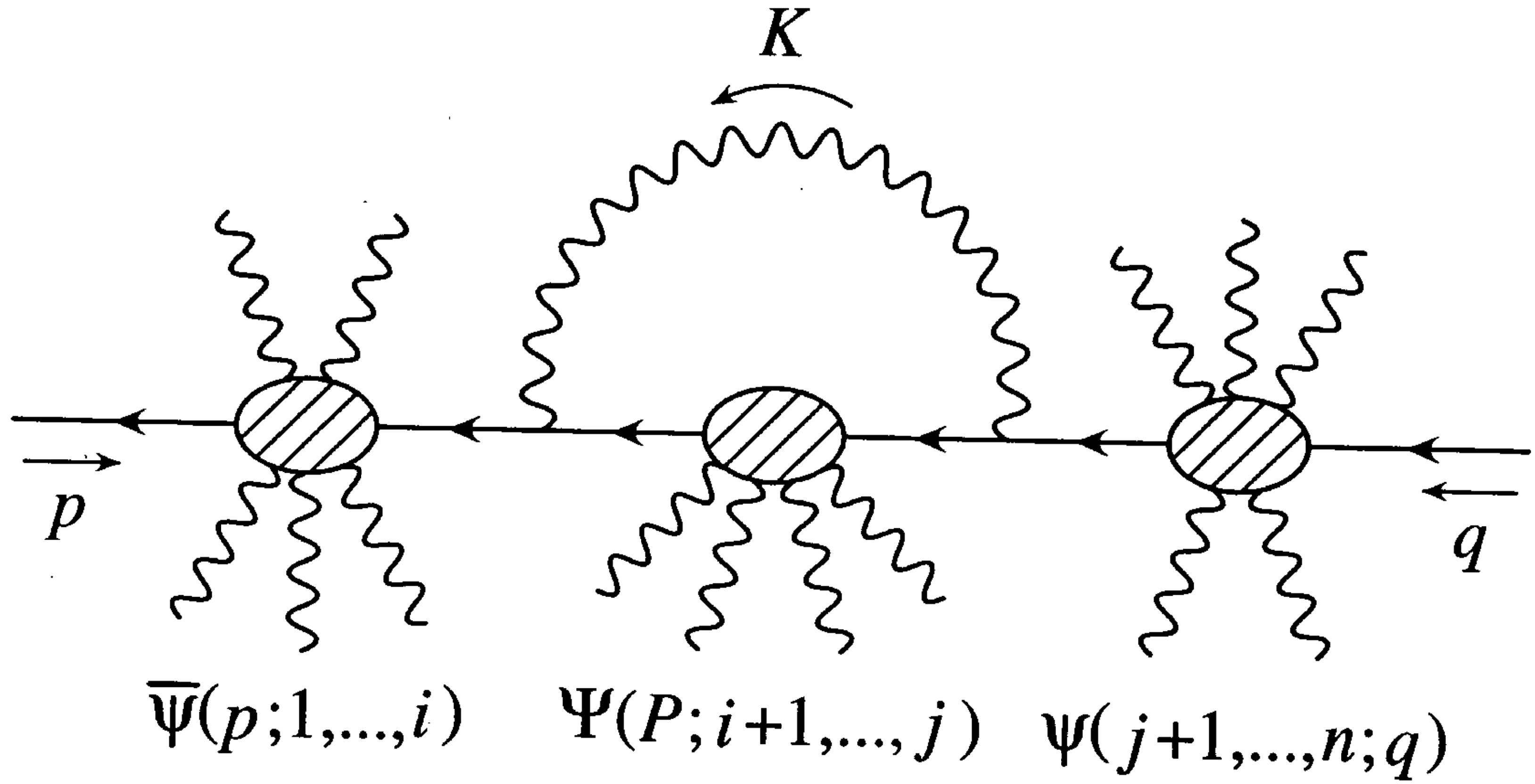
(a)



(b)



Ψ(P; j+1,...,n)
P
p
q
Ψ̄(p;1,...,i)
ψ(i+1,...,j;q)


K
p
q
$\overline{\psi}(p;1,...,i)$
$\Psi(P;i+1,...,j)$
$\psi(j+1,...,n;q)$