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Primordial Black Holes and Generalized Constraints on Chaotic Inflation

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Abstract

Abstract: It is argued that the main quantity of interest in chaotic inflation is the cosmological expansion rate H expressed as a function of the inflaton field ϕ. We derive a general prescription for realising successful inflation in terms of a set of constraints on this function. The formalism is valid for all chaotic inflationary models based on a single scalar field which is minimally coupled to general relativity, so no restrictions on the dynamics of the field are necessary. This technique is used to investigate the possibility that primordial black holes (PBHs) may arise due to adiabatic quantum fluctuations in the inflaton. PBH formation can only be interesting if the amplitude of the fluctuations decreases with increasing mass–scale and this is only possible if the field is accelerating or decelerating sufficiently fast. In this case, limits on the number of PBHs place very interesting constraints on the form of $H(\phi)$ since, together with the *COBE* measurement, they restrict the spectrum of fluctuations over 45 decades of mass. This corresponds to 35 e–foldings of inflationary expansion. If the amplitude of the fluctuations decreases as a power of mass, which is the most interesting situation, then $H(\phi)$ must have a trigonometric form and this allows the constraints to be expressed very simply.

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1 Introduction

The inflationary scenario overcomes the problems of the standard hot big bang model for the early universe by violating the strong energy condition of general relativity at early times [1]. This causes the scale factor of the universe to accelerate and generates reheating when the false vacuum thermalizes, thereby violating the adiabatic assumption which leads to the flatness and horizon problems in the standard model. Inflation is an attractive idea because it suggests that the present state of the observable universe may not depend too strongly on any initial conditions. In the majority of inflationary models, it is assumed that the source of Einstein's field equations is dominated by the potential energy, $V(\phi)$, of a minimally coupled, self-interacting quantum scalar field, ϕ. This field is often referred to as the 'inflaton' due to the current lack of a definite particle physics model and the study of the dynamics of this field is crucial for understanding the physics of the early universe. In the chaotic scenario, which is the subject of the present paper, the field is initially displaced from the global minimum of the potential and proceeds to 'slowly roll' towards it [2]. There are a number of observational constraints that any successful scenario must satisfy and these have been summarized as limits on the potential in ref. [3].

The purpose of the present investigation is to study some of the consequences of dropping the slow-roll approximation. Exact cosmological solutions are few in number and difficult to find when the assumption of slow-roll is relaxed, although some examples include the power law [4] and intermediate inflationary classes [5]. Consequently, if the paradigm is to be extended within the context of a minimally coupled scalar field, a new prescription must be developed which ensures that the general inflationary scenario satisfies the existing constraints. One of the points of this paper is to emphasize that this is best achieved by expressing the expansion rate H during inflation as a function of the scalar field ϕ. This involves using the inflaton as an effective time coordinate and allows the full dynamical behaviour of the field to be investigated in terms of the function $H(\phi)$ without needing to assume that friction terms in the field equations dominate or that the field's kinetic energy is negligible.

In sect. 2 the familiar constraints that any inflationary scenario must satisfy are expressed as restrictions on the function $H(\phi)$. In particular, we derive the condition that there be sufficient inflation and sufficient reheating without exceeding the observed quadrupole anisotropy of the cosmic microwave background radiation (CMBR) [6]. The analysis is valid for an arbitrary $H(\phi)$ (and hence $V(\phi)$) and applies to a *general* chaotic inflationary scenario driven by a minimally coupled scalar field. One is justified in keeping the form of $H(\phi)$ unspecified at this stage in the development of the inflationary scenario because there exist many different particle physics models and the favoured candidate for the unified field theory changes regularly (e.g. supersymmetry, supergravity, superstrings, supermembranes).

In sect. 3 we focus on two more constraints which are associated with the form of the spectrum of perturbations resulting from quantum scalar fluctuations. Firstly, we summarize the constraints on the spectrum imposed by recent anisotropy measurements of the *COBE* satellite [6]. It is important to note that all observations of large-scale galactic structure correspond to mass scales in the range $10^{12}M_\odot$ to $10^{22}M_\odot$ and *COBE*, in particular, only probes scales above $10^{19}M_\odot$. These observations only restrict the form of the spectrum, and hence the form of the inflaton potential, over 10 decades of mass. This is equivalent to only $\ln(10^{10/3}) \approx 8$ of the total number of expansion e–folds during the inflationary epoch. Secondly, we focus on the constraints associated with the possible overproduction of primordial black holes (PBHs). In general, PBH production is important if the spectrum of density perturbations *decreases* with scale. The most important PBH constraints occur at 10^{10} g and 10^{15} g and are due to the photodissociation of deuterium by PBH photons emitted after the nucleosynthesis era and measurements of the γ–ray background respectively. The PBH and *COBE* constraints together restrict the form of the spectrum over 45 decades of mass (viz. 10^{10} g to $10^{22} M_{\odot}$). They are therefore usefully incorporated into the prescription of sect. 2.

In sect. 4 we discuss the conditions under which the PBH constraints apply if such objects are produced by adiabatic density perturbations arising from quantum fluctuations in the inflaton. The alternative possibility of PBH formation via bubble collisions during extended inflation is not considered [44]. We will argue that PBHs may only form if the relative acceleration or deceleration of the field is large compared to its kinetic energy. When roll–over is not slow, this may only apply when the assumption of friction–dominated dynamics is relaxed. In sect. 5 we focus on models in which the amplitude of the fluctuations varies as a power of the mass. We show that this is only possible if $H(\phi)$ has a hyperbolic, trigonometric or exponential form, corresponding to fluctuations which respectively increase or decrease with mass. We examine the latter case in some detail, since it is the one relevant to PBH formation.

There are a number of drawbacks with this approach and these are discussed in sect. 6. Firstly, the inflaton is treated entirely classically and we discuss the stochastic method in which the field is split into a long–wavelength (classical) part and a short–wavelength (quantum) part. Secondly, our treatment is not suitable for discussing the reheating phase or the process whereby inflation ends.

2 Recipe for Successful Generalized Inflation

The three main quantities of interest in the inflationary scenario are the evolution of the scale factor, $a(t)$, with respect to the cosmic time, t, the physics of the model determined by the scalar potential, $V(\phi)$, and the dependence of the primordial fluctuation spectrum on comoving wavenumber, k. For a flat D–dimensional Friedmann universe with a topology $R^1 \otimes S^{D-1}$ and a stress tensor dominated by a single, spatially homogeneous scalar field self–interacting through $V(\phi)$, the energy, momentum and scalar field equations are

$$(D-2)H^2 = \frac{2\kappa^2}{(D-1)}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.1}$$

$$(D-2)\dot{H} = -\kappa^2\dot{\phi}^2 \tag{2.2}$$

$$\ddot{\phi} + (D-1)H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{2.3}$$

where $H \equiv \dot{a}/a$, $\kappa^2 \equiv 8\pi m_P^{-2}$, $m_P \approx 10^{19}$ GeV is the Planck mass, a dot denotes differentiation with respect to cosmic time, units are chosen such that $\hbar = c = 1$ and we assume $D \geq 3$. When $D = 4$ the primordial density spectrum arising from scalar quantum fluctuations is given by

$$A_S(k) = \frac{bH^2}{4\pi^{3/2}|\dot{\phi}|}, \tag{2.4}$$

where the quantities on the right-hand-side are evaluated when the fluctuation first goes outside the horizon and $A_S(k)$ is the amplitude when it re-enters the horizon at $t_{\rm HC}$ during the post inflationary Friedmann phase [4,7]. The term 'horizon' here just means the inverse Hubble distance, so the scale at horizon crossing is determined by the equation $k(t) = a(t)H(t)$. The factor b reflects the evolution of the fluctuations when they are larger than the horizon. $b = 4$ for $t_{\rm HC} < t_{\rm eq}$ and $b = 2/5$ for $t_{\rm HC} > t_{\rm eq}$, where $t_{\rm eq}$ is the time the matter and radiation densities are equal. This corresponds to an horizon mass of $M_{\rm eq} = 10^{15} M_\odot$ [3].

The validity of eq. (2.4) requires discussion. Its derivation assumes that $\ddot{\phi}$ can be neglected and that the scale factor grows quasi-exponentially. In this paper, we will assume that a similar expression holds for the full scalar field dynamics and a general inflationary solution. The application of this problem to power law inflation, $a \propto t^n$ with $n > 1$, was investigated in ref. [4] and it was shown that eq. (2.4) is also correct for this solution. Since any analytical solution can always be expanded as a power series over the narrow range of e-folds relevant for large-scale structure, this suggests that our assumption is valid as a first approximation.

The standard approach in inflation is to specify the physics by choosing an appropriate form for $V(\phi)$ and assuming the friction-dominated and slow-roll conditions, $|\ddot{\phi}| \ll H|\dot{\phi}|$ and $\dot{\phi}^2 \ll V(\phi)$, respectively. Eqs. (2.1)-(2.3) are then solved to determine $\{\phi(t), a(t)\}$ and it is found that $A_S(k)$ is scale-invariant up to a logarithmic term in k for the simplest models [1]. However, it has recently been shown how one may start with any desired form of $a(t)$ or $A_S(k)$ and work backwards to derive the form of the potential [8,9]. In such an analysis, the aim is to determine the potential that gives the theoretical model most closely related to observations rather than identify $V(\phi)$ with a known field theory. However, one should obviously give some physical justification for the form of the potential derived.

Further insight is gained by noting eqs. (2.1)-(2.3) may be combined into a set of first-order equations [10,11]

$$\left(\frac{dH}{d\phi}\right)^2 - \left(\frac{D-1}{D-2}\right)\kappa^2 H^2(\phi) = -\frac{2\kappa^4}{(D-2)^2}V(\phi) \tag{2.5a}$$

$$\kappa^2\dot{\phi} = -(D-2)\frac{dH}{d\phi}. \tag{2.5b}$$

The scalar field is then used as an effective time coordinate and this implies that functions such as $a(t)$ and $A_S(k)$ may be expressed as functions of ϕ. From eqs. (2.4) and (2.5b) we find

$$a(\phi) = a_i \exp\left(-\frac{\kappa^2}{(D-2)}\int_{\phi_i}^{\phi} d\phi' H(\phi')\left(\frac{dH(\phi')}{d\phi'}\right)^{-1}\right) \tag{2.6}$$

$$A_S(\phi) = \frac{b\kappa^2}{8\pi^{3/2}}\frac{H^2(\phi)}{|H'(\phi)|} \tag{2.7a}$$

$$k(\phi) = a(\phi)H(\phi) \tag{2.7b}$$

$$\int_{\phi_i}^{\phi} d\phi' \left(\frac{dH(\phi')}{d\phi'}\right)^{-1} = -\left(\frac{D-2}{\kappa^2}\right)(t - t_i), \tag{2.8}$$

where a prime denotes differentiation with respect to ϕ and eq. (2.6) follows from eq. (2.5b) and the definition of $H(t)$. The solution (2.6) for $D = 4$ was first derived in ref. [11].

It is apparent from eqs. (2.5)–(2.7) that the quantities $\{V(\phi), a(t), A_S(k)\}$ may be determined directly once the form of $H(\phi)$ is known. For this reason, it has been suggested that it is more efficient to begin by specifying the form of $H(\phi)$, rather than $V(\phi)$, $a(t)$ or $A_S(k)$ [12]. Though one might prefer to treat $V(\phi)$ as the fundamental quantity, this method has a number of advantages over existing approaches for generating solutions, since the potential and the dynamics of the scalar field are incorporated into the form of $H(\phi)$ via eqs. (2.5a) and (2.5b). Indeed, the scalar field equation (2.3) is recovered by differentiating eq. (2.5a) with respect to ϕ and substituting in eq. (2.5b). The first advantage is that only two integrations (eqs. (2.6) and (2.8)) are required to find $a(t)$ as opposed to three in existing methods. The second is that the over–damping assumption $|\ddot{\phi}| \ll H|\dot{\phi}|$ need not be made, thereby allowing more general results to be found. Indeed this method has already been used to investigate whether inflationary universes with $\Omega_0 \approx 0.2$ are possible [12,13]. The connection between $V(\phi)$, $a(t)$ and $A_S(k)$ is summarized in Fig. (1), which shows that these quantities are linked by $H(\phi)$.

Figure 1

We now show that it is possible to express all the observational constraints on inflation in terms of limits on $H(\phi)$ by using eqs. (2.5)–(2.7). Consequently, a recipe can be presented for obtaining successful inflation from the form of $H(\phi)$ alone. A similar prescription in terms of $V(\phi)$ was given by Steinhardt & Turner [3] for the case of new inflation but was restricted to the slow–roll regime of the scalar dynamics. The procedure presented here holds for any inflationary scenario driven by a single, minimally coupled scalar field.

a) *$H'(\phi)$ is monotonic:* From eq. (2.5b) the use of ϕ as an effective time coordinate is only valid if $\dot{\phi}$ does not pass through zero. This is because the transformation that leads to equation (2.5b) is invalid when $\dot{\phi}$ changes sign. Therefore this formalism is not suitable for discussing the physics of reheating during the final stages of the inflationary epoch because the field will be oscillating about some global minimum in the potential. Consequently, one must first ensure that the sign of H' remains fixed for consistency.

b) ***Violation of the strong energy condition:*** A necessary and sufficient condition for inflation is that the strong energy condition be violated. This condition may be rewritten as

$$\ddot{a} > 0 \qquad \Longleftrightarrow \qquad \left|\frac{d\ln H(\phi)}{d\phi}\right| < \frac{\kappa}{(D-2)^{1/2}}. \tag{2.9}$$

Eq. (2.9) can be used to specify the values of ϕ at the start of inflation (ϕ_i) or at the end (ϕ_f) depending on the form of $H(\phi)$. We note that inflation becomes more difficult as the dimensionality of the space–time increases, in the sense that the range of parameter space leading to a violation of the strong energy condition decreases for a given functional form for $H(\phi)$. A similar observation was made in ref. [14] for the specific example of inflation in a higher dimensional space–time driven by an exponential potential. In this case, it was shown that inflation will not occur if $D > 6$.

c) *Sufficient inflation*: Eq. (2.6) implies that the number of expansion e–folds since the field had a value ϕ to the end of inflation is

$$N(\phi,\phi_f) \equiv \ln\left(\frac{a(\phi_f)}{a(\phi)}\right) = -\frac{\kappa^2}{(D-2)}\int_{\phi}^{\phi_f} d\phi' H(\phi')\left(\frac{dH(\phi')}{d\phi'}\right)^{-1}. \qquad (2.10)$$

Scales corresponding to the present observable universe will first leave the 'horizon' (i.e. the scale H^{-1}) when the field has some value ϕ_U and then enter it again after inflation has ended. The total number of expansion e–folds between our observable universe leaving the horizon and the end of inflation is therefore $N(\phi_U,\phi_f)$ and the flatness, horizon and smoothness problems are solved if this is 60, although the exact figure depends on the reheat temperature [3]. Smaller scales leave the horizon later and re–enter it earlier.

d) *Quadrupole anisotropy of the cosmic microwave background*: Observations of the quadrupole anisotropy of the CMBR place constraints on the amplitude of scalar perturbations (A_S) and tensor perturbations (A_T) on the current horizon scale [6,15]. The observed temperature quadrupole is $(\Delta T/T)_Q^2 = (0.48 \pm 0.15)^2 \times 10^{-10}$ and assuming such an anisotropy is due to the Sachs–Wolfe effect [16] implies

$$A_S(10^{22}M_\odot) = \frac{\kappa^2}{20\pi^{3/2}}\frac{H^2}{|H'|} \approx 5\times 10^{-6} f_S \qquad (2.11)$$

$$A_T(10^{22}M_\odot) = \frac{\kappa}{4\pi^{3/2}}H \approx 5\times 10^{-6}(1-f_S). \qquad (2.12)$$

The function f_S, satisfying $0 \le f_S \le 1$, determines the relative contribution of the scalar and tensor modes to the anisotropy. Calculation of this quantity is complicated and requires a detailed knowledge of the form of A_S and A_T. In general, however,

$$\frac{A_S}{A_T} = \frac{b\kappa}{2}\frac{H}{|H'|} = \frac{f_S}{1-f_S} \qquad (2.13)$$

and the ratio $|H'|/H$ allows a rough estimate of f_S to be made in many examples. When $M < M_{eq}$ we must take $b = 4$ in eq. (2.13). This implies $A_S > A_T$ if $|H'|/H < 2\kappa$, and since eq. (2.9) implies $|H'|/H < \kappa/\sqrt{2}$, this condition is always satisfied. However, $b = 2/5$ on the quadrupole scale, so $A_S > A_T$ only if $|H'|/H < \kappa/5$ in this case. It is well known that $A_S \gg A_T$ in the limit of slow–roll [4], but the $H(\phi)$ method places an exact upper limit on the ratio of the kinetic and potential energies of the field for this condition to be satisfied. Eq. (2.13) implies that $f_S < 1/2$ for $\kappa/5 < |H'|/H < \kappa/\sqrt{2}$ and this is likely to apply for any model in which there is significant deviation from the slow–roll approximation. In this case the tensor modes dominate the quadrupole measurement and the possible consequences of this for the cold dark matter model of galaxy formation have recently been discussed in a series of papers [17]. It is interesting to note that the scalar and tensor modes only have the same scale dependence when the functional form of $H(\phi)$ is exponential and this is the case which leads to power law inflation.

e) *Reheating and baryogenesis*: At the end of inflation the energy of the scalar field is converted into radiation. Chaotic inflation is not specific enough for a detailed discussion of the reheating process to be given. However, by relating the energy density of the scalar field, $\rho_\phi = (D-2)(D-1)H_f^2/(2\kappa^2)$, to the energy density of the radiation field, $\rho_{\rm rad} = \pi^2 d(D) g(T) T^{4(D-1)/3}/30$, an upper limit on the reheating temperature is obtained. Here $d(D)$ is a weak function of D such that $d(4)=1$, $g(T)$ represents the number of relativistic degrees of freedom and $g(T_{RH}) = O(100)$ in the standard model. The efficiency of the reheating process can be parametrized by the ratio $\epsilon \equiv \rho_{\rm rad}/\rho_\phi$. Baryogenesis can proceed via the decay of supermassive Higgs bosons if the reheat temperature

$$T_{RH} = \left(\frac{15(D-1)(D-2)\epsilon}{8\pi^3 g d(D)}\right)^{3/[4(D-1)]} (H_f m_P)^{3/[2(D-1)]} \tag{2.14}$$

exceeds 10^9 GeV [18]. This constraint is derived by assuming that the lifetime of the Higgs boson created when ϕ decays is longer than H^{-1}. We note that the maximum reheat temperature becomes lower as D increases. When $D=4$, $T_{RH} \approx 0.2(H_f m_P)^{1/2}$ and the constraint (2.12) arising from the gravitational wave spectrum can be used to provide an upper limit on the reheat temperature. Since eq. (2.2) implies $\dot{H} < 0$, we necessarily have $H(\phi_f) < H(\phi_U)$, and so eq. (2.12) may be substituted into eq. (2.14) to yield $T_{\rm RH} \leq 1.4 \times 10^{16} \epsilon^{1/4}$ GeV. Thus T_{RH} must be in the range $10^9 - 10^{16}$ GeV.

f) *Quantum gravity effects*: A semi-classical description of the universe is only possible if $V(\phi_i) \leq m_P^4$ and $\dot{\phi}_i^2 \leq m_P^4$ [19]. Equality corresponds to the Planck epoch which is the earliest time when one can discuss classical fields on a classical space-time. In terms of $H(\phi)$ these constraints become

$$|H'| \leq \frac{8\pi}{(D-2)} \tag{2.15}$$

$$H \leq \left(\frac{24\pi}{(D-1)(D-2)}\right)^{1/2} m_P, \tag{2.16}$$

where eqs. (2.5a) and (2.5b) have been used and equality applies in both equations together. As D increases, these conditions become more stringent.

By following the above prescription it is possible to demonstrate that a period of successful inflation occurs *without* assuming slow-roll and *without* the need to solve the field equations explicitly. Indeed, this code provides a new way of obtaining successful inflation in models where an analytical expression for $a(t)$ cannot be found, i.e. when eq. (2.8) is not invertible. Note that the conditions for quasi-de Sitter expansion (slow-roll) and friction-dominated scalar dynamics may be rewritten as

$$\left|\frac{\dot{H}}{H^2}\right| \ll 1 \qquad \Longleftrightarrow \qquad \left|\frac{H'}{H}\right| \ll \kappa \tag{2.17}$$

$$|\ddot{\phi}| \leq (D-1)H|\dot{\phi}| \qquad \Longleftrightarrow \qquad \left|\frac{H''}{H}\right| \leq \kappa^2 \left(\frac{D-1}{D-2}\right), \tag{2.18}$$

respectively. It should be emphasised that, for some forms of $H(\phi)$, it is possible to have $|H''(\phi)| \approx \kappa^2 H(\phi)$ when $|H'(\phi)| \ll \kappa H(\phi)$, so one may have a quasi-de Sitter solution *without* neglecting the $\ddot{\phi}$ term. In other words, friction-dominated dynamics is not identical to the slow-roll approximation.

With the exception of the density perturbation constraints, we have derived the conditions for successful inflation for arbitrary space–time dimension. We have kept the analysis as general as possible in the hope that some special features will arise in some theories when $D = 4$. In particular, if a given theory only violates the strong energy condition when $D = 4$, this would provide a possible solution to the uniqueness problem of particle physics. It would appear that successful inflation becomes more difficult to achieve as the dimensionality is increased which suggests that lower values of D are favoured.

3 Constraints from PBHs and the CMBR

An important feature of all inflationary scenarios is that they predict the existence of adiabatic density fluctuations, as indicated by eq. (2.4). We have already seen how observations of the quadrupole anisotropy of the microwave background constrain the inflationary model (viz. eqs. (2.11) to (2.13)). However, there are also constraints on the amplitude of the fluctuations on much smaller scales than the current horizon and, depending on the form predicted for $A_S(M)$, these may place more or less interesting restrictions on the scenario. In this section, we discuss the various limits on $A_S(M)$. The most important upper limits on $A_S(M)$ are associated with the CMBR anisotropies on large scales and primordial black hole (PBH) formation on small scales. There are also limits associated with the absence of spectral distortions in the CMBR at intermediate scales. The wide range of scales encompased by these limits provides very stringent constraints on inflationary models in which the amplitude of fluctuations decreases significantly with scale or exhibits peaks over a wide range of scales. The constraints are summarized in Fig. (2) and we now justify them.

Figure 2

Various upper limits can be placed on the CMBR anisotropies, $\Delta T/T$, on angular scales above an arcminute. All of them are in the range $2 - 5 \times 10^{-5}$. These constraints came from satellite and balloons above a few degrees and ground–based radio telescopes on smaller scales. Recently, *COBE* has claimed a positive detection for scales between 10^o and 90^o and, in principle, this gives direct information on the spectrum of density fluctuations at decoupling. However, since the mass associated with an angular scale θ is $M \approx 10^{17}(\theta/\text{degrees})^3 M_\odot$, *COBE* only probes the long wavelength part of the spectrum and not the part associated with galaxies and clusters themselves. Previous analyses of the *COBE* data have assumed that a direct extrapolation to smaller scales can be made unambiguously, but it is plausible that strong features in $A_S(M)$ occur below 10^o. The precise connection between $A_S(M)$ and $\Delta T/T(\theta)$ depends on the nature of the fluctuations (adiabatic, isothermal or isocurvature) and the nature of the dark matter (hot, cold or baryonic). Only observations above 10^o are unambiguous since reionization could modify anisotropies below 10^o. The *COBE* measurements would seem to be consistent with the unbiased cold dark matter scenario, in which the horizon–scale fluctuations $A_S(M)$ are scale invariant [6,20]. However, the large–scale structure data (in particular, evidence from the APM and QDOT surveys and streaming motions) would seem to require some modification [21]. Spatially flat CDM models with $\Omega < 1$ and a cosmological constant may also be consistent with all the data [22]. In view of these uncertainties, the only secure, model-independent constraint is associated with the *COBE* quadrupole anisotropy as shown in Fig. (3).

The CMBR spectral distortion constraints are associated with the fact that the damping of adiabatic density fluctuations at high redshifts ($z > 10^5$) will inject energy into the primordial plasma, thereby inducing a Bose–Einstein spectrum with chemical potential μ. Since *COBE* observations require $\mu < 0.01$, this places the constraints on $A_S(M)$ indicated in Fig. (2). The curves are taken from ref. [23] and correspond roughly to $A_S(M) < \sqrt{\mu}$; the relevant mass range corresponding to the fluctuations which enter the horizon at $z > 10^5$. Daly has also examined this constraint [24].

We now examine the constraint on A_S associated with PBHs. The first one derives from the fact that any PBHs which survive today must certainly have less than the critical density. In the standard radiation-dominated model for the early universe, this implies that the fraction of the universe going into PBHs at time t must be less than $10^{-5}(t/s)^{1/2}$. Since PBHs must have of order the horizon mass, $M_H \approx 10^5(t/s)M_\odot$, at their formation epoch [25, 40], this implies that the fraction of the regions of mass M collapsing at the horizon epoch must satisfy

$$\beta(M) < 10^{-8}\left(\frac{M}{M_\odot}\right)^{1/2} \approx 10^{-17}\left(\frac{M}{10^{15}\mathrm{g}}\right)^{1/2}, \tag{3.1}$$

where 10^{15}g is the mass above which the PBH density will have been unaffected by evaporations [26]. This constraint would be weakened by a factor of 10^7 if the early universe were 'cold' with the microwave background being generated at some late epoch, since the radiation equation of state would then only pertain before 10^{-4}s. However, this possibility now seems unlikely in view of the success of the standard cosmological nucleosynthesis scenario [27].

In order to interpret condition (3.1) as a constraint on $A_S(M)$, we will assume that the primordial fluctuations on a scale M have a Gaussian distribution with rms amplitude $\delta_{\mathrm{rms}}(M)$. If the fluctuations are spherically symmetric and the background equation of state is $p = \gamma\rho$ $(0 < \gamma < 1)$, one then expects [25]

$$\beta(M) \approx \delta_{\mathrm{rms}}(M)\exp\left(-\frac{\gamma^2}{2\delta^2_{\mathrm{rms}}(M)}\right). \tag{3.2}$$

Eq. (3.1) thus places an upper limit on δ_{rms} as a function of M. For $\gamma = 1/3$ (as expected in a radiation-dominated early universe), it is given implicitly by

$$\delta_{\mathrm{rms}}(M) < 0.15\left[17 - \frac{1}{2}\log\left(\frac{M}{10^{15}\mathrm{g}}\right) + \log\delta_{\mathrm{rms}}(M)\right]^{-1/2} \approx 0.2\left[31 - \log\left(\frac{M}{10^{15}\mathrm{g}}\right)\right]^{-1/2}, \tag{3.3}$$

where in the second expression we have anticipated that the log δ_{rms} term is about -1.5. At 10^{15}g, this gives $\delta_{\mathrm{rms}} < 0.04$. The fluctuations can be larger at higher masses but only by a logarithmic factor. For example, $\delta_{\mathrm{rms}} < 0.06$ at $1M_\odot$ and $\delta_{\mathrm{rms}} < 0.08$ at $10^6 M_\odot$.

In fact, the constraints are somewhat stronger just below 10^{15}g on account of quantum emission [28]. In particular, the 100 MeV gamma-ray background measurements imply that 10^{15}g PBHs could have at most 10^{-8} times the critical density [29]. The factor 17 in eq. (3.3) then becomes 25, so the upper limit on δ_{rms} at 10^{15} g drops to 0.03. For PBHs which have evaporated completely, there are also limits on $\beta(M)$ associated with entropy production [30]

$$\beta(M) < 10^{-8}\left(\frac{M}{10^{11}\mathrm{g}}\right)^{-1}, \qquad M < 10^{11}\mathrm{g}, \tag{3.4}$$

and distortion of the microwave background [31]

$$\beta(M) < 10^{-18}\left(\frac{M}{10^{11}\mathrm{g}}\right)^{-1}, \qquad 10^{11}\mathrm{g} < M < 10^{13}\mathrm{g}, \tag{3.5}$$

and cosmological nucleosynthesis constraints

$$\beta(M) < \begin{cases} 10^{-15}\left(\frac{M}{10^{9}\mathrm{g}}\right)^{-1}, & 10^{9}\mathrm{g} < M < 10^{13}\mathrm{g} \\ 10^{-21}\left(\frac{M}{10^{10}\mathrm{g}}\right)^{1/2}, & M > 10^{10}\mathrm{g} \\ 10^{-16}\left(\frac{M}{10^{9}\mathrm{g}}\right)^{-1/2}, & 10^{9}\mathrm{g} < M < 10^{10}\mathrm{g}. \end{cases} \tag{3.6}$$

The last three limits are associated with the increase of the background photon–to–baryon ratio by PBH photons emitted after nucleosynthesis [32], photodissociation of deuterium by such photons [33], and modification of the neutron–to–proton ratio by PBH nucleons emitted before nucleosynthesis [34]. The first limit in Eq. (3.6) is weaker than limit (3.5). The other limits on $\beta(M)$ are summarized in Fig. (2), which is adapted from ref. [47] and the associated constraints on $A_S(M) \equiv \delta_{rms}(M)$, derived from eq. (3.2), are shown in Fig. (3). Note that the deuterium constraint on $A_S(M)$ at 10^{10} g is about 0.03, comparable to the γ–ray constraint at 10^{15}g. It is interesting that the constraints in Figs. (2) and (3) would be modified if the equation of state in the early Universe was ever soft [48], because γ would be very small in Eq. (3.2). In particular, this might occur for a while when the scalar field is oscillating in the minimum of the potential at the end of inflation [49]. However, we neglect this complication here.

Figure 3

When taken together, these constraints are very powerful because they restrict the spectrum over 45 decades of mass. However, it is clear that the PBH constraints are only interesting if the spectrum decreases with scale by a sufficient factor. If the fluctuations can be described by a single power law, $A_S \propto M^{-\alpha}$, then the combination of the *COBE* quadrupole result ($A_S \approx 5 \times 10^{-6}$ at 10^{55}g) and the PBH deuterium constraint ($A_S < 3 \times 10^{-2}$ at 10^{10}g) means that the factor must be 6000 over 45 decades of mass. This requires $\alpha > 0.08$, as indicated by the dashed line in Fig. (3). This is marginally consistent with the *COBE* results since these require α to be in the range $0.1 > \alpha > -0.07$ at the 1–sigma level. The PBH deuterium constraint is therefore slightly stronger than the one derived from the *COBE* detection alone.

However, in the standard cold dark matter model, Liddle *et al.* have argued that $\alpha = -0.08$ provides the best fit to the APM data and this may well exclude any PBH formation [35]. Liddle and Lyth [7] have also derived a stronger constraint on α by combining results from the IRAS/QDOT galaxy survey [45], the POTENT peculiar velocity maps [46] and the *COBE* DMR experiment. They conclude α must be in the range $0.05 > \alpha > -0.05$ at the 2–sigma level. This limit is shown as a dot–dash line in Fig. (3). In these cases, one would require the index α to vary with scale for PBHs to be interesting, becoming positive in at least some mass range above 10^{10} g. It is important to note, however, that these limits on the spectral index are derived from the assumption that the spectrum is a featureless power law and are not strictly valid in more general examples. Indeed, for more general spectra any limits obtained from galaxy surveys and the CMBR only strictly apply above 10^{45} g. We next consider the circumstances under which more complicated spectra may arise.

4 Conditions for the PBH Constraint to Apply

It is usually assumed that the spectrum of density perturbations in the inflationary scenario is nearly scale–invariant. This is a direct consequence of the invariance of de Sitter space under a time–translation. Since the physical size of a fluctuation which crosses the horizon ($\sim H^{-1}$) and the expansion rate of the universe are constant, each scale has a perturbation of the same amplitude. In practice, the assumption of slow–roll leads to a small deviation from de Sitter space and a logarithmic dependence of $A_S(k)$ on k for power law potentials. This can be seen quantitatively by taking the logarithm of eq. (2.7b), differentiating with respect to ϕ and then using eq. (2.6) to obtain

$$(\ln k)' = \frac{H'}{H} - \frac{\kappa^2}{2}\frac{H}{H'}. \tag{4.1}$$

For a quasi–de Sitter space (i.e. for $|H'|/H \ll \kappa$ from eq. (2.17)), the first term on the right–hand–side is negligible, so eq. (2.10) implies $N(\phi) \propto \ln k(\phi)$ and eq. (2.7a) gives $A_S(k) \propto H dN/d\phi$. For example, in the power law model $H \propto \phi^n$, we get $N \propto \phi^2$ and this implies $A_S \propto (\ln k)^{(n+1)/2}$ which is nearly, though not exactly, scale–invariant.

In fact, the definition of inflation given by (2.9) implies that the second term of the right–hand–side of eq. (4.1) always dominates over the first. This suggests that a strong scale–dependence is difficult to achieve for a quasi–exponential expansion since $dN/d\phi$ is nearly constant as a function of k and there is no feature in the functional form of $N(\phi)$ [11]. However, scales of cosmological interest correspond to only a narrow range of $N(\phi)$ and deviations from a logarithmic dependence may be possible if a feature occurs in the potential (or equivalently in $N(\phi)$) [9, 36]. Moreover the *COBE* and PBH constraints together span $\ln(10^{15}) \approx 35$ e–foldings of inflationary expansion. If it is plausible for a feature to occur over cosmological scales, it is certainly reasonable to suppose that a deviation from scale invariance is possible over this significantly wider range.

We now establish whether the spectrum increases or decreases with scale in order to determine when the PBH constraints are important. In the following discussion, we will assume for simplicity that the spectrum does not exhibit any features. If the amplitude increases with scale ($dA_S/dM > 0$), the *COBE* constraint is the most interesting, whereas the PBH constraint may be the most interesting if the amplitude decreases with scale ($dA_S/dM < 0$). The sign of dA_S/dM is easily determined from the functional form of $H(\phi)$ and its first derivative. The form of $A_S(\phi)$ follows from eq. (2.7a) and $H'(\phi)$ determines the sign of $\dot{\phi}(t)$ via eq. (2.5b). This allows us to determine whether $\phi(t)$, and hence $A_S(t)$, increases or decreases with time during the first horizon crossing. If $dA_S/dt > 0$, then $dA_S/dM < 0$ and vice–versa. This simple qualitative test should always be applied when investigating constraints on inflation.

The argument can be made more quantitative in the following way. Since eq. (2.2) requires $\dot{H}(t) < 0$, the scalar perturbation spectrum will only decrease with increasing mass–scale if $|H'|$ also *decreases* as the inflaton evolves. Since

$$dH'/dt = \dot{\phi}H'' = -2H'H''/\kappa^2 \tag{4.2}$$

from eq. (2.5b), this implies that a necessary, but not sufficient, condition for the PBH constraint to apply is that the curvature of $H(\phi)$ be positive definite, i.e. $H'' > 0$. This in turn implies that the potential must be convex. One can conclude, therefore, that PBHs are not interesting if the field lies within the vicinity of a local maximum of $H(\phi)$, and therefore $V(\phi)$, during the relevant horizon crossing. Note that, depending on the sign of H', ϕ may be increasing or decreasing with time. The second time derivative of the field is related to H'' by the expression $\kappa^4\ddot{\phi} = 4H'H''$. Thus, if $H' > 0$ ($H' < 0$), the field must be accelerating (decelerating) for the spectrum to decrease with scale.

In general, it follows by writing $dA_S/dM = (dA_S/d\phi)/(d\phi/dM)$ and differentiating eq. (2.7a) that a necessary and sufficient condition for $dA_S/dM < 0$ is

$$\left(\frac{H'}{H}\right)^2 < \frac{H''}{2H}, \tag{4.3}$$

Since condition (2.9) for the violation of the strong energy condition must also hold, we also infer that a sufficient, but not necessary, condition for a spectrum which decreases with scale is

$$\frac{H''}{H} > \kappa^2. \tag{4.4}$$

From eq. (2.18) this necessarily entails violating the friction–dominated assumption. Since the ratio of $|\ddot{\phi}|$ to $3H|\dot{\phi}|$ is specified by $|H''|/H$, one must therefore consider the full dynamics of the theory by including the contribution from $\ddot{\phi}$ in the scalar field equation if the amplitude of perturbations is to decrease with scale. However, for slow–roll, $|H'|/H \lll \kappa$ and the friction–dominated assumption need not necessarily be violated in all cases. It is important to note, though, that eq. (4.3) necessarily requires some contribution from $|\ddot{\phi}|$.

This analysis suggests that PBH formation will be difficult to achieve in many scenarios for which the inflaton is minimally coupled to gravity, because the form of the potential must be restricted. Consequently, an observation confirming the existence of PBHs would significantly alter our understanding of the inflationary scenario if one assumes that PBHs arise due to the power spectrum of quantum fluctuations in the inflaton. Indeed, eq. (4.3) suggests that it is generally rather difficult to obtain any spectrum which decreases with mass–scale. In contrast, spectra from cosmic strings necessarily contain extra power on smaller scales [37] and this suggests that a determination of the sign of dA_S/dM over cosmological scales may provide strong support for one of these two competing scenarios.

In conclusion, the extension of the prescription discussed in sect. 2 is contained within eqs. (4.2)–(4.4), which indicate when the PBH constraint may be important. Most chaotic models in the flat Friedmann metric lead to a perturbation amplitude that increases with scale and the anisotropy of the CMBR, together with high redshift galaxy surveys, provides the strongest constraint. In the next section we investigate which potentials lead to a power law spectrum which decreases with mass.

5 Constraints on Potentials leading to Power law Fluctuations

The aim of this section is to consider a toy model for which the recipe of sect. 2 and the PBH constraint can be employed to place limits on the parameters of the theory for successful inflation. We adopt a phenomenological approach and begin by deriving the form of $H(\phi)$ which leads to the power law spectrum

$$A_S \propto M^{-\alpha} \propto k^{\beta}, \qquad \beta = 3\alpha. \tag{5.1}$$

This is an interesting case because the *COBE* result is consistent with a power law spectrum, but at present does not determine the sign of β [20]. A second-order differential equation in H can be derived by equating eq. (2.7a) with eq. (5.1), differentiating with respect to ϕ and substituting in eq. (4.1) to eliminate $(\ln k)'$. This gives

$$(2-\beta)\frac{H'^2}{H} - H'' = -\mu H, \tag{5.2}$$

where $\mu \equiv \beta\kappa^2/2$. This simplifies with the use of the identity $2H'' \equiv dH'^2/dH$ to a first-order equation in H':

$$\frac{dH'^2}{dH} - 2(2-\beta)\frac{H'^2}{H} = 2\mu H. \tag{5.3}$$

This equation has the exact integral

$$H'^2 = \left(\frac{\mu}{\beta-1}\right)H^2 + CH^{2(2-\beta)}, \qquad (\beta \neq 1), \tag{5.4}$$

where C is an integration constant. For $1 > \beta > 0$ and $C > 0$, eq. (5.4) can be integrated exactly to yield the trigonometric solution

$$H(\phi) = \lambda \sec^n(\omega\phi), \tag{5.5}$$

where

$$n \equiv \frac{1}{1-\beta}, \qquad \lambda \equiv \left[\frac{\beta\kappa^2}{2(1-\beta)C}\right]^{1/[2(1-\beta)]}, \qquad \omega \equiv \sqrt{\frac{\beta(1-\beta)}{2}}\kappa. \tag{5.6}$$

For $\beta < 0$ or $\beta > 1$, one gets the hyperbolic solution

$$H(\phi) = \lambda \operatorname{sech}^n(\omega\phi), \tag{5.7}$$

if $C < 0$, where

$$n \equiv \frac{1}{1-\beta}, \qquad \lambda \equiv \left[\frac{\beta\kappa^2}{2(\beta-1)|C|}\right]^{1/[2(1-\beta)]}, \qquad \omega \equiv \sqrt{\frac{\beta(\beta-1)}{2}}\kappa. \tag{5.8}$$

In either case, $H(\phi)$ contains two free parameters C and β. Two other solutions of eq. (5.4) are obtained when $C = 0$ or $\beta = 0$. These correspond to

$$H \propto \exp(\pm\omega\phi), \qquad \omega \equiv \sqrt{\frac{\beta}{2(\beta-1)}}\kappa, \qquad (\beta < 0), \tag{5.9}$$

and

$$H = \frac{1}{\sqrt{C}\phi}, \tag{5.10}$$

respectively. The former solution is known to give power law inflation, but corresponds to fluctuations which increase with mass-scale. The latter is the *only* form of $H(\phi)$ which gives *exactly* scale–invariant fluctuations. (The special case of $\beta = 1$ has recently been studied in ref. [50].)

The $H(\phi)$ method therefore leads to the important conclusion that only a hyperbolic, trigonometric or exponential form for $H(\phi)$ can lead to a power law spectrum for the density fluctuations. Here we focus on the trigonometric case because this is the only one which could give rise to PBH formation since the fluctuations decrease with mass. The potential is found from eq. (2.5a) to be

$$V(\phi) = \frac{\lambda^2}{\kappa^2}\left[\left(3 + 2n^2\gamma^2\right)\sec^{2n}\omega\phi - 2n^2\gamma^2\sec^{2n+2}\omega\phi\right], \tag{5.11}$$

where $\gamma \equiv \omega/\kappa = 1/\sqrt{8\pi}$ if $\omega = m_P^{-1}$. Thus $1/\omega$ represents the width of the potential and λ corresponds to the height. This is a generalization of a previous result found in ref. [9], which was valid only in the slow–roll regime. Such an approximation holds when $(n-1) \lll 0$, i.e. $|\beta| \lll 1$. Note that $n\gamma$ is determined by the exponent in the fluctuation spectrum since eq. (5.6) implies

$$n\gamma = \sqrt{\frac{\beta}{2(1-\beta)}}. \tag{5.12}$$

As far as we are aware, there is no particle physics model that predicts a potential exactly of this form, but (5.5) leads to analytical results and may be viewed as an approximation to a more complete theory, at least over some range of ϕ.

Due to the periodic nature of the function (5.5), we may assume $\omega\phi \in [0, \pi/2]$ without any loss of generality. The PBH constraints of sect. (3) imply that β (not to be confused with the function $\beta(M)$) be less than 0.3. From (5.12), this requires $n\gamma < 0.5$, and eq. (5.8) implies that $\gamma > 0.3$ and $n < 1.4$. Hence $H(\phi)$ has a minimum at $\phi = 0$ and tends to $+\infty$ at $\omega\phi = \pi/2$. The potential does not vanish at $\phi = 0$ and one ends up with a non-zero cosmological constant. (This could be remedied by adding a constant in eq. (5.12) but then $A_S(M)$ and $H(\phi)$ would not have the simple forms indicated by eqs. (5.1) and (5.5)). We can now use the analysis of sect. (2) to determine when inflation starts and ends. The strong energy condition is violated when

$$\tan^2\omega\phi < \frac{1}{2n^2\gamma^2} = \frac{1-\beta}{\beta} \tag{5.13}$$

and this implies that inflation may occur for all values of β for which the trigonometric potential pertains $(1 > \beta > 0)$. At the Planck time the value of $\omega\phi$ is specified by eq. (2.16) to be $\sqrt{4\pi}\cos^n\omega\phi_P \approx \lambda/m_P$, so the universe will not be inflating initially if

$$\frac{\lambda}{\sqrt{4\pi}m_P} \stackrel{\cdot}{<} \left(\frac{2n^2\gamma^2}{1+2n^2\gamma^2}\right)^{n/2} = \beta^{1/[2(1-\beta)]}. \tag{5.14}$$

This condition is satisfied for the typical values $\lambda/m_P \approx O(10^{-4})$, $\omega \approx O(m_P^{-1})$ and $n \approx 1$. Thus the field proceeds to roll down the potential but inflation does not start until ϕ reaches the value ϕ_i given by

$$\sin^2\omega\phi_i = \frac{1}{1+2n^2\gamma^2} = 1-\beta. \tag{5.15}$$

For small β, $\omega\phi_i$ must be close to $\pi/2$, so inflation starts near the Planck time.

It is now straightforward to deduce from eq. (2.10) that the horizon problem is solved providing

$$\ln\sin\omega\phi_f \leq -\frac{1}{2}\ln(1+2n^2\gamma^2) - 120n^2\gamma^2, \tag{5.16}$$

where the factor 120 corresponds to 60 inflation e–folds for our universe. Thus $\omega\phi_f \ll 1$ and $H(\phi_f) \approx \lambda$, so eq. (2.14) implies that the reheat temperature is given in terms of λ by

$$\frac{\lambda}{m_P} \approx \frac{10^{-37}}{\epsilon^{1/2}}\left(\frac{T_{\rm RH}}{\rm GeV}\right)^2. \tag{5.17}$$

Note that the factor of 120 in eq. (5.16) itself depends on $T_{\rm RH}$ but this only gives a small correction. This expression can be used, along with constraint (2.12) arising from the contribution of gravitational waves to the CMBR quadrupole anisotropy, to place an upper limit

$$\lambda/m_P \leq 2\times 10^{-5}. \tag{5.18}$$

It is interesting that this result is independent of the efficiency of the reheating process and therefore the direct couplings of the inflaton to the supermassive Higgs bosons responsible for the baryogenesis.

In conclusion we have shown that a potential containing two arbitrary parameters will lead to a scalar perturbation spectrum that exhibits a simple power law. Upper limits on the parameters were obtained directly from the PBH and *COBE* constraints. The 'height' of the potential, as determined by λ, is constrained by the effect of primordial tensor modes on the CMBR. In contrast, the 'width' of the potential depends only on ω and uniquely specifies the spectral index of the *scalar* perturbation spectrum. We are therefore able to constrain the form of the potential (5.11) over approximately 35 e–foldings.

6 Conclusions and Discussion

The main idea behind this work has been to treat the function $H(\phi)$ as the fundamental quantity when studying the evolution of self–interacting scalar fields in the early universe. This allows the full dynamical behaviour of the field to be investigated and a recipe was presented in sect. 2 based on a number of constraints on $H(\phi)$ that any inflationary model must satisfy. This analysis suggests that successful inflation becomes more likely when the dimensionality of the space–time is lower.

It was shown in sect. 3 that additional constraints arising from the formation of PBHs may become important when the assumption of friction–dominated dynamics is relaxed. In general, PBHs may be interesting if the amplitude of density perturbations arising from quantum scalar fluctuations is decreasing with mass–scale. It was argued that the relative acceleration or deceleration of the field must be large compared to the kinetic energy for this to occur. These constraints have not been discussed in the literature because it was thought that the assumption of slow–roll would lead to a spectrum that increased with scale or was very nearly scale–invariant. These additional constraints are incorporated into the recipe of sect. 2 and a specific functional form for $H(\phi)$ was investigated in sect. 5 to illustrate these ideas.

However, there are a number of drawbacks with this approach. It is necessary to assume that the scalar field is a monotonically varying function of t. Consequently, this approach is not suitable for discussing the physics of the reheating phase. Secondly, the inflaton has been treated as a classical quantity. An alternative derivation of eq. (2.5a) is found by multiplying the scalar field equation (2.3) by $\dot{\phi}$ and substituting for the energy density of the field $\rho \equiv \dot{\phi}^2/2 + V(\phi)$ to obtain

$$\dot{\rho} + 3H\dot{\phi}^2 = 0. \tag{6.1}$$

Hence $\rho' = -3H\dot{\phi}$ and this reduces to eq. (2.5a) when $3H^2(\phi) = \kappa^2\rho$ [13]. However, in the stochastic treatment of inflation, the full quantum field Φ is split into long- and short–wavelength components, as defined by the inverse Hubble scale, by writing $\Phi = \phi + q$ [39]. The field equation for Φ reduces to an effective field equation for the coarse–grained field given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\Phi = \phi)}{\partial \phi} = f(\mathbf{x}, t), \tag{6.2}$$

where the quantity $f(\mathbf{x}, t)$ represents the continuous inflow of short–wavelength modes onto ϕ. In practise, ϕ is treated as a classical object that evolves randomly in the presence of the quantum term f. Clearly, the existence of f implies that eq. (6.1) and hence eq. (2.5a) are not valid.

Finally, the exact process whereby inflation ends in the example of sect. (5) has not been discussed in detail. One must argue that the complete potential only approximates to the form necessary for a sufficient period of inflation. After this is achieved the field evolves to a different part of the potential where there may be a stable minimum and reheating can proceed.

Amplitudes which decrease with scale are also possible in super–inflationary scenarios ($\dot{H} > 0$), which arise in some higher–order gravity theories with Lagrangian $f(R)$ [4,41]. It is well known that these theories can be expressed as general relativity plus a scalar field by means of a suitable conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ [42]. The potential of the field, and hence $H(\phi)$, are uniquely specified by $f(R)$. Furthermore, it is easy to express the observational constraints in the $g_{\mu\nu}$ frame in terms of quantities defined in the conformal picture $\tilde{g}_{\mu\nu}$ [43]. Hence the constraints on $H(\phi)$ presented here can be used in principle to constrain these higher–order theories.

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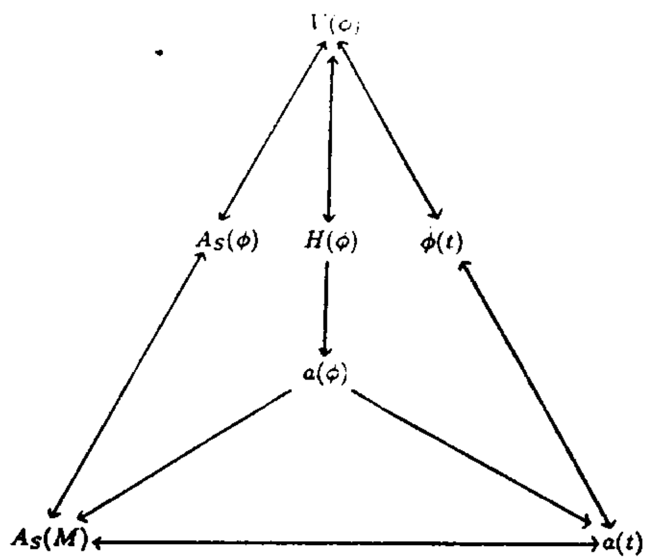
Figure Captions

Figure 1: A triangle of approaches illustrating how the three main quantities of interest in inflation, $\{a(t), A_S(M), V(\phi)\}$, are related and can be obtained from each other once the functional form of one of them has been specified. They are all linked by the functional form of $H(\phi)$. Hence it is natural to view the functional form of $H(\phi)$ as the fundamental quantity in the analysis.

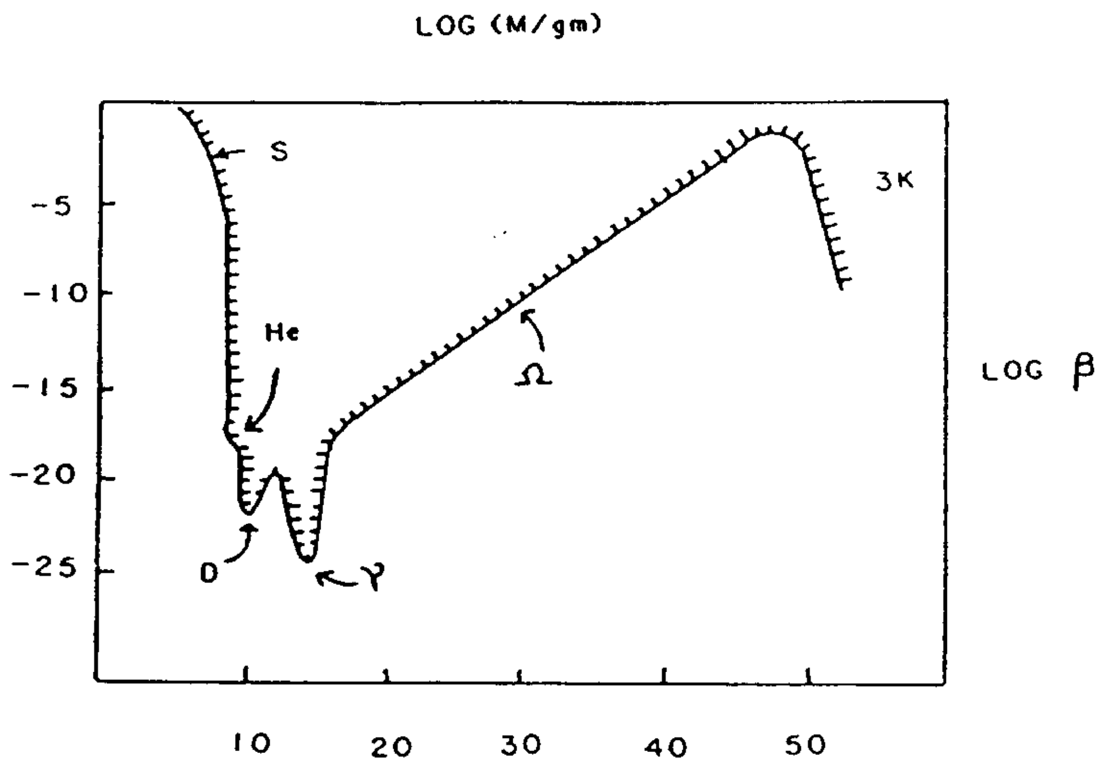
Figure 2: Showing the constraints on $\beta(M)$, the fraction of the universe going into PBHs of mass M, which can be inferred from measurements of the total density parameter (Ω), the gamma-ray background (γ), the primordial helium (He) and deuterium (D) abundances and the photon-to-baryon ratio (S). β must be tiny over most mass ranges.

Figure 3: Showing the constraints on $A_S(M) \equiv \delta_H(M)$ which can be inferred from the constraints on $\beta(M)$ shown in Fig. (2), limits on the spectral distortion in the microwave background and *COBE* measurements of the microwave background anisotropies. If the fluctuations have a power law form, then *COBE* allows the spectrum to lie anywhere in the region bounded by the solid lines, which represent errors at the 1-sigma level. This just includes the dashed line, which corresponds to the maximum slope compatible with the PBH constraints. The dot-dashed line represents the 2-sigma upper limit on the spectral index derived by combining the IRAS/QDOT and POTENT galaxy surveys with *COBE* (see Liddle and Lyth in ref. [7]). The allowed region lies below this line and the possibility of significant PBH production is clearly ruled out if the spectral index is independent of mass-scale.



$As(\phi)$
$H(\phi)$
$\phi(t)$
$a(\phi)$
$As(M)$
$a(t)$

Fig 1.



LOG (M/gm)
S
3K
-5
-10
He
-15
-20
-25
D
γ
Ω
LOG β
10
20
30
40
50

Fig 2

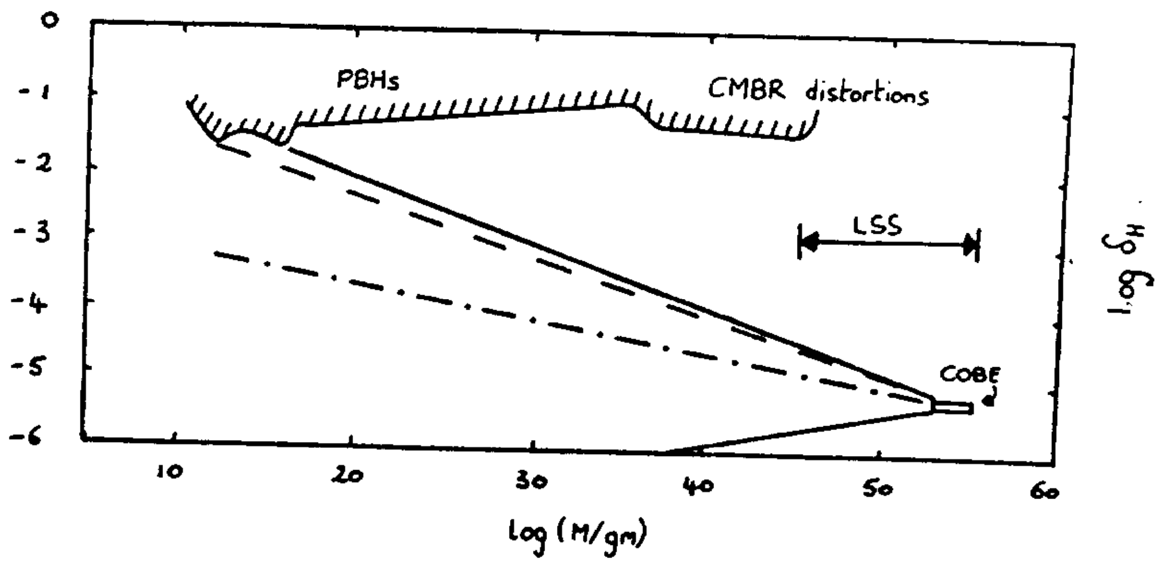


Figure 3.