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A Window to Extra Dimensions Near a Black Hole*

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Abstract

A version of the Rubakov-Shaposhnikov high dimensional model is implemented. Particles are assumed to be confined to a four-dimensional submanifold of a high dimensional space via a potential barrier based on the four-dimensional gravitational field strength. The Schwarzschild black-hole has a window into the higher dimensional space, located outside event.

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1 Introduction

An interesting high dimensional model was introduced a few years ago by Rubakov and Shaposhnikov (RS for short) [1]. Contrarily to the usual assumption in Kaluza-Klein theory, the extra dimensional subspace is not necessarily compact. Particles are trapped in a four dimensional subspace by the action of a potential V. Still, it is admitted that by a high energy process, such as a particle collision, some particles may escape the space-time domain, much like the classic escape velocity problem in mechanics, except by the fact that now the particles would escape the space-time itself. Dimensional reduction results from the breaking of the translational symmetry along the extra dimensions. If the particles acquire sufficiently high energy to break that potential barrier, then the extra dimensions would become visible so to speak, through a 3-surface or "window" in space-time.

As we see, the RS model brings some new ideas into high dimensional theories, but it also opens a number of new problems. To start with, its geometrical setup is unclear and the nature of the potential V should be better understood. In order to make sense, the conservation laws need to be adapted to the proposed high dimensional physics. Finally, there are some topological implications. Would the escaping particles belong to another four dimensional space, connected to the original space-time, or would this process lead to a change in topology?

Several different configurations and applications of the RS model have been proposed[2, 3, 4]. The so called space-time membrane models[5, 6] have some geometrical similarities with the RS model, but their principles are different.

In this note we implement the geometry of the RS model, in terms of the geometry of submanifolds and study the model near a Schwarzschild black hole. We assume that in this situation the dominant contribution to the potential V has gravitational nature, so that it may be described by the gravitational field strength. Using a geometric model for V, we obtain a generalization of the De Broglie energy relation. Particles in the vicinity of a black hole may gain sufficient gravitational energy to overcome that potential barrier, thus escaping from the space time bounds. This process takes place at a spherical 3-surface, located before the horizon, opening a "window" to the extra dimensions[7, 8]. Finally, the region of the high dimensional space, occupied by the escaping particles is described as an extension of the original space-time metric, resembling what would be a classical 4-dimensional wormhole.

Next section gives the conditions for the particles to escape a potential V. Section 3 describes a particular expression of V in terms of the gravitational field strength. In section 4 we examine the situation near a Schwarzschild black hole and describe the geometry of resulting the space-time extension, generated by the escaping particles.

2 Particle Dynamics in Higher Dimensions

From its description, it follows that the natural geometrical setup for the RS model is that of submanifolds of a high dimensional flat space. Therefore, it has a relation with some Kaluza-Klein like models, where the extra dimensions are not necessarily compact.[9, 10, 11].

Consider a 4-dimensional space-time S_4 with metric $\bar{g}_{ij}$, as a locally embedded submanifold of a D-dimensional flat space[1] $M_D, \quad D > 4$. For a particle travelling in M_D with momentum P^μ, its mass is given by the second order Casimir operator of the Poincaré group of that space: $\mu^2 = \eta_{\mu\nu}P^\mu P^\nu$, where $\eta_{\mu\nu}$ denotes the metric of M_D in Cartesian coordinates. Since P^μ has components which are tangent and normal to the space-time S_4, μ^2 may be decomposed as:

$$\mu^2 = m^2 + P_D^2 \tag{1}$$

[1] Unless stated on the contrary, small case Latin indices run from 0 to 3 and capital Latin indices run from 4 to D. All Greek indices run from 0 to D.

where we have denoted by m^2 the operator composed exclusively by the components of P^μ that are tangent to S_4 and P_D^2 contains all remaining terms.

A four-dimensional Riemannian observer can see only the part of (1) restricted to space-time. Denoting $P_D^2\rfloor_{S_4} = \bar{P}_D^2$, $\mu^2\rfloor_{S_4} = \bar{\mu}^2$, for this observer, equation (1) looks like

$$\bar{P}_D^2 = \bar{\mu}^2 - \bar{m}^2. \tag{2}$$

where $\bar{m}$ denotes the mass of the particle as measured by that observer. Using the space-time signature $(+,-,-,-)$, the mass of a particle in curved space-time is [12]

$$\bar{m} = \bar{g}_{ij}\bar{P}^i\bar{P}^j = \bar{g}_{00}(\bar{P}^0)^2 - 2\bar{g}_{0a}\bar{P}^0\bar{P}^a - \bar{g}_{ab}\bar{P}^a\bar{P}^b, \quad a,b = 1..3 \tag{3}$$

If E denotes the total energy of the particle trapped in the space-time by the potential V, then the energy momentum relation for that particle is

$$E - V = \bar{P}^0.$$

Therefore the total non negative energy of a particle initially trapped in space-time is given by

$$E = V + \frac{1}{\sqrt{g_{00}}}\sqrt{\bar{m}^2 + 2\bar{g}_{0a}P^0P^a + \bar{g}_{ab}\bar{P}^a\bar{P}^b}, \quad a,b = 1..3. \tag{4}$$

As stated before, the permanence of the particles in space-time is not necessarily stable, because they may get sufficient energy to escape to the extra dimensions. The threshold condition for this to happen is $E = 0$.

3 Gravitational field strength

In this section we look for an example of the potential V which depends only on the strength of the gravitational field. This quantity may be described as a scalar function of the observable curvature of the space-time, as for example

$$1/\inf|R_{ijkl}| \tag{5}$$

where R_{ijkl} is evaluated in a tetrad basis (see e.g. [13]). Another possible description of the strength of the gravitational field is given by the square root of the above expression, which, as we will see may be associated to a length in the high dimensional space.

Let X^μ denote the Cartesian coordinates of a space-time point in M_D. The equations defining the isometric embedding of S_4 in M_D are:

$$X^\mu_{,i}X^\nu_{,j}\eta_{\mu\nu} = \bar{g}_{ij}, \;\; N^\mu_A X^\nu_{,i}\eta_{\mu\nu} = 0, \;\; N^\mu_A N^\nu_B \eta_{\mu\nu} = g_{AB}, \tag{6}$$

where N_A denote D-4 independent vector fields orthogonal to S_4; $\eta_{\mu\nu}$ are the Cartesian metric components in M_D and $g_{AB} = \epsilon_A \delta_{AB}, \epsilon_A = \pm 1$.

The integrability conditions for (6) are the well know Gauss, Codazzi, Ricci equations of differential geometry, relating the metric to the second fundamental form and the "twisting" vector of S_4, respectively given by:

$$b_{ijA} = -\eta_{\mu\nu}X^\mu{}_{,i}N^\nu_{A,j}, \;\; A_{iAB} = \eta_{\mu\nu}N^\mu_A N^\nu_{B,i}.$$

For our present purposes, it is sufficient to write Gauss' equations

$$\bar{R}_{ijkl} = 2g^{AB}b_{k[iA}b_{j]lB}. \tag{7}$$

The cotangent vector δx^i to a principal line of curvature of the space-time S_4, corresponding to a normal N_A, satisfies the equation [14]

$$(\bar{g}_{ij} - x^A b_{ijA})\delta x^i = 0 \qquad \text{no sum on A.} \tag{8}$$

Therefore x^A must be a solution of

$$det(\bar{g}_{ij} - x^A b_{ijA}) = 0, \quad \text{no sum on A.} \tag{9}$$

For each fixed value of A, there are at most four solutions denoted by $x^A = \rho^A_i$, corresponding to (at most) four independent principal directions δx_i. With these directions we may define a principal tetrad basis (clearly, when the principal directions are not independent, the basis must be complemented [14]).

Consider two principal directions δx^k and δx^l. Writing Gauss' equations, in the basis containing these principal directions, it follows that:

$$R_{ijkl}\, \delta x^k \wedge \delta x^l = 2g^{AB} b_{i[kA} b_{l]jB}\, \delta x^k \wedge \delta x^l.$$

Using the definition (8) and $g^{AB} = 1/g_{AB} = \pm 1$, we obtain

$$R_{ijkl}\, \delta x^k \wedge \delta x^l = \sum_{k,l} \frac{\bar{g}_{ijkl}}{g_{AB}\rho_k^A \rho_l^B}\, \delta x^k \wedge \delta x^l, \tag{10}$$

where we have denoted $\bar{g}_{ijkl} = 2\bar{g}_{i[k}\bar{g}_{l]j}$. Therefore, it follows that for each pair of principal directions the scalar

$$\rho = \frac{1}{\sqrt{\mathrm{Inf}|\frac{\bar{g}_{ijkl}}{g_{AB}\rho_k^A \rho_l^B}|}}, \tag{11}$$

corresponds to a measure of the gravitational field strength, namely the square root of (5), expressed as a length in the high dimensional space. If gravitation is weak, then ρ is large. Strong gravitational fields are associated with small values of ρ.

The RS model does not explain the nature of V. Obviously, gravitation alone cannot explain why particles are confined to four dimensional space-time. If this was the case, then in a weak gravitational field the particles would be free to move around the extra dimensions, rendering these dimensions observable at low energies. Therefore the full expression of the potential V is possibly very complex.

Nonetheless, in the case of a strong gravitational field one may perhaps plausibly assume, that the potential V becomes completely dominated by that field as the predominant interaction (likely to occur at the quantum level). In this case, V may be represented in terms of a scalar function of the gravitational field strength (11).

Using an analogy with the Newtonian potential [15] $V(\rho)$ may be taken to be proportional to $1/\rho$. This is of course a purely intuitive assumption, based on earlier conjectures on the escape problem in a Schwarzschild space-time[16]. The proportionality constant α may be adjusted so that Planck's constant is evidenced:

$$V(\rho) = -\alpha\frac{\hbar}{\rho} \tag{12}$$

4 The Window Near a Black Hole

At the quantum level, we replace the term $\bar{g}_{ab}\bar{P}^a\bar{P}^b$ in (4) by the corresponding expression $(\hbar/\lambda)^2$. Therefore, for a spherically symmetric strong gravitational field, the expression corresponding to 4, using (12) is

$$E = -\alpha\frac{\hbar}{\rho} + \frac{1}{\sqrt{g_{00}}}\sqrt{\bar{m}^2 + (\frac{\hbar}{\lambda})^2}$$

which suggests a generalization of DeBroglie's energy relation for a particle in a spherically symmetric strong gravitational field. In the case of a massless particle we obtain a correction to Planck's energy relation

$$E = -\alpha\frac{\hbar}{\rho} + \frac{\hbar}{\lambda\sqrt{\bar{g}_{00}}}. \tag{13}$$

If $\rho >> \alpha\,\lambda\sqrt{\bar{g}_{00}}$, the first term in (13) becomes negligible and we recover the usual energy relation (with a classical red shift correction to λ).

On the other hand, when ρ is of the order of $\alpha\lambda\sqrt{g_{00}}$, then the first term in (13) cannot be neglected. This suggests that at this scale the quantum behavior of gravitation becomes significant, modifying Planck's relation. At this stage, gravitation contributes positively to the total energy E, producing the necessary energy for the particle to escape to the extra dimensions.

In the case of a Schwarzschild black hole, we have [16]: $\bar{g}_{0a} = 0$, $\sqrt{\bar{g}_{00}} = \sqrt{1 - \frac{2M}{r}}$ and $\rho = \sqrt{\frac{r^3}{2M}}$, so that (13) becomes

$$E = -\alpha\frac{\hbar}{\sqrt{\frac{r^3}{2M}}} + \frac{\hbar}{\lambda\sqrt{1 - \frac{2M}{r}}}. \tag{14}$$

The threshold condition for a particle to escape the black hole with the potential (12) is then

$$f(r) = r^4 - Kr + 2MK = 0, \quad K = 2M\lambda^2\alpha^2, \tag{15}$$

whose real solutions define two spherical surfaces, the "windows", through which the extra dimensions would become visible. The condition for (15) to admit at least two real roots, is that there is a value r_0, such that $\frac{\partial f}{\partial r}\rfloor_{r=r_0} = 0$ and $f(r_0) \leq 0$. This gives a lower bound for the constant α:

$$\alpha \geq \frac{2^4 2M}{3^{3/2}\lambda}, \tag{16}$$

where the equal sign corresponds to the bottom of the potential well r_0.

Equation (15) may be written as

$$r - 2M = \frac{r^4}{2M\lambda^2\alpha^2}. \tag{17}$$

showing that the real roots of (15) are located outside the horizon $r = 2M$.

We may also write (15) as

$$M_{\pm}(r) = \frac{r}{4}[1 \pm \sqrt{1 - \frac{4r^2}{\lambda^2\alpha^2}}]$$

Since M is real and positive, it follows that the roots belong to a finite interval, with an upper bound value given by $\alpha\lambda/2$.

The smaller root r_1 of (15) represents the threshold point for a high energy particle to escape the space-time bound. On the other hand, since we have also $E > 0$ for $r > r_2$, the particles passing through r_2 toward the singularity actually lose energy, until reaching the bottom of the well r_0, where gravitation starts contributing to the escape at r_1. Therefore, only the 3-surface $r = r_1$ in fact corresponds to a window.

Finally, replacing (16) with the equal sign in (17), it follows that the bottom of the potential barrier is located at the point where the gravitational field starts contributing to the energy of the particle:[2]

$$r_0 = \frac{4}{3}2M = 2.66\,M.$$

As an example, consider **Cygnus X-1** as a Schwarzschild black hole with mass $2M \approx 5.02\ 10^5 cm$ [18]. The center of the potential barrier is located at $r_0 = 6.69\,10^5$. For a particle with typical Compton wavelength $\lambda \approx 10^{-13} cm$ and for $\hbar = 2.612\ 10^{-66} cm^2$, we obtain from (16) that $\alpha \geq 3.108\ 10^{18}$. Taking the smallest value of α, we obtain the equation

$$r^4 - 4.849\,10^{18} r + 2.434\,10^{24} = 0,$$

with two real roots: $r_1 = 5.167\,10^5$ and $r_2 = 14.731\,10^5$.

The relative positioning of the roots depend strongly on the values of α. For example, if we chose a large value such as $\alpha = 10^{20}$, then we obtain $K = 0.502\,10^{20}$ and equation (15) becomes

$$r^4 - 0.502\,10^{20} r + 0,252\,10^{26} = 0,$$

whose real roots are now much wider apart: $r_1 = 5.033\,10^5$, $r_2 = 35.036\,10^5$ (fig. 1 shows the relative positions of the roots for the two values of α considered. For clarity the y scales have been slightly adjusted).

To end, we make a brief discussion on the classical geometry of the region of the high dimensional space occupied by the escaping particles, under the hypothesis of the potential (12) near a Schwarzschild black hole.

[2]this value differs slightly from an estimate using Casimir effect considerations [17]

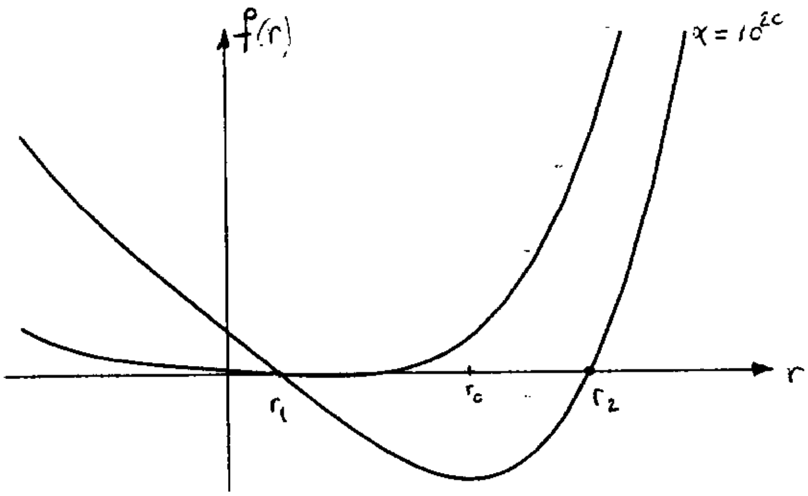

Fig.1: Potential barriers near **Cygnus X-1**

Since that window is located outside the event horizon, that is, within the analytic region of space-time, the flow of particles escaping to the extra dimensions through this surface, produce an analytic extension of the space-time into the high dimensional space. In this space the points of the classical trajectories may be described by [14]:

$$Z^{\mu}(x^i, x^A) = X^{\mu}(x^i) + x^A N^{\mu}_{A}(x^i), \tag{18}$$

where $X^{\mu}(x^i)$ are now the coordinates of the window $r = r_1$ and where x^A the $D-4$ extra coordinates measured along the normal vectors N_A. Expression (18) describes a family of four-dimensional analytic extensions of the space-time in M_D, where each member is defined by a set of values x^A = constant, contacting the original space-time at $r = r_1$, $x^A = 0$.

The geometry of these extensions can be easily calculated from the isometric embedding condition. Using (18):

$$g_{ij} = Z^{\mu}{}_{,i} Z^{\nu}{}_{,j} \eta_{\mu\nu} = \bar{g}^{mn}(\bar{g}_{im} - x^A b_{imA})(\bar{g}_{jn} - x^B b_{jnB}) + x^A x^B g^{MN} A_{iMA} A_{jNB}. \quad (19)$$

The possibility that the submanifold generated by the escaping particles becomes disconnected from the original space-time appears to be classically forbidden [19, 20]. In this respect, it is interesting to notice that the extension metric (19) may become singular at a particular region of the high dimensional space where $det(\bar{g}_{im} - x^A b_{imA}) = 0$ and when A_{ijA} is subjected to the condition $x^A x^B g^{MN} A_{iMA} A_{jNB} \approx 0$. This could in principle induce a topological change in the sense of loss of connectedness.

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